THE PU(2,1) CONFIGURATION SPACE OF FOUR POINTS IN S^3 AND THE CROSS-RATIO VARIETY

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Abstract. The configuration space of four points on the standard CR 3-sphere up to CR-automorphisms is a real four dimensional variety. We prove the existence of natural complex and CR structures on this space.

1. Introduction and Statement of Results

The three dimensional sphere $S^3 \subset \mathbb{C}^2$ is equipped with a natural contact structure

$$D = \mathrm{T}S^3 \cap \mathbb{J}\mathrm{T}S^3,$$

where $\mathbb{J}$ is the standard complex structure induced by multiplication by i in $\mathbb{C}^2$. The standard CR structure on S^3 is obtained by restricting the action of $\mathbb{J}$ to D. The automorphism group of this structure is identified to the Lie group $\mathrm{PU}(2,1)$ which extends to the unit complex ball as its group of biholomorphisms.

The configuration space $\mathcal{C}$ is the set of ordered quadruples consisting of pairwise distinct points on the three dimensional sphere S^3. We consider the diagonal action of the group $\mathrm{PU}(2,1)$ on $\mathcal{C}$ and denote the quotient space $\mathcal{C}/\mathrm{PU}(2,1)$ by $\mathcal{F}$. Our goal in this paper is to describe natural structures defined on $\mathcal{F}$. The motivation comes from the classical case of the configuration space C consisting of ordered quadruples of pairwise distinct points on the Riemann sphere $\mathbb{C}P^1$ and the quotient space $F = C/\mathrm{PSL}(2,\mathbb{C})$. It is well known that F is biholomorphic to $\mathbb{C} \setminus \{0,1\}$ which may be identified to the set of cross-ratios of four pairwise distinct points on $\mathbb{C}P^1$ (see section 4.4 of [B]). We examine the analogue of the above result in the case of the space $\mathcal{F}$.

The action of $\mathrm{PU}(2,1)$ in S^3 is doubly transitive but, up to this action, triples of points in S^3 are distinguished by a real invariant (the angular invariant), introduced by Cartan in [C], see also [G]. The diagonal action of $\mathrm{PU}(2,1)$ is proper but not free on the whole of $\mathcal{C}$. In fact, let $\mathcal{C}'$ be the subset of $\mathcal{C}$ consisting of quadruples (u_1, u_2, u_3, u_4) such that not all u_i lie in the same $\mathbb{C}-$circle ($\mathbb{C}$-circles are topological circles obtained by intersecting the sphere $S^3 \subset \mathbb{C}^2$ with complex planes, see section 2.1.1). Then $\mathrm{PU}(2,1)$ acts properly and freely on $\mathcal{C}'$ and thus the subset $\mathcal{F}' = \mathcal{C}'/\mathrm{PU}(2,1)$ of the quotient configuration space $\mathcal{F}$ is a 4-dimensional real manifold (see section 3 and Proposition 3.4).

By choosing direct sums of invariant distributions on S^3 we may define several CR structures on $\mathcal{C}$ (see Definition 3.5). Fixing such a CR structure, we examine its invariance under the diagonal action by $\mathrm{PU}(2,1)$. This will also enable us to obtain a complex structure on $\mathcal{C}$ according to the following Theorem (see Theorem 3.6 and Proposition 3.16).

IDP was supported by a Marie Curie Reintegration Grant fellowship (Contract No.MEIF-CT-2005-028371) within the 6th Community Framework Programme.

Theorem 1.1. *Let $\mathcal{C}^*$ be the open subset of $\mathcal{C}$ consisting of quadruples (u_1, u_2, u_3, u_4) such that u_1, u_2 do not lie in the same orbit of the stabiliser of u_3, u_4. Then $\mathcal{F}^* = \mathcal{C}^*/\mathrm{PU}(2,1)$ is a two dimensional complex manifold biholomorphic to $(\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}P^1$.*

The CR structure on the quotient is given by

Theorem 1.2. *Let $\mathcal{C}'$ be the open subset of $\mathcal{C}$ consisting of quadruples (u_1, u_2, u_3, u_4) such that u_1, u_2, u_3, u_4 do not lie in the same $\mathbb{C}-$circle. Then there exists a codimension two CR structure defined on $\mathcal{F}' = \mathcal{C}'/\mathrm{PU}(2,1)$.*

We next proceed to examine how the space $\mathcal{F}$ relates to the set of cross-ratios. The notion of the usual cross-ratio of four points in $\mathbb{C}P^1$ was generalised to S^3 by Korányi and Reimann [KR]. The cross-ratio of a configuration of four points in S^3 is not enough to determine completely the quadruple up to $\mathrm{PU}(2,1)$ equivalence. In fact, by permuting the points of the configuration we obtain 24 cross-ratios. Using certain relations between these cross-ratios, we are left with only three of them which we denote by ω_0, ω_1 and ω_2. It is shown in [F] that $\omega_0, \omega_1, \omega_2$ satisfy the following.

$$|\omega_0|^2|\omega_1|^2|\omega_2|^2 = 0, \tag{1.1}$$

$$|\omega_0 - 1|^2 - 1 + |\omega_0|^2(|\omega_1 - 1|^2 - 1) + |\omega_0|^2(|\omega_1|^2(|\omega_2 - 1|^2 - 1) = 0. \tag{1.2}$$

The set of complex cross-ratios is thus a real algebraic variety of $\mathbb{C}^3$ denoted by $\mathcal{X}$. In [PP1] the variety $\mathcal{X}$ is used to define Fenchel-Nielsen coordinates in the complex hyperbolic setting and in [PP2] global geometrical coordinates for $\mathcal{X}$ are introduced.

There exists a natural bijection π between $\mathcal{F}$ and $\mathcal{X}$ (see [F]). It is induced from a CR map $\varpi : \mathcal{C} \to \mathbb{C}^3$ which associates to any quadruple of four pairwise distinct points in S^3 a triple $(\omega_0, \omega_1, \omega_2)$ of cross-ratios. The $\pi-$image of $\mathcal{F}'$ is a subset $\mathcal{X}' = \mathcal{X} \setminus S_{\mathbb{R}}$ where $S_{\mathbb{R}}$ is a one-dimensional subset of $\mathcal{X}$ diffeomorphic to $\mathbb{R} \setminus \{0, 1\}$. In fact we prove (see section 4.1)

Theorem 1.3. *The map π is a CR embedding of $\mathcal{F}'$ onto $\mathcal{X}'$.*

The CR structure on $\mathcal{F}'$ is degenerate in the sense that it has an integrable leaf of codimension two. In fact, the cross-ratio variety $\mathcal{X}$ contains an embedding of $\mathbb{C} \setminus \{0, 1\}$ (we call it $S_{\mathbb{C}}$), the complement of which ($\mathcal{X}'' = \mathcal{X} \setminus S_{\mathbb{C}}$) is a regular CR-structure of codimension two locally equivalent to $S^3 \times \mathbb{R}$. That complement corresponds to the subset $\mathcal{C}'' \subset \mathcal{C}'$ of configurations (u_1, u_2, u_3, u_4) such that u_2, u_3, u_4 do not lie in the same $\mathbb{C}$-circle.

This paper is organised as follows. In section 2 we introduce the basic general definitions and properties of the standard CR structure of S^3. We state Proposition 2.5 which describes how to induce CR structures on quotient spaces of CR manifolds with a group action. In section 3 the configuration space of four points in S^3 is defined and the several complex structures and CR structures are introduced. We next prove Theorem 3.6 and this, together with the coordinates introduced in Proposition 3.16 will prove Theorem 1.1. Theorem 1.2 is also a consequence of Theorem 3.6. The last section deals with the realisation of the $\mathrm{PU}(2,1)$ configuration space as an embedded CR manifold as well as with the proof of Theorem 1.3.

2. Preliminaries

2.1. **The standard CR structure on S^3.** The contents of this section are well known. The reader may consult for instance [J, G].

2.1.1. SU(2, 1) *and the Heisenberg group.* Let $\mathbb{J} : \mathrm{T}\mathbb{C}^2 \to \mathrm{T}\mathbb{C}^2$ be the complex operator defined by multiplication by i. The standard CR structure on S^3 is given by the contact structure $D = TS^3 \cap \mathbb{J}TS^3$ equipped with the restriction $\mathbb{J} : D \to D$. The CR-automorphism group is the group $\mathrm{PU}(2,1)$ which we describe below. Let

$$\mathrm{SU}(2,1) = \{\ g \in \mathrm{SL}(3,\mathbb{C}) \mid \bar{g}^T hg = h\ \},$$

where the Hermitian form h is given by the matrix

$$h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

There is a left action of $\mathrm{SU}(2,1)$ on $\mathbb{C}^3$ which preserves the cone

$$\{\ z \in \mathbb{C}^3 \mid \bar{z}^T hz = 0\ \}$$

whose projectivisation is exactly $S^3 \subset \mathbb{C}P^2$. The center K of $\mathrm{SU}(2,1)$ is a finite cyclic group of order 3 acting trivially on the sphere S^3. We thus define $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/K$.

The *Heisenberg group* $\mathfrak{N}$ is the multiplicative group of pairs $(z,t) \in \mathbb{C} \times \mathbb{R}$ with product

$$(z,t) \cdot (z',t') = (z + z', t + t' + 2\Im(z\overline{z}')).$$

The one point compactification $\overline{\mathfrak{N}} = \mathfrak{N} \cup \infty$ of the Heisenberg group is identified to the sphere S^3 as follows. Any point $p = (z,t) \in \mathfrak{N}$ and the point ∞ are lifted respectively to the following points in $\mathbb{C}^3$:

$$\hat{p} = \begin{bmatrix} \frac{-|z|^2+it}{2} \\ z \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the origin $(0,0)$ is lifted to

$$x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If an involution in $\mathrm{PU}(2,1)$ has fixed points in S^3, then one may show that its fixed point set is a topological circle, called a $\mathbb{C}$-*circle*. Alternatively, using the identification $S^3 = \mathfrak{N} \cup \{\infty\}$, a $\mathbb{C}$-circle may be thought as any circle in S^3 obtained by translating the vertical line $\{(0,t)\} \cup \{\infty\}$ in $\overline{\mathfrak{N}}$ by an element of $\mathrm{PU}(2,1)$.

2.1.2. *Invariants.* The following invariants appear naturally in the context of the CR geometry on S^3.

Definition 2.1 ([C], see also [G])**.** Given any three ordered points p_1, p_2, p_3 in S^3 we define *Cartan's angular invariant* $\mathbb{A}$ as

$$\mathbb{A}(p_1,p_2,p_3) = \arg(-\langle \hat{p}_1, \hat{p}_2\rangle \langle \hat{p}_2, \hat{p}_3\rangle \langle \hat{p}_3, \hat{p}_1\rangle).$$

The invariant $\mathbb{A}$ classifies ordered triples of points in S^3 according to the following.

