REMARKS ON THE HYPERBOLIC GEOMETRY OF PRODUCT TEICHMÜLLER SPACES

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Abstract. Let $\widetilde{T}$ be a cross product of n Teichmüller spaces of Fuchsian groups, $n > 1$. From the properties of Kobayashi metric and from the Royden-Gardiner theorem, $\widetilde{T}$ is a complete hyperbolic manifold. Each two distinct points of $\widetilde{T}$ can be joined by a hyperbolic geodesic segment, which is not in general unique. But when $\widetilde{T}$ is finite dimensional or infinite dimensional of a certain kind, then among all such segments there is only one which enjoys a distinguished property: it is obtained from a uniquely determined holomorphic isometry of the unit disc into $\widetilde{T}$.

If $QC(G)$ is the Quasiconformal Deformation space of a finitely generated Kleinian group G, then since its holomorphic covering is a product of finite dimensional Teichmüller spaces, all the above results hold for $QC(G)$.

1. Introduction And Statement Of Results

Our aim is to study basic aspects of the hyperbolic geometry of products of Teichmüller spaces of Fuchsian groups. This is the geometry obtained by endowing these spaces with the Kobayashi-hyperbolic metric.

Let Γ be a Fuchsian group acting on the upper half plane and denote by $Teich(\Gamma)$ its Teichmüller space. $Teich(\Gamma)$ is a complex manifold and admits various natural metrics. The metric we are interested in is the Teichmüller metric τ_T which measures the distance between two points in terms of maximal dilatations of quasiconformal deformations of Γ. This notion was first introduced by O. Teichmüller in the special case where Γ is of signature $(g, 0)$, that is when Γ uniformises a closed Riemann surface of genus $g > 1$. This metric, later defined for Teichmüller spaces of all kinds of Fuchsian groups, is a complete non-Riemannian metric.

To each complex manifold M is assigned a generally non-Riemannian pseudometric, the Kobayashi pseudometric κ_M, which is such that its group of isometries is exactly the group of biholomorphic automorphisms of M. When κ_M is a metric, M is called a hyperbolic manifold.

The Teichmüller metric is intimately related to the Kobayashi metric and thus to the hyperbolic geometry of $Teich(\Gamma)$. H. Royden proved in 1971 that $\tau_T = \kappa_T$ in the case where Γ is of the first kind and of signature $(g, 0)$, $g > 1$ [15]. Later, this result was extended by F. Gardiner to the case where Γ is arbitrary (Chapter 7 of [5]) and has been known ever since as Royden-Gardiner Theorem (RGT). According to RGT, $Teich(\Gamma)$ is a complete hyperbolic manifold and moreover, when Γ is of the first kind (with the exception of a few cases), the set of biholomorphic automorphisms of $Teich(\Gamma)$ is just $Mod(\Gamma)$, the Modular group of Γ. (The case where Γ is of signature (g, n), $2g + n > 4$, was proved by Earle and Kra. [3] The case where $Teich(\Gamma)$ is infinite dimensional is more complicated, (see §2 below).

Let now $\widetilde{T}$ be the cross product of the Teichmüller spaces $Teich(\Gamma_i)$ of Fuchsian groups Γ_i, $i = 1, ..., n$ for $n > 1$. From the product property of Kobayashi metric [7] we deduce that $\widetilde{T}$ is a

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hyperbolic manifold and

$$\kappa_{\widetilde{T}} = \max_{i=1,..,n} \{\kappa_{T(\Gamma_i)}\} = \max_{i=1,..,n} \{\tau_{T(\Gamma_i)}\}$$

where $\tau_{T(\Gamma_i)}$ is the Teichmüller metric in $Teich(\Gamma_i)$, $i = 1, ..., n$.

The Teichmüller metric $\tau_{\widetilde{T}}$ can be therefore defined in a natural way as

$$\tau_{\widetilde{T}} = \max_{i=1,..,n} \{\tau_{T(\Gamma_i)}\}$$

In this manner the most general version of RGT can be stated as below.

Theorem. *Let $\Gamma_1, ..., \Gamma_n$ be Fuchsian groups and $\widetilde{T}$ be the cross product of their Teichmüller spaces. Then $\widetilde{T}$ is a complete hyperbolic manifold: the Teichmüller and Kobayashi metrics coincide in $\widetilde{T}$.*

It is worth remarking that there are no restrictions for the kind of Fuchsian groups that we consider here.

According to the RGT, holomorphic isometries of $(\widetilde{T}, \tau_{\widetilde{T}})$ are just the holomorphic automorphisms of $\widetilde{T}$. When all Γ_i are of the first kind these automorphisms have been studied extensively by J. Gentilesco in [6]. Since the universal covering space of the Quasiconformal Deformation space $QC(G)$ of a Kleinian group G is a cross product of Teichmüller spaces, he is able to state some results about the group of automorphisms of this deformation space. In particular, it is not a homogeneous space. Our Theorem 4 in §2.3 follows from the generalised version of RGT and Theorem III in [6].

The main part of our work is devoted to the study of holomorphic isometries and geodesics of the hyperbolic manifolds we are concerned with. The model for these manifolds is the hyperbolic unit polydisc Δ^n of $\mathbb{C}^n$, $n > 1$. Given two distinct points in Δ^n there is always a geodesic (with respect to the metric) segment joining them. In fact, one can prove that any two such points can be joined by infinitely many geodesic segments, which all lie in a closed region of Δ^n. Therefore, there is no hope of obtaining a unique (in the classical sense) segment joining any two points of $\widetilde{T}$ even in the finite dimensional case.

We find our way out of this difficulty by observing that among all geodesic segments which join two points of the polydisc there is only one characterised by a special property: it can be obtained by a uniquely determined holomorphic isometry of the disc into the polydisc. (See Proposition 2). Not surprisingly, in this case it coincides with the segment we take when Δ^n is equipped with the standard Bergman metric.

We call this segment maximal, and the geodesic curve upon which it lies a maximal geodesic. Our further study shows that the above picture in the polydisc appears in the class of manifolds we study. Combining our general results with classical statements concerning Teichmüller space we obtain our main result

Theorem. *Let $\widetilde{T}$ be a cross product of Teichmüller spaces $Teich(\Gamma_i)$ of Fuchsian groups Γ_i. For any two distinct points $\widetilde{\phi}$ and $\widetilde{\psi}$ of $\widetilde{T}$ there is a geodesic (with respect to Teichmüller-Kobayashi metric) segment joining them. If $\widetilde{T}$ is finite dimensional then there is a maximal geodesic segment joining $\widetilde{\phi}$ and $\widetilde{\psi}$. The same holds also in the case where $\widetilde{T}$ is infinite dimensional and of a special form.*

Furthermore, $\widetilde{T}$ has a straight space property: maximal geodesics are isometric with the Euclidean real line. We finally wish to remark that as applications to our results here are in the study of the hyperbolic geometry of Moduli spaces of Kleinian Groups. Here we just make a hint in §5.

This work has been carried out while the author was visiting the Department of Mathematics of the University of Crete, Greece.

2. Preliminaries

Definitions and results stated in this section are standard. For instance, we refer the reader to [10] and [13] for an extensive presentation of Teichmüller and Deformation space theory respectively.

2.1. Teichmüller space.

2.1.1. *Definition of Teichmüller space.* Let Γ be a Fuchsian group acting on the upper half plane $\mathbb{U}$. We denote by $\mathbb{L}^\infty(\Gamma)$ the space of invariant differentials for Γ, that is the complex Banach space of measurable complex essentially bounded functions $\mu(z)$ defined on $\mathbb{U}$ and satisfying the transformation law

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$$

for all $\gamma \in \Gamma$ and all $z \in \mathbb{U}$. $Belt(\Gamma)$ is the space of *Beltrami differentials* for Γ, that is the open unit ball of $\mathbb{L}^\infty(\Gamma)$.

Let $\mu \in Belt(\Gamma)$ and denote by f^μ the unique homeomorphism solution (normalised so that it fixes $0, 1$ and ∞) to the Beltrami equation

$$f_{\overline{z}} = \begin{cases} \mu(z) f_z & z \in \mathbb{U} \\ \overline{\mu(\overline{z})} f_z & z \in \widehat{\mathbb{C}} \setminus \mathbb{U} \end{cases}$$

The map f^μ is quasiconformal; an elementary argument shows that $f^\mu \Gamma (f^\mu)^{-1}$ is again a Fuchsian group.

Definition. *Teichmüller space* $Teich(\Gamma)$ is the space of equivalence classes of elements of $Belt(\Gamma)$, where two such differentials μ, ν are equivalent if f^μ, f^ν coincide on the real axis.

2.1.2. *Complex structure and Bers' embedding.* $Teich(\Gamma)$ is a complex manifold modelled on a complex Banach space with $\dim_{\mathbb{C}}(Teich(\Gamma)) = +\infty$ unless Γ is finitely generated and of the first kind. Then $Teich(\Gamma)$ is isomorphic to the Teichmüller space $Teich(\Gamma')$ where Γ' is a Fuchsian group of signature (g, n) : the resulting Riemann surface $\mathbb{U}/\Gamma'$ is analytically finite, i.e of genus $g > 1$ with n punctures. In this case, $\dim_{\mathbb{C}}(Teich(\Gamma)) = 3g - 3 + n$.

