The Roto-Affine group and a lifting theorem for quasiconformal mappings

Georgios A. Moulantzikos

8th January 2018

Acknowledgements

I would first and foremost like to thank my advisor Ioannis D. Platis for introducing me to the topics of this project and his support and guidance during my master studies. I also want to thank professor Konstantin Athanassopoulos and professor Alexandros Kouvidakis for the reading courses I undertook with them.

Contents

1 Introduction **7**

2 Complex Manifolds **9**

2.1 Complex and almost complex structures 9

2.2 Complex vector spaces and complexification 13

3 CR geometry **17**

3.1 Embedded CR manifolds 17

3.2 Abstract CR Manifolds 20

3.3 CR Maps 23

4 Contact, Symplectic and Sub-riemannian manifolds **25**

4.1 Symplectic manifolds 25

4.2 Contact Manifolds 26

4.3 Kahler manifolds 27

4.4 Sub-Riemannian manifolds 28

4.5 Levi form 29

5 Quasiconformal mappings in the Heisenberg Group 31

5.1 The Heisenberg group 31

5.2 Contact, CR and Sub-Riemannian structure 33

5.3 Quasiconformal mappings 35

5.4 The Beltrami equation 36

6 Quasiconformal mappings in the Roto-affine Group 37

6.1 The Roto-affine group 37

6.1.1 First Riemannian structure 39

6.1.2 Contact Structure 40

6.2 Second Riemannian Structure 42

6.2.1 Riemannian submersion from $\mathbb{RA}$ to $\mathbb{L}$ 44

6.3 Horizontal curves and Ehresmann completeness of α 45

6.4 Contact transformations 48

6.5 Contact diffeomorphisms with constant determinant 49

6.6 Lifting Theorem 52

Chapter 1

Introduction

The purpose of this master thesis is to present a brief but self-contained survey of the theory of quasiconformal mappings on the Heisenberg group $\mathbb{H}$ and then describe, in a more detailed and geometric manner, quasiconformal mappings of $\mathbb{H}$ that preserve $\mathbb{V} = \{0\} \times \mathbb{R}$. The main part of this thesis is the study of the Roto-Affine group and the proof of the Lifting theorem (6.13). This is an unpublished result of my advisor, Ioannis D. Platis (see [23]). We should also note that Robin Timsit has obtained the same result with different methods in [24].

The classical theory of quasiconformal mappings was developed first in the Euclidean space $\mathbb{R}^n$ and produced a variety of results, most of them closely connected to topics in Analysis. The theory of quasiconformal mappings on the Heisenberg group emerged after the pioneering articles of Koranyi-Reimann ([18] and [21]). These works constituted a complete framework for the theory of quasiconformal mappings on the Heisenberg group $\mathbb{H}$. In latter developments of that theory, quasiconformal mappings that preserve $\mathbb{V} = \{0\} \times \mathbb{R}$ have appeared as generalizations to $\mathbb{H}$ of classical mappings of the complex plane. Such mappings may be considered as self mappings of $(\mathbb{C} \times \mathbb{R})\setminus\mathbb{V}$, so it is natural to search for any particular structures present. This leads to the definition and study of the Roto-Affine group $\mathbb{RA}$. This project culminates with the proof of the Lifting Theorem (6.13) in Chapter 6.

Our work is organized as follows: Chapters 2,3 and 4 are preparatory. In Chapter 2 we review some basic definitions from the theory of complex manifolds and describe the complexified tangent and cotangent spaces of a manifold. Chapter 3 provides a quick introduction to the theory of CR manifolds. At first we examine the case of embedded CR manifolds and then provide the definition of abstract CR manifolds. This way we proceed from familiar examples to a more abstract setting and our definitions are better motivated. In Chapter 4 we briefly examine the notions of contact, symplectic, kahler and sub-riemannian manifolds. The purpose of this chapter is to provide a reference for terminology that is later used. In the final section, we define the Levi form and describe the relation of contact and sub-riemannian structures on a manifold. Chapter 5 provides a quick survey of the theory of quasiconformal mappings on the Heisenberg group. After defining the Heisenberg group, we describe its contact and sub-riemannian structure. We close this chapter with the definition of quasiconformal mappings and the theorem about the Beltrami equation. Chapter 6, which can be considered as the main part of this work, introduces and studies the Roto-Affine group.

Chapter 2

Complex Manifolds

In this chapter we quickly review the theory of complex and almost complex manifolds. In section 2.1 we define the notions of a complex and almost complex structure and in section 2.2 we describe the tangent and cotangent space of a complex manifold. The main reference is [1].

2.1 Complex and almost complex structures

Definition 2.1. (Holomorphic functions in $\mathbb{C}^n$) Let U be an open subset of $\mathbb{C}^n$. A function $f : U \to \mathbb{C}^n$ is called differentiable at z_0 if

$$\lim_{h \to 0} \frac{1}{h} \left\{ f(z_0^1, ..., z_0^i + h, ..., z_0^n) - f(z_0^1, ..., z_0^i, ..., z_0^n) \right\}$$

exists for every $i = 1, ..., n$. f is called holomorphic on U if f is differentiable at any point on U.

Definition 2.2. (Complex Manifold) A Hausdorff space M is called a complex manifold of complex dimension n, if M satisfies the following properties:

(i) There exists an open covering $\{U_a\}_{a \in A}$ of M and for each a, there exists a homeomorphism

$$\psi_a : U_a \to \psi(U_a) \subset \mathbb{C}^n.$$

(ii) For any two open sets U_a and U_b with nonempty intersections, maps

$$f_{ba} = \psi_b \circ \psi_a^{-1} : \psi_a(U_a \cap U_b) \to \psi_b(U_a \cap u_b),$$

$$f_{ab} = \psi_a \circ \psi_b^{-1} : \psi_b(U_a \cap U_b) \to \psi_a(U_a \cap u_b),$$

are holomorphic.

The set $\{(U_a, \psi_a)\}_{a \in A}$ is called a system of holomorphic coordinate neighbourhoods.

Definition 2.3. Let (U, ψ) be a holomorphic coordinate neighborhood of a complex manifold M. A function $f : U \to \mathbb{C}$ is holomorphic if the function $f \circ \psi^{-1}$ is holomorphic.

Definition 2.4. Let M, N be complex manifolds and (U, ψ) a holomorphic coordinate neighborhood of $x \in M$. A continuous map $\phi : M \to N$ is holomorphic if for any $x \in M$ and for any holomorphic coordinate neighborhood (V, ψ') of N such that $\phi(X) \in V$ and $\phi(U) \subset V, \psi' \circ \phi \circ \psi^{-1} : \psi(U) \to \psi'(V)$ is holomorphic.

Now we recall the definition of an almost complex structure. First we identify a complex number $z = x + iy$ with the element $z = xe_1 + ye_2$ of a real two dimensional vector space V, where (e_1, e_2) denotes the basis of V. Let $J : V \to V$ be the enomorphism defined by

$$Jz = iz = -y + ix.$$

Then we conclude

$$xJe_1 + yJe_2 = J(xe_1 + ye_2) = Jz = -ye_1 + x_2.$$

Therefore, the endomorphism J is determined by

$$Je_1 = e_2, Je_2 = -e_1.$$

Keeping this in mind, we introduce the endomorphism J of the tangent space $T_p(M)$ of a complex manifold M at $p \in M$. Let M be an n-dimensional complex manifold. Identifying the local coordinates $(z^1, ..., z^n)$ with $(x^1, ..., x^n, y^1, ..., y^n)$, we regard M as a $2n$-dimensional differentiable manifold. The tangent space $T_p(M)$ of M at a point $p \in M$ has a natural basis

$$\left\{ (\frac{\partial}{\partial x^1})_p, (\frac{\partial}{\partial y^1})_p, ..., (\frac{\partial}{\partial x^n})_p, (\frac{\partial}{\partial y^n})_p \right\}.$$

For $i = 1, ..., n$ we put

$$J_p \left(\frac{\partial}{\partial x^i} \right)_p = \left(\frac{\partial}{\partial y^i} \right)_p, J_p \left(\frac{\partial}{\partial y^i} \right)_p = \left(\frac{\partial}{\partial x^i} \right)_p$$

Then J_p defines an isomorphism $J_p : T_p(M) \to T_p(M)$. J_p is independent of the choice of holomorphic coordinates and is well defined. Regarding J as a map of the tangent bundle $T(M) = \bigcup_{p \in M} T_p(M)$, we call J the (natural) almost complex structure of M.

Theorem 2.5. *Let M and M' be complex manifolds with almost complex structures J and J' respectively. Then the map $f : M \to M'$ is holomorphic if and only if $f_* \circ J = J' \circ f_*$ where f_* denotes the differential of the map of f.*

Proof. We identify holomorphic coordinates $\mathbf{z} = (z^1, ..., z^n)$ of M with $(\mathbf{x}, \mathbf{y}) = (x^1, ..., x^n, y^1, ..., y^n)$ and holomorphic coordinates $\mathbf{w} = (w^1, ..., w^n)$ of M' with $(\mathbf{u}, \mathbf{v}) = (u^1, ..., u^n, v^1, ..., v^n)$. Then

$$f(\mathbf{z}) = (w^1(\mathbf{z}), ..., w^m(\mathbf{z}))$$

is expressed by

$$f(\mathbf{x}, \mathbf{y}) = (\mathbf{u}(\mathbf{x}, \mathbf{y}), \mathbf{v}(\mathbf{x}, \mathbf{y}))$$

in terms of the real coordinates. Thus we have

$$J' \circ f_* \left(\frac{\partial}{\partial x_i} \right) = \sum_{j=1}^{m} \left(\frac{\partial u^j}{\partial x_i} J' \left(\frac{\partial}{\partial u_j} \right) + \frac{\partial v^j}{\partial x_i} J' \left(\frac{\partial}{\partial v_j} \right) \right) = \sum_{j=1}^{m} \left(\frac{\partial u^j}{\partial x_i} \frac{\partial}{\partial v_j} - \frac{\partial v^j}{\partial x_i} \frac{\partial}{\partial u_j} \right), \tag{2.1}$$

$$J' \circ f_* \left(\frac{\partial}{\partial y_i} \right) = \sum_{j=1}^{m} \left(\frac{\partial u^j}{\partial y_i} J' \left(\frac{\partial}{\partial u_j} \right) + \frac{\partial v^j}{\partial y_i} J' \left(\frac{\partial}{\partial v_j} \right) \right) = \sum_{j=1}^{m} \left(\frac{\partial u^j}{\partial y_i} \frac{\partial}{\partial v_j} - \frac{\partial v^j}{\partial y_i} \frac{\partial}{\partial u_j} \right). \tag{2.2}$$

On the other hand

$$f_* \circ J\left(\frac{\partial}{\partial x_i}\right) = f_*\left(\frac{\partial}{\partial y_i}\right) = \sum_{j=1}^{m}\left(\frac{\partial u^j}{\partial y_i}\frac{\partial}{\partial u_j} + \frac{\partial v^j}{\partial y_i}\frac{\partial}{\partial v_j}\right), \tag{2.3}$$

$$f_* \circ J\left(\frac{\partial}{\partial y_i}\right) = -f_*\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{m}\left(\frac{\partial u^j}{\partial x_i}\frac{\partial}{\partial u_j} + \frac{\partial v^j}{\partial x_i}\frac{\partial}{\partial v_j}\right). \tag{2.4}$$

Comparing (2.1),(2.2) with (2.3),(2.4) yields the Cauchy-Riemann equations

$$\frac{\partial u^j}{\partial x_i} = \frac{\partial v^j}{\partial y_i}, \quad \frac{\partial u_j}{\partial y_i} = -\frac{\partial v_j}{\partial x_i}$$

Consequently, f is holomorphic if and only if $f_* \circ J = J' \circ f_*$

□

Definition 2.6. A differentiable manifold M is said to be an almost complex manifold if there exists a linear map $J : T(M) \to T(M)$ satisfying $J^2 = -id$ and J is said to be an almost complex structure of M.

As we have shown, a complex manifold M admits a naturally induced almost complex structure, given by (1.8), and consequantly M is an almost complex manifold.

Theorem 2.7. *An almost complex manifold M is even-dimensional.*

Proof. Since $J^2 = -id$, for suitable basis of the tangent bundle we have

$$J^2 = \begin{bmatrix} -1 & 0 & ... & 0 \\ 0 & -1 & ... & 0 \\ & & ...... & \\ 0 & ... & ... & -1 \end{bmatrix}$$

Hence, $(-1)^n = \det(J^2) = (\det(J))^2 \geq 0$. Thus, n is even. □

Remark 2.8. We note that an even dimensional differentiable manifold does not necessarily admit an almost complex structure J. It is known, for example that S^4 does not possess an almost complex structure, (see [15], p.217).

The Nijenhuis tensor N of an almost complex structure J is defined by

$$N(X,Y) = J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY],$$

for any $X, Y \in T(M)$.

