OPEN SETS OF MAXIMAL DIMENSION IN COMPLEX HYPERBOLIC QUASI-FUCHSIAN SPACE

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ABSTRACT. Let π_1 be the fundamental group of a closed surface Σ of genus $g > 1$. One of the fundamental problems in complex hyperbolic geometry is to find all discrete, faithful, geometrically finite and purely loxodromic representations of π_1 into $\mathrm{SU}(2,1)$, (the triple cover of) the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. In particular, given a discrete, faithful, geometrically finite and purely loxodromic representation ρ_0 of π_1, can we find an open neighbourhood of ρ_0 comprising representations with these properties. We show that this is indeed the case when ρ_0 preserves a totally real Lagrangian plane.

1. INTRODUCTION

Let Σ be a closed surface of genus $g > 1$ and let $\pi_1 = \pi_1(\Sigma)$ denoted its fundamental group. A specific choice of generators for π_1 is called a *marking*. The collection of marked representations of π_1 into a Lie group G up to conjugation will be denote $\mathrm{Hom}(\pi_1, G)/G$. We give $\mathrm{Hom}(\pi_1, G)/G$ the compact-open topology. This enables us to make sense of what it means for two representations to be close. In the cases we consider, the compact-open topology is equivalent to the l^2-topology on the relevant matrix group. Our main interest in this paper will be the case where $G = \mathrm{SU}(2,1)$ but, before we consider this case, we motivate our discussion by reviewing the better known cases when G is $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$.

Suppose that $\rho : \pi_1 \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is a discrete and faithful representation of π_1. Then $\rho(\pi_1)$ is called *Fuchsian*. Also, $\rho(\pi_1)$ is necessarily geometrically finite and totally loxodromic (if Σ had punctures then this condition would be replaced with type-preserving, which requires that an element of $\rho(\pi_1)$ is parabolic if and only if it represents a peripheral curve). The group $\mathrm{SL}(2,\mathbb{R})$ is a double cover of the group of orientation preserving isometries of the hyperbolic plane. The quotient of the hyperbolic plane by $\rho(\pi_1)$ naturally corresponds to a hyperbolic structure on Σ. The collection of distinct, marked Fuchsian representations, up to conjugacy within $\mathrm{SL}(2,\mathbb{R})$, is the *Teichmüller space* of Σ, denoted $\mathcal{T} = \mathcal{T}(\Sigma) \subset \mathrm{Hom}\big(\pi_1,\, \mathrm{SL}(2,\mathbb{R})\big)/\mathrm{SL}(2,\mathbb{R})$. This has been studied extensively and is known to be a ball of real dimension $6g - 6$. It also has a structure of a complex Banach manifold and is equipped with a Kähler metric (the well known Weil-Petersson metric) of negative holomorphic sectional curvature.

Instead of considering representations of π_1 into $\mathrm{SL}(2,\mathbb{R})$, we may consider representations to $\mathrm{SL}(2,\mathbb{C})$. If such a representation ρ is discrete, faithful, geometrically finite and totally loxodromic then $\rho(\pi_1)$ is *quasi-Fuchsian* (again in the presence of punctures purely loxodromic should be replaced with type-preserving). The collection of distinct, marked quasi-Fuchsian representations, up to conjugation in $\mathrm{SL}(2,\mathbb{C})$ is called *quasi-Fuchsian space* $\mathcal{Q} = \mathcal{Q}(\Sigma) \subset \mathrm{Hom}\big(\pi_1,\, \mathrm{SL}(2,\mathbb{C})\big)/\mathrm{SL}(2,\mathbb{C})$. A quasi-Fuchsian representation corresponds to a three dimensional hyperbolic structure on an interval bundle over Σ. According to a celebrated theorem of Bers [1], $\mathcal{Q}$ may be identified with the product of two copies of Teichmüller space, and so has dimension $12g - 12$. Furthermore, $\mathcal{Q}$ has a

1991 *Mathematics Subject Classification.* 32G05, 32M05.

