A Quadratic surfaces

In this appendix we will study several families of so-called quadratic surfaces, namely surfaces $z = f(x, y)$ which are defined by equations of the type

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Hx + Iy + Jz + K = 0, \qquad \text{(A.1)}$$

with $A, B, C, D, E, F, H, I, J$ and K being fixed real constants and x, y, z being variables. These surfaces are said to be quadratic because all possible products of two of the variables x, y, z appear in (A.1).

In fact, by suitable translations and rotations of the x, y and z coordinate axes it is possible to simplify the equation (A.1) and hence classify all the possible surfaces into the following ten types:

1. Spheres
2. Ellipsoids
3. Hyperboloids of one sheet
4. Hyperboloids of two sheets
5. Cones
6. Elliptic paraboloids
7. Hyperbolic paraboloids
8. Parabolic cylinders
9. Elliptic cylinders
10. Hyperbolic cylinders

It is a requirement of this calculus course that you should be able to recognize, classify and sketch at least some of these surfaces (we will use some of them when doing triple integrals). The best way to do that is to look for identifying signs which tell you what kind of surface you are dealing with. Those signs are:

- The intercepts: the points at which the surface intersects the x, y and z axes.
- The traces: the intersections with the coordinate planes (xy-, yz- and xz-plane).
- The sections: the intersections with general planes.
- The centre: (some have it, some not).
- If they are bounded or not.
- If they are symmetric about any axes or planes.

A.1 Spheres

A sphere is a quadratic surface defined by the equation:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{A.2}$$

the point (x_0, y_0, z_0) in the 3D space is the **centre** of the sphere. The points (x, y, z) in the sphere are all points whose distance to the centre is given by r. Therefore:

(a) The **intercepts** of the sphere with the x, y, z-axes are the points $x_0 \pm r, 0, 0)$, $(0, y_0 \pm r, 0)$ and $(0, 0, z_0 \pm r$.

(b) The **traces** of the sphere are circles or radius r.

(c) The **sections** of the sphere are circles of radius $r' < r$.

(d) The sphere is **bounded**.

(e) Spheres are symmetric about all coordinate planes.

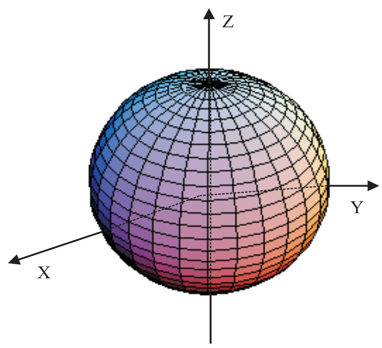
Rule: If we expand the square terms in equation (A.2) we obtain the following equation:

$$x^2 + y^2 + z^2 - 2x_0x - 2y_0y - 2z_0z + x_0^2 + y_0^2 + z_0^2 - r^2 = 0, \tag{A.3}$$

Comparing this equation with the general formula (A.1) we see it has the form

$$x^2 + y^2 + z^2 + Hx + Iy + Jz + K = 0, \tag{A.4}$$

with H, I, J and K being some real constants. Therefore, the rule to recognize a sphere is the following: any quadratic equation such that the coefficients of the x^2, y^2 and z^2 terms are equal and that no other quadratic terms exist corresponds to a sphere.



Sphere centered at the origin.

The sphere is a perfect example of a **surface of revolution**. A surface of revolution is a surface which can be generated by rotating a particular curve about a particular coordinate axis. For example, one way to generate the sphere of the picture above is to take the circle $x^2 + y^2 = 1$ and rotate it about the z-axis.

A.2 Ellipsoids

Ellipsoids are quadratic surfaces parameterized by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{A.5}$$

You can see that for $a = b = c = 1$ we recover the equation of a sphere centered at the origin. Therefore an ellipsoid is a "deformation" of the sphere such that the sphere gets either stretched or squeezed (depending on the values of a, b, c) in the x, y, z-directions.

(a) The **intercepts** of the ellipsoid with the x, y, z-axes are the points $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$.

(b) The **traces** of the ellipsoid are ellipses which satisfy the equations:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.6}$$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.7}$$

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{for } x = 0 \ (yz\text{-plane}). \tag{A.8}$$

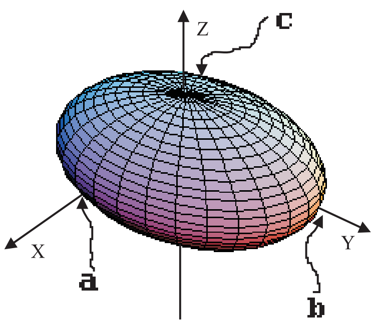
(c) The **sections** of the ellipsoid are also ellipses.