Proposition 2.2 ([C], see also [G]). *There exists an element of* $\mathrm{PU}(2,1)$ *which maps an ordered triple of (distinct) points in* S^3 *to another ordered triple if and only if their corresponding angular invariants are equal. This element is unique unless the triple lies on a* $\mathbb{C}-$*circle.*

A natural cross-ratio is given by the Korányi-Reimann invariant $\mathbb{X}$ introduced in [KR] (see also [G] for a list of its properties).

Definition 2.3. The CR cross-ratio $\mathbb{X} = \mathbb{X}[u_1, u_2, u_3, u_4]$ associated to four pairwise distinct points u_1, u_2, u_3, u_4 in S^3 is

$$\mathbb{X}[u_1, u_2, u_3, u_4] = \frac{\langle \hat{u}_3, \hat{u}_1 \rangle \langle \hat{u}_4, \hat{u}_2 \rangle}{\langle \hat{u}_3, \hat{u}_2 \rangle \langle \hat{u}_4, \hat{u}_1 \rangle}.$$

Both angular invariant and cross-ratio do not depend on the choice of lifts.

2.1.3. *The Lie algebra* $\mathfrak{su}(2,1)$. The elements of the Lie algebra $\mathfrak{g} = \mathfrak{s}u(2,1)$ are matrices of the form

$$\begin{bmatrix} \lambda & -\bar{z} & it \\ -\bar{w} & ri & z \\ is & w & -\bar{\lambda} \end{bmatrix}$$

where $t, r, s \in \mathbb{R}$, $z, w, \lambda \in \mathbb{C}$ and $r = -2\Im(\lambda)$. The Lie algebra $\mathfrak{g}$ is graded, that is

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$$

where

$$\mathfrak{g}^{-2} = \left\{ \begin{pmatrix} 0 & 0 & it \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}^{-1} = \left\{ \begin{pmatrix} 0 & -\bar{z} & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}^{0} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & ri & 0 \\ 0 & 0 & -\bar{\lambda} \end{pmatrix} \right\}, \quad \mathfrak{g}^{1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ -\bar{w} & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}^{2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ is & 0 & 0 \end{pmatrix} \right\}$$

and $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ assuming that $\mathfrak{g}^k = 0$ for $|k| > 2$.

Definition 2.4. We denote by $\mathfrak{h}$ the subalgebra

$$\mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$$

of $\mathfrak{g}$ which is the Lie algebra of the isotropy group H at the origin of $\mathfrak{N}$.

2.2. **Group actions and distributions.** We consider the following general setting. Let M be a smooth manifold and $G \times M \to M$ be a smooth action with isotropy group $H \subset G$ at $x_0 \in M$ and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. Let also v^* be the fundamental vector field corresponding to $v \in \mathfrak{g}$, that is

$$v^*(x) = \frac{de^{tv}x}{dt}.$$

Translations of the tangent space $\mathrm{T}_{x_0}(M)$ and the fundamental vector field are related as follows. For each $g \in \mathfrak{g}$ we have

$$v^*(gx_0) = \frac{de^{tv}gx_0}{dt} = \frac{dgg^{-1}e^{tv}gx_0}{dt} = g\frac{dg^{-1}e^{tv}gx_0}{dt} = g_*((Ad_{g^{-1}}v)^*(x_0))$$

or equivalently, by putting $Ad_{g^{-1}}v = a$ we have

$$(Ad_g a)^*(gx_0) = g_*(a^*(x_0)).$$

From this formula we obtain

$$(Ad_g \mathfrak{h})^*(gx_0) = 0.$$

The following hold.

- If the action is transitive, then G/H may be identified to M and the tangent space $\mathrm{T}_{Id}(G/H) = \mathfrak{g}/\mathfrak{h}$ may be identified to the tangent space $\mathrm{T}_{x_0}(M)$. Moreover, the action of H on $\mathrm{T}_{x_0}(M)$ is the adjoint action: for $v \in \mathfrak{g}$ and $h \in H$,

$$h_* v^*(x_0) = (Ad_h v)^*(x_0).$$

- If the action is proper, then every orbit is closed and the map $G/H \to M$ induced by $g \to gx_0$ is an embedding and the orbit space $Q = M/G$ is Hausdorff.
- If the action is proper and free, then the orbit space is a manifold and the tangent space $\mathrm{T}_x M$ fits into the exact sequence

$$0 \to \mathrm{T}_x(Gx) \to \mathrm{T}_x(M) \to \mathrm{T}_{pr(x)}(Q) \to 0,$$

  where $pr : M \to Q$ is the natural projection.

Suppose now that the action $G \times M \to M$ is proper and free and let D be a distribution defined on M. Denote by $pr : M \to Q = M/G$ the natural projection and for any $x \in M$ consider the exact sequence

$$0 \to D_x \cap \mathrm{T}_x(Gx) \to D_x \to pr_*(D_x) \to 0.$$

Clearly, $\dim(pr_*(D_x)) = \dim\left(D_x/(D_x \cap T_x(Gx))\right)$. Now if an operator L is defined on D, a natural question is whether L induces an operator on the distribution $pr_*(D)$ defined on Q. For that sake, consider an exact sequence of vector spaces

$$0 \to E \to F \to F/E \to 0.$$

Thus, a linear map $L : F \to F$ induces a linear operator $\bar{L} : F/E \to F/E$ if and only if $L(E) \subset E$. All the above are summarised in the following.

Proposition 2.5. *Let M be a manifold and G be a Lie group acting smoothly, properly and freely on M. Let also $pr : M \to Q = M/G$ be the natural projection to the quotient. Suppose that there exists a distribution D defined on M. If $D' = pr_*(D)$ is the projection of D in Q then for each $x \in M$ we have*

$$\dim(pr_*(D_x)) = \dim\left(D_x/(D_x \cap \mathrm{T}_x(Gx))\right). \tag{2.1}$$

Suppose additionally that there is a complex structure $\mathbb{J}$ defined on D. If for each $x \in M$

$$\mathbb{J}(D_x \cap \mathrm{T}_x(Gx)) \subset D_x \cap \mathrm{T}_x(Gx), \tag{2.2}$$

then there exists a complex structure $\mathbb{J}'$ defined on D'.

2.2.1. *Description of the CR structure on S^3 and CR maps.* We may now describe the CR structure on S^3 using the action $\mathrm{PU}(2,1) \times S^3 \to S^3$. We fix the origin of the Heisenberg group $\mathfrak{N}$ and we consider its lift $x_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ as in section 2.1.1. The distribution D is defined by propagating $D_{x_0} = (\mathfrak{g}^{-1})^*(x_0)$, that is $D_{gx_0} = g_*((\mathfrak{g}^{-1})^*(x_0))$. The complex operator $\mathbb{J} : \mathfrak{g}^{-1} \to \mathfrak{g}^{-1}$ is defined by $z \to iz$, that is

$$\begin{bmatrix} 0 & -\bar{z} & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & i\bar{z} & 0 \\ 0 & 0 & iz \\ 0 & 0 & 0 \end{bmatrix}$$

and it induces an operator $\mathbb{J} : D \to D$ as follows: for $v(gx_0) = g_*((a)^*(x_0))$ with $a \in \mathfrak{g}^{-1}$,

$$\mathbb{J}v(gx_0) = g_*((\mathbb{J}a)^*(x_0)) = (Ad_g \mathbb{J}a)^*(gx_0).$$

This is well defined since $Ad_h(\mathbb{J}a) \equiv \mathbb{J}Ad_h(a) \bmod \mathfrak{h}$ for $a \in \mathfrak{g}^{-1}$ and $h \in H$. Finally, we shall also need

Definition 2.6. A CR function $f : S^3 \to \mathbb{C}$ is a function satisfying $df(v + i\mathbb{J}v) = 0$ for all $v \in D$.

In terms of Heisenberg coordinates this can be written as

$$\frac{\partial f}{\partial \bar{z}} - iz\frac{\partial f}{\partial t} = 0.$$

A map $F : S^3 \to \mathbb{C}^n$ given by $F = (f_1, \ldots, f_n)$ is CR if each f_i is a CR function.

3. The $\mathrm{PU}(2,1)$ Configuration Space of Four Points in S^3

Definition 3.1. The configuration space of quadruples of pairwise distinct points of S^3 shall be denoted by $\mathcal{C}$. The quotient $\mathcal{C}/\mathrm{PU}(2,1)$ of $\mathcal{C}$ by the diagonal action of $\mathrm{PU}(2,1)$ shall be denoted by $\mathcal{F}$.

It will turn out shortly that the following subsets of $\mathcal{C}$ and $\mathcal{F}$ are of particular importance.

Definition 3.2. Let $\mathcal{C}'$, $\mathcal{C}^*$, and $\mathcal{C}''$ be the following subsets of $\mathcal{C}$.

(1) $\mathcal{C}'$ is the set of quadruples $(u_1, u_2, u_3, u_4) \in \mathcal{C}$ such that u_i do not all lie in the same $\mathbb{C}$-circle,
(2) $\mathcal{C}^* \subset \mathcal{C}'$ is the set of quadruples $(u_1, u_2, u_3, u_4) \in \mathcal{C}$ such that u_1 and u_2 do not lie in the same orbit of the stabiliser of u_3 and u_4 and
(3) $\mathcal{C}'' \subset \mathcal{C}'$ is the set of quadruples $(u_1, u_2, u_3, u_4) \in \mathcal{C}$ such that u_2, u_3 and u_4 are not in the same $\mathbb{C}$-circle.

Accordingly we define

$$\mathcal{F}' = \mathcal{C}'/\mathrm{PU}(2,1), \quad \mathcal{F}^* = \mathcal{C}^*/\mathrm{PU}(2,1), \quad \mathcal{F}'' = \mathcal{C}''/\mathrm{PU}(2,1).$$

We start by verifying the existence of a manifold structure in $\mathcal{F}'$. For this, we need the following.

Lemma 3.3. *The diagonal action*

$$\mathrm{PU}(2,1) \times \mathcal{C} \to \mathcal{C}$$

is proper.

