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"Mathematics and Fundamental Applications"

LECTURES ON THE CLASSIFICATION OF COMPLEX AND REAL LIE ALGEBRAS

Paris Pamfilos

University of Crete

Lectures given in the framework of the
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Lectures on the Classification of Complex and Real Lie Algebras

Preface

The following notes grew out from a seminar on the classification of real Lie algebras, held at the University of Crete in the WS 1988/89. This seminar was considered as a continuation to the excelent lectures on Lie groups and Lie algebras, given by my friend Hans Samelson, one semester before. The notes were prepared by a non-expert in the field (that's me), to serve as a help, for those which wanted to learn quickly, how the simple real Lie algebras are classified. This, of course, is not an excuse for the errors they contain *, but a warning to the innocent reader.

The reader, which, among other things, has to be patient with the different verses and motos from poets and philosophs figuring everywhere in the text. Incidentaly, 1988 was the 200-th aniversary of Byron's birth, so it was reasonable to honor this great poet (and some others as well) and remember his immortal verses not less than the details of the structure of the Lie algebras.

The list on the next page indicates, I hope, the contents and the way we get to our aim.

Paris Pamfilos, Paleochora, Crete, the 23 February 1991

* I would be very greatefull to those readers (if any) which would point out to me errors and suggestions, to make the text better. They can contact me, writing to: University of Crete, Department of Mathematics, Iraklion-Crete-Greece, P.O. Box 1470. (or call 0030-81-246428)

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[1] K. Nomizu . Linear algebra, Academic Press 1978

[2] H. Samelson. Notes on Lie algebras, Springer 1990

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15. **Nilpotent Lie algebras**
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17. **Semi-simplicity, Cartan's second criterion**
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[3] Humphreys. Introduction to Lie algebras and Representation Theory, Springer 1972

[4] M. Hausner and J. Schwarz. Lie groups, Lie algebras, Gordon and Breach 1968

[5] Chow. Lie groups and Lie algebras. 2 Vols.

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[6] N. Jacobson. Lie algebras, Dover 1979
[7] Seminaire Sophus Lie, Ecole Norm. Sup. 1955
[8] Z. X. Wan. Lie algebras, Pergammon Press 1975
[9] A. Borel and J. de Siebenthal. Les sous groupes fermés de rang maximum .... Comm. M. Hel. 23(1949)
[10] S. Murakami. On the automorphisms of a real semi simple Lie algebra. J. Math. Soc. Japan 4(1952)
[11] S. Murakami. Supplements and corrections to [10]. J. Math. Soc. Japan 5(1953)
[12] S. Murakami. Sur la classification des algèbres de Lie réelles et simples. Osaka J. Math. 2(1965)

I.

Preliminary Linear Algebra

That is the usual method, but not mine-
  My way is to begin with the beginning;
The regularity of my design
  Forbids all wandering as the worst of sinning,
And therefore I shall open with a line
  (Although it cost me half an hour in spinning)
Narrating somewhat of Don Juan's father,
And also of his mother, if you'd rather.

Byron, Don Juan, Canto I, 7

1. Complexification, real forms

The vector spaces, we are concerned here, are real (**R**) or complex (**C**). For a real vector space V, the **complexification** V_C is a complex vector space defined as the set of formal sums

$$V_C = \{X+iY \mid X,Y \in V, i\cdot i = -1\}. \qquad (1)$$

In V_C one defines addition and multiplication by complex numbers through the formulas

$$(X+iY)+(X'+iY') = (X+X')+i(Y+Y'), \qquad (2)$$
$$(u+iv)\cdot(X+iY) = (uX-vY)+i(vX+uY). \qquad (3)$$

Exercise-1 Show that V_C is a complex vector space, with respect to these operations and the C-dimension of this vector space is equal with the **R**-dimension of V. More precisely, show that a **R**-basis of V is also a C-basis of V_C.

Exercise-2 Show that a **R**-linear map $F: V \to V$ can be extended to a C-linear $F_C: V_C \to V_C$, through the definition:

$$F_C(X+iY) = F(X)+iF(Y). \qquad (4)$$

Exercise-3 Each basis of V is a basis of V_C (identify X with X+i0). Show that the matrices of F and F_C, with respect to such a basis, are identical.

Given a complex vector space W, there are many real vector subspaces V of W, whose complexification $V_C = W$. These are called **real forms** of W and are constructed as follows: Take a C-basis $\mathbf{e} = \{e_1,\dots, e_n\}$ of W and consider the <u>real</u> linear spann V_e

$$V_e = \langle e_1,\dots, e_n\rangle = \text{set of all } \mathbf{R}\text{-linear combinations of } \{e_1,\dots, e_n\}. \qquad (5)$$

Exercise-4 Show that V_e is a real vector subspace of the same dimension as W. Show further that $(V_e)_C = W$.

Obviously, different bases of W give different real forms V_e. When, however, the C bases of W $\{e_1,\dots, e_n\}$, $\{e'_1,\dots, e'_n\}$ have a <u>real</u> change-basis matrix T, then the corresponding real forms are identical.

Question-1 Given a C-vector space W and some real form V of W, when is a complex linear map $F : W \to W$, the complex extension of a real linear $F_o : V \to V$?

Question-2 Fix a real form V of W. When is the complex linear $F : W \to W$, the complex extension of a real linear $F_o : V \to V$?

Exercise-5 Show that a linear $F: W \to W$ is extension of the real $F_o : V \to V$, if and only if $F(V) \subset V$. [When this happens, then $F|V = F_o$ is the **R**-linear $V \to V$.] Equivalently, there is a basis of V, with respect to which, F is represented by a real matrix.

Question-3 Find the structure of the set of all linear $F: W \to W$, which are extensions of some **R**-linear $F_o : V \to V$, for some real form V of W.