IDP was supported by a Marie Curie Intra-European fellowship (Contract No.MEIF-CT-2003-500074) within the 6th Community Framework Programme.

rich geometrical and analytic structure. It is a complex manifold of dimension $6g-6$ and it is endowed with a hyper-Kähler metric whose induced complex symplectic form is the complexification of the Weil-Petersson metric on $\mathcal{T}$.

Motivated by these two examples, one may consider representations of π_1 into $\mathrm{SU}(2,1)$ up to conjugation, that is $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. A representation in $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$ is said to be *complex hyperbolic quasi-Fuchsian* if it is discrete, faithful, geometrically finite and totally loxodromic (for surfaces with punctures the last condition should be type-preserving, see [14]). The group $\mathrm{SU}(2,1)$ is a triple cover of the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$. Thus such a representation corresponds to a complex hyperbolic structure on a disc bundle over Σ.

Bowditch has discussed notions of geometrical finiteness for variable negative curvature in [2]. In particular, if Γ is a discrete subgroup of $\mathrm{SU}(2,1)$ and $\Omega \subset \partial\mathbf{H}^2_{\mathbb{C}}$ is the domain of discontinuity of Γ then consider the orbifold $M_C(\Gamma) = \big(\mathbf{H}^2_{\mathbb{C}} \cup \Omega\big)/\Gamma$. Bowditch defines Γ to have property F1, that is Γ is *geometrically finite in the first sense*, if $M_C(\Gamma)$ has only finitely many topological ends, each of which is a parabolic end. In our context, Γ will be totally loxodromic and so will have property F1 provided $M_C(\Gamma)$ is a closed manifold.

The space of all marked complex hyperbolic quasi-Fuchsian representations, up to conjugacy, will be called *complex hyperbolic quasi-Fuchsian space* $\mathcal{Q}_{\mathbb{C}} = \mathcal{Q}_{\mathbb{C}}(\Sigma) \subset \mathrm{Hom}\big(\pi_1,\, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. Compared to Teichmüller space and quasi-Fuchsian space, relatively little is known about complex hyperbolic quasi-Fuchsian space $\mathcal{Q}_{\mathbb{C}}$.

There are two ways to make a Fuchsian representation act on $\mathbf{H}^2_{\mathbb{C}}$. These correspond to the two types of totally geodesic, isometric embeddings of the hyperbolic plane into $\mathbf{H}^2_{\mathbb{C}}$. Namely, totally real Lagrangian planes, which may be thought of as copies of $\mathbf{H}^2_{\mathbb{R}}$, and complex lines, which may be thought of as copies of $\mathbf{H}^1_{\mathbb{C}}$. If a discrete, faithful representation ρ is conjugate to a representation $\rho : \pi_1 \longrightarrow \mathrm{SO}(2,1) < \mathrm{SU}(2,1)$ then it preserves a Lagrangian plane and is called *$\mathbb{R}$-Fuchsian*. If a discrete, faithful representation ρ is conjugate to a representation $\rho : \pi_1 \longrightarrow \mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(1,1)\big) < \mathrm{SU}(2,1)$ then it preserves a complex line and is called *$\mathbb{C}$-Fuchsian*. There is an important invariant of a representation $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ called the *Toledo invariant* denoted $\tau(\rho)$. The main properties of the Toledo invariant are

(i) τ varies continuously with ρ,
(ii) $2-2g \le \tau(\rho) \le 2g-2$, see [3],
(iii) $\tau(\rho) \in 2\mathbb{Z}$, see [12],
(iv) ρ is $\mathbb{C}$-Fuchsian if and only if $|\tau(\rho)| = 2g-2$, see [17],
(v) if ρ is $\mathbb{R}$-Fuchsian then $\tau(\rho) = 0$, see [12].

Further properties of complex hyperbolic representations of surface groups which refer to the Toledo invariant are

(vi) for each even integer t with $2-2g \le t \le 2g-2$ there exists a discrete, faithful representation ρ of π_1 with $\tau(\rho) = t$, see [12],
(vii) if $\tau(\rho_1) = \tau(\rho_2)$ then ρ_1 and ρ_2 lie in the same component of $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$, see [19].