The complex manifold structure of Teichmüller space arises from the *Bers' embedding* which identifies it with an open subset of the complex Banach space $\mathbb{B}(\Gamma)$, the space of *bounded quadratic differentials for* Γ in the lower half plane $\mathbb{L}$. An element $\phi \in \mathbb{B}(\Gamma)$ is a bounded holomorphic function $\phi(z)$ on $\mathbb{L}$, that is

$$\| \phi \|_* = \sup \left\{ \frac{\phi(z)}{y^2},\ z \in \mathbb{L} \right\} < \infty$$

satisfying the transformation law

$$\phi(\gamma(z))(\gamma'(z))^2 = \phi(z)$$

for all $\gamma \in \Gamma$ and for all $z \in \mathbb{L}$.

Teichmüller space may be identified with a subset of $\mathbb{B}(\Gamma)$ which contains the ball $B(0, 2)$ and is contained in the ball $B(0, 6)$. *From now on,* $Teich(\Gamma)$ *will be this subset.*

The mapping

$$\widehat{\Phi} : Belt(\Gamma) \to Teich(\Gamma)$$

sending each μ to the corresponding ϕ is holomorphic and is called the *Bers' projection.*

2.1.3. *Modular group.* We consider the group $\widetilde{M}(\Gamma)$ of all quasiconformal mappings h of $\mathbb{U}$ such that $h \circ \gamma \circ h^{-1} \in \Gamma$ for all $\gamma \in \Gamma$ and the group $\widetilde{M_0}(\Gamma)$ of all quasiconformal mappings g of $\mathbb{U}$ such that $g \circ \gamma \circ g^{-1} = \gamma$ for all $\gamma \in \Gamma$. The *Modular group* $Mod(\Gamma)$ is the quotient $\widetilde{M}(\Gamma)/\widetilde{M_0}(\Gamma)$. There is a homomorphism of $Mod(\Gamma)$ into the group of biholomorphic automorphisms $Aut(Teich(\Gamma))$ of $Teich(\Gamma)$); if h is a representative of a coset, then this coset is mapped into the automorphism γ_h which is such that for every μ

$$\gamma_h(\widehat{\Phi}(\mu)) = \widehat{\Phi}(\nu)$$

where ν is the Beltrami differential of $f^\mu \circ h^{-1}$. Following [2] we shall call such automorphisms *geometric*. The group of geometric automorphisms shall be denoted by $Geom(\Gamma)$.

2.1.4. *Extremal differentials and Teichmüller metric.* Teichmüller's Existence Theorem for extremal quasiconformal mappings of the surface $\mathbb{U}/\Gamma$ states that among all quasiconformal mappings of $\mathbb{U}/\Gamma$ which belong to the same class in $Teich(\Gamma)$ there exists one that is extremal, that is of the smallest maximal dilatation. If f^μ is this mapping then μ satisfies

$$\| \mu \|_\infty = \inf\{\| \nu \|_\infty ; \nu \in Belt(\Gamma), \widehat{\Phi}(\mu) = \widehat{\Phi}(\nu)\}$$

Such a μ is also called extremal.

The Teichmüller metric is defined as follows. In the space of Beltrami differentials $Belt(\Gamma)$ define the *Teichmüller metric* $\tau_{B(\Gamma)}$ by

$$\tau_{B(\Gamma)}(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \overline{\mu}\nu} \right\|_\infty$$

for every μ, ν in $Belt(\Gamma)$. Then the *Teichmüller metric* $\tau_{T(\Gamma)}$ of $Teich(\Gamma)$ is given by

$$\tau_{T(\Gamma)}(\varphi, \psi) = \inf\{\tau_{B(\Gamma)}(\mu, \nu);\ \widehat{\Phi}(\mu) = \varphi,\ \widehat{\Phi}(\nu) = \psi\}$$

for every φ, ψ in $Teich(\Gamma)$.

It follows from Teichmüller's Existence Theorem that $\tau_{T(\Gamma)}$ is a well defined metric. We also note that $\tau_{T(\Gamma)}$ is invariant by the group $Geom(\Gamma)$ of geometric automorphisms of $Teich(\Gamma)$.

A $\mu \in Belt(\Gamma)$ is called uniquely extremal if

$$\widehat{\Phi}(\mu) \neq \widehat{\Phi}(\nu)$$

for every $\nu \in Belt(\Gamma)$ such that $\nu \neq \mu$ and $\| \nu \|_\infty \leq \| \mu \|_\infty$. A uniquely extremal Beltrami differential is also extremal. If $\dim_{\mathbb{C}}(Teich(\Gamma)) < +\infty$, every extremal Beltrami differential is unique in its class due to Teichmüller's Uniqueness Theorem and thus uniquely extremal. It belongs to $Belt_T(\Gamma)$, the space of *Teichmüller differentials for* Γ. This space contains differentials of the form

$$\mu = \lambda \frac{| \phi |}{\phi},\ \lambda \in [0, 1),\ \phi \in \mathbb{Q}(\Gamma),$$

where $\mathbb{Q}(\Gamma)$ is the set of *holomorphic quadratic differentials* for Γ, that is integrable holomorphic functions in $\mathbb{U}$ which satisfy the transformation law:

$$\phi(\gamma(z))(\gamma'(z))^2 = \phi(z)$$

for all $\gamma \in \Gamma$ and for all $z \in \mathbb{U}$.

In the infinite dimensional case, an extremal μ is not necessarily uniquely extremal [11]. But there, uniquely extremal Beltrami differentials μ with the additional property

$$\| \mu \|_\infty = | \mu |$$

almost everywhere, are of particular importance.

2.1.5. *Royden-Gardiner Theorem.* The proof of Royden-Gardiner Theorem (RGT) given in [4] is based on the following Lifting Theorem.

Theorem. ***(Lifting Theorem)**: If $f : \Delta \to Teich(\Gamma)$ is a holomorphic mapping of the unit disc Δ into $Teich(\Gamma)$, then there exists a holomorphic mapping $g : \Delta \to Belt(\Gamma)$ such that*

$$\widehat{\Phi} \circ g = f.$$

If $\mu_0 \in Belt$ and $\widehat{\Phi}(\mu_0) = f(0)$, we can choose g so that $g(0) = \mu_0$.

Theorem. ***(RGT)** : The Teichmüller and Kobayashi metrics of $Teich(\Gamma)$ coincide.*

(For the definition and properties of Kobayashi metric see §3.) Combining RGT and results stated in [2] we have the following.

Theorem. *a) Let $\dim_{\mathbb{C}}(Teich(\Gamma) < +\infty$. Then $Mod(\Gamma)$ is identified to the group $Geom(\Gamma)$ of geometric automorphisms of $Teich(\Gamma)$ and the latter is the full group of biholomorphic isometries $I(Teich(\Gamma))$ of $Teich(\Gamma)$ with respect to $\tau_{T(\Gamma)}$. Thus, it is the full group $Aut(Teich(\Gamma))$ of biholomorphisms of $Teich(\Gamma)$. (A few cases are excluded, see the remark following Theorem 4 below).*

b) Let $\dim_{\mathbb{C}}(Teich(\Gamma) = +\infty$. Then $Geom(\Gamma)$ is identified to $Aut(Teich(\Gamma))$ (and thus to $I(Teich(\Gamma))$ if Γ has the isometry property: If $\mathbb{Q}(S)$ is the space of integrable holomorphic quadratic differentials of $\mathbb{U}/\Gamma$, and X, Y are Riemann surfaces quasiconformally equivalent to S, then every complex linear isometry of $\mathbb{Q}(X)$ onto $\mathbb{Q}(Y)$ is of the form

$$\phi \to af^*(\phi)$$

where $a \in \mathbb{C}$, f is a holomorphic isomorphism of Y onto X and f^* is the pull-back mapping induced by f. Also, if Γ has the isometry property, an extremal μ is a Teichmüller differential.

2.2. **Quasiconformal Deformation space of a Kleinian group.** Let G be a finitely generated, non elementary and torsion free Kleinian group with region of discontinuity $\Omega(G)$ and limit set $\Lambda(G)$. Then,

$$\Omega(G)/G = \bigcup_{i=1}^{n} S_i$$

where S_i, for $i = 1, ..., n$ are analytically finite Riemann surfaces: each S_i is uniformised by a Fuchsian group Γ_i of signature (g_i, n_i) acting on the upper half plane. The collection $\{\Gamma_1, ..., \Gamma_n\}$ is called the *Fuchsian model* of G in $\Omega(G)$.

Definition. The *Teichmüller space* of the Kleinian group G is the cross product

$$\widetilde{T}(G) = \prod_{i=1}^{n} Teich(\Gamma_i)$$

where Γ_i, for $i = 1, ..., n$ is the Fuchsian model for G.

Denote by $\mathbb{L}^\infty(G)$ the space of *invariant differentials* for G, that is the complex Banach space of measurable, complex, essentially bounded functions $\mu(z)$ defined on $\mathbb{C}$ with support on $\Omega(G)$ and satisfying the transformation law:

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$$

for all $\gamma \in G$ and all $z \in \mathbb{C}$. The open unit ball $Belt(G)$ of $\mathbb{L}^\infty(G)$ is the space of *Beltrami differentials for G*.