Proof. We have to show that for each compact $K \subset \mathcal{C}$, the set $\{\ g|\ gK \cap K\ \}$ is also compact. Given a point $(u_1, u_2, u_3, u_4) \in \mathcal{C}$, we may first normalise so that $u_1 = \infty$ and $u_2 = (0, 0)$. Thus we only have to show that the action of the stabiliser $Stab$ of ∞ and $(0, 0)$ on $\mathfrak{N} \setminus \{(0, 0)\} \times \mathfrak{N} \setminus \{(0, 0)\}$ is a proper action. But $Stab$ is the family of dilations $(z, t) \to (\lambda z, |\lambda|^2 t)$ and thus a pair of points in $\mathfrak{N} \times \mathfrak{N}$ is contained in a compact subset if and only if both of them are contained in a set of the form $A_{(m,M)} = \{\ (z, t) \mid m \le |z|^4 + t^2 \le M\ \}$. It can be easily checked that $\{\ g \in Stab \mid gA_{(m,M)} \cap A_{(m,M)}\ \}$ is compact. □

Proposition 3.4. *The set $\mathcal{F}' = \mathcal{C}'/\mathrm{PU}(2, 1)$ is a 4-dimensional real analytic manifold.*

Proof. Clearly, the principal orbits (that is, orbits of maximal dimension) of the action correspond to configurations $(u_1, u_2, u_3, u_4) \in \mathcal{C}$ such that all four points u_i do not lie in the same $\mathbb{C}$-circle. In fact the only configurations having a non-trivial isotropy are the ones in the complement of $\mathcal{C}'$. Thus the action of $\mathrm{PU}(2, 1)$ is free on $\mathcal{C}'$ and by Lemma 3.3 is proper. We conclude that every orbit is an embedded submanifold in $\mathcal{C}'$; hence $\mathcal{F}'$ is Hausdorff and analytic. □

At this point, we wish to make some further comments considering the sets $\mathcal{C}'$ and $\mathcal{C}^*$. In the first place, $\mathcal{C}'$ is a principal $\mathrm{PU}(2, 1)$-bundle over the quotient space $\mathcal{F}'$. On the other hand, the set $\mathcal{C}^*$ is the complement of the set of quadruples (u_1, u_2, u_3, u_4) which are such that there exists an antiholomorphic involution $\phi : S^3 \to S^3$ satisfying $\phi(u_1) = u_2$ and $\phi(u_3) = u_4$ (see [W1]). A description of this set may be obtained as follows. By using Heisenberg coordinates and recalling that the stabiliser of $(0, 0)$ and ∞ is the set of transformations

$$(z, t) \to (\lambda z, |\lambda|^2 t),$$

we obtain that a quadruple $m = ((z_1, t_1), (z_2, t_2), (0, 0), \infty)$ is in $\mathcal{C}^*$ if and only if

$$(|z_1|^2, t_1),\ \ (|z_2|^2, t_2)$$

are linearly independent.

The rest of this section is devoted to showing the existence of a natural complex structure on $\mathcal{F}^*$ and of a natural CR structure on $\mathcal{F}'$. The special properties of the CR structure when this is restricted on $\mathcal{F}'' \subset \mathcal{F}'$ will be fully analysed in section 4.

3.1. **Complex and CR structures: main theorems.** Let $(D, \mathbb{J})$ be the standard CR structure of S^3. From this structure, several canonical distributions may be defined on the set $\mathcal{C}$ and subsequently to the quotient space $\mathcal{F}$. To see this, we fix the point $x_0 \in S^3$ as in section 2.2.1; any point $m = (m_1, m_2, m_3, m_4) \in \mathcal{C}$ may then be written as $m = (g_1x_0, g_2x_0, g_3x_0, g_4x_0)$, where $g_i \in \mathrm{PU}(2, 1)$, $i = 1, \dots, 4$. In the following definition, $\mathfrak{g}^{-1}$ is identified to $(\mathfrak{g}^{-1})^*(x_0)$, where of course $\mathfrak{g} = \mathfrak{su}(2, 1)$.

Definition 3.5. For each $m = (g_1x_0, g_2x_0, g_3x_0, g_4x_0) \in \mathcal{C}$ we set

(1) $D^4_m = D_{g_1x_0} \oplus D_{g_2x_0} \oplus D_{g_3x_0} \oplus D_{g_4x_0} = g_{1*}\mathfrak{g}^{-1} \oplus g_{2*}\mathfrak{g}^{-1} \oplus g_{3*}\mathfrak{g}^{-1} \oplus g_{4*}\mathfrak{g}^{-1}$,
(2) $D^3_m = D_{g_1x_0} \oplus D_{g_2x_0} \oplus D_{g_3x_0} \oplus \{0\} = g_{1*}\mathfrak{g}^{-1} \oplus g_{2*}\mathfrak{g}^{-1} \oplus g_{3*}\mathfrak{g}^{-1} \oplus \{0\}$,
(3) $D^2_m = D_{g_1x_0} \oplus D_{g_2x_0} \oplus \{0\} \oplus \{0\} = g_{1*}\mathfrak{g}^{-1} \oplus g_{2*}\mathfrak{g}^{-1} \oplus \{0\} \oplus \{0\}$,
(4) $D^1_m = D_{g_1x_0} \oplus \{0\} \oplus \{0\} \oplus \{0\} = g_{1*}\mathfrak{g}^{-1} \oplus \{0\} \oplus \{0\} \oplus \{0\}$.

Each D^i defines a distribution in $\mathcal{C}$ on which the natural complex operator $\mathbb{J}$ acts naturally. Therefore, they all give rise to CR structures on $\mathcal{C}$. By permuting the points we obtain a total of 15 different CR structures on $\mathcal{C}$ which turn out to be CR equivalent to the ones in the above list. To see this, consider for instance the map

$$\tilde{\phi} : (x_1, x_2, x_3, x_4) \to (x_2, x_1, x_3, x_4).$$

One checks easily that this is a CR map between the canonical CR structures with support $D \oplus \{0\} \oplus \{0\} \oplus \{0\}$ and $\{0\} \oplus D \oplus \{0\} \oplus \{0\}$ respectively.

Denote by $pr : \mathcal{C}' \to \mathcal{F}'$ the natural projection. For any $m \in \mathcal{C}'$ we have the exact sequence

$$0 \to \mathrm{T}_m(Gm) \to \mathrm{T}_m\mathcal{C}' \to \mathrm{T}_{pr(m)}\mathcal{F}' \to 0$$

and for any distribution D defined on $\mathcal{C}'$, we have the exact sequence

$$0 \to D_m \cap \mathrm{T}_m(Gm) \to D_m \to pr_*(D_m) \to 0.$$

We consider each natural distribution D^i, $i = 1, \ldots, 4$ as in Definition 3.5, restricted on $\mathcal{C}'$. For an arbitrary $m = (u_1, u_2, u_3, u_4) \in \mathcal{C}'$ we set

$$E^i = D^i_m \cap \mathrm{T}_m(Gm).$$

According to Proposition 2.5, $\dim(D^i/E^i) = \dim(pr_*(D^i))$ where $pr_*(D^i)$ is the distribution defined on $\mathcal{F}'$.

If instead of projecting the distributions D^i we had chosen to project another distribution obtained by permuting the points, then this would have not affect the picture, in the sense that the induced CR structure would have been equivalent to one obtained by projecting some D^i. For instance let $\tilde{\phi} : \mathcal{C}' \to \mathcal{C}'$ is a permutation which is CR with respect to a chosen CR structure on $\mathcal{C}'$ and its corresponding permutation on the target. Call the configuration spaces with corresponding CR structures ${\mathcal{C}_1}'$ and ${\mathcal{C}_2}'$ respectively. Than $pr_i : {\mathcal{C}_i}' \to {\mathcal{F}_i}'$ is CR and therefore $pr_2 \circ \tilde{\phi} : {\mathcal{F}_1}' \to {\mathcal{F}_2}'$ defines a CR bijection between the spaces.

From now on, our strategy is to explicitely compute $E^i \cap \mathbb{J}E^i$. In the case where $E^i \cap \mathbb{J}E^i = E^i$, by Proposition 2.5 it follows that there exists a complex structure $\mathbb{J}'$ acting on $pr_*(D^i)$ and preserving $pr_*(D^i)$. Theorems 1.1 and 1.2 will then follow from the next theorem whose proof covers the rest of this section.

Theorem 3.6. (1) *The CR structure D^4 on $\mathcal{C}'$ does not induce a CR structure on $\mathcal{F}'$.*
(2) *The CR structure D^3 on $\mathcal{C}'$ does not induce a CR structure on $\mathcal{F}'$.*
(3) *The CR structure D^2 on $\mathcal{C}^*$ induces a complex structure on $\mathcal{F}^*$.*
(4) *The CR structure D^1 on $\mathcal{C}'$ induces a CR structure on $\mathcal{F}'$.*

Proof. Before we go on carrying out the details of the proof of each case, we need to give a simple description of the tangent space at a point m of the orbit Gm. The proof of the following Lemma is straightforward.

Lemma 3.7. *Consider the diagonal action of $G = \mathrm{PU}(2,1)$ on $\mathcal{C}'$. Then, for each $m \in \mathcal{C}'$,*

$$\mathrm{T}_m(Gm) = \langle X^* \oplus X^* \oplus X^* \oplus X^* \ \ | \ X \in \mathfrak{g}\rangle$$

where X^* denotes the fundamental vector field defined by $X \in \mathfrak{g}$.

Consider an arbitrary $m \in \mathcal{C}'$ and normalise so that

$$m = (u_1, u_2, u_3, u_4) = ((z_1, t_1), (z_2, t_2), (0, 0), \infty)$$

Now it is easy to see that $g_i \in \mathrm{SU}(2, 1)$, $i = 1, \ldots, 4$ where

$$g_1 = \begin{bmatrix} 1 & -\bar{z}_1 & \frac{-|z_1|^2+it_1}{2} \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & -\bar{z}_2 & \frac{-|z_2|^2+it_2}{2} \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g_3 = Id = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

satisfy the relations

$$g_1 x_0 = (z_1, t_1), \quad g_2 x_0 = (z_1, t_2), \quad g_3 x_0 = x_0 = (0, 0), \quad g_4 x_0 = \infty.$$

Since u_i are not in the same $\mathbb{C}$-circle, the orbit Gm is eight dimensional and by Lemma 3.7, $\mathrm{T}_m(Gm)$ is identified to $\mathfrak{g}$. By equation

$$X^*(gx_0) = g_*(Ad_{g^{-1}}X)^*(x_0)$$

(see section 2.2), we obtain that $E^i = D^i_m \cap T_m(Gm)$ are given by

$$\begin{aligned}
E^4 &= \left\{ X \in \mathfrak{g} \mid Ad_{g_i^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 \right\}, \\
E^3 &= E^4 \cap \left\{ X \in \mathfrak{g} \mid Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h} \right\}, \\
E^2 &= E^4 \cap \left\{ X \in \mathfrak{g} \mid Ad_{g_3^{-1}}X = \ Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h} \right\}, \\
E^1 &= E^4 \cap \left\{ X \in \mathfrak{g} \mid Ad_{g_2^{-1}}X = Ad_{g_3^{-1}}X = Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h} \right\},
\end{aligned}$$

where, with some abuse of notation, we have set $E^i = E^i_m$. Finally, recall from section 2.1.3 that an arbitrary element $X \in \mathfrak{g}$ may be written as

$$X = \begin{bmatrix} \lambda & -\bar{z} & ti \\ -\bar{w} & ri & z \\ si & w & -\bar{\lambda} \end{bmatrix}, \tag{3.1}$$

where $r = -2\Im(\lambda)$.