Theorem 2.9 (Newlander-Nirenberg)**.** *Let M be an almost complex manifold with an almost complex structure J. There exists a complex structure on M and J is the almost complex structure which is induced from the complex structure on M if and only if the Nijenhuis tensor N vanishes identically.*

For the proof of this theorem see [16]

2.2 Complex vector spaces and complexification

In this section we recall some results on complex vector spaces applied to tangent and cotangent spaces of complex manifolds. For the tangent space $T_p(M)$ at $p \in M$, we put

$$T_p^{\mathbb{C}}(M) = \{X_p + iY_p : X_p, Y_p \in T_p(M)\}$$

and $T_p^{\mathbb{C}}(M)$ is called the *complexification* of $T_p(M)$. In this way, $T_p^{\mathbb{C}}(M)$ becomes a complex vector space and we can identify $T_p(M)$ with

$$\{X_p + i0_p : X_p \in T_p(M)\}.$$

Now let (M, J) be an almost complex manifold with almost complex structure J. Then J_p can be extended as an isomorphism of $T_p^{\mathbb{C}}(M)$. We define $T_p^{(0,1)}(M)$ and $T_p^{(1,0)}(M)$,respectively, by

$$T_p^{(0,1)}(M) = \{X_p + iJ_pX_p : X_p \in T_p(M)\},$$

$$T_p^{(1,0)}(M) = \{X_p - iJ_pX_p : X_p \in T_p(M)\}.$$

Proposition 2.10. Under the above assumptions,

$$T_p^{\mathbb{C}}(M) = T_p^{(0,1)}(M) \oplus T_p^{(1,0)}(M)$$

where $\oplus$ denotes the direct sum.

Let

$$\begin{aligned} T^{\mathbb{C}}(M) &= \bigcup_{p\in M} T_p^{\mathbb{C}}(M), \\ T^{(0,1)}(M) &= \bigcup_{p\in M} T_p^{(0,1)}(M), \\ T^{(1,0)}(M) &= \bigcup_{p\in M} T_p^{(1,0)}(M) \end{aligned}$$

Definition 2.11. Let D be a distribution on M. We say that D is *involutive* if $[D, D] \subset D$.

Theorem 2.12. *$T^{(0,1)}(M)$ and $T^{(1,0)}$ are involutive if and only if the Nijenhuis tensor N vanishes identically*

Proof. First we note that for $Z \in T^{(0,1)}(M), W \in T^{(1,0)}(M)$, it follows

$$JZ = -iZ, JW = iW$$

and therefore

$$\begin{aligned} N(Z,W) &= J[Z,W] - [JZ,W] - [Z,JW] - J[JZ,JW] \\ &= J[Z,W] + i[Z,W] - i[Z,W] - J[Z,W] = 0. \end{aligned}$$

Let $Z, W \in T^{(0,1)}(M)$. Then

$$\begin{aligned} N(Z,W) &= J[Z,W] - [-iZ,W] - [Z,-iW] - J[-iZ,-iW] \\ &= J[Z,W] + i[Z,W] + i[Z,W] + J[Z,W] \\ &= 2(J[Z,W] + i[Z,W]). \end{aligned}$$

Thus $N(Z,W) = 0$ if and only if $J[Z,W] = -i[Z,W]$, that is, $[Z,W] \in T^{(0,1)}(M)$. In a similar way we can prove the case of $T^{(1,0)}(M)$ which completes the proof. □

Let M be an n-dimensional complex manifold and let $(z^1, ..., z^n)$ be complex coordinates in a neighborhood U of a point p. We regard M as a $2n$-dimensional differentiable manifold with local coordinates $(x^1, y^1, ..., x^n, y^n)$. Then

$$\left\{ (\frac{\partial}{\partial x^1})_p, (\frac{\partial}{\partial y^1})_p, ..., (\frac{\partial}{\partial x^n})_p, (\frac{\partial}{\partial y^n})_p \right\}$$

is a basis of $T_p(M)$. If we put

$$\frac{\partial}{\partial z_i}=\frac{1}{2}(\frac{\partial}{\partial x_i}+i\frac{\partial}{\partial y_i}),\qquad \frac{\partial}{\partial \overline{z}_i}=\frac{1}{2}(\frac{\partial}{\partial x_i}-i\frac{\partial}{\partial y_i})$$

then

$$\left\{(\frac{\partial}{\partial z^1})_p,...,(\frac{\partial}{\partial z^n})_p,(\frac{\partial}{\partial \overline{z}^1})_p,(\frac{\partial}{\partial \overline{z}^n})_p\right\}$$

forms a basis of $T_p^{\mathbb{C}}(M)$. For a natural basis of a tangent space $T_p(M)$ at $p\in M$ we consider its dual basis

$$\left\{(dx^1)_p,(dy^1)_p,...,(dx^n)_p,(dy^n)_p\right\}$$

in the cotangent space $T_p(M)^*$ and we put

$$(dz^i)_p)=(dx^i)_p+i(dy^i)_p$$

,

$$(d\overline{z}^i)_p)=(dx^i)_p-i(dy^i)_p$$

Then

$$\{(dz^1)_p,(d\overline{z}^1)_p,...,(dz^n)_p,(d\overline{z}^n)_p\}$$

is the dual basis of

$$\left\{(\frac{\partial}{\partial z^1})_p,(\frac{\partial}{\partial \overline{z}^1})_p,...,(\frac{\partial}{\partial z^n})_p,(\frac{\partial}{\partial \overline{z}^n})_p\right\}$$

Now, if f is a C^∞ we have

$$df_p=\sum_{i=1}^{n}(\frac{\partial f}{\partial z^i}_p dz_p^1+\frac{\partial f}{\partial \overline{z}^i}_p d\overline{z}_p^i).$$

Definition 2.13. Let r be a positive integer such that $r=p+q$ where p,q are nonnegative integrs. Let ω be an r-form on M spanned by the set $\left\{dz^{i_1}\wedge...\wedge dz^{i_p}\wedge d\overline{z}^{j_1}\wedge...\wedge d\overline{z}^{j_p}\right\}$ where $i_1,...,i_p$ and $j_1,...,j_q$ run over the set of all increasing multi-indices of length p and q. Then ω is called a complex differential form of type (p,q).

Proposition 2.14. Let ω and η be complex differential forms.

(1) If ω is of type (p,q), then $\overline{\omega}$ is of type (q,p).

(2) If ω is of type (p, q) and η is of type (p', q'), then $\omega \wedge \eta$ is of type $(p + p', q + q')$.

Next we can use (1.30) to compute $d\omega$. This computation expresses $d\omega$ as a sum of $(r + 1)$-forms of type $(p + 1, q)$ and of type $(p, q + 1)$, denoted respectively by $\partial\omega$ and $\overline{\partial}\omega$. Thus we obtain two differential operators ∂ and $\overline{\partial}$ and we can write

$$d\omega = \partial\omega + \overline{\partial}\omega, \quad d = \partial + \overline{\partial}.$$

Proposition 2.15. (1) Let ω, η be r forms on M and $a \in \mathbb{C}$. Then we have

$$\begin{aligned} \partial(\omega + \eta) &= \partial\omega + \partial\eta, \\ \overline{\partial}(\omega + \eta) &= \overline{\partial}\omega + \overline{\partial}\eta, \\ \partial(a\omega) &= a\partial\omega, \\ \overline{\partial}(a\omega) &= a\overline{\partial}\omega. \end{aligned}$$

.

(2) For the differential operators $\partial, \overline{\partial}$ and r-form ω, we have

$$\begin{aligned} \partial^2\omega &= 0, \\ (\partial\overline{\partial} + \overline{\partial}\partial)\omega &= 0, \\ \overline{\partial}^2\omega &= 0, \\ \partial\overline{\omega} &= \overline{\overline{\partial}\omega}, \\ \overline{\partial}\overline{\omega} &= \overline{\partial\omega}. \end{aligned}$$

Chapter 3

CR geometry

This chapter provides a quick introduction to the theory of CR manifolds. We start with the simplest case of embedded CR manifolds in section 3.1 and then provide the abstract definition in section 3.2. In the last section we define CR maps and prove a theorem that shows the analogy with holomorphic mappings of complex manifolds. Our references are [2], [10], [11] and [12].

3.1 Embedded CR manifolds

We start this chapter with the definition of an embedded CR manifold, which is the simplest class of CR manifolds. In the next section we will present the definition of an abstract CR manifold. For a smooth submanifold M of $\mathbb{C}^n$, we recall that $T_p(M)$ is the real tangent space of M at a point $p \in M$. In general $T_p(M)$ is not invariant under the complex structure map J for $T_p(\mathbb{C}^n)$. Therefore we give special designation to the largest J-invariant subspace of $T_p(M)$.

Definition 3.1. For a point $p \in M$, the complex tangent space of M at p is the vector space

$$H_p(M) = T_p(M) \cap J(T_p(M))$$

.

The space $H_p(M)$ must be an even dimensional real vector space. We also give special designation to the "other directions" in $T_p(M)$ which do not lie in $H_p(M)$.

Definition 3.2. The totally real part of the tangent space of M is the quotient space

$$X_p(M) = T_p(M)/H_p(M)$$

.

Using the Euclidean inner product on $T_p(\mathbb{R}^{2n})$, we can identify $X_p(M)$ with the orthogonal complement of $H_p(M)$. With this identification $J(X_p(M)) \cap X_p(M) = \{0\}$, because $H_p(M)$ is the largest J-invariant subspace of $T_p(M)$. We have $T_p(M) = H_p(M) \oplus X_p(M)$ and $J(X_p(M))$ is orthogonal to $H_p(M)$. The dimensions $H_p(M)$ and $X_p(M)$ are of crucial importance.

Lemma 3.3. Suppose M is a real submanifold of $\mathbb{C}^n$ of real dimension $2n-d$. Then

$$2n - 2d \leq \dim_{\mathbb{R}} H_p(M) \leq 2n - d$$

$$0 \leq \dim_{\mathbb{R}} X_p(M) \leq d$$

The real dimension of $X_p(M)$ is called the *CR* codimension of M. The lemma states that $\dim_{\mathbb{R}} H_p(M)$ is an even number between $2n - 2d$ and $2n - d$. If M is a real hypersurface, then $d = 1$ and so the only possibility is $\dim_{\mathbb{R}} H_p(M) = 2n - 2$. In particular the dimension of $H_p(M)$ never changes. If $d > 1$ then there are more possibilities as in the following example:

Example 3.4. Let $M = \{z \in \mathbb{C}^n : |z| = 1 \quad \text{and} \quad \text{Im} z_1 = 0\}$. M is just the equator of the unit sphere in $\mathbb{C}^n$. Here $d = 2$ and so

$$2n - 4 \leq dim_{\mathbb{R}} H_p(M) \leq 2n - 2$$

for $p \in M$. At the point $p_1 = (0, 1, ..., 0) \in M$, $T_p(M)$ is spanned over $\mathbb{R}$ by

$$\{\partial/\partial x_1, \partial/\partial y_2, \partial/\partial x_3.\partial/\partial y_3, ..., \partial/\partial x_n, \partial/\partial y_n\}.$$

The vectors $J(\partial/\partial x_1) = \partial/\partial y_1$ and $J(\partial/\partial y_2) = -\partial/\partial x_2$ are orthogonal to $T_{p_1}(M)$ and therefore $\partial/\partial x_1, \partial/\partial y_2$ span $X_{p_1}(M)$.
The vectors $\{\partial/\partial x_3, \partial/\partial y_3, ..., \partial/\partial x_n, \partial/\partial y_n\}$ span the J-invariant subspace

$H_{p_1}(M)$. So in this case $\dim_{\mathbb{R}} H_{p_1}(M) = 2n-4$ and $\dim_{\mathbb{R}} X_{p_1}(M) = 2$. Now consider the point $p_2 = (1, 0, ..., 0) \in M$. Here $T_p(M)$ is spanned (over $\mathbb{R}$) by

$$\{\partial/\partial x_2, \partial/\partial y_2, ..., \partial/\partial x_n, \partial/\partial y_n\}$$

which is J-invariant. Therefore $H_{p_2}(M) = T_{x_2}(M)$ and $X_{p_2}(M) = \{0\}$. In this case $\dim_{\mathbb{R}} H_{p_2}(M) = 2n-2$ and $\dim X_{p_2}(M) = 0$.

In the above example the dimension of $H_p(M)$ varies with p. The basic requirement of a CR manifold is that $\dim_{\mathbb{R}} H_p(M)$ is independent of $p \in M$.

Definition 3.5. A submanifold M of $\mathbb{C}^n$ is called an embedded CR manifold or a CR submanifold of $\mathbb{C}^n$ if $\dim_{\mathbb{R}} H_p(M)$ is independent of $p \in M$.

Example 3.6.

- Any real hypersurface in $\mathbb{C}^n$ is a CR submanifold of $\mathbb{C}^n$.
- Another class of CR submanifolds is the class of complex submanifolds of $\mathbb{C}^n$. For a complex submanifold M, the real tangent space is already J-invariant and so $T_p(M) = H_p(M)$.
- Another example of a CR submanifold is a totally real submanifold, which is on the opposite end of the spectrum from a complex manifold.