Let $\mu \in Belt(G)$ and denote by w^{μ} the unique homeomorphism solution (normalised so that it fixes $0, 1$ and ∞) to the Beltrami equation

$$w_{\overline{z}} = \mu(z) w_z$$

in $\widehat{\mathbb{C}}$. w^{μ} is quasiconformal and wGw^{-1} is again a Kleinian group.

Definition. *Quasiconformal Deformation space* $QC(G)$ *of* G is the set of equivalence classes of Beltrami differentials, where two such differentials μ, ν are equivalent if w^{μ}, w^{ν} coincide on the limit set of G.

$QC(G)$ admits a natural complex structure. The natural surjection

$$\Phi : Belt(G) \to QC(G)$$

given by

$$\Phi(\mu) = [\mu]$$

for each $\mu \in Belt(G)$, is holomorphic.

The complex manifold structure of $QC(G)$ is given by Bers' Theorem [1], [8], [13]:

Theorem. ***(Bers' Theorem)****: If* G *is a finitely generated, torsion free Kleinian group with Fuchsian model* $\{\Gamma_1, ..., \Gamma_n\}$ *then there exists a holomorphic covering*

$$\Psi : \widetilde{T}(G) \to QC(G)$$

with holomorphic covering group a fixed point free subgroup $L_1 \times ... \times L_n$ *of* $Mod(\Gamma_1) \times ... \times Mod(\Gamma_n)$. *The mapping*

$$\Psi_G : Teich(\Gamma_1)/L_1 \times ... \times Teich(\Gamma_n)/L_n \to QC(G)$$

is biholomorphic and $QC(G)$ *is a simply connected complex manifold.*

Note. a) Denote by pr_i the natural projections $Teich(\Gamma_i) \to Teich(\Gamma_i)/L_i$, $i = 1, ..., n$. Then

$$\Psi = \Psi_G \circ pr$$

where $pr = (pr_1, ..., pr_n)$.

b) If $\Omega(G)$ consists of simply connected components then Ψ is a biholomorphism.

2.3. **Kobayashi Metric And Hyperbolic Manifolds.** For the following definition and results concerning the Kobayashi metric we follow [4]. We also refer to [9] and [7] for further details.

Let M be a complex open Banach manifold and denote by $H(\Delta, M)$ the set of holomorphic mappings from the unit disc into M. The Kobayashi function

$$\delta_M : M \times M \to [0, +\infty]$$

is defined for every $p, q \in M$ by

$$\delta_M(p, q) = \inf\{d_{\Delta}(0, t)\}$$

where the infimum is taken over all $f \in H(\Delta, M)$ such that $f(0) = p$ and $f(t) = q$.

If M, N are complex Banach manifolds and $f : M \to N$ is a holomorphic mapping, then

$$\delta_N(f(p), f(q)) \leq \delta_M(p, q)$$

for every $p, q \in M$. The equality holds if f is biholomorphic.

Definition. The *Kobayashi (pseudo)metric* κ_M is the largest (pseudo)metric on M such that

$$\kappa_M(p, q) \leq \delta_M(p, q)$$

for all $p, q \in M$.

If δ_M is a metric then $\delta_M = \kappa_M$ and M is called a *hyperbolic manifold.* If κ_M is complete, then M is a *complete hyperbolic* manifold.

It is clear that

a) the Kobayashi metric is distance decreasing: if M, N are hyperbolic manifolds and $f : M \to N$ is a holomorphic mapping, then

$$\kappa_N(f(p), f(q)) \leq \kappa_M(p, q)$$

for every $p, q \in M$.

b) The group of holomorphic isometries $I(M)$ of a hyperbolic manifold M, is just the group $Aut(M)$ of its biholomorphic automorphisms.

The following also holds [7].

Theorem 1. *Let M_i, $i = 1, ..., n$ be hyperbolic manifolds and $\widetilde{M}$ be their cross product. Then $\widetilde{M}$ satisfies the product property: for each $\widetilde{p} = (p_1, ...p_n)$ and $\widetilde{q} = (q_1, ..., q_n)$ in $\widetilde{M}$ we have*

$$\kappa_{\widetilde{M}}(\widetilde{p}, \widetilde{q}) = \max_{i=1,...,n}\{\kappa_{M_i}(p_i, q_i)\}.$$

Denote by $\widetilde{\Delta}$ the unit polydisc of $\mathbb{C}^n$

$$\{\widetilde{z} = (z_1, ..., z_n) \in \mathbb{C}^n : \mid z_i \mid < 1, i = 1, ...n\}.$$

Then

$$\kappa_{\widetilde{\Delta}}(\widetilde{z}, \widetilde{w}) = d_{\widetilde{\Delta}}(\widetilde{z}, \widetilde{w}) = \max_{i=1,...,n} d_{\Delta}(z_i, w_i)$$

(cf. [9] p. 47). The metric $d_{\widetilde{\Delta}}$ coincides with the Bergman metric defined for $\widetilde{\Delta}$ only in the case when $\widetilde{\Delta} = \Delta$.

We shall also need ([9] Proposition 1.6 and Theorem 4.7):

Theorem 2. *Let M' and M be complex manifolds and $\pi : M' \to M$ be a holomorphic covering. Then*

a)

$$\kappa_M(p, q) = \inf\{\kappa_{M'}(p', q')\}$$

where the infimum is taken over all $p', q' \in M'$ such that $\pi(p') = p$ and $\pi(q') = q$ and

b) M' is (complete) hyperbolic if and only if M is (complete) hyperbolic.

Theorems 1 and 2 have immediate consequences to the manifolds we study.

Let $\Gamma_1, ..., \Gamma_n$ be Fuchsian groups acting on the upper half plane,

$$\widetilde{B} = \prod_{i=1}^{n} Belt(\Gamma_i)$$

be the cross product of the spaces of their Beltrami differentials and

$$\widetilde{T} = \prod_{i=1}^{n} Teich(\Gamma_i)$$

be the cross product of their Teichmüller spaces. RGT and Theorem 1 imply

Theorem 3. *$\widetilde{B}$ and $\widetilde{T}$ are complete hyperbolic manifolds:*

$$\kappa_{\widetilde{B}}(\widetilde{\mu}, \widetilde{\nu}) = \max_{i=1,..n}\{\kappa_{B(\Gamma_i)}(\mu_i, \nu_i)\} = \max_{i=1,..n}\{\tau_{B(\Gamma_i)}(\mu_i, \nu_i)\}$$

for every $\widetilde{\mu} = (\mu_1, ..., \mu_n), \widetilde{\nu} = (\nu_1, ..., \nu_n) \in \widetilde{B}$ and

$$\kappa_{\widetilde{T}}(\widetilde{\phi}, \widetilde{\psi}) = \max_{i=1,...,n}\{\kappa_{T(\Gamma_i)}(\phi_i, \psi_i)\} = \max_{i=1,...,n}\{\tau_{T(\Gamma_i)}(\phi_i, \psi_i)\}$$

for every $\widetilde{\phi} = (\phi_1, ..., \phi_n)$, $\widetilde{\psi} = (\psi_1, ..., \psi_n)$ *elements of* $\widetilde{T}$. *Furthermore,*

$$\kappa_{\widetilde{T}}(\widetilde{\phi}, \widetilde{\psi}) = \inf\{\kappa_{\widetilde{B}}(\widetilde{\mu}, \widetilde{\nu}) : \widetilde{\Phi}(\widetilde{\mu}) = \widetilde{\phi}, \widetilde{\Phi}(\widetilde{\nu}) = \widetilde{\psi}\}.$$

Suppose that $\widetilde{T}$ is finite dimensional and denote by $\widetilde{M}$ the cross product

$$\prod_{i=1}^{n} Mod(\Gamma_i) = \prod_{i=1}^{n} Geom(\Gamma_i)$$

of the Modular groups of the Fuchsian groups Γ_i.

The following may be immediately deduced from Theorem 3 and a result due to Gentilesco ([6], Theorem III). It describes the full group of holomorphic self isometries $I(\widetilde{T})$ of $(\widetilde{T}, \tau_{\widetilde{T}})$ in the finite dimensional case.

Theorem 4. *$\widetilde{M}$ acts properly discontinuously as a subgroup of holomorphic isometries of $(\widetilde{T}, \kappa_{\widetilde{T}})$ when $\widetilde{T}$ is finite dimensional. If the signature (g_i, n_i) of each Γ_i satisfies $2g_i + n_i > 4$ then $I(\widetilde{T})$ and thus $Aut(\widetilde{T})$, is the semi direct product of $\widetilde{M}$ by the finite subgroup H, where H is generated by elements h_{ij} defined as follows: let $f_{ij} : Teich(\Gamma_i) \to Teich(\Gamma_j)$ be a chosen biholomorphic map. (Such a map exists if Γ_i, Γ_j are of the same signature). Then $h_{ij} = (k_1, ..., k_n)$ where $k_i = f_{ij}$, $k_j = f_{ij}^{-1}$ and $k_r = id$ for $r \neq i, j$.*

Remark. It is worth discussing the exceptional cases to the above theorem. When Γ_i is of signature $(2, 0)$, $I(Teich(\Gamma_i)) \simeq Mod(\Gamma_i)/\mathbb{Z}_2$. If Γ_i is of signature $(0, 3)$ then $Teich(\Gamma_i)$ is a single point. If it is of signature $(1, 2)$ then $Teich(\Gamma_i)$ is biholomorphic to a Teichmüller space of a group of signature $(0, 5)$. Finally, if Γ_i is of signature $(1, 1)$ or $(0, 4)$, then its Teichmüller space is conformally equivalent to the upper half plane. For details, see [3].