Fix a basis $\mathbf{e} = \{e_1,\dots, e_n\}$ of W. Then, all other bases of W are given by bases $\mathbf{e}'$ resulting from $\mathbf{e}$, by right multiplication with a matrix $g \in GL(n, \mathbf{C})$:

$$e' = e \cdot g, \quad g \in GL(n, \mathbf{C}). \tag{6}$$

The matrix of the **C**-linear F, with respect to this basis, is given by $h \in GL(n,\mathbf{R})$, if F is extension of some real linear map on the real form defined by e' .

$$F(e') = e' \cdot h.$$

Thus, such an extension is defined by a pair of two matrices $(g,h) \in GL(n,\mathbf{C}) \times GL(n,\mathbf{R})$. Two pairs (g,h) and (g',h') define the same extension if and only if, the matrix $t = g^{-1} \cdot g'$ is real and $h' = t^{-1} \cdot h \cdot t$.

$$(g,h) \approx (g',h') \Leftrightarrow \quad t = g^{-1} \cdot g' \text{ is real, and } h' = t^{-1} \cdot h \cdot t, \tag{7}$$

defines an equivalence relation in the set of pairs of matrices and shows that each extension of a real linear map on some real form of W, corresponds to some orbit of $GL(n, \mathbf{C}) \times GL(n,\mathbf{R})$, under the action of $GL(n,\mathbf{R})$:

$$(t, (g,h)) \to \quad (g' = g \cdot t \,,\, h' = t^{-1} \cdot h \cdot t). \tag{8}$$

Thus, the set of complex extensions of real linear maps on real forms of W, is in one-to-one correspondence with the space of orbits of this action:

$$GL(n, \mathbf{C}) \times GL(n,\mathbf{R})/GL(n,\mathbf{R}). \tag{9}$$

Real forms are in 1-1 correspondence with **conjugatiations** i.e. involutive conjugate-linear maps of the complex vector space V:

$$\begin{aligned} &\Phi: V \to V, \\ &\Phi(X+aY) = \Phi(X) + \bar{a}\Phi(Y), \text{ for every } a \in \mathbf{C}, \\ &\Phi \circ \Phi = I. \end{aligned} \tag{10}$$

Given Φ, the corresponding real form V is the (real) vector space of fixed points of Φ:

$$V_\Phi = \{X \in V | \ \Phi(X) = X\}.$$

Inversely, given the real form W, one defines the corresponding conjugation by:

$$F(X+iY) = X-iY, \text{ for every } X, Y \in W.$$

Notice that the composition $\Phi \circ \Phi'$ of two conjugations is a complex isomorphism of the vector space V. Consequently, every conjugation Φ' of V is of the form $\Phi' = \Phi \circ g$, where g is an isomorphism of V and Φ is some fixed conjugation of V.

Nothing so difficult as a beginning
  In poesy, unless perhaps the end;
For oftentimes when Pegasus seems winning
  The race, he sprains a wing, and down we tend,
Like Lucifer when hurl'd from heaven for sinning;
  Our sin the same, and hard as his to mend,
Being pride, which leads the mind to soar too far,
Till our own weakness shows us what we are.

Byron, Don Juan, Canto IV

2. Realification

Each complex vector space W is at the same time a real vector space, with respect to the same addition and multiplication by scalars (the only difference is that we restrict ourselves to real scalars). When $\mathbf{e} = \{e_1,...,e_n\}$ is a basis of W, then

$$\{e_1,...,e_n, ie_1,...,ie_n\} \tag{1}$$

is a basis of W, considered as a real vector space. We denote this vectorspace by $W_\mathbf{R}$ and call it the **realification** of W. Obviously

$$\dim_\mathbf{R} W_\mathbf{R} = 2\dim_\mathbf{C} W . \tag{2}$$

Exercise-1 Show that "multiplication by i" in $W_\mathbf{R}$, defines a **R**-linear map

$$J : W_\mathbf{R} \rightarrow W_\mathbf{R}, \text{ with } J^2 = -I. \tag{3}$$

The matrix of this linear map, with respect to the basis (1) is

$$J = \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & -1 & \dots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & 0 & 0 & \dots & -1 \\ \hline 1 & \dots & 0 & 0 & \dots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & 1 & 0 & \dots & 0 \end{array}\right) \tag{4}$$

Each **C**-linear map $F : W \rightarrow W$, may be considered as an **R**-linear $F_\mathbf{R} : W_\mathbf{R} \rightarrow W_\mathbf{R}$, which satisfyies

$$F_\mathbf{R}(i \cdot X) = i \cdot F_\mathbf{R}(X), \text{ for all } X \in W, \tag{5}$$

which, by the definition of J, is equivalent with

$$F_\mathbf{R} \cdot J = J \cdot F_\mathbf{R}. \tag{6}$$

Exercise-2 Let $g = X+iY \in GL(n,\mathbf{C})$ be the matrix of the linear map $F : W \rightarrow W$, with respect to the basis **e** (X, Y are real matrices, but not necesserily in $GL(n,\mathbf{R})$). Then the corresponding $F_\mathbf{R} : W_\mathbf{R} \rightarrow W_\mathbf{R}$, has with respect to the basis (1), the matrix

$$g_\mathbf{R} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}. \tag{7}$$

Exercise-3 Show that the matrices of the form (7) are precisely the real $2n \times 2n$ matrices, which commute with J, defined by (4).

Exercise-4 Show that each matrix of the form (7) defines a **C**-linear map $F : W \rightarrow W$, having, with respect to the basis e, the matrix $g = X+iY$.