Definition 3.7. A submanifold M of $\mathbb{C}^n$ is said to be totlly real if $H_p(M) = \{0\}$, for each $p \in M$.

Lemma 3.8. Suppose that M is a CR submanifold of $\mathbb{C}^n$. Then

1. $H_p^{(0,1)} \cap H_p^{(1,0)} = \{0\}$ for each $p \in M$,
2. The subbundles $H^{(0,1)}$ and $H^{(1,0)}$ are involutive.

Proof. The proof of the first part follows from the fact that the intersection of eigenspaces of any linear map corresponding to different eigenvalues is always trivial. For the second part we first note that

$$H^{(1,0)}(M) = T^{\mathbb{C}}(M) \cap \{T^{(1,0)}(\mathbb{C}^n)|_M\}.$$

The bundle $T^{(1,0)}(\mathbb{C}^n)$ is involutive because the Lie bracket of any two vector fields spanned by $\{\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}\}$ is again spanned by $\{\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}\}$. In addition

$T^{\mathbb{C}}$ is involutive beacause the tangent bundle of any manifold is involutive. So, $H^{(1,0)}(M)$ is involutive as desired. Finally as $H^{(0,1)}(M) = \overline{H^{(1,0)}}(M)$, $H^{(0,1)}(M)$ is also involutive. □

The above lemma is important because Properties 1 and 2 will be used to define an abstract CR manifold in the next section.

3.2 Abstract CR Manifolds

So far we have been dealing with CR submanifolds of $\mathbb{C}^n$. In this section we define the concept of an abstract CR manifold which requires no mention of an ambient $\mathbb{C}^n$ or complex manifold. Properties 1 and 2 of Lemma 2.7 make no mention of a complex stucture on $\mathbb{C}^n$ other than to define the space $H^{(1,0)}(M)$. Therefore we define an abstract CR manifold to be a manifold together with a subbundle of $T^{\mathbb{C}}(M)$ which satisfies the above two properties.

Definition 3.9 (Complex CR structure)**.** Let M be a $(2p+s)$-dimensional real manifold and let $T^{\mathbb{C}}(M)$ be its complexified tangent bundle. A CR structure of codimension s in M is a complex p-dimensional smooth subbundle H of $T^{\mathbb{C}}(M)$ such that:

(i) $H \cap \bar{H} = 0$.

(ii) H is involutive, that is for any vector fields Z and W in H we have $[Z, W] \in H$.

Another way to define an abstract CR structure based on Theorem 1.11 is the following:

Definition 3.10 (Real CR structure)**.** Suppose that M is a $(2p+s)$-dimensional real manifold. A CR structure of codimension s in M is a pair (D, J) where D is a $2p$-dimensional smooth subbundle of $T(M)$ and J is a bundle automorphism of D such that:

(i) $J^2 = -id$ and

(ii) if X and Y are sections of D then the same holds for

$$[X,Y]-[JX,JY],[JX,Y]+[X,JY] \tag{3.1}$$

and moreover

$$J([X,Y]-[JX,JY])=[JX,Y]+[X,JY]. \tag{3.2}$$

The following theorem shows that these two definitions are actually equivalent.

Theorem 3.11. *A differentiable manifold has a real CR structure if and only if it has a complex CR structure.*

Proof. Suppose M has a real CR structure (D,J). Then we define

$$H=\{X-iJX; X\in\mathscr{X}(D)\}. \tag{3.3}$$

Of course we have $H\cap\bar{H}=0$. Moreover, if we take $U=X-iJX$ and $V=Y-iJY$ from H we obtain

$$[U,V]=[X,Y]-[JX,JY]-i\{[X,JY]+[JX,Y]\}. \tag{3.4}$$

Taking account of (3.2) , (3.4) becomes

$$[U,V]=[X,Y]-[JX,JY]-iJ\{[X,Y]-[JX,JY]\}. \tag{3.5}$$

Thus by using (3.1) and (3.3) we obtain that $[U,V]$ belongs to H. Consequently M has a complex CR structure. Conversely, suppose M has a complex CR structure. Then we define the distribution D by

$$D=\{X=\mathrm{Re}(U); U\in H\} \text{ and } J:D\to D \tag{3.6}$$

given by

$$JX=\mathrm{Re}(iU), \text{where } X=\mathrm{Re}(U) \text{ and } U\in H. \tag{3.7}$$

Then it is easy to check that we have $J^2=-id$. On the other hand by using (3.6) and (3.7) we get

$$[JX,JY]-[X,Y]=-\mathrm{Re}([U,V])\in\mathscr{X}(D). \tag{3.8}$$

where $X = \mathrm{Re}(U)$ and $Y = \mathrm{Re}(V)$, that is, condition 3.1 is satisfied. Substituting Y by JY in (3.8) we obtain

$$[JX, Y] + [X, JY] = \mathrm{Re}(i[U, V]) \in \mathscr{X}(D). \tag{3.9}$$

The condition (3.2) follows from (3.8) and (3.9). The proof is complete.

□

By this theorem we can say that we have a **CR structure** on M either when M has a real CR structure or a complex CR structure. A manifold with a CR-structure is called a **CR manifold**. A special class of CR manifolds are generic submanifolds of complex manifolds.

Definition 3.12. Let M be a complex manifold of complex dimension m and let N be a submanifold of M of real dimension n. Then, denoting by J the almost complex structure on M we have $[JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0$ for any $X, Y \in T(M)$. Now let $D_p = T_p(N) \cap J(T_p(N)), p \in N$ so that D_p is the maximal invariant subspace of $T_p(N)$ under the action of J. Then we say that N is a generic submanifold of M if $D : p \to D_p \subset T_p(N)$ is a distribution on N.

Theorem 3.13. *Each generic submanifold of a complex manifold is a CR-manifold.*

Proof. Let J' be the restriction of J to the distribution D. We will show that (D, J') is a CR structure on N. First, we consider a coordinate neighborhood U on N and take a complementary distribution D' to D on U. Denote by P and P' the projection morphisms of $T(N)$ to D and respectively D'. Then

$$X = PX + P'X, \text{for any X} \in \mathscr{X}(T(N)). \tag{3.10}$$

Next we see that $JPX = J'PX \in \mathscr{X}(D)$ and $JP'X$ is not tangent to N, otherwise D is not the maximal holomorphic distribution on N. Using the fact that $[JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0$ for any $X, Y \in T(M)$ and 3.10, we obtain

$$[JX, JY] - [X, Y] - JP[JX, Y] + [X, JY] = 0 \tag{3.11}$$

and

$$P'[JX, Y] + [X, JY] = 0 \text{ for any X, Y} \in \mathscr{X}(D). \tag{3.12}$$

Replacing X by JX in (3.12) we obtain (3.1) and then (3.2) is a consequence of (3.11) taking account of (3.10) and (3.12). □

3.3 CR Maps

Definition 3.14. Suppose (M, H) and (N, H') are CR structures. A C^1 map $F : M \to N$ is called a CR map if $f_*\{H\} \subset H'$.

Theorem 3.15. *Suppose (M, H) and (N, H') are CR structures. Let $J_M : H(M) \to H(M)$ and $J_N : H(N) \to H(N)$ be the associated complex structure maps. A C^1 map $f : M \to N$ is a CR map if and only if for each $x \in m$, $f_*\{H_p(M)\} \subset H_{f(p)}(N)$ and $J_N \circ f_*(p) = f_*(p) \circ J_M$ on $H_p(M)$.*

Proof. Assume first that f is a CR map as in Definition 2.13. For $X \in H$

$$f_*(X + \overline{X}) = f_*(X) + \overline{f_*(X)}.$$

Since $f_*(X) \in H'$, $f_*(X + \overline{X})$ is an element of $H(N)$. Moreover, H and $\overline{H}$ are the i and $-i$ eigenspaces for J_M. Therefore

$$f_*(J_M(X + \overline{X})) = f_*(iX - i\overline{X}) = i(f_*(X) - \overline{f_*(X)}).$$

Since $f_*(X)$ is an element of H', which is is the i eigenspace of J_N the above equation becomes

$$f_*(J_M(X + \overline{X})) = J_N(f_*(X + \overline{X}).$$

Thus, $J_N \circ f_*(p) = f_*(p) \circ J_M$, as desired. For the converse: Each element X in H can be written as

$$X = Y - iJ_M Y,$$

where $Y = \frac{1}{2}(X + \overline{X}) \in H(M)$. We also have

$$f_*(X) = f_*(X) - if_*(J_M X) = f_*(X) - iJ_N f_*(X).$$

H' is generated by vectors of the form $Y - iJ_N Y$ for $Y \in H(N)$. Since $f_*(X)$ belongs to $H(N)$ the above equation shows that $f_*(X)$ belongs to H', as desired. □

One of the characterizations of holomorphic mappings between two complex manifolds is that the derivative commutes with the complex structures. The point of the above theorem is that the analogous characterization holds for CR maps between CR manifolds.

We now make some brief comments concerning the embeddability of abstract CR manifolds. The Newlander-Nirenberg theorem states that a manifold with an involutive almost complex structure is a complex manifold and complex manifolds can locally be embedded into $\mathbb{C}^n$ by definition. The analogous question for CR manifolds is then the following: if (M, H) is an abstract CR structure, does there exist a locally defined diffeomorphism $F : M \to \mathbb{C}^n$ so that $F(M)$ is a CR submanifold of $\mathbb{C}^n$ with $F_*H = H^{(1,0)}(F(M))$? In the case that M is real analytic, the answer to this question is positive. However, in the case that M is only smooth this is not always possible as shown by a counterexample by Nirenberg. CR manifolds that can be realized as real hypersurfaces of a certain complex manifold are called realizable. The interested reader can look in [2],[3].

Chapter 4

Contact, Symplectic and Sub-riemannian manifolds

This chapter provides a quick review of basic definitions and results of the theory of contact, symplectic and sub-riemannian manifolds. In the last section we describe how the notions of CR, contact and sub-riemannian structures are related. The main reference for sections 4.1-4.3 is [14]. For section 4.4 our main reference is [8].

4.1 Symplectic manifolds

Let V be an $m-$dimensional vector space over $\mathbb{R}$ and $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. The map Ω is *s*kew symmetric if $\Omega(u,v) = -\Omega(v,u)$ for all $u, v \in V$.

Definition 4.1. A skew symmetric bilinear map Ω is *symplectic* or *nondegenerate* if the linear map $\tilde{\Omega} : V \to V^*$ defined by $\tilde{\Omega}(v)(u) = \Omega(v,u)$ is bijective.

Definition 4.2. The 2-form ω is *symplectic* if ω is closed and ω_p is symplectic for all $p \in M$.

Definition 4.3. A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example 4.4. Let $M = \mathbb{R}^n$ with coordinates $x_1, ..., x_n, y_1, ..., y_n$. The form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic

Definition 4.5. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $g : M_1 \to M_2$ be a diffeomorphism. Then g is a *symplectomorphism* if $g^*\omega_2 = \omega_1$

The following theorem locally classifies symplectic manifolds up to symplectomorphism:

Theorem 4.6. *(Darboux) Let (M, ω) be a $2m$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart*

$$(U, x_1, ..., x_n, y_1, ..., y_n, p)$$

centered at p such that on U

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

4.2 Contact Manifolds

Definition 4.7. A *contact element* on a manifold M is a point $p \in M$, called the *contact point*, together with a tangent hyperplane at p, $H_p \subset T_pM$.

If (p, H_p) is a contact element then $H_p = \ker \omega_p$ with $\omega_p : T_pM \to \mathbb{R}$ linear.

Definition 4.8. A *contact structure* on M is a smooth field of tangent hyperplanes $H \subset TM$, such that for any locally defining 1-form ω, we have $d\omega|_H$ nondegenerate. The pair (M, H) is then called a contact manifold and ω is called a local contact form.

At each $p \in M$, $T_pM = \ker \omega_p \oplus \ker d\omega_p$. It follows that any contact manifold is odd dimensional.

Proposition 4.9. Let H be a field of tangent hyperplanes on M and $\dim M = 2n+1$. Then H is a contact structure if and only if $\omega \wedge (d\omega)^n \neq 0$ for every locally defining 1-form ω.

Example 4.10. On $\mathbb{R}^3$ with coordinates (x, y, z), consider $\omega = xdy + dz$. Then

$$\omega \wedge (d\omega) = (xdy + dz) \wedge (dx \wedge dy) = dx \wedge dy \wedge dz \neq 0$$

, hence ω is a contact form on $\mathbb{R}^3$. The corresponding field of hyperplanes $H = \ker \omega$ at $(x, y, z) \in \mathbb{R}^3$ is

$$H_{(x,y,z)} = \left\{ v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} | \omega(v) = bx + c = 0 \right\}$$

The following theorem is analogous to the Darboux theorem for symplectic manifolds:

Theorem 4.11. *Let (M, H) be a contact manifold and $p \in M$. Then there exists a coordinate system $(U, x_1, ..., x_n, y_1, ..., y_n, z)$ centered at p such that on U $\omega = \sum x_i dy_i + dz$ is a local contact form for H.*

Let (M, H) be a contact manifold with a contact form ω. There exists a unique vector field R on M such that $i_R d\omega = 0, i_R \omega = 1$. The vector field R is called the *Reeb vector field* determined by ω.