We conjecture that an analogous result holds in the infinite dimensional case at least when all Γ_i have the isometry property (see §2).

Conjecture. *Let Γ_i, for $i = 1, ..., n$ be Fuchsian groups which have the isometry property and $\widetilde{T}$ be the cross product of their Teichmüller spaces. Also let*

$$\widetilde{G} = \prod_{i=1}^{n} Geom(\Gamma_i)$$

be the cross product of the groups of their geometric automorphisms. Then the group of biholomorphic isometries $I(\widetilde{T})$ of $\widetilde{T}$ with respect to the Kobayashi metric is the semi-direct product of $\widetilde{G}$ by the finite subgroup H, where H is generated by elements h_{ij} defined as in Theorem 4.

We believe that this conjecture can be proved following the line of the proof of Theorem III in [6].

Now let G be a finitely generated, torsion free and non elementary Kleinian group with non empty region of discontinuity. Let $\{\Gamma_1, ..., \Gamma_n\}$ be its Fuchsian model and

$$\widetilde{T}(G) = \prod_{i=1}^{n} Teich(\Gamma_i)$$

be its *Teichmüller space*. By Theorem 3, $\widetilde{T}(G)$ is a complete hyperbolic manifold. Additionally, the group $I(\widetilde{T}(G))$ is described in the following.

Theorem 5. *Let $(\widetilde{T}(G), \kappa_{\widetilde{T}})$ as before and $\widetilde{M}(G)$ be the cross product*

$$\prod_{i=1}^{n} Mod(\Gamma_i) = \prod_{i=1}^{n} Geom(\Gamma_i)$$

of the Modular groups of the Fuchsian groups Γ_i. If the type (g_i, n_i) of each Γ_i satisfies $2g_i + n_i > 4$ then $I(\widetilde{T}(G))$ is the semi-direct product of $\widetilde{M}(G)$ by the finite subgroup H, where H is generated by elements h_{ij} as in Theorem 4.

By Bers' Theorem, $QC(G)$ is a complex manifold, thus admits a Kobayashi pseudometric. Moreover, Theorems 1, 2 and 3 immediately deduce

Theorem 6. *Let $QC(G)$ be the Quasiconformal Deformation space of a finitely generated, torsion free and non elementary Kleinian group G with Fuchsian model $\{\Gamma_1, ..., \Gamma_n\}$. If $\widetilde{T}(G)$ is the Teichmüller space of G with hyperbolic metric $\kappa_{\widetilde{T}}$ and κ_Q is the Kobayashi pseudometric on $QC(G)$, then for each $[\mu], [\nu] \in QC(G)$,*

$$\kappa_Q([\mu], [\nu]) = \inf\{\kappa_{\widetilde{T}}(\widetilde{\phi}, \widetilde{\psi})\}$$

where the infimum is taken over all $\widetilde{\phi}, \widetilde{\psi} \in \widetilde{T}(G)$ such that $\Psi(\widetilde{\phi}) = [\mu]$, $\Psi(\widetilde{\psi}) = [\nu]$ and Ψ is the holomorphic covering $\widetilde{T}(G) \to QC(G)$ given by Bers' Theorem. Thus, $QC(G)$ is a complete hyperbolic manifold. Moreover, if the region of discontinuity $\Omega(G)$ consists of simply connected components then its complete hyperbolic metric κ_Q is given for each $[\mu], [\nu] \in QC(G)$ by

$$\kappa_Q([\mu], [\nu]) = \kappa_{\widetilde{T}}(\Psi^{-1}([\mu]), \Psi^{-1}([\nu])).$$

3. Holomorphic Isometries And Geodesics Of Hyperbolic Manifolds

3.1. **Holomorphic isometries.** Let M, N be hyperbolic manifolds and $H(N, M)$ be the set of holomorphic mappings of N into M.

Definition 1. A mapping $f \in H(N, M)$ is called a *holomorphic isometry* if for every $t, s \in N$

$$\kappa_M(f(t), f(s)) = \kappa_N(t, s).$$

We denote the set of holomorphic isometries by $I(N, M)$.

If $N = M$, then the holomorphic isometries are exactly the biholomorphisms of M. To describe completely the group $I(M)$ of a hyperbolic manifold M is not a trivial task at all. The following classical result concerning the special case of the polydisc will be needed later ([14], Proposition 3, p.68).

Theorem 7. *Let $(\widetilde{\Delta}, d_{\widetilde{\Delta}})$ be the hyperbolic manifold*

$$\{\widetilde{z} = (z_1, ..., z_n) \in \mathbb{C}^n : \mid z_i \mid < 1, i = 1, ...n\}$$

with hyperbolic metric

$$d_{\widetilde{\Delta}}(\widetilde{z}, \widetilde{w}) = \max_{i=1,\dots,n} d_{\Delta_i}(z_i, w_i).$$

Then for each $F \in I(\widetilde{\Delta})$ there exists a permutation

$$\sigma : (1, ..., n) \to (1, ..., n)$$

of the integers from 1 *to* n*, real numbers* $\theta_1, ..., \theta_n$*, and complex numbers* $a_1, ..., a_n$*,* $\mid a_i \mid < 1$ *for each* $i = 1, ..., n$*, such that*

$$F(z_1, ..., z_n) = \left(e^{i\theta_1} \frac{z_{\sigma(1)} - a_1}{1 - \overline{a_1} z_{\sigma(1)}}, ..., e^{i\theta_n} \frac{z_{\sigma(n)} - a_n}{1 - \overline{a_n} z_{\sigma(n)}} \right).$$

Proposition 1. *Let* M_i $i = 1, ..., n$ *be hyperbolic manifolds and* $\widetilde{M}$ *be their cross product with hyperbolic metric*

$$\kappa_{\widetilde{M}} = \max_{i=1,...,n} \{\kappa_{M_i}\}.$$

Then: a) A mapping $f : \Delta \to \widetilde{M}$*, where*

$$f(z) = (f_1(z), ..., f_n(z))$$

belongs to $I(\Delta, \widetilde{M})$ *if each* $f_i \in I(\Delta, M_i)$*.*

b) If $f_i \in I(\Delta, M_i)$ *then* $F : \widetilde{\Delta} \to \widetilde{M}$ *defined by*

$$F(\widetilde{z}) = (f_1(z_1), ..., f_n(z_n))$$

belongs to $I(\widetilde{\Delta}, \widetilde{M})$*.*

Proof. a) Since each f_i is a holomorphic isometry, we have that for every $t, s \in \Delta$

$$\kappa_{M_i}(f_i(t), f_i(s)) = d_\Delta(t, s).$$

Now

$$\begin{aligned} \kappa_M(f(t), f(s)) &= \max_{i=1,...,n} \{\kappa_{M_i}(f_i(t), f_i(s))\} \\ &= d_\Delta(t, s), \end{aligned}$$

which proves a).

To prove b) consider arbitrary $\widetilde{z}, \widetilde{w} \in \widetilde{\Delta}$. Then

$$\begin{aligned} \kappa_{\widetilde{M}}(F(\widetilde{z}), F(\widetilde{w})) &= \max_{i=1,...,n} \{\kappa_{M_i}(f_i(z_i), f_i(w_i))\} \\ &= \max_{i=1,...,n} \{d_{\Delta_i}(z_i, w_i)\} \\ &= d_{\widetilde{\Delta}}(\widetilde{z}, \widetilde{w}). \end{aligned}$$

□

We shall use the above results to describe the set $I(\Delta, \widetilde{\Delta})$. In the first place, Proposition 2 a) induces that this set contains elements of the form

$$f(z) = (f_1(z), ..., f_n(z))$$

where f_i are holomorphic automorphisms of the disc, $i = 1, ..., n$. If we also require $f(0) = \widetilde{0}$, then from Schwarz's Lemma we obtain

$$f(z) = (e^{i\theta_1} z, ..., e^{i\theta_n} z).$$

We shall now track down all the other elements $f = (f_1, ..., f_n)$ of $I(\Delta, \widetilde{\Delta})$ which satisfy the condition $f(0) = \widetilde{0}$. Since all f_i are holomorphic self-mappings of the disc and $f_i(0) = 0$, Schwarz's Lemma implies

$$\mid f_i(z) \mid \leq \mid z \mid .$$

But there is at least one $z_0 \in \Delta$, $z_0 \neq 0$, such that $\mid f_j(z_0) \mid = \mid z_0 \mid$ for some j, otherwise f would not be a holomorphic isometry. Thus, again from Schwarz's Lemma,

$$f_j(z) = e^{i\theta} z$$

for at least this j. Take any other f_i which satisfies

$$\mid f_i(z) \mid < \mid z \mid$$

and let m_i be the order of the root 0 of f_i, $m_i \geq 1$. Then we can write

$$f_i(z) = z^{m_i} h_i(z)$$

for some holomorphic function of the disc h_i, with $h_i(0) \neq 0$. We consider an arbitrary but fixed $z \in \Delta$, and an $r \in (0,1)$ such that $\mid z \mid < r$. Then $h_i(z)$ is holomorphic in the disc $\{\mid z \mid < r\}$ and continuous in the closed disc $\{\mid z \mid \leq r\}$. Applying the Maximum Principle, we have

$$\left| \frac{f_i(z)}{z^{m_i}} \right| = \mid h_i(z) \mid \leq \max_{|\zeta|=r} \left| \frac{f_i(\zeta)}{\zeta^{m_i}} \right| < \frac{1}{r^{m_i-1}}.$$

If $m_i = 1$, then $\mid h_i(z) \mid \leq 1$. If $m_i > 1$ we let $r \to 1$ to obtain

$$\mid h_i(z) \mid = \left| \frac{f_i(z)}{z^{m_i}} \right| \leq 1$$

where equality holds only if

$$f_i(z) = c_i z^{m_i}$$

where $\mid c_i \mid = 1$ $(m_i > 1)$. Therefore $f_i(z) = z^{m_i} h_i(z)$ is either of the above form or such that $\mid h_i(z) \mid < 1$.