We remark that in the case where Σ has cusps then, in fact, $\tau(\rho)$ is a real number in the interval $\big[\chi(\Sigma),\, -\chi(\Sigma)\big]$ and for any real number t in this interval there exists a discrete, faithful representation ρ of $\pi_1(\Sigma)$ with $\tau(\rho) = t$, see [14]. Moreover, Dutenhefner and Gusevskii [4] have constructed an example of a discrete, faithful, type-preserving representation of the fundamental group of a particular punctured surface whose limit set is a wild knot. This means that it cannot be in the same component of the space of discrete faithful representations as a Fuchsian representation. It may well be possible to extend this example to the case of closed surfaces, which would lead to questions about the number of components of complex hyperbolic quasi-Fuchsian space (Xia's result [19], given in (vii) above, does not involve discreteness).

An immediate consequence of (i) and (iii) is that τ is locally constant and, together with (iv), implies that given a $\mathbb{C}$-Fuchsian representation ρ_0 any nearby representation ρ_t is also $\mathbb{C}$-Fuchsian. This result is known as the Toledo-Goldman rigidity theorem [17], [10]. In fact, the component of $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$ with $|\tau| = 2g-2$ has dimension $8g-6$ and the other components have dimension $16g-16$ (see Theorem 6 of [10]).

In [15] we begin with any $\mathbb{R}$-Fuchsian representation ρ_0 and we consider nearby representations ρ_t in $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. The main result of [15] is

Theorem 1.1. *Let Σ be a closed surface of genus g with fundamental group $\pi_1 = \pi_1(\Sigma)$. Let $\rho_0 : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ be an $\mathbb{R}$-Fuchsian representation of π_1. Then there exists an open neighbourhood $U = U(\rho_0)$ of ρ_0 in $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$ so that any representation ρ_t in U is complex hyperbolic quasi-Fuchsian (that is discrete, faithful, geometrically finite and totally loxodromic).*

Corollary 1.2. *There are open sets of dimension $16g-16$ in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.*

Up to now, families of complex hyperbolic quasi-Fuchsian groups have only been constructed by varying a particular geometrical construction, see for example [13], [14], [6], [7], [8], [16]. By contrast, in this paper we only use the hypothesis that ρ_t and ρ_0 are nearby representations. From this information we must make a geometrical construction of a fundamental domain. To go from algebra to geometry (and back again) we use the following theorem of Falbel and Zocca [9].

Theorem 1.3. *Any element C of $\mathrm{SU}(2,1)$ may be written as $C = \iota_1 \circ \iota_0$ where ι_0 and ι_1 are involutions fixing Lagrangian planes R_0 and R_1 respectively. Moreover*

(i) *$C = \iota_1 \circ \iota_0$ is loxodromic if and only if R_0 and R_1 are disjoint;*
(ii) *$C = \iota_1 \circ \iota_0$ is parabolic if and only if R_0 and R_1 intersect in exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;*
(iii) *$C = \iota_1 \circ \iota_0$ is elliptic if and only if R_0 and R_1 intersect in at least one point of $\mathbf{H}^2_{\mathbb{C}}$.*

We prove Theorem 1.1 by first constructing a fundamental domain $\mathbf{\Delta}_0$ in $\mathbf{H}^2_{\mathbb{C}}$ for $\rho_0(\pi_1)$ and then showing that for any other representation ρ_t sufficiently close to ρ_0 we may construct a fundamental domain $\mathbf{\Delta}_t$ for $\rho_t(\pi_1)$. By sufficiently close, we mean that there exists an $\epsilon > 0$ so that the generators of $\rho_t(\pi_1)$ are ϵ-close to the generators of $\rho_0(\pi_1)$ in the l^2-topology on $\mathrm{SU}(2,1)$.

Constructing fundamental domains in complex hyperbolic space is challenging because, unlike the case of constant curvature, there are no totally geodesic real hypersurfaces. Thus, before constructing a fundamental polyhedron we must choose the class of real hypersurfaces containing its faces. The most usual method of constructing a fundamental domain in complex hyperbolic space involves domains whose boundary is made up of pieces of bisectors. In particular, this is the case for the construction of Dirichlet domains. This idea goes back to Giraud and was developed further by Mostow and Goldman (see [11] and the references therein), and see [13], [14] for other examples of fundamental domains bounded by bisectors. Other classes of hypersurfaces used to build fundamental domains are $\mathbb{C}$-spheres [9] and $\mathbb{R}$-spheres [16] (for the relationship between $\mathbb{C}$-spheres and $\mathbb{R}$-spheres see [8]).