Definition 4.12. A *contactomorphism* is a diffeomorphism f of a contact manifold (M, H) which preserves the contact structure, that is $f_* H = H$.

4.3 Kahler manifolds

Definition 4.13. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called *compatible* with ω if the assignement

$$x \mapsto g_x : T_xM \times T_xM \to \mathbb{R}$$

$$g_x(u, v) = \omega_x(u, Jv)$$

is a riemannian metric on M.

Definition 4.14. An almost complex structure J on a manifold M is called *integrable* if and only if J is induced by a complex manifold structure on M.

Definition 4.15. A *Kahler manifold* is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure. The symplectic form ω is then called a *Kahler form.*

Proposition 4.16. Let M be a complex manifold and $\rho \in C^\infty(M; \mathbb{R})$ be *strictly plurisubharmonic* (on each complex chart, the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\right)$ is positive-definite at all $p \in M$). Then

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho$$

is Kahler.

4.4 Sub-Riemannian manifolds

A sub-Riemannian manifold is a manifold that has measuring restrictions, that is we are allowed to measure the magnitude of vectors only for a distinguished subset of vectors called *horizontal vectors.* The precise definition is the following:

Definition 4.17. A sub-Riemannian manifold is a real manifold M of dimension n together with a nonintegrable distribution D of rank $k < n$ endowed with a sub-Riemannian metric g (i.e. an assignement $g_p : D_p \times D_p \to \mathbb{R}$ for all $p \in M$, which is a positive definite, nondegenerate, inner product).

D is called the *horizontal distribution* and the vectors $v \in D_p$ are called *horizontal vectors* at p. A curve $\gamma : [a, b] \to M$ is called a *horizontal curve* if it is absolutely continuous and $\dot{\gamma}(s) \in D_{\gamma(s)}$ for all $s \in [a, b]$,that is the velocity vector of the curve belongs to the horizontal distribution. The metric g is called the sub-riemannian metric and ca be used to define the length of horizontal curves by

$$l(\gamma) = \int_a^b g(\dot{\gamma}(s), \dot{\gamma}(s))^{1/2} ds.$$

Definition 4.18. Let p and q be two points on the manifold M. If there is a horizontal curve joining them, $\gamma : [a, b] \to M$,$\gamma(a) = p, \gamma(b) = q$ then the sub-Riemannian distance between p and q is defined by

$$d_c(p, q) = \inf \{l(\gamma) : \gamma \quad \text{horizontal curve joining} \quad p \quad \text{and} \quad q\}.$$

d_c is also called the Carnot-Caratheodory distance of the sub-Riemannian manifold. In general there is no assurance that there is a horizontal curve joining any two given points. This is the subject of the famous Chow-Rashevskii theorem that deals with horizontal connectivity. Let T_pM be the tangent space of the manifold M at p. Consider the following sequence of subspaces of the space T_pM:

$$\begin{aligned} D_p^1 &= D_p, \\ D_p^2 &= D_p^1 + [D_p, D_p^1], \\ &\vdots \\ D_p^{n+1} &= D_p^n + [D_p, D_p^n]. \end{aligned}$$

Definition 4.19. The distribution D is said to be bracket generating at the point $p \in M$ if there is an integer $r \geqslant 1$ such that $D_p^r = T_pM$. The integer r is called the step of the sub-riemannian manifold (M, D, g) at the point p. (See [8], page 41 and [7], page 47.)

Theorem 4.20. *(Chow-Rashevskii) If D is a bracket generating distribution on a connected manifold M, then any two points can be joined by a horizontal piecewise curve.*

4.5 Levi form

In this section we study how the notions of CR, contact and sub-riemannian structures are related. We concentrate on domains of $\mathbb{C}^2$. Let us start with a definition:

Definition 4.21. Let Ω be a connected open set in $\mathbb{C}^n$ with boundary $\partial\Omega$. The *domain* Ω is said to have C^k boundary, if there exists a k times continuously differentiable function ρ defined on a neighborhood U of the boundary of Ω such that:

$$\Omega \cap U = \{z \in U : \rho(z) < 0\}$$

,

$$\nabla\rho \neq 0 \text{ on the boundary of } \Omega.$$

We call the function ρ a C^k *defining function* for Ω.

Now let Ω be a domain in $\mathbb{C}^2$ with C^2 boundary and let ρ be its defining function. Then

$$\Omega = \left\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) < 0\right\}.$$

and

$$\partial\Omega = \left\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = 0\right\}.$$

Let $H = \ker \partial\rho =< Z >, \quad Z \in T^{(1,0)}_{\partial\Omega}(\mathbb{C}^2)$ where $\partial\rho = \left(\frac{\partial\rho}{\partial z_1}, \frac{\partial\rho}{\partial z_2}\right)$. If $[Z, Z] = 0$, then H is a formally integrable CR-structure.

The *Levi form* of Ω is equal to $i\partial\overline{\partial}\rho$ and its matrix is:

$$L = \begin{bmatrix} \frac{\partial^2\rho}{\partial z_1 \partial \overline{z_1}} & \frac{\partial^2\rho}{\partial z_1 \partial \overline{z_2}} \\ \frac{\partial^2\rho}{\partial z_2 \partial \overline{z_1}} & \frac{\partial^2\rho}{\partial z_2 \partial \overline{z_2}} \end{bmatrix}.$$

If $Z \cdot L \cdot \overline{Z}^T < 0$ the CR-structure H is totally non-integrable. Equivalently, if $Z = \frac{1}{2}(X - iY)$ with $X, Y \in T_{\partial\Omega}(\mathbb{C}^2)$ then $[X, Y] \notin H$. Therefore we define a sub-riemannian metric in $\partial\Omega$ as follows:

$$< X, X >=< Y, Y >= 1, \quad < X, Y >= 0.$$

We also define the horizontal length of a curve $\gamma : [a, b] \to \partial\Omega$ as follows:

$$l_h(\gamma) = \int_a^b < \dot\gamma(t), \dot\gamma(t) >^{\frac{1}{2}} dt.$$

The Carnot-Caratheodory distance is defined as $d_{cc}(p, q) = \inf_\gamma l_h(\gamma)$ where the infimum is taken over all horizontal absolutely continuous curves γ joining p to q. Finnaly, we define a contact form for Ω as:

$$\omega = \mathrm{Im}\partial\rho = \frac{\partial\rho - \overline{\partial}\rho}{2i}$$

Then, $\omega(X) = \omega(Y) = 0$ and there is a unique $T \in T_{\partial\Omega}(\mathbb{C}^2), \quad T \notin H, \quad T \in \ker \partial\overline{\partial}\rho$ such that $\omega(T) = 1$.

Chapter 5

Quasiconformal mappings in the Heisenberg Group

In this chapter we study the Heisenberg group. In section 5.1 we define the Heisenberg group. We also define the Koranyi-Cygan metric and explain why it cannot be derived from a Riemannian metric. In section 5.2 we apply the results of section 4.5 in the case of the Heisenberg group and we study its contact, CR and sub-riemannian structure. Finally, sections 5.3 and 5.4 provide a quick introduction to the definitions and main results for the theory of quasiconformal mappings in the Heisenberg group. For more details and proofs see [18], [19], [20], [21] and [22].

5.1 The Heisenberg group

Definition 5.1. The Heisenberg group $\mathbb{H}$ is the analytic, nilpotent Lie group whose underlying manifold is $\mathbb{R}^3$ and whose Lie algebra $\mathfrak{h}$ is graded as $\mathfrak{h} = V_1 \oplus V_2$ where V_1 has dimension 2 and V_1 has dimension 1. Also following commutator relations hold: $[V_1, V_1] = V_2$ and $[V_1, V_2] = [V_2, V_2] = 0$.

In what follows we are going to use a specific model for the Heisenberg group, that is we will consider $\mathbb{H}$ as the set $\mathbb{C} \times \mathbb{R}$ with multiplication $*$ given by

$$(z,t) * (w,s) = (z+w, t+s+2\mathrm{Im}(\bar{w}z))$$

Let:

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

The vector fields X, Y, T form a basis for the Lie algebra of left-invariant vector fields and the grading of $\mathfrak{h}$ is the following:

$$V_1 = \mathrm{span}_{\mathbb{R}}\{X, Y\} \quad \text{and} \quad V_2 = \mathrm{span}_{\mathbb{R}}\{T\}$$

The Koranyi-Cygan metric d_h is defined by the relation:

$$d_h((z_1,t_1),(z_2,t_2) = |(z_1,t_1)^{-1} * (z_2,t_2)|.$$

The metric d_h is invariant under left translations, conjugation and rotations around the vertical axis $V = \{0\} \times \mathbb{R}$. Left translations are defined by:

$$T_{(\zeta,s)}(z,t) = (\zeta,s) * (z,t),$$

conjugation j is defined by:

$$j(z,t) = (\bar{z}, -t),$$

and rotations are defined by

$$R_\theta(z,t) = (ze^{i\theta}, t).$$

Left translations, conjugations and rotations form the group of Heisenberg isometries. We consider two other kinds of transformations, dilations defined by:

$$D_\delta(z,t) = (\delta z, \delta^2 t),$$

and inversion defined by:

$$I(z,t) = (z(a(z,t))^{-1}, -t|a(z,t)|^{-2}).$$

where $a(z,t) = -|z|^2 + it$. Composites of Heisenberg isometries, dilations and inversion form the similarity group of $\mathfrak{h}$.

The Koranyi-Cygan metric cannot be derived from a Riemannian metric as $d_h(0,t) = |t|^{\frac{1}{2}}$ and the limit $\lim_{t\to 0} \frac{d_h(0,t)}{|t|}$ does not exist. This is what makes the theory of quasiconformal mappings on the Heisenberg group different from the classical theory of quasiconformal mappings.

5.2 Contact, CR and Sub-Riemannian structure

In this section we are going to identify the Heisenberg group with the boundary of the Siegel domain. The defining function for the Siegel domain S is $\rho = 2\mathrm{Re} z_1 + |z_2|^2$. Hence we have that

$$S = \left\{(z_1, z_2) \in \mathbb{C}^2 : 2\mathrm{Re} z_1 + |z_2|^2 < 0\right\}$$

and

$$\partial S = \left\{(z_1, z_2) \in \mathbb{C}^2 : 2\mathrm{Re} z_1 + |z_2|^2 = 0\right\}.$$

Straightforward calculations show that:

$$\begin{aligned} \partial\rho &= dz_1 + \overline{z_2} dz_2, \\ \overline{\partial}\rho &= d\overline{z_1} + z_2 d\overline{z_2}, \\ \partial\overline{\partial}\rho &= dz_2 \wedge d\overline{z_2}. \end{aligned}$$

We also set $H = < \overline{z_2}\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} >$. Now the Levi form for the Siegel domain is:

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$Z \cdot L \cdot \overline{Z}^T = \begin{bmatrix} \overline{z_2} & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} z_2 \\ -1 \end{bmatrix} > 0.$$

This means that H is a totally non integrable CR-structure.

Let us now describe the contact form of the Heisenberg group. First, consider the following two mappings:

$$\begin{aligned} \phi &: \mathbb{H} \mapsto \partial S, \phi(\zeta, t) = (-|\zeta|^2 + it, \sqrt{2}\zeta), \\ \psi &: \partial S \mapsto \mathbb{H}, \psi(z_1, z_2) = (\frac{z_2}{\sqrt{2}}, \mathrm{Imz_1}) \in \mathbb{H}. \end{aligned}$$

One verifies that

$$\phi \circ \psi = id.$$

We now have:

$$\begin{aligned}\omega &= \mathrm{Im}\partial\rho = dy_1 + \mathrm{Im}(\overline{z_2}dz_2)\\ &= dt + 2\mathrm{Im}(\overline{\zeta}d\zeta)\\ &= dt + 2xdx - 2ydy.\end{aligned}$$

Then, it follows that $Z = \frac{\partial}{\partial\zeta} + i\overline{\zeta}\frac{\partial}{\partial t}$ and for the corresponding real vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

where $[X, Y] = T$. Notice also that $\omega(T) = 1$. As before we define a sub-riemannian metric using

$$< X, X >=< Y, Y >= 1, \quad < X, Y >= 0.$$

For a curve $\gamma : [a, b] \to \partial S$ we have

$$\begin{aligned}\dot{\gamma}(s) &= \dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} + \dot{z}(s)\frac{\partial}{\partial z}\\ &= \dot{x}(s)(X + 2y(s)T) + \dot{y}(s)(Y - 2x(s)T) + \dot{t}(s)T\\ &= \dot{x}(s)X + \dot{y}(s)Y + (\dot{t}(s) - 2x(s)\dot{y}(s) + 2y(s)\dot{x}(s))T\end{aligned}$$

It follows that for horizontal curves we have:

$$< \dot{\gamma}, \dot{\gamma} >= (\dot{x})^2 + (\dot{y})^2$$

and for such curves the horizontal length is given by:

$$l_h(\gamma) = \int_1^b \sqrt{(\dot{x})^2 + (\dot{y})^2} ds.$$

The Carnot-Caratheodory distance is defined as before, that is, $d_{cc}(p, q) = \inf_\gamma l_h(\gamma)$ where the infimum is taken over all horizontal absolutely continuous curves γ joining p to q. The Carnot-Caratheodory metric is left invariant and homogeneous with respect to dilations. Moreover the Carnot-Caratheodory metric and the Heisenberg metric are equivalent in the sense that there exists positive constants C_1, C_2 such that:

$$C_1 d_h(\xi, \eta) \leqslant d_{cc}(\xi, \eta) \leqslant C_2 d_h(\xi, \eta).$$

(See [4]).