Note here that from the Pick's formulation of Schwarz's Lemma, we also have

$$d_\Delta(h_i(0), h_i(z)) \leq d_\Delta(0, z).$$

Finally, if $f(0) = \widetilde{z} \neq \widetilde{0}$ then $f = F^{-1} \circ g$, where F is an automorphism of the polydisc mapping $\widetilde{z}$ to $\widetilde{0}$ (cf. Theorem 7) and g is an isometry of the above described form.

We conclude from the above discussion that in general, given two points $\widetilde{z}$ and $\widetilde{w}$ in the polydisc with $a = \tanh d_{\widetilde{\Delta}}(\widetilde{z}, \widetilde{w})$ there can be a whole family of holomorphic isometries of the disc into the polydisc which map 0 to $\widetilde{z}$ and a to $\widetilde{w}$. The following proposition states that there is a unique such isometry which satisfies two additional conditions.

Proposition 2. *Let $\widetilde{z} = (z_1, ..., z_n)$, $\widetilde{w} = (w_1, ..., w_n)$ be any two distinct points in $\widetilde{\Delta}$ and $a = \tanh d_{\widetilde{\Delta}}(\widetilde{z}, \widetilde{w})$, $a_i = \tanh d_\Delta(z_i, w_i)$, $i = 1, ..., n$. There exists a unique $f \in I(\Delta, \widetilde{\Delta})$, $f = (f_1, ..., f_n)$, which satisfies the following conditions:*

(A) $f(0) = \widetilde{z}$ and $f(a) = \widetilde{w}$.

(B) The image $f_i([0, a])$ is the geodesic segment of Δ joining z_i and w_i, $i = 1, ..., n$.

(C) For each i such that $a_i < a$ the quantities $d_\Delta(z_i, f_i(a_i))$ are maximal in the sense that if there exists a $g = (g_1, ..., g_n)$ in $I(\Delta, \widetilde{\Delta})$ different from f and satisfying conditions (A) and (B) then

$$d_\Delta(z_i, f_i(a_i)) > d_\Delta(z_i, g_i(a_i))$$

unless $f_i(a_i) = g_i(a_i)$.

Proof. For simplicity, we shall omit the exceptional case where $z_i = w_i$ for some i. Suppose first that $\widetilde{z} = \widetilde{0} = (0, ..., 0)$ and $\widetilde{w} = \widetilde{a} = (a_1, ..., a_n)$. We distinguish two cases:

a) $a_1 = ... = a_n = a$. According to our previous discussion, the only $f = (f_1, ..., f_n)$ in $I(\Delta, \widetilde{\Delta})$ which satisfies condition (A) is $f(z) = (z, ..., z)$.

Indeed, if $f_j(z) = e^{i\theta} z$ for some j, then $f_j(a) = a$ implies that $f_j(z) = z$. If now $f_i(z) = z^{m_i} h_i(z)$ for some i, consider first the case where $| h_i(z) |< 1$.

Then $f_i(a) = a$ implies

$$a^{m_i - 1} h_i(a) = 1$$

which is impossible. If $f_i(z) = c_i z^{m_i}$ then $f_i(a) = a$ implies $c_i = 1$ and $m_i = 1$ which is also a possibility that cannot occur. Thus for all i, $f_i(z) = z$ and therefore f clearly satisfies condition (B). Condition (C) is vacuous in this case.

b) $a_i \neq a_j$ for at least two different i, j. We claim that f is given by

$$f(z) = \left(\frac{a_1}{a} z, \frac{a_2}{a} z, ..., \frac{a_{n-1}}{a} z, \frac{a_n}{a} z \right).$$

Clearly $f \in I(\Delta, \widetilde{\Delta})$ and satisfies conditions (A) and (B). We check that it is the only one which also satisfies condition (C).

Let $g = (g_1, ..., g_n) \in I(\Delta, \widetilde{\Delta})$ be different from f and such that it satisfies conditions (A) and (B). Then, at least one g_i is the identity and all others are of the form

$$g_i(z) = c_i z^{m_i}$$

where c_i is of modulus 1 and $m_i > 1$ or is of the form

$$g_i(z) = z^{m_i} h_i(z)$$

where $m_i > 0$, h_i is a holomorphic function of the disc, $h_i(0)$ is not 0 and $| h_i(z) |< 1$.

In the first case condition (B) implies that $c_i = 1$ and from $g_i(a) = a^{m_i} = a_i$ we obtain $m_i = (\lg a_i)/(\lg a)$. Now $m_i > 2$ implies $a_i > a^2$ and thus

$$g_i(a) = a^{m_i} < a^2 < a_i$$

which is impossible. Thus $m_i = 2$, $a_i = a^2$ and $g_i(z) = z^2$.

We now have

$$d_\Delta(0, g_i(a_i)) = d_\Delta(0, a_i^2) = d_\Delta(0, a^4)$$

On the other hand,

$$d_\Delta(0, f_i(a_i)) = d_\Delta(0, a_i^2/a) = d_\Delta(0, a^3)$$

and we are done with this case.

Suppose now that $g_i(z) = z^{m_i} h_i(z)$ as above. Then from condition (A) and since h_i is bounded above by 1, we have

$$h_i(a) = \frac{a_i}{a^{m_i}} < 1.$$

Thus $a_i < a^{m_i}$ and moreover, $h_i(t)$ is strictly increasing in $[0, a]$, therefore $h_i(a_i) < h_i(a)$. Now,

$$\begin{aligned} d_\Delta(0, g_i(a_i)) &= d_\Delta(0, a_i^{m_i} h_i(a_i)) \\ &< d_\Delta(0, a_i^{m_i} h_i(a)) \\ &< d_\Delta(0, (a_i/a)^{m_i} a_i) \\ &< d_\Delta(0, (a_i/a) a_i) \\ &= d_\Delta(0, f_i(a_i)) \end{aligned}$$

and we are done in all cases.

Let now $\widetilde{z}, \widetilde{w}$ be arbitrary, and $a_i = \tanh d_\Delta(z_i, w_i)$. There exists an automorphism $F = (F_1, ..., F_n)$ of the polydisc, mapping $\widetilde{z}$ to $\widetilde{0}$ and $\widetilde{w}$ to $\widetilde{a} = (a_1, ..., a_n)$. Namely,

$$F(\zeta_1, ..., \zeta_n) = \left(e^{i\theta_1} \frac{\zeta_1 - z_1}{1 - \overline{z_1}\zeta_1}, ..., e^{i\theta_n} \frac{\zeta_n - z_n}{1 - \overline{z_n}\zeta_n} \right)$$

where

$$e^{i\theta_i} = a_i \frac{1 - \overline{z_i} w_i}{w_i - z_i}.$$

Set $f = F^{-1} \circ f_0$ where f_0 is the unique element of $I(\Delta, \widetilde{\Delta})$ obtained in a). Then f clearly satisfies conditions (A) and (B).

Let $g \in I(\Delta, \widetilde{\Delta})$, $g = (g_1, ..., g_n)$ different from f which also satisfies conditions (A) and (B). Suppose further that there exists an i such that

$$d_\Delta(z_i, f_i(a_i)) \leq d_\Delta(z_i, g_i(a_i))$$

and $f_i(a_i) \neq g_i(a_i)$. Since $f_i = F_i^{-1} \circ (f_0)_i$ and F_i is a holomorphic isometry of the disc, by applying F_i to the above inequality we obtain

$$d_\Delta(0, (f_0)_i(a_i)) \leq d_\Delta(0, (F_i \circ g_i)(a_i)).$$

for some i. This can not happen unless $(f_0)_i = F_i \circ g_i$ for all i. But then $g_i = F_i^{-1} \circ (f_0)_i = f_i$ and therefore $f_i(a_i) = g_i(a_i)$ which is a contradiction. Thus such a g can not exist and therefore f satisfies condition (C). □

3.2. **Geodesics.**

Definition 2. Let M be a hyperbolic manifold with Kobayashi metric κ_M and $c : [a, b] \to M$ be a continuous mapping. We call the image $J = c([a, b])$ a *geodesic segment* $[\overline{p, q}]$ joining the *endpoints* $c(a) = p, c(b) = q$ *of* J if for every $t_1 \leq t_2 \leq t_3 \in [a, b]$ the following holds:

$$\kappa_M(c(t_1), c(t_3)) = \kappa_M(c(t_1), c(t_2)) + \kappa_M(c(t_2), c(t_3)).$$

It is *unique* if for every $r \in M$ not in $[\overline{p, q}]$ we have

$$\kappa_M(p, r) + \kappa_M(r, q) > \kappa_M(p, q).$$

A *geodesic curve* γ is the image of a continuous mapping $c : (-\infty, +\infty) \to M$ such that the image of every closed subinterval is a geodesic segment.