(d) The ellipsoid is **bounded** (all points (x, y, z) in the ellipsoid correspond to finite values of x, y and z.

(e) The **centre** of the ellipsoid in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The ellipsoid is symmetric about all coordinate planes.

Rule: A quadratic equation such that the coefficients of the x^2, y^2 an z^2 terms are different from each other and all positive and such that no other quadratic terms exist corresponds to an ellipsoid.



Ellipsoid centered at the origin.

A.3 Hyperboloids of one sheet

A hyperboloid of one sheet is parameterized by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{A.9}$$

(a) The **intercepts** of the hyperboloid of one sheet with the x, y, z-axes are the points $(\pm a, 0, 0)$, $(0, \pm b, 0)$. Notice that there is no intersection with the z-axis. The reason is that if we set $x = y = 0$ in the equation (A.9) we obtain the condition $z^2 = -c^2$ which admits no real solution for real c.

(b) The **traces** of the hyperboloid of one sheet are ellipses in the xy-plane

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.10}$$

and hyperbolas in the xz- and yz-planes:

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.11}$$

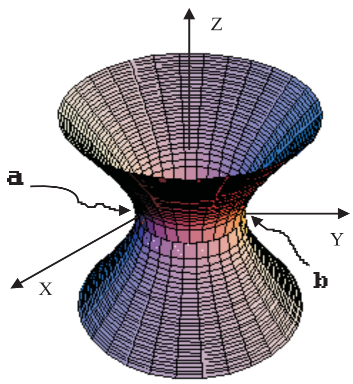
$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{for } x = 0 \ (yz\text{-plane}). \tag{A.12}$$

(c) The **sections** of the hyperboloid of one sheet are ellipses for planes parallel to the xy plane and hyperbolas for planes parallel to the yz- and xz-planes.

(d) The hyperboloid of one sheet is **not bounded**.

(e) The **centre** of the hyperboloid of one sheet in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The hyperboloid of one sheet is symmetric about all coordinate planes.



Hyperboloid of one sheet centered at the origin.

A.4 Hyperboloid of two sheets

A hyperboloid of two sheets is a surface generated by the points satisfying the equation

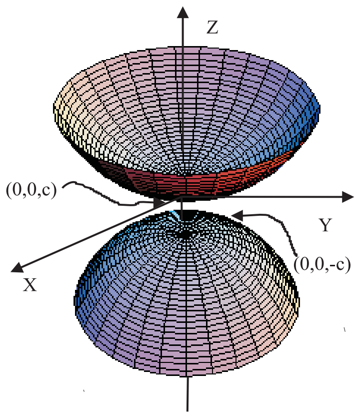
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1. \tag{A.13}$$

(a) The **intercepts** of the hyperboloid of two sheets with the x, y, z-axes are the points $(0, 0, \pm c)$. There are no intersections with the x, y-axes. The reason is that if we set $z = x = 0$ in the equation (A.9) we obtain the condition $y^2 = -b^2$ which can not be fulfilled for any real values of y and b. Analogously if we set $z = y = 0$ we obtain $x^2 = -a^2$ which also has no real solutions.

(b) In this case we have two sheets, in contrast to all examples we have seen so far. The reason is that the equation (A.13) implies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1. \tag{A.14}$$

This equation can only be solved if $z^2/c^2 - 1 \geq 0$, which implies that $|z| \geq c$.



Hyperboloid of two sheets centered at the origin.

(c) The **traces** of the hyperboloid of two sheets are hyperbolas in the xz- and yz-planes:

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.15}$$

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \quad \text{for } x = 0 \ (yz\text{-plane}). \tag{A.16}$$

(d) The **sections** of the hyperboloid of two sheets are hyperbolas for any planes parallel to the xz- or yz-planes and ellipses for planes parallel to the xy-plane with $|z| > c$.

(e) The hyperboloid of two sheets is also **not bounded**.

(f) The **centre** of the hyperboloid of two sheets in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(g) The hyperboloid of two sheets is again symmetric about all coordinate planes.

General rule: Any quadratic surface such that: the coefficients of x^2, y^2 and z^2 are different, one of the coefficients is negative and two of the coefficients are positive, and no other quadratic terms appear describes a hyperboloid. In addition, if the constant term has negative sign the hyperboloid has two sheets whereas if the constant term has positive sign the hyperboloid has only one sheet.