5.3 Quasiconformal mappings

Let $f : G \to H$ be a homeomorphism where G is an open connected set in H. We consider the following functions

$$L_f(p,r) = \sup_{d(p,q)=r} d(f(p), f(q))$$

$$l_f(p,r) = \inf_{d(p,q)=r} d(f(p), f(q))$$

which are defined when $d(p, \partial G) > r$. Also let's set

$$H_f(p) = \limsup_{r\to 0} \frac{L_f(p,r)}{l_f(p,r)}$$

$$J_f(p) = \limsup_{r\to 0} \frac{|f(b_{r,p})|}{B_{r,p}}$$

where $B_{r,p}$ is the Heisenberg ball $\{q \in \mathbb{H} : d(p,q) < r\}$.

Definition 5.2. A homeomorphism $f : G \to \mathbb{H}$ is a quasiconformal mapping if H_f is uniformly bounded in the domain G. If in addition

$$ess \sup_{p\in G} |H_f(p)| = ||H_f||_\infty \le K$$

then f is called a K-quasiconformal mapping.

It is proved in [21] that for smooth mappings of the Heisenberg group the following definition is equivalent to 5.2. The smoothness requirement can also be relaxed considerably.

Definition 5.3. A smooth K-quasiconformal mapping $f : M \to M'$ between strictly pseudocnvex CR manifolds is a C^2-contact transformation such that $f^*\theta' = \lambda\theta$ with $\lambda > 0$ and

$$\lambda K_{-1} L(X,X) \le L'(f_*X, f_*X) \le \lambda K L(X,X)$$

for all $X \in HM$.

5.4 The Beltrami equation

Theorem 5.4. *The C^2-diffeomorphism f is a K-quasiconformal mapping if and only if there exists a complex valued function μ with $|\mu| \leq \frac{K-1}{K+1}$ such that $f_I = f_1 + if_2$ and $f_{II} = f_3 + i|f_I|^2$ satisfy the equations*

$$\overline{\boldsymbol{Z}} f_I = \mu \boldsymbol{Z} f_I \tag{5.1}$$

$$\overline{\boldsymbol{Z}} f_{II} = \mu \boldsymbol{Z} f_{II}. \tag{5.2}$$

Moreover, if f is a contact quasiconformal mapping then (5.1) implies (5.2).

Chapter 6

Quasiconformal mappings in the Roto-affine Group

This chapter contains the main result of the thesis. We start with the definition of the Roto-affine group and the description of the left invariant vector fields in section 6.1. Then, we define a riemannian structure on the Roto-affine group which gives a riemannian isometry between the Roto-affine group and $\mathbb{L} \times S^1$. Section 6.1 ends with the description of the contact structure of the Roto-affine group. In section 6.2 we define an alternative riemannian structure on the Roto-affine group which makes the Koranyi map a riemannian submersion from the Roto-affine group to the hyperbolic plane. In section 6.3 we describe the relation between horizontal curves of the hyperbolic plane and horizontal curves on the Roto-affine group. We proceed with the proofs of some propositions regarding contact transformations in sections 6.4 and 6.5. The last section is devoted to the statement and proof of the Lifting theorem.

6.1 The Roto-affine group

Definition 6.1. We denote by $\mathbb{RA}$ the set $\mathbb{C}_* \times \mathbb{R}$, with group law

$$(z, t) \star (z', t') = (zz', t + t'|z|^2), \tag{6.1}$$

and we call $\mathbb{RA}$ the roto-affine group.

$\mathbb{RA}$ is a non Abelian group, its neutral element is $(1, 0)$ and the inverse of (z, t) is $(1/z, -t/|z|^2)$. It is also a Lie group with underlying manifold $\mathbb{C}_* \times \mathbb{R}$. Indeed the map

$$\begin{aligned} \mathbb{RA} \times \mathbb{RA} &\to \mathbb{RA}, \\ (p, q) &\mapsto p^{-1} \star q. \end{aligned}$$

can be written in coordinate form as

$$(z, t, w, s) \mapsto (\frac{w}{z}, \frac{s-t}{|z|^2}),$$

which is clearly smooth. To detect the left-nvariant vector fields of $\mathbb{RA}$, we fix a left translation

$$F(z, t) = L_{(w,s)}(z, t) = (wz, s + t|w|^2),$$

and we consider the complex matrix DF of the differential F_*:

$$DF = \begin{bmatrix} w & 0 & 0 \\ 0 & \bar{w} & 0 \\ 0 & 0 & |w|^2 \end{bmatrix}.$$

Now we can check that the vector fields

$$Z^* = z\frac{\partial}{\partial z}, \quad \bar{Z}^* = \bar{z}\frac{\partial}{\partial \bar{z}}, \quad T^* = |z|^2\frac{\partial}{\partial t},$$

are all left-invariant for $\mathbb{RA}$. Indeed, for the vector field Z^* we have

$$F_*(Z) = wz\frac{\partial}{\partial z} = Z \circ F.$$

The verification for $\bar{Z}^*$ and T^* is similar. The vector field $Z^*, \bar{Z}^*, T^*$ form a basis for the tangent space of $\mathbb{RA}$ which satisfy the following bracket relations:

$$[Z^*, \bar{Z}^*] = 0, \quad [Z^*, T^*] = [\bar{Z}^*, T^*] = T^*.$$

The corresponding real basis is

$$X^* = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y^* = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad T^* = (x^2 + y^2)\frac{\partial}{\partial t},$$

so that

$$Z^* = \frac{1}{2}(X^* - iY^*), \bar{Z}^* = \frac{1}{2}(X^* + iY^*),$$

The bracket relations for X^*, Y^* and T^* are:

$$[X^*, T^*] = 2T^*, [X^*, Y^*] = [Y^*, T^*] = 0.$$

Also, since $\det(DF) = |w|^4$ we have that the Haar measure of $\mathbb{RA}$ is:

$$dm = \frac{dx \wedge dy \wedge dt}{(x^2+y^2)^2} = \frac{i}{2}\frac{dz \wedge d\bar{z} \wedge dt}{|z|^2}.$$

6.1.1 First Riemannian structure

By declaring that the vector fields X^*, Y^*, T^* form an orthonormal basis we obtain a left-invariant Riemannian metric g^* for $\mathbb{RA}$:

$$\begin{aligned} g^*(X^*, X^*) &= g^*(Y^*, Y^*) = g^*(T^*, T^*) = 1, \\ g^*(X^*, Y^*) &= g^*(X^*, T^*) = g^*(Y^*, T^*) = 0. \end{aligned}$$

Using

$$\begin{aligned} X^* &= x\partial_x + y\partial_y, \\ Y^* &= -y\partial_x + x\partial_y, \\ T^* &= |z|^2\partial_t, \end{aligned}$$

we find that

$$\partial_x = \frac{xX^* - yY^*}{|z|^2},$$

$$\partial_y = \frac{xY^* + yX^*}{|z|^2},$$

$$\partial_t = \frac{1}{|z|^2}T^*.$$

We may now write the metric tensor g^* as:

$$g^* = \frac{dx^2 + dy^2}{|z|^2} + \frac{dt^2}{|z|^4}.$$

Now let $\mathbb{L} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ be the left half plane and S^1 the unit circle. Consider the map $\mathfrak{A} : \mathbb{RA} \to \mathbb{L} \times S^1$ given by

$$\mathfrak{A}(z, t) = (-|z|^2 + it, \arg(z))$$

with inverse

$$\mathfrak{A}^{-1}(\zeta, \phi) = (\sqrt{-\mathrm{Re}(\zeta)}e^{i\phi}, \mathrm{Im}(\zeta)).$$

This map identifies $\mathbb{RA}$ with the cylinder $\mathbb{L} \times S^1$ and we have the following proposition:

Proposition 6.2. If $g = ds^2 = ds_h^2 + d\phi^2$ is the Riemannian product metric in $\mathbb{L} \times S^1$, where $ds_h = \frac{|d\zeta|}{-2\mathrm{Re}(\zeta)}$ is the hyperbolic metric in $\mathbb{L}$ and $|d\phi|$ is the intrinsic Riemannian metric in S^1 the map $\mathfrak{A}$ is an isometry between $(\mathbb{RA}, g^*)$ and $(\mathbb{L} \times S^1, g)$.

Proof. Consider the Jacobian matrix $D\mathfrak{A}$ of the differential $\mathfrak{A}_*$:

$$D\mathfrak{A} = \begin{bmatrix} -2x & -2y & 0 \\ 0 & 0 & 1 \\ \frac{y}{|z|^2} & -\frac{x}{|z|^2} & 0 \end{bmatrix}.$$

This gives $\mathfrak{A}_* X^* = 2\xi\partial_\xi$, $\mathfrak{A}_* Y^* = -\partial_\phi$, $\mathfrak{A}_*(2T^*) = -2\xi\partial_\eta$. Since the above vector fields are orthonormal for g we obtain $\mathfrak{A}^* g = g^*$ and the proof is complete. □

6.1.2 Contact Structure

Now we will study the contact structure of the roto-affine group. We will consider the follwing 1-form

$$\omega^* = \frac{\omega}{2|z|^2},$$

which arises naturally from the contact form ω of the Heisenberg group. The following proposition describes this form.

Proposition 6.3. The manifold $(\mathbb{RA}, \omega^*)$ is contact. Explicitely:

1. The form ω^* is left invariant.
2. If dm is the Haar measure for $\mathbb{RA}$ then $dm = \omega^* \wedge d\omega^*$.
3. The kernel ω^* is generated by the left invariant vector fields
$$\mathbf{X} = X^* = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$
$$\mathbf{Y} = Y^* - 2T^* = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - 2|z|^2\frac{\partial}{\partial t}.$$
where X^*, Y^*, T^* are the basis of $T(\mathbb{RA})$.
4. The Reeb vector field for ω^* is
$$\mathbf{T} = Y^* = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$
5. The only non trivial bracket relation between **X,Y,T** is
$$[\mathbf{X}, \mathbf{Y}] = 2(\mathbf{Y} - \mathbf{T}).$$

Proof. To prove (1) fix a $(w, s) \in \mathbb{RA}$ and consider a left translation
$$f(z,t) = T_{(w,s)}(z,t) = (wz, s + t|w|^2),$$
Then
$$\begin{aligned} f^*(\omega^*) &= \frac{d(s+t|w|^2) + 2\mathrm{Im}(\bar{w}\bar{z}d(wz)}{2|w|^2|z|^2} \\ &= \frac{|w|^2(dt + 2\mathrm{Im}(\bar{z}dz))}{2|w|^2|z|^2} \\ &= \omega^*. \end{aligned}$$
To prove (2) we observe first that
$$\omega^* = \frac{dt}{2|z|^2} + d(\mathrm{Im}(\mathrm{Log}z)).$$

Therefore

$$d\omega^* = d(\frac{dt}{2|z|^2}) = -i\frac{\bar{z}dz + zd\bar{z}}{2|z|^4} \wedge dt = -\frac{xdx + ydy}{|z|^4} \wedge dt.$$

It follows that

$$\omega^* \wedge d\omega^* = -i\frac{dz \wedge d\bar{z} \wedge dt}{2|z|^4} = \frac{dx \wedge dy \wedge dt}{|z|^4} = dm.$$

Now (3) is clear. **X,Y,T** are left invariant, linearly independent and

$$\omega^*(\mathbf{X}) = \omega^*(\mathbf{Y}) = 0.$$

As for (4) we have that $\omega^*(\mathbf{T}) = 1$ and we can verify that $d\omega^*(\mathbf{T}, X^*) = d\omega^*(\mathbf{T}, Y^*) = d\omega^*(\mathbf{T}, T^*) = 0$. Therefore $d\omega^*(\mathbf{T}, X) = 0$ for all $X \in T(\mathbb{RA})$. To conclude the proof we show the bracket relation:

$$[\mathbf{X}, \mathbf{Y}] = [X^*, Y^* - 2T^*] = -2[X^*, T^*] = -4T^* = 2(\mathbf{Y} - \mathbf{T}).$$

□

6.2 Second Riemannian Structure

From the contact form ω^* of $\mathbb{RA}$ we will define a second Riemannian structure on $\mathbb{RA}$ which turns it into a contact Riemannian manifold with a significant property. We will denote this metric tensor by g and define it by declaring the left-invariant vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{T}$ an orthonormal basis:

$$g(\mathbf{X}, \mathbf{X}) = g(\mathbf{Y}, \mathbf{Y}) = g(\mathbf{T}, \mathbf{T}) = 1, \quad g(\mathbf{X}, \mathbf{Y}) = g(\mathbf{X}, \mathbf{T}) = g(\mathbf{Y}, \mathbf{T}) = 0.$$