Exercise-5 With the previous notations, show that

$$\det(g_R) = |\det(X+iY)|^2. \tag{8}$$

$$\left[\det\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix} = \det\begin{pmatrix}X-iY & -Y\\ Y+iX & X\end{pmatrix} = (i)^n\det\begin{pmatrix}-iX-Y & iY\\ Y+iX & 0\end{pmatrix} = (i)^n\det\begin{pmatrix}-iX-Y & -iY\\ 0 & X+iY\end{pmatrix}\right]$$

Exercise-6 Let V be a real vector space of dimension m and assume that there is a linear map $J : V \rightarrow V$, with $J^2 = -I$. Show that there is a basis of V, such that J, with respect to this basis, has the matrix representation (Jordan form)

$$J = \begin{pmatrix} \boxed{\begin{matrix}0 & -1\\ 1 & 0\end{matrix}} & & & \\ & \boxed{\begin{matrix}0 & -1\\ 1 & 0\end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix}0 & -1\\ 1 & 0\end{matrix}} \end{pmatrix}.$$

Conclude that $\dim V = 2n$. Changing to a suitable basis, show that the same linear map has the matrix representation given by (4).

Exercise-7 With the assumptions of the preceding exercise, show that $V = W_R$, where W is a suitable complex vector space (namely V with complex multiplication $(u+iv)X = uX+vJ(X)$).

An **R**-linear $J : V \rightarrow V$, with $J^2 = -I$, is called a **complex structure** on the real vector space V. From the preceding Exercise, we see that V, having a complex structure, must have even dimension. The exercises give also the proof of the

Theorem *Each realification W_R, of a C-vector space W, has a complex structure and inversely, if a real vector space has a complex structure, then it is the realification of a complex vector space.*

In mathematics, when we discriminate between lines, planes, and solids, we find that rectangular planes result from straight lines, and cubic magnitudes from rectangular planes. The Receptive accomodates itself to the qualities of the Creative and makes them its own. Thus a square develops out of a straight line and a cube out of a square. This is compliance with the laws of the Creative; nothing is taken away, nothing added. Therefore the Receptive has no need of a special purpose of its own, nor of any effort; yet everything turns out as it should.

I Ching, The Receptive, p. 13

3. Complex structures

Let V be a real vector space of 2n dimensions and $J : V \to V$ a **complex structure**, i.e. a linear map satisfying $J^2 = -I$. We define a multiplication of the elements of V by complex numbers, through the rule

$$(u+iv)\cdot X = uX+vJ(X). \tag{1}$$

The same set V, with the old addition and the new multiplication becomes a complex vector space of dimension n. Denote this vector space by V_J.

Exercise-1 Show that V_J is actually a complex vector space of dimension n.

Exercise-2 Each complex linear $F: V_J \to V_J$ defines an **R**-linear $F_{\mathbf{R}} : V \to V$, with the property

$$F_{\mathbf{R}} \cdot J = J \cdot F_{\mathbf{R}} . \tag{2}$$

And inversely, each $F_{\mathbf{R}}$ satisfying (2), defines a C-linear F on the corresponding V_J.

Exercise-3 Given two complex structures J, J' on the real vectrospace V, show that there exists an **R**-linear and invertible $F : V \to V$, with the property

$$F\cdot J = J'\cdot F . \tag{3}$$

[Use Ex-2, $V_J \approx V_{J'} \approx \mathbf{C}^n$]

If we choose a C-basis $\{e_1,...,e_n\}$ of V_J, $<e_1,...,e_n>_{\mathbf{R}}$ is a real form of V_J:

$$<e_1,...,e_n>_{\mathbf{R}} \oplus <Je_1,...,Je_n>_{\mathbf{R}} = V. \tag{4}$$

If however the $e_1,...,e_n$ are only <u>real</u> independent, the last equation may be false. Furthermore, a C-basis $\{e_1,...,e_n\}$ of V_J may be not a C-basis of $V_{J'}$, for another structure J' of V. F.e. it may happen that $J'(e_1) = e_2$.

Question-1 Find the structure of the set of all possible complex structures on a real vector space of dimension 2n.

Question-2 Find the structure of the set of all possible complex structures J, which admit a given set of n linearly independent vectors $\{e_1,...,e_n\}$ in V, as a C-basis for the correspond ing V_J.

The first question is easy to answer. In fact, identify by means of a basis, V with $\mathbf{R}^{2n}$.

Ex-3 shows that $GL(2n,\mathbf{R})$ operates, by conjugation, transitively on the set S of complex structures. Taking as J the matrix (2) in §2, we find that the isotropy group of the operation, at this point, coincides (Ex-2 - Ex-4 in §2) with $GL(n,\mathbf{C})$. Thus, the set of all complex structures on V is isomorphic with the homogeneous space $GL(2n,\mathbf{R})/GL(n,\mathbf{C})$, where the complex matrices in $GL(n,\mathbf{C})$ are identifyied with the real matrices of the form

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

in $GL(2n,\mathbf{R})$. We proved the

Theorem-1 *The set of complex structures on the real 2n-dimensional vectorspace V, is isomorphic to the homogeneous space $GL(2n,R)/GL(n,C)$.*

The second question may be handled as follows. Fix a complex structure J and let $\{e_1,...,e_n\}$ be a C-basis of V_J. Let J' be another complex structure. Then, the same basis is also C-basis of $V_{J'}$ if and only if, $\{e_1,...,e_n\} \cup \{J'e_1,...,J'e_n\}$ is a basis of V. Thus, the change bases matrix from $\{e_1,...,e_n\} \cup \{Je_1,...,Je_n\}$ to $\{e_1,...,e_n\} \cup \{J'e_1,...,J'e_n\}$, which has the form

$$\begin{pmatrix} I & S \\ 0 & T \end{pmatrix}$$

must be invertible, which is equivalent with the condition $\det(T) \neq 0$. Looking the n last columns of the preceding matrix, we see that they define an open subset of the Stiefel manifold of n-frames in $\mathbf{R}^{2n}$. Thus, the set, we are looking for, is isomorphic with such an open subset (a chart) of this manifold.

Dieser Mann arbeitete an einem System der Naturgeschichte, worin er die Tiere nach der Form der Exkremente geordnet hatte. Er hatte drei Klassen gemacht: die zylindrischen, sphärischen und kuchenförmigen.