We now proceed to examine each case of the Theorem separately.

Case 1: D^4.

Lemma 3.8. *Let g_3, g_4 and X be as above. Then*

(1) $Ad_{g_3^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$ *if and only if* $t = 0$.
(2) $Ad_{g_4^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$ *if and only if* $s = 0$.

Lemma 3.9. *Suppose that $Ad_{g_3^{-1}}X, Ad_{g_4^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$. Then $Ad_{g_i^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$ if and only if*

$$\Im\left((\lambda - z_i w)\frac{|z_i|^2 + it_i}{2} + \bar{z}_i z\right) = 0, \quad i = 1, 2.$$

Proof. Since

$$X = \begin{bmatrix} \lambda & -\bar{z} & 0 \\ -\bar{w} & -2i\Im\lambda & z \\ 0 & w & -\bar{\lambda} \end{bmatrix}, \tag{3.2}$$

we have

$$Ad_{g_4^{-1}}X = \begin{bmatrix} -\bar{\lambda} & w & 0 \\ z & 2i\Im\lambda & -\bar{w} \\ 0 & -\bar{z} & \lambda \end{bmatrix}. \tag{3.3}$$

Now for each $i = 1, 2$,

$$Ad_{g_i^{-1}}X = \begin{bmatrix} \lambda - \bar{z}_i\bar{w} & -w\frac{|z_i|^2+it_i}{2} - \bar{z}_i\bar{\lambda} + \bar{z}_i^2\bar{w} - \bar{z} & 2i\Im\left((\lambda - z_i w)\frac{|z_i|^2+it_i}{2} + \bar{z}_i z\right) \\ -\bar{w} & -2i\Im(\lambda - \bar{z}_i\bar{w}) & \bar{w}\frac{|z_i|^2-it_i}{2} + z_i\lambda - z_i^2 w + z \\ 0 & w & -\bar{\lambda} + z_i w \end{bmatrix} \tag{3.4}$$

and this completes the proof. ☐

Lemma 3.9 enables us to compute the dimension of $E^4_m = D^4_m \cap T_m(Gm)$. In fact we have

Lemma 3.10. *Given an arbitrary point $m \in \mathcal{C}'$ then $\dim(E^4_m) = \dim(D^4_m \cap T_m(Gm)) = 4$.*

Proof. Conditions of Lemma 3.9 may be written in real terms as

$$2y_i x - 2x_i y - t_i\lambda_1 - |z_i|^2\lambda_2 + (t_i x_i + y_i|z_i|^2)u + (-t_i y_i + x_i|z_i|^2)v = 0,$$

where $z = x + iy$, $w = u + iv$ and $\lambda = \lambda_1 + i\lambda_2$. Since $m \in \mathcal{C}'$, with no loss of generality we may choose an element of the orbit such that $x_1 = 1$ and $y_1 = 0$. In this case we are only left with the equations

$$\begin{aligned} &-2y - t_1\lambda_1 - \lambda_2 + t_1 u + v = 0, \\ &2y_2 x - 2x_2 y - t_2\lambda_1 - |z_2|^2\lambda_2 + (t_2 x_2 + y_2|z_2|^2)u + (-t_2 y_2 + x_2|z_2|^2)v = 0. \end{aligned}$$

One may then easily verify that the rank of the above system of equations is 2. ☐

Proof of (1). We shall show that

$$E^4 \cap \mathbb{J}E^4 = \{0\}.$$

Note that for $X^* \in E^4$,

$$\mathbb{J}X^*(g_i x_0) = g_{i*}(\mathbb{J}Ad_{g_i^{-1}}X)^*(x_0)$$

for all g_i. Therefore $\mathbb{J}X^* \in E^4$ if and only if for all g_i, $\mathbb{J}Ad_{g_i^{-1}}X \equiv Ad_{g_i^{-1}}Y \bmod \mathfrak{h}$ for some $Y \in \mathfrak{g}^{-1} \oplus \mathfrak{h}$. Thus if such a Y exists, then it should be of the form

$$Y = \begin{bmatrix} \lambda' & i\bar{z} & 0 \\ -i\bar{w} & -2i\Im(\tilde{\lambda}) & iz \\ 0 & -iw & -\bar{\lambda}' \end{bmatrix},$$

where $\lambda' = i(\lambda - 2z_1 w) = i(\lambda - 2z_2 w)$. Observe that if $z_1 \neq z_2$ then Y is defined only if $w = 0$. Using Lemma 3.9 we then have

$$\Im\left(\lambda\frac{|z_i|^2 + it_i}{2} + \bar{z}_i z\right) = 0, \quad i = 1, 2.$$

This is equivalent to the system

$$2y + \lambda_1 t_1 + \lambda_2 = 0,$$

$$2x_2 y - 2y_2 x + \lambda_1 t_2 + \lambda_2 |z_2| = 0.$$

The rank of the system is less than two only if $|z_1|^2 + it_1$ and $|z_2|^2 + it_2$ are linearly dependent. We conclude that $\dim(E^4_m \cap \mathbb{J}E^4_m) = 0$ at points of $\mathcal{C}'$. □

Case 2: D^3. The following condition is obtained from the condition in lemma 3.9 by making $w = 0$.

Lemma 3.11. *Suppose $Ad_{g_3^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$ and $Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h}$. Then $Ad_{g_i^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$ if and only if*

$$\Im\left(\lambda\frac{|z_i|^2 + it_i}{2} + \bar{z}_i z\right) = 0, \quad i = 1, 2.$$

Proof. By Lemma 3.8,

$$X = \begin{bmatrix} \lambda & -\bar{z} & 0 \\ -\bar{w} & ri & z \\ 0 & w & -\bar{\lambda} \end{bmatrix}$$

where $r = -2\Im(\lambda)$. Equation 3.3 and relation

$$Ad_{g_4^{-1}}X = 0 \text{ modulo } \mathfrak{h}$$

then yield $w = 0$. Therefore, from equations 3.4 we have

$$Ad_{g_i^{-1}}X = \begin{bmatrix} \lambda & -\bar{z}_i\bar{\lambda} - \bar{z} & 2i\Im\left(\lambda\frac{|z_i|^2+it_i}{2} + \bar{z}_i z\right) \\ 0 & -2i\Im(\lambda) & z_i\lambda + z \\ 0 & 0 & -\bar{\lambda} \end{bmatrix}, \quad i = 1, 2.$$

□

Lemma 3.12. *Given an arbitrary point $m \in \mathcal{C}'$ then* $\dim(E^3_m) = \dim(D^3_m \cap T_m(Gm)) = 2$.

Proof. As in the proof of case (1), we first write the conditions of Lemma 3.11 in real terms as

$$2y_i x - 2x_i y - t_i\lambda_1 - |z_i|^2\lambda_2 = 0.$$

Here, $z = x + iy$ and $\lambda = \lambda_1 + i\lambda_2$. The rank of the above system is two and therefore $\dim(E^3_m) = 2$. □

Proof of (2). We use a similar argument to that in the proof of case (1) to obtain $E^3 \cap \mathbb{J}E^3 = \{0\}$. □

Case 3: D^2. The following condition is obtained from the condition in Lemma 3.9 by making $w = 0$ and $z = 0$.

Lemma 3.13. *Suppose* $Ad_{g_3^{-1}}X = Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h}$. *Then* $Ad_{g_i^{-1}}X \in \mathfrak{g}^{-1} \bmod \mathfrak{h}$ *if and only if*

$$\Im\left(\lambda\frac{|z_i|^2 + it_i}{2}\right) = 0, \quad i = 1, 2.$$

Proof. We use Lemma 3.8 and the relation $Ad_{g_3^{-1}}X = Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h}$ to write $X \in \mathfrak{g}$ as

$$X = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & ri & 0 \\ 0 & 0 & -\bar{\lambda} \end{bmatrix},$$

where $r = -2\Im(\lambda)$. Therefore,

$$Ad_{g_i^{-1}}X = \begin{bmatrix} \lambda & -\bar{z}_i\bar{\lambda} & 2i\Im\left(\lambda\frac{|z_i|^2+it_i}{2}\right) \\ 0 & -2i\Im(\lambda) & z_i\lambda \\ 0 & 0 & -\bar{\lambda} \end{bmatrix}, \quad i = 1, 2$$

and the result follows. □

From Lemma 3.13 we obtain

Lemma 3.14. *Given* $m \in \mathcal{C}^*$, *then* $\dim(D^2_m \cap T_m(Gm)) = 0$.

Proof of (3). It follows directly by making use of Proposition 2.5. □

Case 4: D^1. Using Lemma 3.8 and the relations $Ad_{g_2^{-1}}X = Ad_{g_3^{-1}}X = Ad_{g_4^{-1}}X = 0 \bmod \mathfrak{h}$ we have $X = 0$. We immediately obtain

Lemma 3.15. *Given an arbitrary point* $m \in \mathcal{C}'$ *then* $\dim(D^1_m \cap \mathrm{T}_m(Gm)) = 0$.

Proof of (4). It follows directly by making use of Proposition 2.5. □

The proof of Theorem 3.6 is now complete.

□

In the next section we shall describe the complex structure on $\mathcal{F}^*$ explicitly. This will complete the proof of Theorem 1.1.

3.2. **Coordinate description of the complex structure.** Let $m = (u_1, u_2, u_3, u_4) \in \mathcal{C}'$. We may normalise so that

$$u_1 = (z_1, t_1) \quad u_2 = (z_2, t_2) \quad u_3 = 0 \quad u_4 = \infty$$

where z_1 and z_2 are not simultaneously zero. Let $H' \subset \mathfrak{N} \times \mathfrak{N}$ be the configuration space of two distinct points in the Heisenberg group such that none of them is $(0, 0)$ and not both of them are contained in the vertical axis $\{(0, t)\}$. Define $H' \to \mathbb{C}P^1 \times \mathbb{C}P^1$ by

$$((z_1, t_1), (z_2, t_2)) \to ([\ |z_1|^2 - it_1, |z_2|^2 - it_2\ ], [\ z_1, z_2\ ]). \tag{3.5}$$

It is a CR map and its image is contained in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Moreover, it is clearly invariant under the action of $\mathbb{C}^*$ (that is the stabiliser of 0 and ∞ given by $\lambda.(z,t) = (\lambda z, |\lambda|^2 t)$), therefore it projects to a well defined map on the quotient space $H'/\mathbb{C}^*$, namely

$$p : \mathcal{C}' \to \mathbb{C}P^1 \times \mathbb{C}P^1.$$

Proposition 3.16.