The horizontal bundle $H(\mathbb{RA})$ is generated by the vector fields $\mathbf{X}, \mathbf{Y} \in \ker(\omega^*)$. Define an almost complex operator J in $H(\mathbb{RA})$ by the relations

$$J\mathbf{X} = \mathbf{Y}, \quad J\mathbf{Y} = -\mathbf{X}.$$

For arbitrary $\mathbf{U}, \mathbf{V} \in H(\mathbb{RA})$ we define a Riemannian tensor g by the relation

$$g(J\mathbf{U}, \mathbf{V}) = \frac{1}{2}d\omega^*(\mathbf{U}, \mathbf{V})$$

For the basis $\mathbf{X}, \mathbf{Y}$ of $H(\mathbb{RA})$ we thus have

$$g(\mathbf{X}, \mathbf{X}) = g(\mathbf{Y}, \mathbf{Y}) = 1, \quad g(\mathbf{X}, \mathbf{Y}) = g(\mathbf{Y}, \mathbf{X}) = 0.$$

We may extend this product to the whole tangent bundle $T(\mathbb{RA})$ by the relation

$$\omega^*(\mathbf{U}) = g(\mathbf{U}, \mathbf{T}).$$

In this way we obtain

$$g(\mathbf{X}, \mathbf{T}) = g(\mathbf{Y}, \mathbf{T}) = 0, g(\mathbf{T}, \mathbf{T}) = 1.$$

On the other hand, by setting $J\mathbf{T} = 0$ we obtain

$$g(\mathbf{T}, \mathbf{X}) = g(\mathbf{T}, \mathbf{Y}) = 0.$$

Solving the system

$$\begin{aligned} x\partial_x + y\partial_y &= \mathbf{X}, \\ -y\partial_x + x\partial_y - 2|z|^2\partial_t &= \mathbf{Y}, \\ -y\partial_x + x\partial_y &= \mathbf{T}, \end{aligned}$$

we have

$$\partial_x = \frac{1}{|z|^2}(x\mathbf{X} - y\mathbf{T}), \partial_y = \frac{1}{|z|^2}(x\mathbf{T} + y\mathbf{X}), \partial_t = \frac{1}{2|z|^2}(\mathbf{T} - \mathbf{Y}).$$

Consequently the metric tensor is given in terms of coordinates

$$ds^2 = \frac{dx^2 + dy^2}{|z|^2} + \frac{dt^2 + 2xdydt - 2ydxdt}{2|z|^4} = \frac{dx^2 + dy^2 + \omega^* dt}{|z|^2}.$$

If ∇ is a Riemannian connection on a 3-manifold and X_1, X_2, X_3 is an orthonormal basis of vector fields, then Koszul's formula for ∇ is

$$-2g(Z, \nabla_Y X) = g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z),$$

where X, Y, Z run through X_1, X_2, X_3. Using this and the equation

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

we calculate straightforwardly:

$$\begin{aligned} \nabla_{\mathbf{X}}\mathbf{X} &= 0, \nabla_{\mathbf{Y}}\mathbf{X} - 2\mathbf{Y} + \mathbf{T}, \nabla_{\mathbf{T}}\mathbf{X} = \mathbf{Y}, \\ \nabla_{\mathbf{X}}\mathbf{Y} &= -\mathbf{T}, \nabla_{\mathbf{Y}}\mathbf{Y} = 2\mathbf{X}, \nabla_{\mathbf{T}}\mathbf{Y} = -\mathbf{X}, \\ \nabla_{\mathbf{X}}\mathbf{T} &= \mathbf{Y}, \nabla_{\mathbf{Y}}\mathbf{T} = -\mathbf{X}, \nabla_{\mathbf{T}}\mathbf{T} = 0. \end{aligned}$$

Denote by R the curvature tensor:

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

We have

$$\begin{aligned} R(\mathbf{X},\mathbf{Y})\mathbf{X} &= -7\mathbf{Y}, \\ R(\mathbf{X},\mathbf{T})\mathbf{X} &= \mathbf{T} - 2\mathbf{Y}, \\ R(\mathbf{Y}.\mathbf{T})\mathbf{Y} &= 2\mathbf{Y} + \mathbf{T}. \end{aligned}$$

The sectional curvature of the planes spanned by$\mathbf{X}, \mathbf{Y}$ is thus

$$K(\mathbf{X},\mathbf{Y},\mathbf{X},\mathbf{Y}) = g(R(\mathbf{X},\mathbf{Y})\mathbf{X},\mathbf{Y}) = -7,$$

the sectional curvature of the planes spanned by$\mathbf{X}, \mathbf{T}$ is

$$K(\mathbf{X},\mathbf{T},\mathbf{X},\mathbf{T}) = g(R(\mathbf{X},\mathbf{T})\mathbf{X},\mathbf{T}) = 1,$$

and the same holds for the sectional curvature of the planes spanned by $\mathbf{Y}, \mathbf{T}$:

$$K(\mathbf{Y},\mathbf{T},\mathbf{Y},\mathbf{T}) = g(R(\mathbf{Y},\mathbf{T})\mathbf{Y},\mathbf{T}) = 1.$$

6.2.1 Riemannian submersion from $\mathbb{RA}$ to $\mathbb{L}$

We now show that the second Riemannian metric of $\mathbb{RA}$ is exactly the one that for which the Korányi map is a Riemannian submersion.

Proposition 6.4. Let $\alpha : \mathbb{RA} \to \mathbb{L}$ be the Koranyi map, $\alpha(z,t) = -|z|^2 + it$. We consider $\mathbb{RA}$ endowed with the metric g and $\mathbb{L}$ with the usual hyperbolic metric g_h. Then

$$\alpha : \mathbb{RA} \to \mathbb{L}$$

is a Riemannian submersion.

Proof. Let $\zeta = \xi + i\eta$ be the complex coordinate on $\mathbb{L}$. Then

$$g_h = ds^2 = \frac{d\xi^2 + d\eta^2}{4\xi^2},$$

and the vector fields

$$\Xi = -2\xi\partial_\xi, \quad \mathrm{H} = -2\xi\partial_\eta,$$

is an orthonormal frame for $T(\mathbb{L})$. The matrix $D\alpha$ of the differential $\alpha_* : T(\mathbb{R}\mathbb{A}) \to T(\mathbb{L})$ is

$$D\alpha = \begin{bmatrix} -2x & -2y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Straightforward calculations then show that $\mathbf{X}, \mathbf{Y}, \mathbf{T}$ are α-related to Ξ, H. In particular we have

$$\alpha_*(\mathbf{X}) = \Xi, \quad \alpha_*(\mathbf{Y}) = \mathrm{H}, \quad \alpha_*(\mathbf{T}) = 0.$$

Therefore $\ker(\alpha_*) =< \mathbf{T} >$. Hence

$$\ker^\perp(\alpha_*) =< \mathbf{X}, \mathbf{Y} >$$

It follows immediately that

$$\alpha_* : \ker^\perp(\alpha_*) \to T(\mathbb{L}),$$

is a linear isometry, that is, α is a Riemannian submersion. □

6.3 Horizontal curves and Ehresmann completeness of α

Let $\gamma : [a, b] \to \mathbb{C}_* \times \mathbb{R}$ be an absolutely continuous curve. If $\dot\gamma$ is the tangent vector field along γ we may write

$$\begin{aligned}
\dot\gamma(s) &= \dot x(s)\partial_x + \dot y(s)\partial_y + \dot t(s)\partial_t \\
&= \frac{\dot x(s)}{|z(s)|^2}(x(s)\mathbf{X} - y(s)\mathbf{T}) + \frac{\dot y(s)}{|z(s)|^2}(x(s)\mathbf{T} + y(s)\mathbf{X}) + \frac{\dot t(s)}{2|z(s)|^2}(\mathbf{T} - \mathbf{Y}) \\
&= \frac{x(s)\dot x(s) + y(s)\dot y(s)}{|z(s)|^2}\mathbf{X} - \frac{\dot t(s)}{2|z(s)|^2}\mathbf{Y} + \frac{\dot t(s) + 2(x(s)\dot y(s) - y(s)\dot x(s))}{2|z(s)|^2}\mathbf{T}.
\end{aligned}$$

Therefore

$$\begin{aligned}
<\dot{\gamma}, \mathbf{X}>_{\gamma(s)} &= \mathrm{Re}(\frac{\dot{z}(s)}{z(s)}),\\
<\dot{\gamma}, \mathbf{Y}>_{\gamma(s)} &= \mathrm{Im}(\frac{\dot{z}(s)}{z(s)}) - \frac{\dot{t}(s) + 2\mathrm{Im}(\overline{z(s)}\dot{z}(s))}{2|z(s)|^2},\\
<\dot{\gamma}, \mathbf{T}>_{\gamma(s)} &= \frac{\dot{t}(s) + 2\mathrm{Im}(\overline{z(s)}, \dot{z}(s))}{2|z(s)|^2}
\end{aligned}$$

and the length of γ with respect to the metric g is defined by

$$l_{RA}(\gamma) = \int_a^b \sqrt{<\dot{\gamma}, \mathbf{X}>^2_{\gamma(s)} + <\dot{\gamma}, \mathbf{Y}>^2_{\gamma(s)} + <\dot{\gamma}, \mathbf{T}>^2_{\gamma(s)}}.$$

Now the following proposition is straightforward

Proposition 6.5. If $\gamma(s) = (z(s), t(s)), z(s) = x(s) + iy(s)$ then γ is horizontal if and only if

$$<\dot{\gamma}, \mathbf{T}>_{\gamma(s)}= 0,$$

for a.e s. In this case

$$l_{RA}(\gamma) = \int_a^b \frac{\dot{z}(s)}{z(s)}.$$

If γ is horizontal, let $\gamma^* : [a, b] \to \mathbb{L}$ defined by $\gamma^* = \alpha \circ \gamma$ where α is the Koranyi map. Then for the hyperbolic length of γ^* we have:

$$\begin{aligned}
l_h(\gamma^*) &= \int_a^b \frac{|\gamma^*(s)|}{2\mathrm{Re}(\gamma^*(\mathrm{s}))} ds\\
&= \int_a^b \frac{|2\mathrm{Re}(\overline{z(s)}\dot{z}(s)) - 2i\mathrm{Im}(\overline{z(s)}\dot{z}(s))|}{2|z(s)|^2}\\
&= \int_a^b \frac{|\overline{z(s)}\dot{z}(s)|}{|z(s)|^2}\\
&= l_{\mathbb{RA}}(\gamma).
\end{aligned}$$

Let $\pi : M \to B$ be a submersion and H a distribution on M supplementary to $V = \ker(\pi_*)$. The distribution H is Ehresmann complete if for any path $\gamma*$ in B with starting point p^*, and any $p \in \pi^{-1}(p^*)$ there exists a horizontal lift γ of γ^* in M ($\pi \circ \gamma = \gamma^*$) starting from p. The following proposition shows that the horizontal distribution $H =< \mathbf{X}, \mathbf{Y} >$ is Ehresmann complete.

Proposition 6.6. Suppose that $\gamma^* : [a,b] \to \mathbb{L}$ is an absolutely continuous curve starting from $p^* \in \mathbb{L} = \gamma^*(a)$. Then for every $p \in \alpha^{-1}(p^*)$ there exists a horizontal $\gamma : [a,b] \to \mathbb{RA}$ starting from p and such that $\alpha \circ \gamma = \gamma^*$ and $l_{\mathbb{RA}}(\gamma) = l_h(\gamma^*)$. Here, α is the Koranyi map.