Since bisectors are rather badly adapted to $\mathbb{R}$-Fuchsian representations, we have chosen to introduce a new class of hypersurfaces. Just as bisectors are foliated by slices that are complex lines so our hypersurfaces are foliated by Lagrangian planes. These hypersurfaces resemble a pack of (infinitely many) playing cards, each Lagrangian plane representing a card. Therefore we call we call such hypersurfaces *packs*. Explicitely, we have

Proposition 1.4. *Let R_0 and R_1 be disjoint Lagrangian planes in $\mathbf{H}^2_{\mathbb{C}}$ and let ι_0 and ι_1 be the respective inversions. Consider $C = \iota_1\iota_0$ (which is loxodromic map by Theorem 1.3) and its powers C^x for each $x \in \mathbb{R}$. Then:*

(i) *ι_x defined by $C^x = \iota_x\iota_0$ is inversion in a Lagrangian plane $R_x = C^{x/2}(R_0)$.*
(ii) *R_x intersects the complex axis L_C of C orthogonally in a geodesic γ_x.*
(iii) *The geodesics γ_x are the leaves of a foliation of L_C.*
(iv) *For each $x \neq y \in \mathbb{R}$, R_x and R_y are disjoint.*

Definition 1.5. Given disjoint Lagrangian planes R_0 and R_1, then for each $x \in \mathbb{R}$ let R_x be the Lagrangian plane constructed in Proposition 1.4. Define

$$P = P(R_0, R_1) = \bigcup_{x\in\mathbb{R}} R_x.$$

Then P is a real analytic 3-submanifold which we call the *pack* determined by R_0 and R_1. We call $\gamma = \mathrm{Ax}(\iota_1\iota_0)$ the *spine* of P and the Lagrangian planes R_x for $x \in \mathbb{R}$ the *slices* of P.

The boundaries of packs are foliated by $\mathbb{R}$-circles and so are closely related to Schwartz' $\mathbb{R}$-spheres [16] and examples of packs (with no twist) were introduced by Will [18], who calls them $\mathbb{R}$-balls. Both Schwartz and Will use these objects to construct fundamental domains. The relationship between bisectors and packs is an example of the duality, which resembles mirror symmetry, between complex and real objects in complex hyperbolic space, see the discussion in the introduction to [8]. The polyhedra $\mathbf{\Delta}_0$ and $\mathbf{\Delta}_t$ we construct have boundaries that are made up of pieces of packs. In order to show that $\rho_t(\pi_1)$ is complex hyperbolic quasi-Fuchsian we use a version of Poincaré's polyhedron theorem for such polyhedra (this should be compared with [5]).

2. Sketch of the proof of the main theorem

2.1. **A fundamental polyhedron for an $\mathbb{R}$-Fuchsian group.** Let Σ be a closed surface of genus $g > 1$ and let ρ_0 be any $\mathbb{R}$-Fuchsian representation of π_1, the fundamental group of Σ. We denote the image of ρ_0 by $\Gamma_0 = \rho_0(\pi_1) < \mathrm{SU}(2,1)$. Without loss of generality, we suppose that Γ_0 preserves $R_{\mathbb{R}}$ and so $\Gamma_0 < \mathrm{SO}(2,1)$. Consider the action of Γ_0 on $R_{\mathbb{R}}$ and let Δ_0 be a fundamental hyperbolic polygon for this action with $4g$ sides $s^{(1)}, \ldots, s^{(4g)}$. Let $v^{(1)}, \ldots, v^{(4g)}$ denote the vertices of Δ_0. We adopt the convention that $s^{(k)}$ has endpoints $v^{(k)}$ and $v^{(k+1)}$ and superscripts are taken mod $4g$. Conjugating if necessary, we suppose that $v^{(1)}$ is the origin o. By construction, there are $4g$ elements of Γ_0, denoted $A_0^{(1)}, \ldots, A_0^{(4g)}$ that pair the sides of Δ according to the following rules:

(i) For $j = 0, \ldots, g-1$ the map $A_0^{(4j+1)}$ sends the side $s^{(4j+1)}$ to the side $s^{(4j+3)}$ and the map $A_0^{(4j+2)}$ sends the side $s^{(4j+2)}$ to the side $s^{(4j+4)}$. Thus $A_0^{(4j+1)} = \big(A_0^{(4j+3)}\big)^{-1}$ and $A_0^{(4j+2)} = \big(A_0^{(4j+4)}\big)^{-1}$.
(ii) There are no reflection relations and only one cycle relation:

$$\prod_{j=0}^{g-1} A_0^{(4j+2)}\left(A_0^{(4j+1)}\right)^{-1}\left(A_0^{(4j+2)}\right)^{-1}A_0^{(4j+1)} = I. \tag{2.1}$$

For this polygon, it is straightforward to verify that side conditions analogous to (S.1) to (S.6) are satisfied. In this case, each codimension 2 face is a point, namely one of $v^{(1)}, \ldots, v^{(4g)}$. This condition replaces (E.1). With this change, (E.2) is also satisfied. Thus we could have used the classical Poincaré polygon theorem to verify that Δ_0 is a fundamental domain for the action of Γ_0 on $R_{\mathbb{R}}$. Moreover, as (2.1) generates all relations in π_1 we see that ρ_0 is faithful. In particular, Γ_0

has no elliptic elements. Since there are no tangencies between faces of Δ_0 we also see that Γ_0 contains no parabolics. Hence it is totally loxodromic.
Let $\boldsymbol{\Delta}_0 = \Pi_{\mathbb{R}}^{-1}(\Delta_0)$ be the inverse image of the polygon Δ_0 under projection onto $R_{\mathbb{R}}$ (see Section 6.1.1 of [18] where Will constructs fundamental domains for $\mathbb{R}$-Fuchsian triangle groups and punctured torus groups in a similar way). We claim that $\boldsymbol{\Delta}_0$ satisfies the conditions (S.1) to (S.6), (E.1) and (E.2). Thus Poincaré's Theorem will imply that $\boldsymbol{\Delta}_0$ is a fundamental domain for the action of Γ_0 on $\mathbf{H}^2_{\mathbb{C}}$. We now show how to check the conditions. The edges of $\boldsymbol{\Delta}_0$ are the Lagrangian planes $R_0^{(k)} = \Pi_{\mathbb{R}}^{-1}(v^{(k)})$. In particular, $R_0^{(1)} = \Pi_{\mathbb{R}}^{-1}(o) = R_{\mathbb{J}}$. Thus condition (E.1) is satisfied. The sides of $\boldsymbol{\Delta}_0$ are $S_0^{(k)} = \Pi_{\mathbb{R}}^{-1}(v^{(k)})$ for $k = 0, \ldots, 4g$. These are each pieces of the pack $P_0^{(k)}$ determined by the Lagrangian planes $R_0^{(k)}$ and $R_0^{(k+1)}$.
It is easy to prove that $\Pi_{\mathbb{R}}$ commutes with any element of $\mathrm{SO}(2,1)$, and so for $j = 0, \ldots, g-1$ the map $A_0^{(4j+1)}$ sends the side $S_0^{(4j+1)}$ to the side $S_0^{(4j+3)}$ and the map $A_0^{(4j+2)}$ sends the side $S_0^{(4j+2)}$ to the side $S_0^{(4j+4)}$. Thus the side conditions (S.1) to (S.6) are automatically satisfied. The condition (E.2) is therefore satisfied: there is only one cycle of vertices and the cycle transformation is given by (2.1) with $m = 1$. Using a suitable version of Poincaré's theorem, we see that $\boldsymbol{\Delta}_0$ is indeed a fundamental domain for Γ_0.
By construction, for any $k = 1, \ldots, 4g$ the edge $R_0^{(k)}$ is the image of $R_0^{(1)} = R_{\mathbb{J}}$ under some fixed word in the generators $A_0^{(1)}, \ldots, A_0^{(4g)}$. In fact this word comprises the last n letters of the relation (2.1) for some n. We denote this word by $B_0^{(k)}$. For example $B_0^{(1)}$ is the identity, $B_0^{(4)} = A_0^{(1)}$ and $B_0^{(3)} = \big(A_0^{(2)}\big)^{-1} A_0^{(1)}$. There is a homotopy class of loops $\beta_k \in \pi_1$ so that $B_0^{(k)} = \rho_0(\beta_k)$. Clearly $B_0^{(k)}$ is loxodromic for each k. So there is a constant $K > 0$ so that $\mathrm{tr}(B_0^{(k)}) \geq 3 + K > 3$ for all k.