Lichtenberg, Sudelbücher, p. 344

4. Bilinear forms

So are called the bilinear maps F of a K-vector space V ($K = \mathbf{C}, \mathbf{R}$):

$$F : V \times V \to K .$$

The main problem of this section is the "reduction to canonical form". This is connected with the matrix representations of the bilinear form, with respect to the various bases of V.

Given a basis $a = \{a_1,...,a_n\}$ of V and $x = (x_1,...,x_n)$ identifyied with $x_1 a_1 + ... + x_n a_n$, we have

$$F(x,y) = x \cdot A \cdot y^t , \text{ with the matrix } A = [F(a_j,a_k)] \tag{1}$$

and y^t denoting the column vector or transpose of $(y_1,...,y_n)$. A is the representation matrix of the bilinear form, with respect to the basis **a**. Changing from the basis a to another basis **b**, $b = a \cdot g$, with $g \in GL(n,K)$ (i.e. $b_i = \sum a_j \cdot g_{ji}$), we get for the corresponding matrix representations the relation

$$B = [F(b_i,b_j)] = g^t \cdot A \cdot g. \tag{2}$$

Problem-1 Find the simplest possible matrix representation for a given bilinear form.

Problem-2 Find the simplest possible matrix representation of the bilinear form F, allowing not all $g \in GL(n,K)$, but only those which belong to some subgroup G of GL(n,K).

Problem-3 Given the bilinear form F, find the group of all linear endomorphisms of V, which preserve the form, i.e. all linear $f: V \to V$, which satisfy

$$F(f(a),f(b)) = F(a,b), \text{ for all } a,b \in V. \tag{3}$$

The **rank** of F is defined to be the rank of the matrix A. The transformation rule (2) shows that this definition is independent of the particular basis used.

Let V* denote the **dual** vector space of V. V* is the set of all linear $h : V \to K$. Given a bilinear form F, there is a natural linear map

$$i_F: V \to V^*, \text{ given by } i_F(a)(b) = F(a,b), \text{ for all } a,b \in V. \tag{4}$$

Exercise-1 Let $a = \{a_1,...,a_n\}$ be a basis of V and $\{a_1^*,...,a_n^*\}$ be the dual basis in V* (i.e. $a_i^*(a_j) = \delta_{ij}$). Show that the matrix representing i_F, with respect to these bases coincides with A. Conclude that the rank of the bilinear F is identical with the rank of the linear i_F.

The dimension of the kernel of i_F is called the **nullity** of the bilinear form. The vector subspace:

$$\text{kern } i_F = \{a \in V \mid F(a,b) = 0, \text{ for all } b \in V\}, \tag{5}$$

is called **nullspace** or **radical** of the bilinear form.

F is called **non-degenerate**, when it has zero nullity or equivalently, its rank = dimV, hence the map $i_F: V \to V^*$ is an isomorphism.

Exercise-2 Show that the nullity = dimV - rank F. Complete a basis $\{a_{k+1},...,a_n\}$ of the nullspace to a basis $\{a_1,...,a_n\}$ of V. Show that the corresponding representation matrix of F has the form

$$A = \begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} \\ a_{k1} & \cdots\cdots & a_{kn} \\ 0 & \cdots\cdots & 0 \end{pmatrix}. \qquad (6)$$

The restriction of F on a complement W of the nullspace of F, isn't in general non-degenerate. When however F has symmetry properties, the restriction on such a complement is non-degenerate. The most important cases are the following:

- F **symmetric** : F(a,b) = F(b,a), for all **a, b** $\in$ V, V real or complex vector space.
- F **skew-symmetric** : F(a,b) = - F(b,a), for all **a, b** $\in$ V, V real or complex vector space.
- More general : F(a,b) = F(A(b),a), for all **a, b** $\in$ V, V real or complex vector space and A is a linear automorphism of V (usually an involution $A^2 = I$), which necessarily preserves F (F(a,b) = F(A(a),A(b)), for all **a, b**).

Exercise-3 Show that for symmetric blinear forms, (6) becomes:

$$A = \left(\begin{array}{ccc|cc} a_{11} & \cdots & a_{1k} & 0 & 0 \\ & A' & & & \\ a_{k1} & \cdots & a_{kk} & 0 & 0 \\ \hline 0 & & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 \end{array}\right).$$

in which the matrix A' is symmetric and non-singular.

Exercise-4 Show that for skew-symmetric bilinear forms the matrix in (6) takes the preceding form, where A' is now antisymmetric and non singular, hence k is even.

[Skew-symmetric matrices of odd dimensions are singular]

The two preceding exercises show, that the restriction of a symmetric or skew-symmetric bilinear form on a complement of the nullspace of the bilinear form, defines a non-degenerate bilinear form, of the same kind, on that complement.

Given a subspace W of V, the **conjugate space** or **orthogonal** space of W, denoted by $W^\perp$ is the subspace of V defined by

$$W^\perp = \{a \in V \mid F(a,b) = 0 \text{ for every } b \in W\}. \qquad (7)$$

Note that $W^\perp$ may contain the whole W, f.e. when W = nullspace of F, then $W^\perp = V$. A subspace W which satisfies $W \subset W^\perp$ is called **isotropic** with respect to F.

Exercise-5 Show that the conjugate of a subspace W contains always the nullspace of F.

Exercise-6 Show that $\dim(W^\perp) \geq n - \dim W$ (n = dimV). [look at the homogeneous system of linear equations corresponding to a basis of W]

Exercise-7 Show that the restriction of F on W is non-degenerate, if and only if

$$W \cap W^{\perp} = \{0\}. \tag{8}$$

Exercise-8 Show that, when the restriction of F on W is non-degenerate then

$$W \oplus W^{\perp} = V. \tag{9}$$

[use $\dim(W+W^{\perp}) = \dim W + \dim(W^{\perp}) - \dim(W \cap W^{\perp})$ and Ex-6,7]

Of particular interest are the **positive (negative) definite** symmetric forms, which, by definition, satisfy

$$F(x,x) > 0 \;\; (\text{resp.} < 0) \text{ for all } x \neq \mathbf{0}. \tag{10}$$

A positive definite symmetric form on V is called an **inner product** or a **metric** on V.