Proof. Set $\gamma^*(s) = \zeta(s), \gamma^*(a) = \zeta_0$. Then

$$\alpha^{-1}(\zeta_0) = \{((\mathrm{Re}^{1/2}(-\zeta_0)e^{i\theta}, \mathrm{Im}(\zeta_0))|\theta \in \mathbb{R}\}.$$

Pick a $\theta_0 \in \mathbb{R}$ and define $\gamma : [a,b] \to RA$ from the relation

$$\gamma(s) = (z(s), t(s)) = (\mathrm{Re}^{1/2}(\zeta(s))e^{i\theta(s)}, \mathrm{Im}(\zeta(s))),$$

where

$$\theta(s) - \theta_0 = \int_a^s \frac{\mathrm{Im}(\dot{\zeta}(u))}{2\mathrm{Re}(\zeta(u))} du.$$

It is clear that γ starts from $((\mathrm{Re}^{1/2}(-\zeta_0)e^{i\theta}, \mathrm{Im}(\zeta 0))$ and we show that γ is horizontal:

$$\begin{aligned} \frac{\dot{t}(s)}{2|z(s)|^2} &= \frac{\mathrm{Im}(\zeta(\dot{s}))}{2\mathrm{Re}(\zeta(s))} \\ &= -\dot{\theta}(s) = -\arg(\dot{z}(s)). \end{aligned}$$

If γ^* is closed $\gamma^*(a) = \gamma^*(b) = \zeta_0 \in \mathbb{L}$ then by setting $\zeta = \xi + i\eta$ we have

$$\theta(b) - \theta(a) = \theta(b) - \theta_0 = \int_a^b \frac{\mathrm{Im}(\dot{\zeta}(s))}{2\mathrm{Re}(\zeta(s))} = \int_{\gamma^*} \frac{d\eta}{2\xi}.$$

Applying Stokes' Theorem we get

$$\theta(b) = -\int\int_{int(\gamma^*)} \frac{d\xi \wedge d\eta}{2\xi^2} = -2\mathrm{Area_h}(\mathrm{int}(\gamma^*)).$$

Thus

$$\begin{aligned} \gamma(a) &= (\mathrm{Re}^{1/2}(-\zeta_0)e^{i\theta_0}, \mathrm{Im}(\zeta_0)), \\ \gamma(b) &= (\mathrm{Re}^{1/2}(-\zeta_0)e^{i(\theta_0 - 2\mathrm{Area_h}(\mathrm{int}(\gamma^*))}, \mathrm{Im}(\zeta_0)). \end{aligned}$$

The last statement of the proposition is obvious. □

6.4 Contact transformations

Let $F = (f_I, f_3)$ be an orientation preserving diffeomorphism of $\mathbb{RA}$ which is a contact transformation, that is $F^*\omega^* = \lambda^*\omega^*$ for some positive function λ^*. In other words:

$$d\arg(f_I) + \frac{df_3}{2|f_I|^2} = \lambda^*\omega^*.$$

We set $f_{II} = -|f_I|^2 + if_3$. We observe that a basis for the cotangent space comprises the forms

$$\phi, \quad \overline{\phi} \quad \text{and} \quad \omega^*,$$

where

$$\phi = -\frac{d\alpha(z,t)}{2|z|^2}$$

and α is the Korányi map. Now, the Jacobian matrix of the differential F_* may be expressed as follows:

$$\begin{bmatrix} <\phi, F_*\mathbf{Z}> & <\phi, F_*\overline{\mathbf{Z}}> & <\phi, F_*\mathbf{T}> \\ <\overline{\phi}, F_*\mathbf{Z}> & <\overline{\phi}, F_*\overline{\mathbf{Z}}> & <\overline{\phi}, F_*\mathbf{T}> \\ <\omega^*, F_*\mathbf{Z}> & <\omega^*, F_*\overline{\mathbf{Z}}> & <\omega^*, F_*\mathbf{T}> \end{bmatrix} = \begin{bmatrix} -\frac{\mathbf{Z}f_{II}}{2|f_I|^2} & -\frac{\overline{\mathbf{Z}}f_{II}}{2|f_I|^2} & -\frac{\mathbf{T}f_{II}}{2|f_I|^2} \\ -\frac{\overline{\mathbf{Z}f_{II}}}{2|f_I|^2} & -\frac{\overline{\mathbf{Z}f_{II}}}{2|f_I|^2} & -\frac{\overline{\mathbf{T}f_{II}}}{2|f_I|^2} \\ 0 & 0 & \lambda^* \end{bmatrix}.$$

We prove the above equality. In the first place

$$<\phi, F_*\mathbf{Z}> = (F^*(\frac{d|z|^2 - idt}{2|z|^2}))(\mathbf{Z}) = -\frac{df_{II}}{2|f_I|^2}(\mathbf{Z}) = -\frac{\mathbf{Z}f_{II}}{2|f_I|^2}.$$

and analogously for the other coefficients of the first two rows. For the third row:

$$\begin{aligned} <\omega^*, F_*\mathbf{Z}> &= (F^*\omega^*)(\mathbf{Z}) = \lambda^*\omega(\mathbf{Z}) = 0, \\ <\omega^*, F_*\overline{\mathbf{Z}}> &= (F^*\omega^*)(\overline{\mathbf{Z}}) = \lambda^*\omega(\overline{\mathbf{Z}}) = 0, \\ <\omega^*, F_*\mathbf{T}> &= (F^*\omega^*)(\mathbf{T}) = \lambda^*\omega(\mathbf{T}) = \lambda^*, \end{aligned}$$

since F is contact. These equalities induce the contact conditions:

$$\begin{aligned} \frac{\mathbf{Z}f_3}{2|f_I|^2} + \mathbf{Z}\arg(f_I) &= 0, \\ \frac{\overline{\mathbf{Z}}f_3}{2|f_I|^2} + \overline{\mathbf{Z}}\arg(f_I) &= 0, \\ \frac{\mathbf{T}f_3}{2|f_I|^2} + \mathbf{T}\arg(f_I) &= \lambda^*. \end{aligned}$$

From the contact conditions we immediately obtain that the Jacobian matrix of the differential F_* may also be written as:

$$\begin{bmatrix} \mathbf{Z}\mathrm{Log}(f_I) & \overline{\mathbf{Z}}\mathrm{Log}(f_I) & \mathbf{T}\mathrm{Log}(f_I) - i\lambda^* \\ \mathbf{Z}\mathrm{Log}(\overline{f_I}) & \overline{\mathbf{Z}}\mathrm{Log}(\overline{f_I}) & \mathbf{T}\mathrm{Log}(\overline{f_I}) - i\lambda^* \\ 0 & 0 & \lambda^* \end{bmatrix}.$$

Proposition 6.7. The Jacobian determinant J_F satisfies $J_F = (\lambda^*)^2$. Moreover

$$\begin{aligned} \lambda^* &= |\mathbf{Z}\mathrm{Log}(f_I)|^2 - |\overline{\mathbf{Z}}\mathrm{Log}(f_I)|^2 \\ &= 2\mathrm{Im}(\mathbf{Z}\mathrm{Log}(|f_I|)\overline{\mathbf{Z}}\arg(f_I)) \\ &= -\frac{\mathrm{Im}(\mathbf{Z}\log(|f_I|)\overline{\mathbf{Z}}f_3}{|f_I|^2}. \end{aligned}$$

Proof. We prove the first equality. The middle equality follows from straightforward calculations and the last equality follows from the contact conditions. We have

$$\begin{aligned} |\mathbf{Z}\mathrm{Log}(f_I)|^2 - |\overline{\mathbf{Z}}\mathrm{Log}(f_I)|^2 &= \det\begin{pmatrix} <\phi, F_*\mathbf{Z}> & <\phi, F_*\overline{\mathbf{Z}}> \\ <\overline{\phi}, F_*\mathbf{Z}> & <\overline{\phi}, F_*\overline{\mathbf{Z}}> \end{pmatrix} \\ &= <\phi, F_*\mathbf{Z}><\overline{\phi}, F_*\overline{\mathbf{Z}}> - <\phi, F_*\overline{\mathbf{Z}}><\overline{\phi}, F_*\mathbf{Z}> \\ &= (\phi\wedge\overline{\phi})(F_*\mathbf{Z}, F_*\overline{\mathbf{Z}}) \\ &= F^*(\phi\wedge\overline{\phi})(\mathbf{Z}, \overline{\mathbf{Z}}) \\ &= F^*(-id\omega^*)(\mathbf{Z}, \overline{\mathbf{Z}}) \\ &= -i(d\lambda^*\wedge\omega^* + \lambda^* d\omega^*)(\mathbf{Z}, \overline{\mathbf{Z}}) \\ &= -i\lambda^* d\omega^*((\mathbf{Z}, \overline{\mathbf{Z}}) = \lambda^*. \end{aligned}$$

□

6.5 Contact diffeomorphisms with constant determinant

Lemma 6.8. Let $h : \mathbb{RA} \to \mathbb{C}$ be a C^1 function. Then if $h = h(\zeta)$ we have

1. $\mathbf{Z}h = 2\mathrm{Re}(\zeta)h_\zeta, \overline{\mathbf{Z}}h = 2\mathrm{Re}(\zeta)h_{\overline{\zeta}}$ and $\mathbf{T}h = 0$.
2. $h = h(\zeta)$ if and only if $\mathbf{T}h = 0$.

Proposition 6.9. Suppose that $F = (f_I, f_3)$ is an orientation preserving, C^2 contact diffeomorphism of $\mathbb{RA}$. Then λ^* is constant if and only if the function $f_{II} : \mathbb{RA} \to \mathbb{L}, f_{II} = -|f_I|^2 + if_3$ depends only on $\zeta = |z|^2 + it$. Moreover, in that case we have $\arg(f_I) = \arg(z) + \phi(\zeta)$, for some C^2 function $\phi : \mathbb{RA} \to \mathbb{R}$.

Proof. Suppose first that $F = (f_I, f_3)$ is an orientation preserving, C^1 contact diffeomorphism of $\mathbb{RA}$ with constant λ^*. Then the mapping

$$\left(\frac{f_I}{\sqrt{\lambda^*}}, \frac{f_3}{\lambda^*}\right),$$

is also contact and has determinant one. Thus we may always normalize and in what follows we suppose that $\lambda^* = 1$. Taking differentials at both sides of the relation (6.1) we obtain

$$\frac{d|f_I|^2 \wedge df_3}{2|f_I|^4} = d\omega^*$$

which can also be written as

$$d\log(|f_I|) \wedge df_3 = |f_I|^2 d\omega^*.$$

Taking differentials at both sides we get

$$d|f_I|^2 \wedge d\omega^* = 0.$$

Using

$$d|f_I|^2 = \mathbf{Z}(|f_I|^2)\phi + \overline{\mathbf{Z}}(|f_I|^2)\overline{\phi} + \mathbf{T}(|f_I|^2)\omega^*$$

and

$$d\omega^* = i\phi \wedge \phi^*,$$

we get

$$0 = i\mathbf{T}(|f_I|^2)dm.$$

Using the previous lemma we conclude that $|f_I|$ depends only on ζ. On the other hand, we have

$$0 = id\omega^*(\mathbf{(}Z), \mathbf{T}) = i(F^*d\omega^*)(\mathbf{Z}, \mathbf{T}) = id\omega^*(F_*\mathbf{Z}, F_*\mathbf{T}) = i\phi \wedge \overline{\phi}(F_*\mathbf{Z}.F_*\mathbf{T}).$$

Thus

$$\begin{aligned} 0 &= \phi \wedge \overline{\phi}(F_*\mathbf{Z}, F_*\mathbf{T}) \\ &= < \phi, F_*\mathbf{Z} >< \overline{\phi}, F_*\mathbf{T} > - < \phi, F_*\mathbf{T} >< \overline{\phi}, F_*\mathbf{Z} > \\ &= \frac{1}{4|f_I|^2}(\mathbf{Z}f_{II}\mathbf{T}\overline{f_{II}} - \mathbf{Z}\overline{f_{II}}\mathbf{T}f_{II}). \end{aligned}$$

But

$$\mathbf{T}f_{II} = i\mathbf{T}f_3, \quad \mathbf{T}\overline{f_{II}} = -i\mathbf{T}f_3.$$

Thus

$$-\frac{i\mathbf{T}f_3}{4|f_I|^2}(\mathbf{Z}f_{II} + \mathbf{Z}\overline{f_{II}}) = 0.$$

Now $\mathbf{Z}f_{II} + \mathbf{Z}\overline{f_{II}} \neq 0$ because if that was the case at some point then λ^* would vanish at this point. Therefore $\mathbf{T}f_3 = 0$ which proves our first claim. We now prove that f_{II} depends only on ζ that is $\mathbf{T}f_{II} = 0$ and λ^* is constant. For this we consider the differential of λ^*:

$$d\lambda^* = \mathbf{Z}\lambda^*\phi + \overline{\mathbf{Z}}\lambda^*\overline{\phi} + \mathbf{T}\lambda^*\omega^*$$

and we will show that this is equal to zero. We have

$$\begin{aligned} F^*d\omega^* = d(F^*\omega^*) = d(\lambda^*\omega^*) &= d\lambda^* \wedge \omega^* + \lambda^* d\omega^*, \\ &= \mathbf{Z}\lambda^*\phi \wedge \omega^* + \overline{\mathbf{Z}}\lambda^*\overline{\phi} \wedge \omega^* + i\lambda^*\phi \wedge \phi^*. \end{aligned}$$

Now in the first place:

$$F^*d\omega^*(\mathbf{Z}, \mathbf{T}) = \mathbf{Z}\lambda^*\overline{\phi} \wedge \omega^*(\mathbf{Z}, \mathbf{T}) = \mathbf{Z}\lambda^*.$$

On the other hand

$$\begin{aligned} F^*d\omega^*(\mathbf{Z}, \mathbf{T}) &= d\omega^*(F_*\mathbf{Z}, F_*\mathbf{T}) \\ &= i\phi \wedge \phi^*(F_*\mathbf{Z}, F_*\mathbf{T}) \\ &= \frac{i}{4|f_I|^2}(\mathbf{Z}f_{II}\mathbf{T}\overline{f_{II}} - \mathbf{Z}\overline{f_{II}}\mathbf{T}f_{II}) \\ &= 0. \end{aligned}$$

Thus $\mathbf{Z}\lambda^* = 0$ and similarly $\overline{\mathbf{Z}}\lambda^* = 0$. Finally we have

$$\lambda^* = \frac{|\mathbf{Z}f_{II}|^2 - |\overline{\mathbf{Z}}f_{II}|^2}{4|f_I|^2}.$$

It suffices to prove that $\mathbf{TZ}f_{II} = 0$. Indeed

$$\mathbf{TZ}f_{II} = -\mathbf{ZT}f_{II} = 0.$$

and the proof is complete. □

Definition 6.10. A bijection $F : \mathbb{RA} \to \mathbb{RA}$ is called circles-preserving if for every $\zeta \in \mathbb{L}$ there exists an $\eta \in \mathbb{L}$ such that

$$F(a^{-1}(\zeta)) = a^{-1}(\eta).$$

That is, a circles preserving mapping $F = (f_I, f_3)$ preserves the fibers of the Koranyi map a. Such a mapping defines a bijection $F : \mathbb{L} \to \mathbb{L}$ by the rule

$$f \circ a = a \circ F = f_{II}.$$

We immediately have the following:

Corollary 6.11. An orientation preserving C^2 contact diffeomorphism $F = (f_I, f_3)$ of $\mathbb{RA}$ has constant λ^* if and only if F is circles preserving.