A.5 Cones

A cone is a quadratic surface whose points fulfil the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 = 0. \tag{A.17}$$

Comparing (A.17) with the equations for the hyperboloids of one and two sheet we see that the cone is some kind of limiting case when instead of having a negative or a positive number on the l.h.s. of the quadratic equation we have exactly 0.

(a) The only **intercept** of the cone with the x, y, z-axes is the origin of coordinates $(0, 0, 0)$.

(b) The **traces** of the cone are lines in the xz- and yz-planes

$$z = \pm x/a \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.18}$$

$$z = \pm y/b \quad \text{for } x = 0 \ (yz\text{-plane}), \tag{A.19}$$

and the origin $(0, 0)$ in the xy-plane:

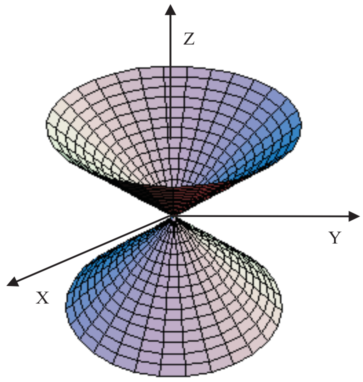
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad \text{for } z = 0 \ (xy\text{-plane}). \tag{A.20}$$

(c) The **sections** of the cone are lines for any planes parallel to the xz- pr yz-planes and ellipses (or circles if $a = b$) for planes parallel to the xy-plane.

(d) The cone is **not bounded**.

(e) The **centre** of the cone in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The cone is symmetric about all coordinate planes.



Cone centered at the origin.

Rule: Any quadratic surface such that: the coefficients of x^2, y^2 and z^2 are different, one of the coefficients is negative and two of the coefficients are positive, no other quadratic terms appear and no constant term appears describes a cone.

A.6 Elliptic paraboloids

A quadratic surface is said to be an elliptic paraboloid is it satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z. \tag{A.21}$$

(a) The only **intercept** of the elliptic paraboloid with the x, y, z-axes is the origin of coordinates $(0, 0, 0)$.

(b) The **traces** of the paraboloid are parabolas in the xz- and yz-planes

$$z = \frac{x^2}{a^2} \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.22}$$

$$z = \frac{y^2}{b^2} \quad \text{for } x = 0 \ (yz\text{-plane}), \tag{A.23}$$

and the origin $(0, 0)$ in the xy-plane corresponding to the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.24}$$

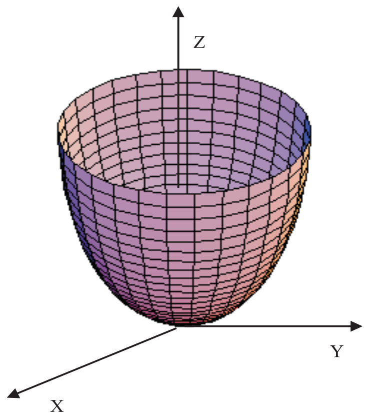
(c) The **sections** of the elliptic paraboloid with any planes parallel to the xz- or yz-planes are parabolas. The sections with planes parallel to the xy-plane are ellipses (or circles if $a = b$).

(d) The paraboloid is **not bounded** from above.

(e) The **centre** of the paraboloid in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The elliptic paraboloid is symmetric about the xz- and yz-planes.

Rule: Any quadratic surface which contains: only linear terms in one of the variables (in our example z), quadratic terms in the other two variables with coefficients of the same sign and no constant term is an elliptic paraboloid.



Elliptic paraboloid centered at the origin.

A.7 Hyperbolic paraboloids

A hyperbolic paraboloid is defined by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z, \tag{A.25}$$

(a) The only **intercept** of the hyperbolic paraboloid with the x, y, z-axes is the origin of coordinates $(0, 0, 0)$.

(b) The **traces** of the paraboloid are parabolas in the xz- and yz-planes

$$z = \frac{x^2}{a^2} \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.26}$$

$$z = -\frac{y^2}{b^2} \quad \text{for } x = 0 \ (yz\text{-plane}), \tag{A.27}$$

and two lines in the xy-plane corresponding to the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.28}$$

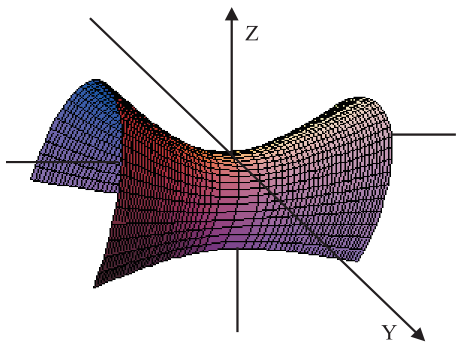
(c) The **sections** of the hyperbolic paraboloid with planes parallel to the xz- or yz-planes are parabolas. The sections with planes parallel to the xy-plane are hyperbolas.