Exercise-11 Show that $<X,Y>=\mathrm{trace}(X^tY)$, defines an inner product on $M(n,n;\mathbf{R})$ (= the vector space of $n \times n$ real matrices). Show that the subspaces of symmetric $M_s(n,n;\mathbf{R})$ and skew-symmetric matrices $M_{ss}(n,n;\mathbf{R})$ are orthogonal complements of each other, with respect to this metric.

Theorem-1 *When F is symmetric, we can split V in a direct sum of a positive subspace, a negative subspace and the nullspace of F.*

By "positive" (resp. negative) we mean a subspace, where the restriction of F is positive (resp. negative) definite.

To prove the theorem consider a positive subspace W^+ of maximal dimension and denote by W' the conjugate of W^+. Then $F(v,v) \leq 0$, for all $v \in W'$. In fact, if there where $v \in W'$ with $F(v,v) \geq 0$, then for all $u \in W^+$ we would have $F(u+v,u+v) = F(u,u)+F(v,v) > 0$, and the dimension of the positive W^+ wouldn't be maximal. Consider now the restriction F' of F on W' and take there a negative subspace W^- of F', of maximal dimension. Finally take W^o to be the conjugate of W^- in W' (with respect to F'). It is trivial to show that

$$F(v,v) = 0, \text{ for all } v \in W^o. \tag{*}$$

It is also trivial to show that

$$W^+ \oplus W^- \oplus W^o = V. \tag{**}$$

By construction, W is maximal positive and the splitting is orthogonal. Using this splitting we see easily that W^o is the nullspace of F. W^- is maximal negative in V. In fact, if there where some W^* negative with more dimensions than W^-,then the intersection $W^- \oplus W^o \cap W^*$ would be non zero. This gives a contradiction and completes the proof.

Although the spaces W^+ and W^- are not uniquely defined, their dimensions n_+ and n_- are invariantly defined. These numbers enter into the following

Theorem-2 *Given a symmetric bilinear form F on the real n-dimensional vectorspace V, there is a basis, with respect to which, F has diagonal form. The number of positive entries in the diagonal is n_+ and the number of negative is n_-. The remaining entries in the diagonal are zero and their number is $n_o = \dim W^o$.*

The theorem is a consequence of The-1, and the well-known diagonalization theorem for symmetric matrices. This reduction is obtained by orthogonal matrices, which satisfy $g^t = g^{-1}$.

Exercise-9 With the previous notations, show that $W^+ \oplus W^-$ can be any complement of the nullspace of F.

Skew-symmetric real bilinear forms have an analogous theory to the symmetric ones. Ex-4 shows that such bilinear forms have even rank. The same exercise shows also that any complement of the nullspace has even dimension and the restriction of the bilinear form on that complement is a non-degenerate skew-symmetric bilinear form.

Exercise-10 Show that the eigenvalues of a real skew-symmetric matrix A are pairs of conjugate pure imaginary numbers $\{\pm i\mu\}$.

For each such pair, and corresponding eigenvectors (of the complexification $\mathbf{C}^n$ of $\mathbf{R}^n$) X, $\overline{X}$ define the real vectors

$$U = (1/2)(X+\overline{X}), \quad V = (1/2i)(X-\overline{X}), \text{ for which}$$
$$AU = (1/2)(AX+A\overline{X}) = (1/2)(i\mu X-i\mu\overline{X}) = -\mu V,$$
$$AV = (1/2i)(AX-A\overline{X}) = (1/2i)(i\mu X+i\mu\overline{X}) = \mu U,$$

and $\langle U,V\rangle=0$ (for the canonical inner product of $\mathbf{R}^n$). For the orthogonal complement W of the vector subspace, spanned by U and V, we have $AW \subset W$. Thus, inductively, as in the symmetric case, one proves the

Theorem-3 *For every real skew-symmetric matrix A, there is an orthogonal matrix g, such that the matrix g^tAg has the form*

$$\left(\begin{array}{cccccccc|c}
0 & -\mu_1 & & & & & & & \\
\mu_1 & 0 & & & & & & & \\
 & & 0 & -\mu_2 & & & & & \\
 & & \mu_2 & 0 & & & & & \\
 & & & & \ddots & & & & 0 \\
 & & & & & & 0 & -\mu_k & \\
 & & & & & & \mu_k & 0 & \\
\hline
 & & & & 0 & & & & 0
\end{array}\right),$$

where $\pm i\mu_r$ are the non-zero eigenvalues of the matrix and 2k is its rank.

Diagonal matrices and skew-symmetric matrices of the previous kind can be simplified by transforming A to g^tAg, for a convenient g, thus giving the proof of the

Theorem-4 *For every real symmetric bilinear form $F : V\times V \to R$, there is a basis of V, with respect to which the representation matrix of F has the diagonal canonical form*

$$\begin{pmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & & -1 & & & \\
 & & & & \ddots & & \\
 & & & & & -1 & \\
 & & & & & & 0 \\
 & & & & & & & \ddots \\
 & & & & & & & & 0
\end{pmatrix}, \qquad \begin{array}{l} n_+ = \text{number of } +1, \\ n_- = \text{number of } -1, \\ n_0 = \text{number of } 0. \end{array}$$

Analogously, when F is skew-symmetric, then there is a basis of V, with respect to which the representation matrix of F has the canonical form

$$\left(\begin{array}{ccccc|ccc} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & -1 & & & \\ & & & 1 & 0 & & & \\ \hline & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{array}\right) , \qquad \begin{array}{l} k = \text{number of blocks,} \\ n_0 = \text{number of } 0. \end{array}$$

Corollary *Two symmetric (resp. skew-symmetric) $n \times n$ real matrices are equivalent ($B = g^t A g$, for some $g \in GL(n;R)$) if and only if they have the same diagonal (resp. block-diagonal) canonical form.*

Besonders aber haben die Hegelianer, in Folge ihrer ausgezeichneten Unwissenheit und philosophischen Rohheit, ihn, unter dem, aus der vorkantischen Zeit wieder hervorgeholten, Namen "Geist und Natur", von neuem in Gang gebracht, unter welchem sie ihn ganz naiv auftischen, als hätte es nie einen Kant gegeben und giengen wir noch, mit Allongenperücken geziert, zwischen geschorenen hecken umher, indem wir, wie Leibnitz, im Garten zu Herrenhausen (Leibn. ed. Erdmann p. 755), mit Prinzessinen und Hofdamen philosophirten, über "Geist und Natur", unter letzterer die geschorenen Hecken, unter ersterem den Inhalt der Perücken verstehend.