(1) *The image of* p *is* $((\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}P^1) \cup V^*$ *where*

$$V^* = \left\{ \ (|\lambda|^2, \lambda) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid \lambda \in \mathbb{C} \setminus \{0,1\} \ \right\}.$$

(2) $p : \mathcal{C}^* \to (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}P^1$ *is a surjective CR map. It induces a biholomorphism* $\mathcal{F}^* = \mathcal{C}^*/\mathrm{PU}(2,1) \to (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}P^1$.

Proof. We show that the quotient map $H'/\mathbb{C}^* \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is a biholomorphism outside the set $((1,t_1),(\lambda,|\lambda|^2 t_1))$, $\lambda \neq 0,1$. Observe that this set is diffeomorphic to $\mathbb{R} \times (\mathbb{C} \setminus \{0,1\})$ and its image is the set $\{(|\lambda|^2,\lambda) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid \lambda \in \mathbb{C} \setminus \{0,1\}\ \}$ which is diffeomorphic to $\mathbb{C} \setminus \{0,1\}$. In fact, given $(z_1,z_2) \in \mathbb{C}^2$ and

$$\omega_0 = \frac{|z_2|^2 - it_2}{|z_1|^2 - it_1} = a + ib,$$

we obtain $t_1 b = |z_2|^2 - a|z_1|^2$ and $t_2 = at_1 - b|z_1|^2$. Therefore, if $b \neq 0$, this determines t_1 and t_2. If $b = 0$ then $|z_2|^2 = a$, $t_2 = |z_2|^2 t_1$ and t_1 may be chosen arbitrarily. $\square$

An explicit description of the map $p : \mathcal{C}^* \to \mathbb{C}P^1 \times \mathbb{C}P^1$ in terms of the cross-ratio parameters (see next section) can be obtained by writing

$$u_1 = (z_1,t_1) \quad u_2 = (z_2,t_2) \quad u_3 = (0,0) \quad u_4 = \infty.$$

Using formulae in [F] we have

- $\frac{|z_2|^2 - it_2}{|z_1|^2 - it_1} = \omega_0 = \mathbb{X}[u_4,u_3,u_1,u_2]$
- $\frac{z_1}{z_2} = \frac{1+\omega_1(\omega_2 - 1)}{\omega_0\omega_1\omega_2 + 1}$, where $\omega_1 = \mathbb{X}[u_4,u_2,u_3,u_1]$ and $\omega_2 = \mathbb{X}[u_2,u_3,u_4,u_1]$.

4. The Cross-Ratio Variety

4.1. **The cross-ratio variety and the space $\mathcal{F}$.** In $\mathbb{C}^3$ we consider the equations

$$f_1(\omega_0,\omega_1,\omega_2) = |\omega_0|^2|\omega_1|^2|\omega_2|^2 - 1 = 0, \tag{4.1}$$

$$f_2(\omega_0,\omega_1,\omega_2) = |\omega_0 - 1|^2 - 1 + |\omega_0|^2(|\omega_1 - 1|^2 - 1) + |\omega_0|^2|\omega_1|^2(|\omega_2 - 1|^2 - 1) = 0, \tag{4.2}$$

which define a real algebraic variety of $\mathbb{C}^3$.

Definition 4.1. We call the solution set of 4.1 and 4.2 the *cross-ratio variety* and we denote it by $\mathcal{X}$.

As in the case of the configuration space $\mathcal{C}$ there are certain subsets of $\mathcal{X}$ which are of particular importance.

Definition 4.2. Let $\mathcal{X}_{\mathbb{R}}$ be the set of the real points of $\mathcal{X}$. We call

$$S_{\mathbb{R}} = \left\{(\omega_0, \omega_1, \omega_2) \in \mathcal{X}_{\mathbb{R}} : \omega_2 + \frac{1}{\omega_1} = 1, \quad \omega_1 + \frac{1}{\omega_0} = 1, \quad \omega_0 + \frac{1}{\omega_2} = 1\right\} \tag{4.3}$$

the *real singular set of* $\mathcal{X}$ and we call

$$S^* = \{(\omega_0, \omega_1, \omega_2) \in \mathcal{X} : \omega_0 > 0\ \} \cup S_{\mathbb{R}} \tag{4.4}$$

the *complex singular set of* $\mathcal{X}$. We also call

$$S_{\mathbb{C}} = \left\{(\omega_0, \omega_1, \omega_2) \in \mathcal{X} : \omega_2 + \frac{1}{\omega_1} = 1, \quad \omega_1 + \frac{1}{\omega_0} = 1, \quad \omega_0 + \frac{1}{\omega_2} = 1\right\} \tag{4.5}$$

the *CR singular set of* $\mathcal{X}$.

One may check that $S_{\mathbb{R}}$ is diffeomorphic to $\mathbb{R} \setminus \{0,1\}$, $S^* \setminus S_{\mathbb{R}}$ is diffeomorphic to $\mathbb{R} \times (\mathbb{C} \setminus \{0,1\})$ and $S_{\mathbb{C}}$ is biholomorphic to $\mathbb{C} \setminus \{0,1\}$ (see Proposition 4.4).

Our goal is to prove Theorem 1.3, i.e. there exists a CR-embedding of $\mathcal{F}'$ onto $\mathcal{X}'$. We start with the following Proposition, [F], which shows the intimate relation between the space $\mathcal{F} = \mathcal{C}/\mathrm{PU}(2,1)$ and the cross-ratio variety $\mathcal{X}$.

Proposition 4.3. *There exists a bijection π between $\mathcal{F}$ and $\mathcal{X}$.*

The bijection π is defined by the relation $\pi \circ pr = \varpi$ where $\varpi : \mathcal{C} \to \mathcal{X}$ is the surjection given by

$$\begin{aligned} \varpi((u_1, u_2, u_3, u_4)) &= (\omega_0, \omega_1, \omega_2) \\ &= (\mathbb{X}[u_3, u_4, u_2, u_1], \mathbb{X}[u_4, u_2, u_3, u_1], \mathbb{X}[u_2, u_3, u_4, u_1]) \,. \end{aligned}$$

and $pr : \mathcal{C} \to \mathcal{F}$ is the natural projection to the quotient.

Proposition 4.4. *Let $\varpi : \mathcal{C} \to \mathcal{X}$ be as in Proposition 4.3. The following hold.*

(1) $\varpi : \mathcal{C} \to \mathbb{C}^3$ *is a CR map.*
(2) $\varpi(\mathcal{C}') = \pi(\mathcal{F}') = \mathcal{X}' = \mathcal{X} \setminus S_{\mathbb{R}}$.
(3) $\varpi(\mathcal{C}^*) = \pi(\mathcal{F}^*) = \mathcal{X}^* = \mathcal{X} \setminus S^*$.
(4) $\varpi(\mathcal{C}'') = \pi(\mathcal{F}'') = \mathcal{X}'' = \mathcal{X} \setminus S_{\mathbb{C}}$. *Moreover,*

$$S_{\mathbb{C}} = \{(\omega_0, \omega_1, \omega_2) \in \mathcal{X} : \omega_0\omega_1\omega_2 = -1\} \,.$$

Proof. With no loss of generality we may suppose that

$$u_1 = (z, t), \quad u_2 = (w, s), \quad u_3 = (0, 0), \quad u_4 = \infty.$$

Then,

$$\omega_0 = \mathbb{X}[u_3, u_4, u_2, u_1] = \mathbb{X}[u_4, u_3, u_1, u_2] = \frac{|w|^2 - si}{|z|^2 - ti},$$

$$\omega_1 = \mathbb{X}[u_4, u_2, u_3, u_1] = \frac{|z|^2 - ti + |w|^2 + si - 2z\bar{w}}{|w|^2 + si},$$

$$\omega_2 = \mathbb{X}[u_2, u_3, u_4, u_1] = \overline{\mathbb{X}[u_4, u_1, u_2, u_3]} = \frac{|z|^2 - ti}{|z|^2 - ti + |w|^2 + si - 2\bar{w}z}.$$

Using the above relations, we may verify that ϖ is indeed a CR map and this proves (1). To prove (4), we observe that

$$\omega_0\omega_1\omega_2 = \frac{|w|^2 - si}{|w|^2 + si}$$

and it is clear that $\omega_0\omega_1\omega_2 = -1$ if and only if $w = 0$, that is if and only if u_2, u_3, u_4 lie in the same $\mathbb{C}$-circle. Suppose now that $(\omega_0, \omega_1, \omega_2) \in S_{\mathbb{C}}$. Then

$$\omega_0\omega_1\omega_2 = (\omega_0 - 1)\omega_2 = (\omega_0 - 1)\frac{1}{1-\omega_0} = -1,$$

and thus $w = 0$. On the other hand, if $w = 0$ we obtain

$$\omega_0 = \frac{-si}{|z|^2 - ti}, \quad \omega_1 = \frac{|z|^2 - ti + si}{si}, \quad \omega_2 = \frac{|z|^2 - ti}{|z|^2 - ti + si}.$$

Simple computations show that $(\omega_0, \omega_1, \omega_2) \in S_{\mathbb{C}}$ and thus (4) is proved. Another proof follows from the identity $\omega_0\omega_1\omega_2 = e^{2i\mathbb{A}(u_2,u_3,u_4)}$ where $\mathbb{A}$ is Cartan's angular invariant.