The next proposition states that holomorphic isometries map geodesic segments (resp. curves) to geodesic segments (resp. curves).

Proposition 3. *Let M, N be hyperbolic manifolds and $F : M \to N$ be a holomorphic isometry.*

a) If $p, q \in M$ and $[\overline{p, q}]$ is a geodesic segment joining points p, q of M then $F([\overline{p, q}])$ is a geodesic segment joining $F(p), F(q)$ in N.

b) If $c : (-\infty, +\infty) \to M$ defines a geodesic of M, then $c' = F \circ c$ defines a geodesic of N.

Proof. It suffices to prove a). Suppose that c defines the geodesic segment $[\overline{p, q}]$ of M and let $t_1 \leq t_2 \leq t_3 \in [a, b]$. Then, $F([\overline{p, q}])$ is defined by $c' = F \circ c$ and

$$\begin{aligned}
\kappa_N(c'(t_1), c'(t_3)) &= \kappa_N((F \circ c)(t_1), (F \circ c)(t_3)) \\
&= \kappa_M(c(t_1), c(t_3)) \\
&= \kappa_M(c(t_1), c(t_2)) + \kappa_M(c(t_2), c(t_3)) \\
&= \kappa_N((F \circ c)(t_1), (F \circ c)(t_2)) + \kappa_N((F \circ c)(t_2), (F \circ c)(t_3)) \\
&= \kappa_N(c'(t_1), c'(t_2)) + \kappa_N(c'(t_2), c'(t_3)).
\end{aligned}$$

□

Uniqueness of geodesics in a hyperbolic manifold is not in general ensured by any condition concerning the metric. For instance, completeness of the metric may not be enough.

For reasons of clarification, we shall describe below a standard failure of uniqueness in the case of the hyperbolic polydisc $(\widetilde{\Delta}, d_{\widetilde{\Delta}})$. It is well known that there is a unique geodesic segment joining any two points of the polydisc when the latter is equipped with the Bergman metric. Unfortunately, the picture in our case is not that good.

Proposition 4. *Let $(\widetilde{\Delta}, d_{\widetilde{\Delta}})$ be the hyperbolic polydisc. Then for each two distinct $\widetilde{z}, \widetilde{w} \in \widetilde{\Delta}$, $\widetilde{z} = (z_1, ..., z_n)$, $\widetilde{w} = (w_1, ..., w_n)$, there exists a unique geodesic segment joining them only if and only if all distances $d_\Delta(z_i, w_i)$ are equal. Otherwise there exist infinitely many geodesic segments joining these two points.*

Proof. The result follows immediately from the proof of Proposition 2. We only note that in the case where all distances $d_\Delta(z_i, w_i)$ are equal, then the hyperbolic metric coincides with the Bergman metric and the induced geodesic segment is unique in the standard sense.

□

Notice also that apart from geodesic segments joining two points in the polydisc which are obtained by holomorphic isometries, there also exist others which are not of this kind. We shall give an example in the case $n = 2$ where we can find at least two such segments. The modifications needed for the general case will then be apparent.

We suppose that $\widetilde{z} = (0,0) = \widetilde{0}$ and $\widetilde{w} = (a_1, a_2) = \widetilde{a}$, $a_i \in (0,1)$. We shall treat the case when $a_1 < a_2$ and $a_2 - a_1 > a_1$. All other cases may be carried out in the same manner.. We consider the euclidean segments

$$[\overline{(0,0),(0,a_1)}]_e \cup [\overline{(0,a_1),(a_1,a_2)}]_e$$

and

$$[\overline{(0,0),(a_1,a_1)}]_e \cup [\overline{(a_1,a_1),(a_1,a_2)}]_e$$

joining $(0,0)$ and (a_1, a_2). Straightforward calculations show that these segments are also geodesic with respect to the hyperbolic metric and can not be the images of any $f \in I(\Delta, \widetilde{\Delta})$.

More generally speaking, given a hyperbolic manifold M with Kobayashi metric κ_M, then to each two distinct points p, q of M we may assign a subset $G(p,q)$ of M defined by

$$G(p,q) = \{r \in M; \kappa_M(p,r) + \kappa_M(r,q) = \kappa_M(p,q)\}.$$

$G(p,q)$ enjoys several properties, a number of which we state below. Their proof is straightforward.

(1) $G(p,q)$ is a closed subset of M and is always different from the null set since it contains at least p, q.

(2)
$$M = \bigcup_{p,q \in M, p \neq q} G(p,q).$$

(3) Every geodesic segment joining p and q lies entirely in $G(p,q)$, and from each point of $G(p,q)$ passes at least one geodesic segment.

(4) $diam(G(p,q)) = \max\{\kappa_M(r_1, r_2); r_1, r_2 \in G(p,q)\} = \kappa_M(p,q)$.

(5) If $F \in I(M)$ is such that $F(p) = p'$ and $F(q) = q'$, then $F(G(p,q)) = G(p', q')$.

It is therefore natural to ask if among all segments in $G(p,q)$, there exists a unique one characterised by a special property. We have seen in Proposition 2 that such a segment actually exists in the case of the polydisc.

In the rest of this section we study more general hyperbolic manifolds in which this situation also appears.

Definition 3. A hyperbolic manifold M with hyperbolic metric κ_M is called an *I-manifold* if the following condition holds.

(A) For every two different points p,q in M there is a holomorphic isometry $f \in I(\Delta, M)$ such that $f(0) = p$ and $f(a) = q$ where $a = \tanh \kappa_M(p,q)$.

It follows from the definition that any two points of an I-manifold can be joined by at least one geodesic segment. For our purposes we shall also need

Definition 4. An I-manifold M is called an *IU-manifold* if

a) for every two different points p,q in M there is a *unique* holomorphic isometry $f \in I(\Delta, M)$ such that $f(0) = p$ and $f(a) = q$ where $a = \tanh \kappa_M(p,q)$ and

b) if there is an $h \in H(\Delta, M)$ such that $h(0) = p$, $h(a) = q$ and for each $t \in (0,a)$,

$$h(t) \in [\overline{p,q}] = [\overline{f(0), f(a)}],$$

then $h \equiv f$.

Note. An interesting question arising from the definition is for which manifolds condition b) is always satisfied. It is easy to see that condition b) implies $h = f$ on $[0,a]$. Indeed for each $t \in (0,a)$ there is an s in $(0,t]$ such that $h(t) = f(s)$. (s might not be in (t,a): by the distance decreasing property $\kappa_M(p, h(t)) \leq d_\Delta(0,t)$).

Now,

$$\begin{aligned} d_\Delta(0,a) &= \kappa_M(p,h(t)) + \kappa_M(h(t),q) \\ &\leq d_\Delta(0,s) + d_\Delta(t,a) \end{aligned}$$

and since

$$d_\Delta(0,a) = d_\Delta(0,t) + d_\Delta(t,a)$$

we have

$$d_\Delta(0,t) \leq d_\Delta(0,s).$$

Since $s \leq t$ only equality can hold true, and thus $s = t$ and $h(t) = f(t)$.

Following a standard analytic continuation argument one can prove that when M is (equivalent to) a bounded domain in $\mathbb{C}^n$ (or more generally in an n dimensional complex Banach space) then $h = g$ everywhere in the disc.

The next propositions will be needed in the next section.

Proposition 5. *Let $\widetilde{M}$ be a cross product of I-manifolds M_i, $i = 1, ..., n$ with hyperbolic metric $\kappa_{\widetilde{M}} = \max\{\kappa_{M_i}\}$. Then $\widetilde{M}$ is an I-manifold.*

Proof. Let $\widetilde{p} = (p_1, ..., p_n)$, $\widetilde{q} = (q_1, ..., q_n)$ be points in $\widetilde{M}$ and $a_i = \tanh \kappa_{M_i}(p_i, q_i)$, $a = \tanh \kappa_{\widetilde{M}}(\widetilde{p}, \widetilde{q})$. Assume for simplicity that all a_i are different from 0. Since all M_i are I-manifolds, there exist $f_i \in I(\Delta, M_i)$ such that $f_i(0) = p_i$ and $f_i(a_i) = q_i$.