6.6 Lifting Theorem

In this section we are going to prove that if a smooth diffeomorphism of $\mathbb{L}$ is symplectic with respect to the Kahler form of $\mathbb{L}$ then it can be lifted to a smooth contact circles-preserving diffeomorphism of $\mathbb{RA}$ with Jacobian determinant $\lambda^* = 1$. As a corollary we will obtain that a symplectic quasiconformal mapping of $\mathbb{L}$ can be lifted to a circles-preserving quasiconformal map of $\mathbb{RA}$. We start with the following lemma:

Lemma 6.12. Let $f : \mathbb{L} \to \mathbb{L}$ be a smooth symplectic diffeomorphism. Then there exists a function $\psi : \mathbb{L} \to \mathbb{R}$ such that

$$\psi_\zeta = \frac{i}{4\mathrm{Re}(\zeta)} + \frac{(\mathrm{Im}(f))_\zeta}{2\mathrm{Re}(f)}.$$

Proof. Since f is symplectic we have

$$\mathrm{Re}^2(f)(\zeta) = \mathrm{Re}^2(\zeta) J_f(\zeta). \tag{6.2}$$

We have

$$\begin{aligned} J_f &= |f_\zeta|^2 - |f_{\overline{\zeta}}|^2 \\ &= |(\mathrm{Re}(f))_\zeta + i(\mathrm{Im}(f))_\zeta|^2 - |(\mathrm{Re}(f))_{\overline{\zeta}} + i(\mathrm{Im}(f))_{\overline{\zeta}}|^2 \\ &= (\mathrm{Re}(f))_\zeta + i(\mathrm{Im}(f))_\zeta|^2 - (\mathrm{Re}(f))_\zeta - i(\mathrm{Im}(f))_\zeta|^2 \\ &= -4\mathrm{Im}((\mathrm{Re}(f))_\zeta(\mathrm{Im}(f))_{\overline{\zeta}}) \\ &= 2i((\mathrm{Re}(f))_\zeta(\mathrm{Im}(f))_{\overline{\zeta}}) - (\mathrm{Re}(f))_{\overline{\zeta}}(\mathrm{Im}(f))_\zeta). \end{aligned}$$

Therefore we write (6.2) again as

$$\mathrm{Re}^2(f)(\zeta) = 2i\mathrm{Re}^2(\zeta)((\mathrm{Re}(f))_\zeta(\mathrm{Im}(f))_{\overline{\zeta}}) - (\mathrm{Re}(f))_{\overline{\zeta}}(\mathrm{Im}(f))_\zeta). \tag{6.3}$$

We next set

$$g(\zeta) = \frac{i}{4\mathrm{Re}(\zeta)} + \frac{(\mathrm{Im}(f))_\zeta}{2\mathrm{Re}(f)}.$$

Then

$$g_{\overline{\zeta}} = -\frac{i}{8\mathrm{Re}^2(\zeta)} + \frac{(\mathrm{Im}(f))_{\zeta\overline{\zeta}}\mathrm{Re}(f) - (\mathrm{Re}(f))_{\overline{\zeta}}(\mathrm{Im}(f))_\zeta}{8\mathrm{Re}^2(f)}$$

and

$$\overline{g}_\zeta = \frac{i}{8\mathrm{Re}^2(\zeta)} + \frac{(\mathrm{Im}(f))_{\zeta\overline{\zeta}}\mathrm{Re}(f) - (\mathrm{Re}(f))_\zeta(\mathrm{Im}(f))_{\overline{\zeta}}}{8\mathrm{Re}^2(f)}.$$

Using (6.3) we find that

$$\overline{g}_\zeta - g(\zeta) = 0.$$

We next consider the real 1-form

$$\beta = g(\zeta)d\zeta + \overline{g(\zeta)}d\overline{\zeta}.$$

By the previous equation we have that β is closed and hence exact by Poncare's Lemma. It follows that $\beta = d\psi$ for some real function ψ and the proof is complete □

Theorem 6.13. *Let* $F : \mathbb{L} \times S^1 \to \mathbb{L} \times S^1, F = (f, \Theta)$

$$f = f(\zeta), \Theta(\zeta, \theta) = \theta + \psi(\zeta)$$

be an orientation preserving bundle diffeomorphism. Then the map $G : \mathbb{RA} \to \mathbb{RA}, G = (g_I, g_3)$,

$$g_I(z,t) = |z| J_f^{\frac{1}{4}}(-|z|^2 + it) e^{i(\arg(z)+\psi(-|z|^2+it))}, g_3(z,t) = \mathrm{Im}(f(-|z|^2 + it)),$$

is an orientation preserving, circles preserving contactomorphism of $\mathbb{RA}$ *if* f *is symplectic with respect to the Kahler form of* $\mathbb{L}$.

Proof. In the first place, $g_{II}(z,t) = -|z|^2 J_f^{\frac{1}{4}}(\zeta) + i\mathrm{Im}(f)(\zeta) = f(\zeta)$. Moreover:

$$\lambda^* = \frac{J_{f_{II}}}{4|f_I|^4} = \frac{4\mathrm{Re}^2(\zeta)J_f}{4\mathrm{Re}^2(\zeta)J_f} = 1.$$

Now

$$\begin{aligned} \frac{\mathbf{Z} f_3}{2|f_I|^2} + \mathbf{Z}\arg(f_I) &= \frac{2\mathrm{Re}(\zeta)(\mathrm{Im}(f(\zeta)))_\zeta}{-2\mathrm{Re}(\zeta)J_f^{\frac{1}{2}}} + \frac{1}{2i} + 2\mathrm{Re}(\zeta)_{\psi_\zeta} \\ &= -\mathrm{Re}(\zeta)\frac{(\mathrm{Im}(f(\zeta)))_\zeta}{\mathrm{Re}(f)} + \frac{1}{2i} + 2\mathrm{Re}(\zeta)\psi_\zeta \\ &= 0. \end{aligned}$$

Here we used (6.12).Similarly

$$\frac{\overline{\mathbf{Z}} f_3}{2|f_I|^2} + \overline{\mathbf{Z}}\arg(f_I) = 0$$

and finally

$$\frac{\mathbf{T} f_3}{2|f_I|^2} + \mathbf{T}\arg(f_I) = \mathbf{T}\arg(z) = 1.$$

□

Corollary 6.14. A symplectic quasiconformal self-map of $\mathbb{L}$ with Beltrami differential μ_f may be lifted to a contact circles-preserving quasiconformal map G of $\mathbb{RA}$ such that

$$\mu_G(z,t) = -\frac{z}{\bar{z}} \cdot \mu_f(\alpha(z,t)),$$

where α is the Korányi map.

Proof. If $f : \mathbb{L} \to \mathbb{L}$ is quasiconformal, then its Beltrami differential

$$\mu_f = \frac{f_{\overline{\zeta}}}{f_\zeta}$$

is essentially bounded by a constant $k \in [0, 1)$. For such an f which is also symplectic, we consider its lift G from Theorem 6.13. Since

$$g_I(z,t) = \frac{1}{2i}(f(\alpha(z,t)) - \overline{f(\alpha(z,t))}),$$

we have by chain rule:

$$\begin{aligned}
\mathbf{Z}g_I(z,t) &= \frac{1}{2i}\left(f_\zeta(\alpha(z,t))\mathbf{Z}\alpha + f_{\overline{\zeta}}(\alpha(z,t))\overline{\mathbf{Z}}\alpha\right) = i\overline{z}f_\zeta(\alpha(z,t)),\\
\overline{\mathbf{Z}}g_I(z,t) &= \frac{1}{2i}\left(f_\zeta(\alpha(z,t))\mathbf{Z}\overline{\alpha} + f_{\overline{\zeta}}(\alpha(z,t))\overline{\mathbf{Z}}\overline{\alpha}\right) = izf_{\overline{\zeta}}(\alpha(z,t).
\end{aligned}$$

Therefore

$$\mu_G(z,t) = \frac{\overline{\mathbf{Z}}g_I(z,t)}{\mathbf{Z}g_I(z,t)} = -\frac{z}{\overline{z}} \cdot \frac{f_{\overline{\zeta}}(\alpha(z,t))}{f_\zeta(\alpha(z,t))} = -\frac{z}{\overline{z}} \cdot \mu_f(\alpha(z,t)).$$

The proof is thus concluded $\square$

Bibliography

[1] Mirjana Djoric, *CR Submanifolds of Complex Projective space*, Springer, Developments in Mathematics (Book 19), (2009).

[2] Albert Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex*, CRC Press, Studies in Advanced Mathematics (Book 1), (1991).

[3] Roger Penrose, *Physical space-time and nonrealizable CR-structures*, Bull. Amer. Math. Soc. (N.S.) Volume 8, Number 3 (1983), 427-448.

[4] Capogna, Luca; Danielli, Donatella; Pauls, Scott; Tyson, Jeremy, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Progress in Mathematics. 259. Berlin: Birkhauser, (2007).

[5] Luca Capogna; Puqi Tang, *Uniform domains and quasiconformal mappings on the Heisenberg group*, Manuscripta Mathematica, **86**, 267-281, (1995).

[6] William M. Goldman, *Complex Hyperbolic Geometry*, Clarendon Press, Oxford Mathematical Monographs, (1999).

[7] Richard Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, American Mathematical Society, Mathematical Surveys and Monographs (Book 91), (2006).

[8] Ovidiu Calin; Der-Chen Chang, *Sub-Riemannian Geometry: General Theory and Examples*, Cambridge University Press, Encyclopedia of Mathematics and its Applications (Book 126), (2009).

[9] Ovidiu Calin; Der-Chen Chang; Peter Greiner, *Geometric Analysis on Heisenberg Group and its Generalizations*, American Mathematical Society, AMS/IP Studies in Advanced Mathematics (Book 40), (2007).

[10] Howard Jacobowitz, *An introduction to CR structures*, American Mathematical Soc., (1990)

[11] Aurel Bejancu, *Geometry of CR-Submanifolds*, Spinger, Mathematics and its Applications (Book 23), (1986).

[12] E.M. Chirka, *Introduction to the geometry of CR-manifolds*, Russ. Math. Surv. **46** 95, (1991).

[13] Loring W. Tu, *Introduction to Manifolds*, Springer, Universitext, (2011).

[14] Ana Cannas da Silva, *Lectures on Symplectic Geometry*, Springer, Lecture Notes in Mathematics (Book 1764), (2008).

[15] Norman Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton Mathematical Series (Book 14), (1951).

[16] Newlander A. Nirenberg L., *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391404.

[17] Andrew McInerney, *First Steps in Differential Geometry* Springer, Undergraduate Texts in Mathematics, (2013).

[18] A. Koranyi and H.M. Reimann, *Foundations for the theory of quasiconformal mappings on the Heisenberg group*, Advances in Math. **111** (1995), 1-87

[19] A. Koranyi and H.M. Reimann, *Quasiconformal mappings on CR manifolds*, Lecture Notes in Math. **1422** (1990), 59-75

[20] A. Koranyi, *Quasiconformal mappings in several complex variables*

[21] A. Koranyi and H.M. Reimann, *Quasiconformal mappings on the Heisenberg group*, Invent. Math. **80** (1985), 309-338.

[22] Ioannis D. Platis, *Quasiconformal mappings on the Heisenberg group: An overview*, Handbook of Teichmller Theory Vol. VI, (2016), 375–396.

[23] Angel Cano and Ioannis D. Platis, *Lifting Quasiconformal mappings from the hyperbolic plane to the Heisenberg group*, https://www.researchgate.net/project/Lifting-of-quasiconformal-maps-from-the-hyperbolic-plane-to-the-Heisenberg-group

[24] Robin Timsit, *Geometric construction of quasiconformal mappings in the Heisenberg group*, Arxiv,1612.01422 [math:DG]