In order to prove (2), we observe that $S_{\mathbb{R}} \subset S_{\mathbb{C}}$, hence $w = 0$ for any $(\omega_0, \omega_1, \omega_2) \in S_{\mathbb{R}}$. Thus $\omega_0 = \mathbb{X}[u_3, u_4, u_1, u_2] = \frac{-si}{|z|^2 - ti}$ which is real if and only if $z = 0$ and therefore our assertion follows. The proof of (3) is similar. This completes the proof. □

Proposition 4.5. *The map $\pi : \mathcal{F}' \to \mathbb{C}^3$ is a CR map of rank four.*

Proof. Recall that ϖ is a CR map. The rest of the proof lies on the computation of the rank of the Jacobian matrix $D\varpi$. Without loss of generality, we may suppose that $w \neq 0$ (if $w = 0$ a permutation of u_1 and u_2 will not change the rank). We may also choose $w = 1$ by normalising using an appropriate dilation. ϖ is given then by

$$\omega_0 = \frac{1 - si}{|z|^2 - ti},$$

$$\omega_1 = \frac{|z|^2 - ti + 1 + si - 2z}{1 + si},$$

$$\omega_2 = \frac{|z|^2 - ti}{|z|^2 - ti + 1 + si - 2z}.$$

Writing $z = x + iy$, the map ϖ is given in real terms by

$$(x, y, t, s) \to (\Re(\omega_0), \Im(\omega_0), \Re(\omega_1), \Im(\omega_1), \Re(\omega_2), \Im(\omega_2)).$$

The Jacobian matrix $D\varpi$ is a 6×4 matrix and after a simple calculation one shows that the subdeterminant obtained by deleting the fifth and sixth line (we denote it by d_{56}) vanishes if and only if

$$s(x^2 + y^2) = 2tx - t + 2(x^2 + y^2)y.$$

If $x = y = 0$ then $t = 0$ (in order to have a vanishing determinant) but this is impossible otherwise the point $(z = x + iy, t)$ would coincide with $(0, 0)$. We can analyse therefore the other determinants supposing that $s = \frac{2tx - t + 2(x^2+y^2)y}{x^2+y^2}$. The subdeterminant d_{45} (obtained by deleting the fourth and fifth line) vanishes if and only if $x = 1/2$ or $x^2y + tx + y^3 = 0$. In the first case, a computation shows that d_{35} does not vanish. In the second case we may suppose that $x \neq 0$ (otherwise $y = 0$ and d_{56} would be non-vanishing). Therefore the only possibility for the matrix to have rank less than four is if

$$t = -\frac{y(x^2 + y^2)}{x}$$

and

$$s = \frac{y}{x}.$$

Now, computing d_{35} with those values we obtain

$$d_{35} = \frac{8x^5}{(1 - 2x + x^2 + y^2)(x^2 + y^2)^5}$$

which vanishes only if $x = 0$. But we already observed that this cannot happen.

□

We shall also need the following Theorem which we only state here for the moment; its proof is presented in section 4.3.

Theorem 4.6. *Let $\mathcal{X}' = \mathcal{X} \setminus S_{\mathbb{R}}$. Then $\mathcal{X}'$ admits a 4-dimensional real analytic manifold structure. Moreover it has a codimension two CR structure.*

Proof of Theorem 1.3. The proof follows immediately from the fact that $\pi : \mathcal{F}' \to \mathcal{X}'$ is a CR map of rank 4 (Proposition 4.5) and that $\mathcal{X}'$ has a codimension two CR structure (Theorem 4.6). □

4.2. **CR structures of codimension 2 in varieties of $\mathbb{C}^3$.** Let $\mathcal{V}$ be the real subvariety of $\mathbb{C}^3$ defined by the equations $f_i(z_0, z_1, z_2) = 0$, where $f_i : \mathbb{C}^3 \to \mathbb{R}$, $i = 1, 2$ are smooth functions. If the Jacobian matrix

$$D = \begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_0} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_0} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

is of constant rank 2 everywhere in $\mathcal{V}$, then $\mathcal{V}$ is a (real) submanifold of $\mathbb{C}^3$ and $\dim(\mathcal{V}) = 4$. The tangent space of $\mathcal{V}$ at a point $\omega = (\omega_0, \omega_1, \omega_2) \in \mathcal{V}$ is

$$\mathrm{T}_\omega(\mathcal{V}) = \{X \in \mathrm{T}(\mathbb{C}^3) : (df_i)_\omega(X) = 0, i = 1, 2\} = \bigcap_{i=1}^{2} \ker(df_i)_\omega.$$

Denote by $\mathbb{J}$ the natural complex structure of $\mathbb{C}^3$. For each $\omega \in \mathcal{V}$ define the *maximal complex subspace* $\mathcal{E}_\omega \subset \mathrm{T}_\omega(\mathcal{V})$ to be

$$\mathcal{E}_\omega = \mathrm{T}_\omega(\mathcal{V}) \cap \mathbb{J}\mathrm{T}_\omega(\mathcal{V}). \tag{4.6}$$

The complex subbundle $\mathcal{E} = \{(\omega, \mathcal{E}_\omega), \omega \in \mathcal{V}\}$ is called the *maximal complex subbundle of* $\mathrm{T}(\mathcal{V})$. Setting

$$d^c f_i(X) = -df_i(\mathbb{J}X),$$

we may equivalently write

$$\mathcal{E}_\omega = \left(\bigcap_{i=1}^{2} \ker(df_i)_\omega\right) \bigcap \left(\bigcap_{i=1}^{2} \ker(d^c f_i)_\omega\right). \tag{4.7}$$

We are mostly interested on the dimension of each $\mathcal{E}_\omega$. Let $\mathbb{J}$ act on $\mathcal{E}$ and consider

$$\mathcal{E}_\omega^{(1,0)} = \{Z \in \mathrm{T}^{(1,0)}(\mathbb{C}^3) : \partial(f_i)_\omega(Z) = 0, i = 1, 2\} = \bigcap_{i=1}^{2} \ker(\partial f_i)_\omega, \tag{4.8}$$

where

$$\partial(f)_\omega = \sum \left(\frac{\partial f}{\partial z_i}\right)_\omega (dz_i)_\omega. \tag{4.9}$$

The collection of all $\mathcal{E}_\omega^{(1,0)}$ is the subbundle

$$\mathcal{E}^{(1,0)} = \{(\omega, \mathcal{E}_\omega^{(1,0)}), \omega \in \mathcal{V}\}$$

of $\mathrm{T}(\mathcal{V})^{\mathbb{C}} = \mathrm{T}(\mathcal{V}) \otimes \mathbb{C}$. Observe that $Z = X - i\mathbb{J}X \in \mathcal{E}^{(1,0)}$ if and only if $X \in \mathcal{E}$. This follows immediately from the relation

$$\begin{aligned} 2\partial f_i(Z) &= (df_i + id^c f_i)(X - i\mathbb{J}X) \\ &= (df_i(X) + d^c f_i(\mathbb{J}X)) + i(d^c f_i(X) - df_i(\mathbb{J}X)). \end{aligned}$$

We show now that the dimension of $\mathcal{E}$ is related to the the rank of the matrix

$$D^{(1,0)} = \begin{bmatrix} \frac{\partial f_1}{\partial z_0} & \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_0} & \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix}.$$

Proposition 4.7. *For each $\omega \in \mathcal{V}$, then the following holds.*

1. *If* $\mathrm{rank}(D^{(1,0)}) = 1$ *at* ω *then* $\dim(\mathcal{E}_\omega^{(1,0)}) = 2$ *and* $\dim(\mathcal{E}_\omega) = 4$.
2. *If* $\mathrm{rank}(D^{(1,0)}) = 2$ *at* ω *then* $\dim(\mathcal{E}_\omega^{(1,0)}) = 1$ *and* $\dim(\mathcal{E}_\omega) = 2$.

Proof. The dimension of $\mathcal{E}_\omega^{(1,0)}$ is the dimension of the solution set of the equations

$$\partial(f_i)_\omega = \sum_{i=0}^{2} \left(\frac{\partial f_i}{\partial z_i}\right)_\omega (dz_i)_\omega = 0. \tag{4.10}$$

Consider the subdeterminants

$$D_{z_0,z_1} = \left|\frac{\partial(f_1, f_2)}{\partial(z_0, z_1)}\right|, \quad D_{z_1,z_2} = \left|\frac{\partial(f_1, f_2)}{\partial(z_1, z_2)}\right|, \quad D_{z_2,z_0} = \left|\frac{\partial(f_1, f_2)}{\partial(z_2, z_0)}\right|.$$

Suppose first that $\mathrm{rank}(D^{(1,0)}) = 1$ at ω. Then all the subdeterminants vanish at ω and with no loss of generality we may suppose that $(f_1)_{z_0}(\omega) \neq 0$. Then the system 4.10 reduces to the single equation

$$(dz_0)_\omega = -\frac{(f_1)_{z_1}(\omega)}{(f_1)_{z_0}(\omega)}(dz_1)_\omega - \frac{(f_1)_{z_2}(\omega)}{(f_1)_{z_0}(\omega)}(dz_2)_\omega.$$

Thus a basis for $\mathcal{E}_\omega^{(1,0)}$ consists of the (1,0) vectors

$$(Z_1)_\omega = (f_1)_{z_0}(\omega)\left(\frac{\partial}{\partial z_1}\right)_\omega - (f_1)_{z_1}(\omega)\left(\frac{\partial}{\partial z_0}\right)_\omega,$$

$$(Z_2)_\omega = (f_1)_{z_0}(\omega)\left(\frac{\partial}{\partial z_2}\right)_\omega - (f_1)_{z_2}(\omega)\left(\frac{\partial}{\partial z_0}\right)_\omega.$$

It follows that a basis for $\mathcal{E}_\omega$ is

$$\{X_1, X_2, \mathbb{J}X_1, \mathbb{J}X_2\}_\omega$$

where $Z_i = X_i - i\mathbb{J}X_i$.

Suppose now that the rank of $\mathcal{E}_\omega^{(1,0)}$ is 2. With no loss of generality we may suppose that D_{z_1,z_2} is not zero at ω. We may then solve 4.10 to obtain

$$(dz_1)_\omega = \frac{D_{z_2,z_0}(\omega)}{D_{z_1,z_2}(\omega)}(dz_0)_\omega, \quad (dz_2)_\omega = \frac{D_{z_1,z_0}(\omega)}{D_{z_1,z_2}(\omega)}(dz_0)_\omega.$$

Thus, a basis for $\mathcal{E}_\omega^{(1,0)}$ consists of the single vector

$$Z_\omega = D_{z_1,z_2}(\omega)\left(\frac{\partial}{\partial z_0}\right)_\omega + D_{z_2,z_0}(\omega)\left(\frac{\partial}{\partial z_1}\right)_\omega + D_{z_0,z_1}(\omega)\left(\frac{\partial}{\partial z_2}\right)_\omega$$

and a basis for $\mathcal{E}_\omega$ is $\{X, \mathbb{J}X\}_\omega$ where $Z = X - i\mathbb{J}X$. □

Definition 4.8. Suppose that $\mathrm{rank}(D^{(1,0)}) = 2$ everywhere on a subset $\mathcal{W}$ of a 4-dimensional variety $\mathcal{V}$ of $\mathbb{C}^3$. Then $(\mathcal{E}, \mathbb{J})$ is called a *CR structure of codimension 2* defined on $\mathcal{W}$. The subset $S_\mathbb{C} = \mathcal{V} \setminus \mathcal{W}$ is called the *CR singular set of* $\mathcal{V}$.