Schopenhauer, Parerga ... II, p. 117

5. Complex quadratic forms

In C^n, the canonical hermitean inner product is defined by

$$<x,y> = x_1\bar{y}_1+...+x_n\bar{y}_n \quad . \qquad (1)$$

It is linear with respect to the first variable and conjugate-linear with respect to the second variable. More general, a **hermitian quadratic form** on the complex vectorspace V, is a map $F:V\times V\rightarrow C$, which is linear with respect to the first variable and satisfyies $F(y,x)=\bar{F}(x,y)$, hence is conjugate linear with respect to the second variable. Analogously a **skew-hermitian quadratic form** on V is a map $F:V\times V\rightarrow C$, which is linear with respect to the first variable and satisfies $F(y,x)=-\bar{F}(x,y)$, hence is conjugate-linear, with respect to the second variable.

Given a basis $a = \{a_1,...,a_n\}$ of V and $x = (x_1,...,x_n)$ identified with $x_1a_1+...+x_na_n$, we have

$$F(x,y) = x\cdot A\cdot\bar{y}^t \ , \ \text{with the matrix } A = [F(a_j,a_k)] \qquad (2)$$

and $\bar{y}^t$ denoting the column vector or transpose of $(\bar{y}_1,...,\bar{y}_n)$. A is the representation matrix of the quadratic form, with respect to the basis a. Changing from the basis a to another basis b, $b = a\cdot g$, with $g \in GL(n,C)$ (i.e. $b_i = \sum a_j\cdot g_{ji}$), we get for the corresponding matrix representations the relation :

$$B = [F(b_i,b_j)] = g^t\cdot A\cdot\bar{g}. \qquad (3)$$

Exercise-1 Show that F is hermitian (resp. skew-hermitian) if and only if, the above matrix A is hermitian symmetric i.e. $a_{ji} = \bar{a}_{ij}$ (resp. hermitian skew-symmetric $a_{ji} = -\bar{a}_{ij}$).

Exercise-2 Show that the set of hermitian $M_h(n,n;\mathbf{R})$ (resp. skew-hermitian $M_{sh}(n,n;\mathbf{R})$) $n\times n$ matrices is a real n^2-dimensional subspace of $M(n,n;C)$=set of $n\times n$ complex matrices. Show further that "multiplication by i" is a real isomorphism of $M_h(n,n;\mathbf{R})$ onto $M_{sh}(n,n;\mathbf{R})$. Show that $M_h(n,n;\mathbf{R})$ and $M_{sh}(n,n;\mathbf{R})$ are real forms of $M(n,n;C)$, and that each complex ma trix $A \in M(n,n;C)$ can be uniquely written as a sum $A=(1/2)(A+\bar{A}^t)+(1/2)(A-\bar{A}^t)$ of a hermitian and a skew-hermitian matrix.

The subset U(n) of matrices in GL(n;C), which preserve (1) is a group, called the **uni-**

tary group. This group is characterized by the following equivalent conditions :

1) $<Ax,Ay> = <x,y>$, for every $x,y \in C^n$. (4)
2) $\overline{A}^t = A^{-1}$ (5)
3) the columns of A build an orthonormal basis of C^n. (6)

Theorem-1 *For every hermitian matrix A, there is a unitary matrix g, with the property*

$$g^{-1}Ag = M = diag(\mu_1,...,\mu_n) \quad \text{(diagonal matrix)}. \qquad (7)$$

The entries of M are the eigenvalues of A, and the columns of g are corresponding eigenvectors.

To prove the theorem, take **u** to be unit eigenvector of A. Then A leaves $<u>^\perp$ (the orthogonal complement of **u**) invariant. Proceed, by induction, with the restriction of A on this complement.

Of particular interest are the **positive (negative) definite** hermitian forms, which by definition satisfy

$$F(x,x) > 0 \;\; (\text{resp.} < 0) \text{ for all } x \neq 0. \qquad (8)$$

A positive definite hermitian form on V is called a **hermitian inner product** or a **hermitian metric** on V.

Exercise-3 Show that $<X,Y>=trace(X^t\overline{Y})$, defines a hermitian inner product on M(n,n;C). Show that the subspaces of hermitian matrices $M_h(n,n;\mathbf{R})$ and skew-hermitian matrices $M_{sh}(n,n;C)$ are orthogonal complements of each other, with respect to this metric.

Theorem-2 *When F is hermitian, we can split V in a direct sum of a positive subspace, a negative subspace and the nullspace of F.*

By "positive" (resp. negative) we mean a subspace, where the restriction of F is positive (resp. negative) definite. The proof is analogous to that of Theorem-1 in §4. We get again a splitting of V

$$W^+ \oplus W^- \oplus W^o = V. \qquad (9)$$

W^+ is a positive subspace of maximal dimension, W^- is a negative subspace of maximal dimension and W^o is the nullspace of F.