(d) The paraboloid is **not bounded**.

(e) The **centre** of the paraboloid in the picture is the origin of coordinates which in this case is called a **saddle point**. It can be changed by shifting x, y, z by constant amounts.

(f) The hyperbolic paraboloid is symmetric about the xz- and yz-planes.

Rule: A hyperbolic paraboloid has the same features as the elliptic paraboloid with the only difference that the coefficients of the quadratic terms (x^2, y^2 in our example) have opposite signs.



Hyperbolic paraboloid centered at the origin.

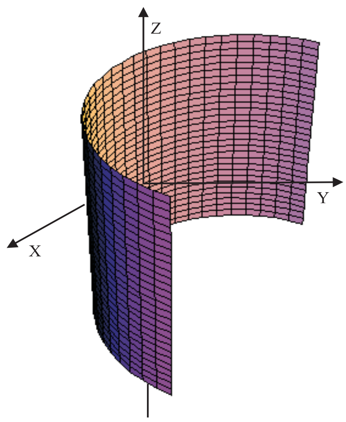
A.8 Parabolic cylinders

A parabolic cylinder is a quadratic surface whose points satisfy the equation

$$x^2 = 4cy. \tag{A.29}$$

This kind of surface is very simple, since the equation above does not depend on z. Since (A.29) describes a parabola in the xy-plane, the surface we are describing here is generated by the translation of the parabola (A.29) along the z-direction.

Rule: This kind of surface is easy to recognize, since its equation does not depend explicitly on one of the variables and is the equation of a parabola in the other two variables.



Parabolic cylinder centered at the origin.

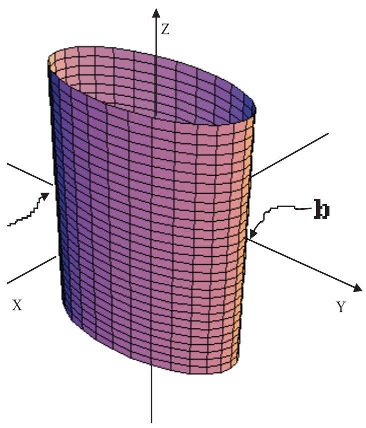
A.9 Elliptic cylinders

An elliptic cylinder is a quadratic surface described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{A.30}$$

As in the previous case, this surface does not depend explicitly on the coordinate z. Equation (A.30) describes an ellipse in the xy-plane (or a circle if $a = b$). Therefore an elliptic cylinder is the surface generated by the translation of the ellipse (A.30) along the z direction.

Rule: Again we have here a surface easy to recognize, since its equation does not depend explicitly on one of the variables and is the equation of an ellipse in the other two variables.



Elliptic cylinder centered at the origin.

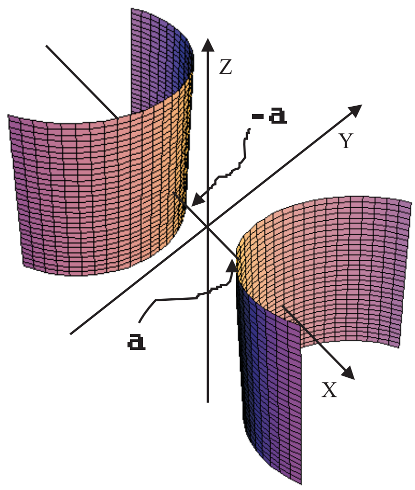
A.10 Hyperbolic cylinders

A hyperbolic cylinder is a quadratic surface parameterized by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{A.31}$$

The equation above describes a hyperbola in the xy-plane. Therefore this surface is simply generated by the translation of the hyperbola (A.31) along the z-direction.

Rule: The rule to recognize this kind of surface will be the same as in the previous case, with the difference that instead of an ellipse we have now a hyperbola.



Hyperbolic cylinder centered at the origin.

General rule: In general, cylinders are surfaces whose corresponding quadratic equation does not involve z explicitly. Therefore we must be told that they are in 3D to recognize a cylindrical surface.

References

[1] R. A. Adams, Calculus: A complete course (Addison Wesley).

[2] T. Apostol, Calculus (Wiley).

[3] S. N. Salas and E. Hille, Calculus: One and several variables (Wiley).

[4] H. Anton, Calculus (Wiley).