The CR-structure $\mathcal{E} = \{X, \mathbb{J}X\}$ is called *Levi flat* if it is integrable. By Frobenius' Theorem, this happens if and only if $[X, \mathbb{J}X] \in \mathcal{E}$. Equivalently, $[Z, \bar{Z}] = 0$ which happens if and only if the Levi forms $\partial\overline{\partial} f_i$ satisfy the condition $\partial\overline{\partial} f_i(Z, \overline{Z}) = 0$, $i = 1, 2$. Denote by $[\partial\overline{\partial} f_i]_{j,k}$ the complex 3×3 matrices of $\partial\overline{\partial} f_i$. For $i = 1, 2$, and $\omega \in \mathcal{V}'$ consider the functions

$$L_i(\omega) = (\partial\overline{\partial} f_i)_\omega(Z_\omega, \overline{Z}_\omega) = \begin{bmatrix} D_{z_1,z_2} & D_{z_2,z_0} & D_{z_0,z_1} \end{bmatrix}_\omega [\partial\overline{\partial} f_i]_\omega \begin{bmatrix} D_{\overline{z_1},\overline{z_2}} \\ D_{\overline{z_2},\overline{z_0}} \\ D_{\overline{z_0},\overline{z_1}} \end{bmatrix}_\omega. \tag{4.11}$$

We thus have that integrability of $\mathcal{E}$ is equivalent to the vanishing of the functions 4.11.

4.3. **Real manifold and CR structure of $\mathcal{X}$.** This section is mainly devoted to the proof of Theorem 4.6. For this we shall need two computational Lemmas.

Lemma 4.9. *Let $\omega = (\omega_0, \omega_1, \omega_2) \in \mathcal{X}$. Then*

$$D^{(1,0)}(\omega) = \begin{bmatrix} \frac{\partial f_1}{\partial z_0} & \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_0} & \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix}_\omega = \begin{bmatrix} \frac{1}{\omega_0} & \frac{1}{\omega_1} & \frac{1}{\omega_2} \\ \frac{\overline{\omega_0}}{\omega_0} & \frac{\overline{\omega_1}|\omega_2-1|^2-1}{|\omega_1|^2|\omega_2|^2} & \frac{1}{\omega_2}\left(1 - \frac{1}{\overline{\omega_2}}\right) \end{bmatrix}.$$

Thus $\mathrm{rank}(D^{(1,0)}) = 2$ *everywhere in* $\mathcal{X}'' = \mathcal{X} \setminus S_\mathbb{C}$ *where*

$$S_\mathbb{C} = \left\{\omega = (\omega_0, \omega_1, \omega_2) \in \mathcal{X} : \omega_1 + \frac{1}{\omega_0} = 1, \omega_0 + \frac{1}{\omega_2} = 1, \omega_2 + \frac{1}{\omega_1} = 1\right\} \tag{4.12}$$

is the CR singular set 4.5 of $\mathcal{X}$.

Proof. In the first place, the holomorphic derivatives of f_1 at $\mathbb{C}^3$ are

$$\frac{\partial f_1}{\partial z_0} = \overline{z_0}|z_1|^2|z_2|^2, \quad \frac{\partial f_1}{\partial z_1} = \overline{z_1}|z_0|^2|z_2|^2, \quad \frac{\partial f_1}{\partial z_2} = \overline{z_2}|z_1|^2|z_0|^2$$

and the elements of the first row of $D^{(1,0)}(\omega)$ are obtained using the equation $f_1(\omega)=0$. The holomorphic derivatives of f_2 at $\mathbb{C}^3$ are

$$\frac{\partial f_2}{\partial z_0} = \overline{z_0} - 1 + \overline{z_0}(|z_1-1|^2-1) + \overline{z_0}|z_1|^2(|z_2-1|^2-1),$$
$$\frac{\partial f_2}{\partial z_1} = |z_0|^2(\overline{z_1}|z_2-1|^2-1),$$
$$\frac{\partial f_2}{\partial z_2} = |z_0|^2|z_1|^2(\overline{z_2}-1).$$

Using $f_2(\omega)=0$ we obtain the elements of the second row of $D^{(1,0)}(\omega)$. Now, the Jacobian subdeterminants

$$D_{z_i,z_j} = \left|\frac{\partial(f_1,f_2)}{\partial(z_i,z_j)}\right|, \quad i,j=0,1,2 \quad i\neq j.$$

calculated at ω are

$$D_{z_1,z_2}(\omega) = \frac{1}{\omega_1\omega_2|\omega_2|^2}\left(\overline{\omega}_2 + \frac{1}{\overline{\omega}_1} - 1\right), \tag{4.13}$$

$$D_{z_2,z_0}(\omega) = \frac{1}{\omega_0\omega_2}\left(\overline{\omega}_0 + \frac{1}{\overline{\omega}_2} - 1\right), \tag{4.14}$$

$$D_{z_0,z_1}(\omega) = \frac{\overline{\omega}_0}{\omega_1}\left(\overline{\omega}_1 + \frac{1}{\overline{\omega}_0} - 1\right). \tag{4.15}$$

The rank of $D^{(1,0)}$ is 1 at points of $\mathcal{X}$ where all subdeterminants 4.13, 4.14 and 4.15 vanish simultaneously. This completes the proof. □

Lemma 4.9 allows us to compute the Jacobian matrix

$$D(\omega) = \begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_0} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_0} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2}, \end{bmatrix}_\omega$$

for each $\omega=(\omega_0,\omega_1,\omega_2)\in\mathcal{X}$, $\omega_i=\xi_i+i\eta_i$ by using the equations

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i}{\partial z_i} + \frac{\partial f_i}{\partial \bar{z}_i}, \quad \frac{\partial f_i}{\partial y_j} = i\left(\frac{\partial f_i}{\partial z_i} - \frac{\partial f_i}{\partial \bar{z}_i}\right).$$

Lemma 4.10. *Let $\omega=(\omega_0,\omega_1,\omega_2)\in\mathcal{X}$. Then*

$$D(\omega) = \begin{bmatrix} \frac{2\xi_0}{|\omega_0|^2} & \frac{2xi_1}{|\omega_1|^2} & \frac{2\xi_2}{|\omega_2|^2} & -\frac{2\eta_0}{|\omega_0|^2} & -\frac{2\eta_1}{|\omega_1|^2} & -\frac{2\eta_2}{|\omega_2|^2} \\ \frac{2(\xi_0^2-\eta_0^2)}{|\omega_0|^2} & \frac{2(|\omega_2-1|^2\xi_1-1)}{|\omega_1|^2|\omega_2|^2} & \frac{2(\xi_2-1)}{|\omega_2|^2} & -\frac{4\xi_0\eta_0}{|\omega_0|^2} & \frac{2\eta_1|\omega_2-1|^2}{|\omega_1|^2|\omega_2|^2} & \frac{2\eta_2}{|\omega_2|^2} \end{bmatrix}.$$

The rank of $D(\omega)$ is everywhere 2 in $\mathcal{X}'=\mathcal{X}-S_{\mathbb{R}}$ where

$$S_{\mathbb{R}} = \left\{\omega=(\omega_0,\omega_1,\omega_2)\in\mathcal{X}_{\mathbb{R}} : \omega_1+\frac{1}{\omega_0}=1, \omega_0+\frac{1}{\omega_2}=1, \omega_2+\frac{1}{\omega_1}=1\right\}$$

is the real singular set 4.3 of $\mathcal{X}$.

Proof. We only prove that $\mathrm{rank}(D) = 2$ at points of $\mathcal{X}'$. Suppose there exists a $\omega \in \mathcal{X}$ such that $\mathrm{rank}(D(\omega)) = 1$. Then, all Jacobian subdeterminants

$$D_{x_i,x_j} = \left|\frac{\partial(f_1,f_2)}{\partial(x_i,x_j)}\right|, \quad D_{x_i,y_j} = \left|\frac{\partial(f_1,f_2)}{\partial(x_i,y_j)}\right|, \quad D_{y_i,y_j} = \left|\frac{\partial(f_1,f_2)}{\partial(y_i,y_j)}\right|, \tag{4.16}$$

calculated at ω must vanish. In the first place, it is easy to see that $D_{x_0,y_0}(\omega) = 0$ if and only if $\eta_0 = 0$. We then have

$$D_{x_0,x_2}(\omega) = \frac{4\xi_0}{|\omega_0\omega_2|^2}(-\xi_0\xi_2 + \xi_2 - 1) = 0.$$

ξ_0 can not be zero, otherwise $\omega_0 = 0$. On the other hand, ξ_2 can not be zero, otherwise $D_{x_0,x_2}(\omega) = -2\xi_0/|\omega_0\omega_2|^2 = 0$ would deduce $\xi_0 = 0$. Thus we obtain that at points where $D(\omega)$ has rank 1 we must have

$$\xi_0 + \frac{1}{\xi_2} = 1.$$

Now, we observe that for all points $\omega = (\omega_0, \omega_1, \omega_2) \in \mathcal{X}$,

$$2\Re\left(\frac{1}{\omega_0} + \omega_2\right) \geq 1 \tag{4.17}$$

and equality holds if and only if $\omega_1 = \xi_1 = 1$. This inequality is induced immediately when we write the second defining equation 4.2 of $\mathcal{X}$ equivalently as

$$2\Re\left(\frac{1}{\omega_0} + \omega_2\right) = |\omega_0|^2|\omega_1 - 1|^2 + 1.$$

Now

$$D_{x_0,y_2}(\omega) = \frac{4\eta_2}{\xi_0^3}(1 + \xi_2) = 0$$

implies that $eta_2 = 0$ or $\xi_2 = -1$. In the latter case we have $\xi_0 = 2$, which contradicts inequality 4.17. Thus $\eta_2 = 0$. Next we have

$$\begin{aligned} D_{x_0,y_1}(\omega) &= 4\xi_0\eta_1((\xi_2 - 1)^2 + \xi_0\xi_2^2) = 0 \\ &= 4\xi_0\xi_2^2\eta_1\left(\left(1 - \frac{1}{\xi_2}\right)^2 + \xi_0\right) \\ &= 4\xi_0^2\xi_2^2\eta_1(\xi_0 + 1). \end{aligned}$$