We distinguish two cases:

a) $a_1 = ... = a_n = a$. Then $\widehat{f} = (f_1, ..., f_n)$ belongs to $I(\Delta, \widetilde{M})$ as this follows from Proposition 3 a) and clearly $\widehat{f}(0) = \widetilde{p}$ and $\widehat{f}(a) = \widetilde{q}$.

b) $a_i \neq a_j$ for at least one pair of distinct i, j. Define $F : \widetilde{\Delta} \to \widetilde{M}$ by

$$F(z_1, ..., z_n) = (f_1(z_1), ..., f_n(z_n)).$$

Then from Proposition 3 b) we have that $F \in I(\widetilde{\Delta}, \widetilde{M})$, $F(0, ..., 0) = \widetilde{p}$ and $F(a_1, ..., a_n) = \widetilde{q}$. Let $g : \Delta \to \widetilde{\Delta}$ given by

$$g(z) = \Big(\frac{a_1}{a}z, \frac{a_2}{a}z, ..., \frac{a_{n-1}}{a}z, \frac{a_n}{a}z\Big)$$

be the holomorphic isometry of the disc into the polydisc as in the proof of Proposition 2. Here $a = \max\{a_i; i = 1, ..., n\}$. Let $\widehat{f} = F \circ g$. Then,

$$\widehat{f}_i(z) = f_i(\frac{a_i}{a}z), \quad \widehat{f}(0) = F(g(0)) = F(0, ..., 0) = \widetilde{p}, \quad \widehat{f}(a) = F(g(a)) = F(a_1, ..., a_n) = \widetilde{q}$$

and $\widehat{f} \in I(\Delta, \widetilde{M})$. Thus $\widetilde{M}$ is a I-manifold. □

Note. The isometry $\widehat{f} = (\widehat{f_1}, ..., \widehat{f_n})$ constructed above satisfies an additional property

(B) for each $i = 1, ..., n$ $\widehat{f}_i(\overline{[0, a}] = \overline{[f_i(0), f_i(a_i)}]$ where the latter is the geodesic segment joining p_i and q_i in M_i obtained by f_i.

Indeed, if $t \in (0, a]$, $\widehat{f}_i(t) = f_i(\frac{a_i}{a}t) \in \overline{[f_i(0), f_i(a_i)}]$ since $\frac{a_i}{a}t \leq a_i$. Conversely, if $r \in \overline{(f_i(0), f_i(a_i)}]$, there exists a $t \in (0, a_i]$ such that $r = f_i(t) = \widehat{f}_i(\frac{a}{a_i}t)$ and $\frac{a}{a_i}t \leq a$.

The following generalises Proposition 2.

Proposition 6. *Let $\widetilde{M}$ be a cross product of IU-manifolds M_i, $i = 1, ..., n$ with hyperbolic metric $\kappa_{\widetilde{M}} = \max\{\kappa_{M_i}\}$. Then for each two points $\widetilde{p} = (p_1, ..., p_n)$, $\widetilde{q} = (q_1, ..., q_n)$ of $\widetilde{M}$, there exists a unique element $\widehat{f} \in I(\Delta, \widetilde{M})$, $\widehat{f} = (\widehat{f_1}, ..., \widehat{f_n})$ such that:*

(A) $\widehat{f}(0) = \widetilde{p}$ and $\widehat{f}(a) = \widetilde{q}$ where $a = \tanh \kappa_{\widetilde{M}}(p, q)$.

(B) For each i and for each $t \in (0, a]$,

$$\widehat{f}_i(t) \in \overline{[f_i(0), f_i(a_i)}]$$

where $\overline{[f_i(0), f_i(a_i)}]$ is the geodesic segment joining p_i and q_i in M_i determined by the unique $f_i \in I(\Delta, M_i)$ which are such that $f_i(0) = p_i$ and $f_i(a_i) = q_i$, $a_i = \tanh \kappa_{M_i}(p_i, q_i)$.

(C) For each i such that $a_i < a$ the quantities $\kappa_{M_i}(p_i, \widehat{f}_i(a_i))$ are maximal in the sense that if there exists a $h = (h_1, ..., h_n)$ in $I(\Delta, \widetilde{M})$ different from $\widehat{f}$ and satisfying conditions (A) and (B) then

$$\kappa_{M_i}(p_i, \widehat{f}_i(a_i)) > \kappa_{M_i}(p_i, h_i(a_i))$$

unless $h_i(a_i) = \widehat{f}_i(a_i)$.

Proof. We construct $\widehat{f}$ as in Proposition 5 and distinguish two cases:

a) $a_1 = ... = a_n$. Then,

$$\widehat{f} = (f_1, ..., f_n)$$

and we shall show that $\widehat{f}$ is the unique element of $I(\widetilde{\Delta}, \widetilde{M})$ which satisfies (A) and (B).

Indeed, if $h = (h_1, ..., h_n) \in I(\widetilde{\Delta}, \widetilde{M})$ satisfies (A) and (B), then since $h_i \in H(\Delta, M_i)$, $h_i(0) = p_i$ $h_i(a) = q_i$ and $h_i(t) \in [\overline{f_i(0), f_i(a_i)}]$ for all i, we obtain that $h_i \equiv f_i$ as this follows from Definition 4 b).

b) $a_i \neq a_j$ for at least one pair of distinct i, j. For each i,

$$\widehat{f}_i(z) = f_i((a_i/a)z).$$

Let again $h = (h_1, ..., h_n) \in I(\widetilde{\Delta}, \widetilde{M})$ different from $\widehat{f}$ and such that it satisfies (A) and (B). Suppose first that some $h_j \in I(\Delta, M_j)$. Then by (B), $a_i = a$ and thus h_j is identical to f_j which in this case is identical to $\widehat{f}_i$. So let us suppose that there is some $h_i \notin I(\Delta, M_i)$ and therefore $a_i < a$. Then if $t \in (0, a_i]$,

$$\kappa_{M_i}(p_i, h_i(t)) \leq d_\Delta(0, t) = \kappa_{M_i}(p_i, f_i(t)).$$

For $t = a_i$ we then have

$$\begin{aligned} \kappa_{M_i}(p_i, h_i(a_i)) &\leq \kappa_{M_i}(p_i, f_i(a_i)) \\ &< \kappa_{M_i}(p_i, f_i(a)) \\ &= \kappa_{M_i}(p_i, \widehat{f}_i(a_i)) \end{aligned}$$

and the proof is complete. □

Definition 5. If $\widehat{f}$ satisfies the conditions of Proposition 6 then it is called *the maximal geodesic isometry of* Δ *into* $\widetilde{M}$ (assigned to the points $\widetilde{p}, \widetilde{q}$).

The geodesic segment induced by $\widehat{f}$ is called *the maximal geodesic segment* joining $\widetilde{p}, \widetilde{q}$ and the geodesic $\widehat{f}((\overline{-1, 1}))$ *the maximal geodesic passing through* $\widetilde{p}$ and $\widetilde{q}$.

Proposition 4 may now restated as follows:

Proposition 7. *Let* $(\widetilde{\Delta}, d_{\widetilde{\Delta}})$ *be the hyperbolic polydisc. Then for each* $\widetilde{z}, \widetilde{w} \in \widetilde{\Delta}$ $\widetilde{z} = (z_1, ..., z_n)$, $\widetilde{w} = (w_1, ..., w_n)$ *there exists a maximal geodesic segment joining them. This segment is identical to the one obtained when* $\widetilde{\Delta}$ *is equipped with the standard Bergman metric.*

4. Application To $\widetilde{T}$, $\widetilde{T}(G)$ And $QC(G)$.

We start with the following theorems which can be found in [4] (Theorems 5 and 6 respectively):

Theorem 8. *Let* Γ *be a Fuchsian group,* $f \in H(\Delta, Teich(\Gamma))$ *and* $t_0 \in \Delta$*. Suppose that*

$$\tau_{T(\Gamma)}(f(t_0), f(t)) = d_\Delta(t_0, t)$$

for some $t \in \Delta$*. Then* $f \in I(\Delta, Teich(\Gamma))$*.*

Theorem 9. *Suppose that* $0 \neq \mu \in Belt(\Gamma)$ *is extremal. Then the following are equivalent:*

a) μ *is uniquely extremal and* $\| \mu \|_\infty = | \mu |$ *almost everywhere,*

b) there is only one geodesic segment joining 0 *and* $\widehat{\Phi}(\mu)$*,*

c) there is only one $f \in I(\Delta, Teich(\Gamma))$ *such that* $f(0) = 0$ *and* $f(\| \mu \|_\infty) = \widehat{\Phi}(\mu)$*, and*

d) there is only one $g \in H(\Delta, Belt(\Gamma))$ *such that* $g(0) = 0$ *and* $\widehat{\Phi}(g(\| \mu \|_\infty)) = \widehat{\Phi}(\mu)$*.*

Theorem 10. *Let $Teich(\Gamma)$ be the Teichmüller space of a Fuchsian group Γ and $\tau_{T(\Gamma)}$ be the Teichmüller-Kobayashi metric on $Teich(\Gamma)$. Given two distinct points ϕ, ψ in $Teich(\Gamma)$ there exists a geodesic segment joining them. If $Teich(\Gamma)$ is finite dimensional or if each non-zero extremal $\mu \in Belt(\Gamma)$ is uniquely extremal with $\| \mu \|_\infty = | \mu |$ almost everywhere, then this segment is unique.*

Proof. (Outline). It suffices to assume that $\psi = 0$. By Teichmüller's Existence Theorem there exists an extremal Beltrami differential $0 \neq \mu \in Belt(\Gamma)$, such that $\widehat{\Phi}(\mu) = \phi$. The image of $f : [0, \| \mu \|_\infty] \to Teich(\Gamma)$ where for each t in $[0, \| \mu \|_\infty]$,

$$f(t) = \widehat{\Phi}(t\mu / \| \mu \|_\infty)$$

is a geodesic segment joining 0 and ϕ. We can extend f in the obvious way to a holomorphic mapping of the unit disc into $Teich(\Gamma)$ and then from Theorem 8 we have that $f \in I(\Delta, Teich(\Gamma))$. If $Teich(\Gamma)$ is finite dimensional then by Teichmüller's Uniqueness Theorem each extremal Beltrami differential $\mu \in Belt(\Gamma)$ is uniquely extremal and is of the form

$$\mu = \lambda \frac{| \varphi |}{\varphi}$$

where $\lambda \in [0, 1)$ and $\varphi \in \mathbb{Q}(\Gamma_i)$ i.e. it is a Teichmüller differential. Therefore condition a) of Theorem 9 is satisfied and we obtain the result. In the infinite dimensional case, condition a) is satisfied by assumption. □

Let f be as in the proof of Theorem 10. The f-image of the unit disc is called a *Teichmüller disc* centred at ϕ. Additionally, there is an isometry of $\mathbb{R}$ onto $f(\overline{(-1, 1)})$ and therefore every Teichmüller geodesic is isometric to the real Euclidean line. Therefore $Teich(\Gamma)$ is a *straight* space.