2.2. **The variation of the polyhedron.** Let $\Gamma_t = \rho_t(\pi_1) < \mathrm{SU}(2,1)$ be a point in the representation variety $\mathrm{Hom}\big((\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. We will only consider representations that are close to Γ_0. To make this notion precise, for $k = 2, \ldots, 4g$ let $B_t^{(k)} = \rho_t(\beta_k)$ (here $\beta_k \in \pi_1$ is the homotopy class of loops for which $\rho_0(\beta_k) = B_0^{(k)}$ as described above). Then, given $\epsilon = \epsilon(t) > 0$ the representation ρ_t is said to be *ϵ-close to ρ_0* if for each $k = 2, \ldots, 4g$ we have

$$\left\| B_t^{(k)} - B_0^{(k)} \right\| < \epsilon$$

measured using the l^2-norm on $\mathrm{SU}(2,1)$. In the same way, for $k = 1, \ldots, 4g$ let α_k be the homotopy class of loops in π_1 so that $A_0^{(k)} = \rho_0(\alpha_k)$. Then we define $A_t^{(k)} = \rho_t(\alpha_k)$.
Our goal will be to show that there exists an ϵ depending only on ρ_0 so that all representations ρ_t that are ϵ-close to ρ_0 are complex hyperbolic-quasi-Fuchsian. In order to achieve this goal we will construct a domain $\boldsymbol{\Delta}_t$ and by Poincaré's Theorem we may show that $\boldsymbol{\Delta}_t$ is a fundamental domain for $\Gamma_t = \rho_t(\pi_1)$. Moreover, this will also imply that ρ_t is faithful, and Γ_t is totally loxodromic and geometrically finite. In other words, Γ_t is complex hyperbolic quasi-Fuchsian. We begin by constructing the edges of $\boldsymbol{\Delta}_t$. Let $R_t^{(1)} = R_{\mathbb{J}}$, the totally imaginary Lagrangian plane. For $k = 2, \ldots, 4g$ we define $R_t^{(k)}$ to be the Lagrangian plane

$$R_t^{(k)} = B_t^{(k)}\big(R_t^{(1)}\big) = B_t^{(k)}(R_{\mathbb{J}}). \tag{2.2}$$

Theorem 2.1. *There exists $\epsilon_1 = \epsilon_1(\rho_0) > 0$ so that if $\epsilon < \epsilon_1$ then the Lagrangian planes $R_t^{(1)}, \ldots, R_t^{(4g)}$ are disjoint.*

Suppose that the disjoint Lagrangian planes $R_0^{(k)}$ and $R_0^{(k+1)}$ are edges of $\mathbf{\Delta}_0$ in the boundary of the side $S_0^{(k)}$. Then we define the corresponding side $S_t^{(k)}$ of $\mathbf{\Delta}_t$ as follows. From Theorem 2.1 we see that the Lagrangian planes $R_t^{(k)}$ and $R_t^{(k+1)}$ are disjoint, and so determine a pack $P_t^{(k)}$. Define the side $S_t^{(k)}$ to be that part of $P_t^{(k)}$ lying between $R_t^{(k)}$ and $R_t^{(k+1)}$. We emphasise that once we have defined the Lagrangian planes $R_t^{(k)}$, the construction of $S_t^{(k)}$ is canonical. Thus, since the side pairing maps match the edges $R_t^{(k)}$ they automatically match the sides $S_t^{(k)}$.