Although the spaces W^+ and W^- are not uniquely defined, their dimensions n_+ and n_- are invariantly defined. These numbers enter into the following

Theorem-3 *Given a hermitian form F on the complex n-dimensional vectorspace V, there is a basis, with respect to which, F has diagonal form. The number of positive entries in the diagonal is n_+ and the number of negative is n_-. The remaining entries in the diagonal are zero and their number is $n_o = \dim W^o$.*

The theorem is a consequence of The-2 and The-1. This reduction is obtained by unitary matrices, satisfying $\overline{g}^t = g^{-1}$. Simplifying further with diagonal matrices we get the

Theorem-4 *For every hermitian form $F : V \times V \to C$, there is a basis of V, with respect to which the representation matrix of F has the diagonal canonical form*

$$diag(1, \dots, 1, -1, \dots, -1, 0, \dots, 0),$$

where n_+=number of +1's, n_-= number of -1's, n_0= number of 0's.

Exercise-4 With the previous notations, show that $W^+ \oplus W^-$ can be any complement of the nullspace of F.

Exercise-5 Show that with each eigenvalue μ of a skew-hermitian matrix A, also $-\overline{\mu}$ is an eigenvalue of A.

Theorem-5 *Let A be a skew-hermitian matrix. Then there is a unitary matrix g, such that* $g^t A\overline{g} = i\cdot M$, *where M is a diagonal matrix, as in the The-4.*

The proof follows from The-1, by noticing that iA is hermitian.

A complex space V, endowed with a hermitian inner product <x,y>, is called a **Hermitian** vector space. In such a space we can define and use geometric concepts such as the **length** or **norm** $||x|| = <x,x>^{1/2}$, **distance** $||x-y||$ etc. The following exercises formulate the main properties of these notions.

Exercise-6 (**Schwarz inequality**) Show that for any pair of vectors holds

$$|<x,y>| \le ||x||\cdot||y|| , \tag{10}$$

and the equality holds precisely in the case where y is a complex multiple of x.

[Start with $<\mu x-y,\mu x-y>\ge 0$, for real t and $\mu = t(\overline{<x,y>}/|<x,y>|)$, when $<x,y>\neq 0$. The case $<x,y>=0$ is trivial].

Exercise-7 (**Triangle inequalities**) Show that for any pair of vectors hold the inequalities

$$||x+y|| \le ||x|| + ||y|| , \tag{11}$$

$$|\,||x|| - ||y||\,| \le ||x-y|| . \tag{12}$$

The analogon of hermitian (resp. skew-hermitian) matrices in a hermitian vector space are the **self-adjoint** (resp. **antiself-adjoint**) operators $A:V\to V$. In general for every (complex) linear $A:V\to V$, there is a unique operator A* satisfying $<Ax,y> = <x,A^*y>$. The operator is called self-adjoint precisely when A*=A (resp. antiself-adjoint, when A*=-A).

Exercise-8 Show that the matrix representing A* with respect to an orthonormal basis of V is the conjugate traspose of the matrix representing A, with respect to the same basis. Conclude that an operator A is self-adjoint (resp. antiself-adjoint) if his matrix-representations with respect to orthonormal bases of V are hermitian (resp. skew-hermitian).

Symmetric and skew-symmetric <u>bilinear</u> forms on a complex vectorspace V can be "diagonalized" as in the real case. Suppose first that $F: V\times V\to C$, is a symmetric bilinear form. Then, there is some v for which $F(v,v)\neq 0$ [else we would have F(x,y)=0 for all x,y]. Then F is non-degenerate on <v>, hence, according to Ex-8, p. 4-2, the conjugate space (orthogonal with respect to F) $<v>^{\perp}$ is a complement of <v>. Repeating the reasoning with the restriction of F on this complement, we proceed inductively to the proof of the

Theorem-6 *For every complex bilinear symmetric form F on the complex space V, there is a basis, with respect to which F has the representation matrix*

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \quad k = \text{the number of 1's} = \text{the rank of F}.$$

Suppose now that F is skew symmetric. Then there are non-zero x, y such that $F(x,y)\neq 0$ (unless $F=0$). Then F restricted on the space W, spanned by these two vectors, is non-degenerate, hence the conjugate space $W^{\perp}$ to W is a complement of W. Proceeding with the restriction of F on this complement, as in the symmetric case, we prove in an analogous way the

Theorem-7 *For every skew-symmetric complex bilinear form F on a complex vector space V, there is a basis, with respect to which F has the representation matrix*

$$\left(\begin{array}{ccc|ccc}
\begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & & & \\
& \ddots & & & & \\
& & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & \\ \hline
& & & 0 & & \\
& & & & \ddots & \\
& & & & & 0
\end{array}\right),$$

k = number of blocks, ($2k$=rank of F).
n_0= number of 0's
= nullity of F.

> Je mehr nun aber Einem die Furcht Ruhe läßt, desto mehr beunruhigen ihn die Wünsche, die Begierden und Ansprüche. Goethes so beliebtes Lied, "ich hab' mein' Sach auf nichts gestellt" [»Vanitas! Vanitatum vanitas!«] besagt eigentlich, daß erst nachdem der Mensch aus allen möglichen Ansprüchen herausgetrieben und auf das nackte, kahle Daseyn zurückgewiesen ist, er derjenigen Geistesruhe theilhaft wird, welche die Grundlage des menschlichen Glückes ausmacht, indem sie nöthig ist, um die Gegenwart, und somit das ganze Leben, genießbar zu finden.
>
> A. Schopenhauer, Aphorismen p. 454

6. Pairs of quadratic forms and Jordan normal form

The main problem for two quadratic forms $F_i : V \times V \to C$, $(i=1,2)$, is that of simultaneous diagonalization. Even in the case of two hermitian forms, the problem is non-trivial. A general result in this direction is the following:

Theorem-1[1] *If for two hermitian matrices A, B and all $x \neq 0$ in C^n, we do not have both $\langle Ax,x\rangle = 0$ and $\langle Bx,x\rangle = 0$ ($\langle \dots,\dots \rangle$ denoting the standard hermitian product in C^n) simultaneously, then A and B are simultaneously diagonable.*