If $\xi_0 = -1$ then $\xi_2 = 1/2$ and we contradict inequality 4.17. Thus $\eta_1 = 0$. Finally, the vanishing of $D_{x_0,x_1}(\omega)$ and $D_{x_1,x_2}(\omega)$ is equivalent to the conditions

$$\xi_1 + \frac{1}{\xi_0} = 1, \quad \xi_2 + \frac{1}{\xi_1} = 1.$$

This completes the proof. ∎

Proof of Theorem 4.6. The proof that $\mathcal{X}'$ is a manifold follows immediately from Lemma 4.10: The rank of the Jacobian matrix D of f_1, f_2 is always 2 at each $\omega \in \mathcal{X}$ unless $\omega \in S_{\mathbb{R}}$. In order to prove that the induced CR structure is of codimension two we analyse first $\mathcal{X}''$ and argue that the structure extends to $\mathcal{X}'$ by continuity. Observe that from Lemma 4.9 we have that the rank of the matrix $D^{(1,0)}$ is always 2 in $\mathcal{X}'' = \mathcal{X} \setminus S_{\mathbb{C}}$. Following the discussion of section 4.2 and according to Definition 4.8 given there, a CR structure $\mathcal{E}$ of codimension two is defined on $\mathcal{X}'' = \mathcal{X} - S_{\mathbb{C}}$. More explicitely, for each $\omega \in \mathcal{X}''$ the space $\mathcal{E}_\omega^{(1,0)}$ is generated by the single vector

$$Z_\omega = D_{z_1,z_2}(\omega)\left(\frac{\partial}{\partial z_0}\right)_\omega + D_{z_2,z_0}(\omega)\left(\frac{\partial}{\partial z_1}\right)_\omega + D_{z_0,z_1}(\omega)\left(\frac{\partial}{\partial z_2}\right)_\omega \tag{4.18}$$

where $D_{z_1,z_2}(\omega)$, $D_{z_2,z_0}(\omega)$, $D_{z_0,z_1}(\omega)$ are given by 4.13, 4.14 and 4.15 respectively. $\square$

4.4. **The Levi form.** In this section we compute the Levi form the CR structure $(\mathcal{E}, \mathbb{J})$ of Theorem 4.6. Recall from equations 4.11 that for $i = 1, 2$, and $\omega \in \mathcal{X}''$

$$L_i(\omega) = \begin{bmatrix} D_{z_1,z_2} & D_{z_2,z_0} & D_{z_0,z_1} \end{bmatrix}_\omega [\partial\overline{\partial} f_i]_\omega \begin{bmatrix} D_{\overline{z_1},\overline{z_2}} \\ D_{\overline{z_2},\overline{z_0}} \\ D_{\overline{z_0},\overline{z_1}} \end{bmatrix}_\omega .$$

We begin with the neccessary calculations first.

Lemma 4.11. *Let $\omega \in \mathcal{X}$. The complex 3×3 matrices $[\partial\overline{\partial} f_i]_{j,k}$ of the Levi forms $\partial\overline{\partial} f_i$, $i = 1, 2$ are given in ω respectively by*

$$[\partial\overline{\partial} f_1]_\omega = \begin{bmatrix} \frac{1}{|\omega_0|^2} & \frac{1}{\omega_1\overline{\omega_0}} & \frac{1}{\omega_2\overline{\omega_0}} \\ \frac{1}{\omega_0\overline{\omega_1}} & \frac{1}{|\omega_1|^2} & \frac{1}{\omega_2\overline{\omega_1}} \\ \frac{1}{\omega_0\overline{\omega_2}} & \frac{1}{\omega_1\overline{\omega_2}} & \frac{1}{|\omega_2|^2} \end{bmatrix},$$

$$[\partial\overline{\partial} f_2]_\omega = \begin{bmatrix} 2\Re(\omega_0)|\omega_1|^2|\omega_2|^2 & \omega_0(\overline{\omega_1}|\omega_2-1|^2-1) & \omega_0|\omega_1|^2(\overline{\omega_2}-1) \\ \overline{\omega_0}(\omega_1|\omega_2-1|^2-1) & |\omega_0|^2|\omega_2-1|^2 & |\omega_0|^2\omega_1(\overline{\omega_2}-1) \\ \overline{\omega_0}|\omega_1|^2(\omega_2-1) & \overline{\omega_1}|\omega_0|^2(\omega_2-1) & |\omega_0|^2|\omega_1|^2 \end{bmatrix}.$$

Proof. This follows after calculating all partial derivatives of the second order $\frac{\partial^2 f_i}{\partial z_j \overline{z_k}}$ $i = 1, 2$, $j, k = 0, 1, 2$ at ω. The calculations are straightforward and therefore omitted. $\square$

We shall also need the following elementary Lemma.

Lemma 4.12. *Let $w = \begin{bmatrix} w_0 & w_1 & w_2 \end{bmatrix} \in \mathbb{C}^3$ and*

$$A = \begin{bmatrix} |a|^2 & b\bar{a} & c\bar{a} \\ a\bar{b} & |b|^2 & c\bar{b} \\ a\bar{c} & b\bar{c} & |c|^2 \end{bmatrix}, \quad a, b, c \in \mathbb{C}. \tag{4.19}$$

Then $wA\bar{w}^T = |aw_0 + bw_1 + cw_2|^2$.

Proposition 4.13. *Let $L_i, i = 1, 2$ be as above. Then*

(1) $L_1(\omega) = 0$ *for every* $\omega = (\omega_0, \omega_1, \omega_2) \in \mathcal{X}$.
(2) $L_2(\omega) = |\omega_0|^2|1 + \omega_0\omega_1\omega_2|^2 > 0$ *for every* $\omega = (\omega_0, \omega_1, \omega_2) \in \mathcal{X}''$.

Proof. In the first place, for each $\omega \in \mathcal{X}$ observe that $[\partial\overline{\partial} f_1]_\omega$ is of the form 4.19 and furthermore, the following symmetric condition holds:

$$\omega_1\omega_2 D_{z_1,z_2}(\omega) + \omega_0\omega_2 D_{z_2,z_0}(\omega) + \omega_1\omega_0 D_{z_0,z_1}(\omega) = 0. \tag{4.20}$$

Thus

$$\begin{aligned} L_1(\omega) &= \left| \frac{D_{z_1,z_2}(\omega)}{\omega_0} + \frac{D_{z_2,z_0}(\omega)}{\omega_1} + \frac{D_{z_0,z_1}(\omega)}{\omega_2} \right|^2 \\ &= \left|\omega_1\omega_2 D_{z_1,z_2}(\omega) + \omega_0\omega_2 D_{z_2,z_0}(\omega) + \omega_1\omega_0 D_{z_0,z_1}(\omega)\right|^2 \\ &= 0. \end{aligned}$$

The calculation of $L_2(\omega)$ is more complicated. One may write first

$$[\partial\overline{\partial} f_2] = [\partial\overline{\partial} f_2]_1 + [\partial\overline{\partial} f_2]_2$$

where

$$[\partial\overline{\partial} f_2]_1 = \begin{bmatrix} |\omega_1|^2|\omega_2-1|^2 & \omega_0\overline{\omega_1}|\omega_2-1|^2 & \omega_0|\omega_1|^2(\overline{\omega_2}-1) \\ \overline{\omega_0}\omega_1|\omega_2-1|^2 & |\omega_0|^2|\omega_2-1|^2 & |\omega_0|^2\omega_1(\overline{\omega_2}-1) \\ \overline{\omega_0}|\omega_1|^2(\omega_2-1) & \overline{\omega_1}|\omega_0|^2(\omega_2-1) & |\omega_0|^2|\omega_1|^2 \end{bmatrix},$$

$$[\partial\overline{\partial} f_2]_1 = \begin{bmatrix} 1-2\Re(\omega_1) & -\omega_0 & 0 \\ -\overline{\omega_0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, $[\partial\overline{\partial} f_2]_1$ is of the form 4.19 therefore, putting $W = \begin{bmatrix} D_{z_1,z_2} & D_{z_2,z_0} & D_{z_0,z_1} \end{bmatrix}_\omega$ we have

$$\begin{aligned} L_2(\omega) &= W[\partial\overline{\partial} f_2]_1\overline{W}^T + W[\partial\overline{\partial} f_2]_2\overline{W}^T \\ &= |\omega_1(\omega_2-1)D_{z_1,z_2}(\omega) + \omega_0(\omega_2-1)D_{z_2,z_0}(\omega) + \omega_0\omega_1 D_{z_0,z_1}(\omega)|^2 + \\ &\quad (1-2\Re(\omega_1))|D_{z_1,z_2}(\omega)|^2 - 2\Re(\overline{\omega_0}D_{z_1,z_2}(\omega)D_{\overline{z_2},\overline{z_0}}(\omega)) \\ &= |\omega_1 D_{z_1,z_2}(\omega) + \omega_0 D_{z_2,z_0}(\omega)|^2 + \\ &\quad (1-2\Re(\omega_1))|D_{z_1,z_2}(\omega)|^2 - 2\Re(\overline{\omega_0}D_{z_1,z_2}(\omega)D_{\overline{z_2},\overline{z_0}}(\omega)) \\ &= |(\omega_1-1)D_{z_1,z_2} + \omega_0 D_{z_2,z_0}|^2 \\ &= \frac{|\omega_0\omega_1 D_{z_0,z_1} + \omega_2 D_{z_1,z_2}|^2}{|\omega_2|^2} \\ &= \frac{|\omega_0|^2}{|\omega_2|^2}\,|1+\omega_0\omega_1\omega_2|^2, \end{aligned}$$

where we have used 4.20 to derive the third and the penultimate equality. We conclude that $L_2(\omega)$ is always positive unless $\omega_0\omega_1\omega_2 = -1$. But then $\omega \in S_{\mathbb{C}}$ as this follows from Proposition 4.4. The proof is thus complete. □

We finally state

Proposition 4.14. *The set $\mathcal{X}''$ admits a codimention one foliation. Its leaves are three dimensional manifolds, each of which is locally diffeomorphic to the Heisenberg group.*

Proof. Let $\mathcal{E} = \{X, \mathbb{J}X\}$ be the CR structure of $\mathcal{X}''$ and consider the distribution

$$\mathcal{D} = \{X, \mathbb{J}X, [X, \mathbb{J}X]\}$$

where $X = Z + \overline{Z}$ where Z is as in equation 4.18. Observe that $\ker d^c f_1 = \mathcal{D}'$ and moreover

$$\partial\overline{\partial} f_1 = \left|\frac{1}{\omega_0}dz_0 + \frac{1}{\omega_1}dz_1 + \frac{1}{\omega_2}dz_2\right|^2 = \partial f_1 \wedge \overline{\partial} f_1.$$

From Frobenius' theorem, $\mathcal{D}$ is integrable if and only if

$$dd^c f_1 \wedge d^c f_1 = 0.$$

In fact,

$$\begin{aligned} dd^c f_1 \wedge d^c f_1 &= -2\partial\overline{\partial} f_1 \wedge (\partial f_1 - \overline{\partial} f_1) \\ &= -2(\partial f_1 \wedge \overline{\partial} f_1 \wedge \partial f_1 - \partial f_1 \wedge \overline{\partial} f_1 \wedge \overline{\partial} f_1) \\ &= 0. \end{aligned}$$

The proof is thus complete. □

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