Theorem 2.2. *There exists $\epsilon_2 = \epsilon_2(\rho_0)$ with $0 < \epsilon_2 < \epsilon_1$ so that for all $\epsilon < \epsilon_2$ we have:*

(i) *the sides $S_t^{(1)}, \ldots, S_t^{(4g)}$ only intersect in the Lagrangian planes $R_t^{(1)}, \ldots, R_t^{(4g)}$;*
(ii) *the combinatorial pattern of this intersection is the same as that for the faces of $\mathbf{\Delta}_0$;*
(iii) *there is a $\lambda > 0$ so that disjoint sides of $\mathbf{\Delta}_t$ are at least a distance λ apart.*

It follows that $\mathbf{\Delta}_t$ satisfies the conditions of Poincaré's theorem, and so is a fundamental domain for Γ_t. Thus Γ_t is discrete, faithful, totally loxodromic and is geometrically finite. This has proved our main theorem.

References

[1] L. Bers; *Spaces of Kleinian groups.* Several Complex Variables I, Lecture Notes in Mathematics **155** (1970) 9–34.
[2] B.H. Bowditch; *Geometrical finiteness with variable negative curvature.* Duke Math. J. **77** (1995) 229–274.
[3] A. Domic & D. Toledo; *The Gromov norm of the Kähler class of symmetric domains.* Math. Ann. **276** (1987) 425–432.
[4] F. Dutenhefner & N. Gusevskii; *Complex hyperbolic Kleinian groups with limit set a wild knot.* Topology **43** (2004) 677–696.
[5] D.B.A. Epstein & C. Petronio; *An exposition of Poincaré's polyhedron theorem.* L'Enseignement Mathématique, **40** (1994) 113–170.
[6] E. Falbel & P.-V. Koseleff; *Flexibility of the ideal triangle groups in complex hyperbolic geometry.* Topology, **39** (2000) 1209–1223.
[7] E. Falbel & P.-V. Koseleff; *Rigidity and flexibility of triangle groups in complex hyperbolic geometry.* Topology, **41** (2002) 767–786.
[8] E. Falbel & J.R. Parker; *The moduli space of the modular group in complex hyperbolic geometry.* Invent. Math., **152** (2003) 57–88.
[9] E. Falbel & V. Zocca; *A Poincaré's polyhedron theorem for complex hyperbolic geometry.* J. reine angew. Math., **516** (1999) 133–158.
[10] W.M. Goldman; *Representations of fundamental groups of surfaces.* Geometry and Topology, ed. J. Alexander & J. Harer, Lecture Notes in Mathematics **1167** (1985) 95–117.
[11] W.M. Goldman; Complex Hyperbolic Geometry. Oxford University Press, 1999.
[12] W.M. Goldman, M.E. Kapovich & B. Leeb; *Complex hyperbolic manifolds homotopy equivalent to a Riemann surface.* Comm. Anal. Geom. **9** (2001) 61–95.
[13] W.M. Goldman & J.R. Parker; *Complex hyperbolic ideal triangle groups.* J. reine angew. Math., **425** (1992) 71–86.
[14] N. Gusevskii & J.R. Parker; *Complex hyperbolic quasi-Fuchsian groups and Toledo's invariant.* Geometriae Dedicata, **97** (2003) 151–185.
[15] J.R. Parker & I.D. Platis; *Open sets of maximal dimension in complex hyperbolic quasi-Fuchsian space.* Preprint 2004, `http://maths.dur.ac.uk/d̃ma0jrp/img/OpenSet.pdf`.
[16] R.E. Schwartz; *Degenerating the complex hyperbolic ideal triangle groups.* Acta Math., **186** (2001) 105–154.
[17] D. Toledo; *Representations of surface groups in complex hyperbolic space.* J. Differential Geometry, **29** (1989) 125–133.
[18] P. Will; *Punctured torus and Lagrangian triangle groups in* PU(2, 1), preprint 2004, `http://www/institut.math.jussieu.fr/p̃reprints/`.
[19] E.Z. Xia; *The moduli of flat* PU(2, 1) *structures on Riemann surfaces.* Pacific J. Maths., **195** (2000) 231–256.

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