Theorem 10 can be restated as below.

Theorem 11. *Let $Teich(\Gamma)$ be the Teichmüller space of a Fuchsian group Γ and $\tau_{T(\Gamma)}$ be the Teichmüller-Kobayashi metric on $Teich(\Gamma)$. Then $Teich(\Gamma)$ is an I-manifold. If $Teich(\Gamma)$ is finite dimensional or infinite dimensional and such that each non-zero extremal $\mu \in Belt(\Gamma)$ is uniquely extremal with $\| \mu \|_\infty = | \mu |$ almost everywhere, then it is an IU-manifold.*

Proof. We only have to check condition b) of Definition 4. In the finite dimensional case this is automatically satisfied as this follows from the note following Definition 4. But in general, we can work as follows.

Let as before 0 and ϕ points of $Teich(\Gamma)$ and suppose that $\phi = \widehat{\Phi}(\mu)$ where μ is uniquely extremal and $\| \mu \|_\infty = | \mu |$ almost everywhere. Suppose that there exists an $h \in H(\Delta, Teich(\Gamma))$ such that $h(0) = 0$ and $h(\| \mu \|_\infty) = \phi = \widehat{\Phi}(\mu)$. From Lifting Theorem in §2, there exists a $g \in H(\Delta, Belt(\Gamma))$ such that $\widehat{\Phi} \circ g = h$ and we can choose it so that $g(0) = 0$. Then g also satisfies

$$(\widehat{\Phi} \circ g)(\| \mu \|_\infty) = h(\| \mu \|_\infty) = \phi = \widehat{\Phi}(\mu)$$

and thus it is the uniquely defined mapping of Theorem 9 d). Again, Lifting Theorem induces that the mapping f in the proof of Theorem 10 is lifted to the same g and $\widehat{\Phi} \circ g = f$. Thus $f = h$ and our assertion is proved. □

We now state the main theorem of this section:

Theorem 12. *A cross product $\widetilde{T}$ of Teichmüller spaces $Teich(\Gamma_i)$ of Fuchsian groups Γ_i is an I-manifold: Given two distinct points $\widetilde{\phi} = (\phi_1, ..., \phi_n)$ and $\widetilde{\psi} = (\psi_1, ..., \psi_n)$ of $\widetilde{T}$ there is an $\widehat{f} \in I(\Delta, \widetilde{T})$ such that $\widehat{f}(0) = \widetilde{\phi}$ and $\widehat{f}(a) = \widetilde{\psi}$ where*

$$a = \tau_{\widetilde{T}}(\widetilde{\phi}, \widetilde{\psi}) = \max_{i=1,\dots,n} \{a_i = \tau_{T(\Gamma_i)}(\phi_i, \psi_i)\}.$$

Therefore $[\overline{\widetilde{\phi},\widetilde{\psi}}] = \widehat{f}([0,a])$ *is a geodesic segment joining them.*

If $\widetilde{T}$ *is finite dimensional or any extremal Beltrami differential* μ_i *of* $Belt(\Gamma_i)$ *is uniquely extremal and satisfies* $\| \mu_i \|_\infty = | \mu_i |$ *almost everywhere, then there is a maximal geodesic segment joining* $\widetilde{\phi}$ *and* $\widetilde{\psi}$.

Proof. For each i, $Teich(\Gamma_i)$ is an I-manifold as this follows from the proof of Theorem 10. From Proposition 5 we also deduce that $\widetilde{T}$ is also an I-manifold. Now if each $Teich(\Gamma_i)$ is finite dimensional, or each extremal Beltrami differential is uniquely extremal and satisfies the condition of the theorem, then again from Theorem 10 it is an IU-manifold. Our last assertion is thus obtained by Proposition 6. □

Corollary 1. *Let* $\widetilde{T}(G)$ *(resp.* $QC(G)$*) be the Teichmüller (resp. the Quasiconformal Deformation) space of a finitely generated Kleinian group with hyperbolic metric* $\kappa_{\widehat{T}}$ *(resp.* κ_Q*). For any two distinct points, there is a maximal geodesic segment joining them.*

We now wish to describe the form of the maximal geodesic isometry $\widehat{f}$ when $\widetilde{\psi} = \widetilde{0}$ and $\widetilde{\phi} = (\phi_1, ..., \phi_n)$ and $\dim(\widetilde{T}) < +\infty$. From the general construction of $\widehat{f}$ in Proposition 6 and the proof of Theorem 10, we obtain that

$$\widehat{f}(z) = \widetilde{\Phi}\left(\frac{\mu_1}{\| \mu \|_\infty} z, ..., \frac{\mu_n}{\| \mu \|_\infty} z\right)$$

where $\widetilde{\Phi} = (\widehat{\Phi}_1, ..., \widehat{\Phi}_n)$, μ_i are the unique extremal differentials such that $\widehat{\Phi}_i(\mu_i) = \phi_i$ and

$$\| \mu \|_\infty = \max_{i=1,\dots,n} \{\| \mu_i \|_\infty\}.$$

Thus the $\widehat{f}$-image of Δ is an n dimensional complex manifold which is a cross product of Teichmüller discs centred at ϕ_i and possibly single points. We call this set a *Teichmüller* $n-$*polydisc centred at* $\widetilde{\phi}$.

Finally, $\widetilde{T}$ has a straight space property. Let $\widehat{f}((\overline{-1,1}))$ be the maximal geodesic of $\widetilde{T}$ passing throughout the origin and $\widetilde{\phi}$. It is straightforward to prove that the mapping

$$\mathbb{R} \to \widehat{f}((\overline{-1,1}))$$

such that

$$x \to \widehat{f}(\tanh x)$$

for every $x \in \mathbb{R}$ is an isometry of the real Euclidean line into $\widetilde{T}$.

5. A Note On Moduli Spaces

Let as usual $\widetilde{T}(G)$ be the Teichmüller space of a Kleinian group G and $\{\Gamma_1, ..., \Gamma_n\}$ be the Fuchsian model for G. Assume further that the signature (g_i, n_i) of Γ_i is so that $2g_i + n_i > 4$ for each $i = 1, ..., n$. Following Gentilesco, we call such a G *non-exceptional.*

As it follows from Theorem 4, the Modular group

$$\widetilde{M}(G) = \prod_{i=1}^{n} Mod(\Gamma_i)$$

acts properly discontinuously on $\widetilde{T}(G)$ as a subset of hyperbolic isometries. Define now the Moduli space of G to be

$$\widetilde{R}(G) = \widetilde{T}(G)/\widetilde{M}(G).$$

It is clear that

$$\widetilde{R}(G) = \prod_{i=1}^{n} Riem(\Gamma_i)$$

where $Riem(\Gamma_i) = Teich(\Gamma_i)/Mod(\Gamma_i)$ is the classical Moduli space of Γ_i. This space is an orbifold and an irreducible normal complex space. The Teichmüller-Kobayashi metric $\kappa_{\widetilde{T}}$ projects to a complete hyperbolic metric in $\widetilde{R}(G)$ by virtue of Proposition 6.3 in [9]. It has been shown by C. McMullen in [12] that $Riem(\Gamma)$ admits a complete Kähler metric g comparable to the Teichmüller-hyperbolic metric, with respect to which it has finite volume.

Proposition 8. *The Moduli space $\widetilde{R}(G)$ of a non exceptional Kleinian group G admits a complete Kähler metric $\mathbf{g}$ comparable to the hyperbolic metric. With respect to this metric, $\widetilde{R}(G)$ is of finite volume $vol_{\mathbf{g}}$. Explicitly, if vol_{g_i} is the finite volume of the Moduli space of each Γ_i, then*

$$vol_{\mathbf{g}} = vol_{g_1} \times ... \times vol_{g_n}.$$

Proof. Just set $\mathbf{g} = g_1 \times ... \times g_n$ that is the product Riemannian metric. We firstly prove that $\mathbf{g}$ is comparable to the hyperbolic metric $\kappa_{\widetilde{R}}$ of $\widetilde{R}(G)$. To do so, it suffices to prove comparability in the universal covers. Let $\widetilde{\mathbf{g}} = \widetilde{g}_1 \times ... \times \widetilde{g}_n$ be the metric in $\widetilde{T}(G)$. Since $\widetilde{g}_i$ is comparable to $\tau_{T(\Gamma_i)}$ for each i, we easily obtain that $\widetilde{\mathbf{g}}$ is comparable to $\tau_{\widetilde{T}(G)}$. Now $\mathbf{g}$ is complete and Kähler, and by Fubini's Theorem we obtain the desired result. □

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