There is a special case where the two forms can be simultaneously diagonalized. This happens when one of them, say $F=F_1$, is hermitian and positive definite and the other $G=F_2$, is **normal** with respect to F. By this we mean the following: V endowed with F becomes a hermitian space, we change from F to the notation $\langle x,y \rangle$. Then

$$G(x,y) = \langle Ax,y \rangle , \tag{1}$$

defines a linear operator $A : V \to V$. G is called normal with respect to F ($=\langle \dots,\dots \rangle$), when $A^*A=AA^*$. In elementary linear algebra these operators are characterized by the fact, that they are completely diagonalizable, with respect to an orthonormal basis of V. Thus, we "proved" the

Theorem-2 *When $F_i : V\times V \to C$, $(i=1,2)$, are two forms, one of which is hermitian positive definite and the other is normal with respect to the first, then there is a basis of V, with respect to which, both forms have diagonal matrix-representations.*

Important special cases of the preceding theorem are these for which the normal form is, more specific, hermitian, skew-hermitian, symmetric or skew-symmetric. In all these cases the corresponding operators A are normal and we have more specific diagonalization theorems.

I turn now to a brief discussion of the "Jordan normal form". This is closely related to the "Jordan-Chevalley" decomposition $A = S+N$ of an operator A in a (unique) commuting pair S, N of a semi simple S and a nilpotent operator N. The standard example is

$$A = S + N = \begin{pmatrix} d & & \\ 1 & \ddots & \\ & 1 & d \end{pmatrix} = \begin{pmatrix} d & & \\ & \ddots & \\ & & d \end{pmatrix} + \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}, \quad SN = NS .$$

The general case is reduced to this one by finding a basis, with respect to which, the given

1. Becker R. I., Necessary and sufficient conditions for the simultaneous diagonability of two quadratic forms. Linear Algebra and its applications 30(1980), pp. 129-139.

operator is represented by blocks A, as above. We recall that **semi simple** is a linear operator $A:V\rightarrow V$ which has the property: for every A-invariant subspace W of V, there is an A-invariant complement W'. For Complex vector spaces, we can prove, by induction, that semi simplicity is equivalent to diagonability of the operator. For real vector spaces, diagonable operators are of course semi simple, but not the inverse. For example, orthogonal matrices represent obviously semi simple operators, but aren't diagonable. Any way, **nilpotent** are called operators which satisfy $A^k = 0$, for some positive integer k (called **degree** of nilpotency of A). A key ingredient in the proof of Jordan's normal form theorem is the decomposition of the vectorspace V in A-invariant **cyclic subspaces** W(a) of a nilpotent operator A. By this we mean subspaces which have a basis of the form

$$a, A(a), \ldots, A^d(a)\neq 0, \quad A^{d+1}(a)=0, \quad \text{with } d+1\leq k=\text{degree of } A.$$

With respect to such a basis, the operator $A|W(a)$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}$$

and considering a decomposition of V in such subspaces we get a representation of A in "block-diagonal-form" $A=\text{diag}(A_1, \ldots, A_m)$, where each block has the preceding form.

If A is not nilpotent, then we use the minimal polynomial $p(x) = a_r x^r + \ldots + a_0$, which is the minimum-degree polynomial satisfying $p(A)=0$. When V is complex, p(x) decomposes into linear factors:

$$p(x) = (x-\lambda_1)^{d_1}(x-\lambda_2)^{d_2}\ldots(x-\lambda_m)^{d_m},$$

and V itself decomposes into the direct sum of "generalized eigenspaces"

$$V(\lambda_s) = \text{kern}\,[(A-\lambda_s I)^{d_s}].$$

$V(\lambda_s)$ is an A-invariant subspace in which the operator $B=A-\lambda_s I$ is nilpotent, with nilpotency degree d_s and we apply the preceding analysis to B. This is roughly the way we get the Jordan normal form. More details can be found in the book of Nomizu[2].

2. K. Nomizu, Linear Algebra, Academic Press, 1978.

Aber sollte die Natur, aus bloßer Verstocktheit, ewig vor unserer Frage verstummen? Ist sie nicht, wie alles Große, offen, mittheilend und sogar naiv? Kann daher ihre Antwort je aus einem andern Grunde fehlen, als weil die Frage verfehlt war, schief war, von falschen Voraussetzungen ausgieng, oder gar einen Widerspruch herbergte?
Schopenhauer, Parerga ... II, p. 106

7. Schur's lemma

This lemma deals with families of linear maps $F = \{A: V \to V\}$ on a vector space V. V is called **irreducible** with respect to F, when there is no non-zero subspace W, different from V, such that $A(W) \subset W$, for all A in F. We call also a subspace W, with $A(W) \subset W$, for all A in F, an **F-invariant** subspace of V. The following formulation is due to Nomizu[3].

Lemma (Schur) *Let V_i (i=1,2) two irreducible spaces with respect to the families F_i (i=1,2). Let also $C: V_1 \to V_2$ be a linear map, such that the equation*

$$CA = BC, \tag{1}$$

i) has a solution $B \in F_2$, for every given $A \in F_1$, and

ii) has a solution $A \in F_1$, for every given $B \in F_2$

Then, C is either the zero map or an isomorphism.

The lemma is a direct corollary of the fact, that $W_1 = \text{Kern}C$ and $W_2 = \text{Im}C$ are, correspondingly, F_1-invariant and F_2-invariant. In fact, $x \in W_1$, $A \in F_1 \Rightarrow CAx = BCx = 0$ (for some B), hence $Ax \in W_1$, and analogously the invariance of W_2. The irreducibility of the two families implies $W_1 = \{0\}$ or $W_1 = V_1$ etc.

A special case is the one with $V_1 = V_2 = V$, $F_1 = F_2 = F$ and $C: V \to V$ a linear map commuting with every $A \in F$. Then, every eigenspace of C is F-invariant. In fact,

$$(C-\mu I)x = 0 \text{ and } A \in F \quad \Rightarrow$$
$$(C-\mu I)Ax = A(C-\mu I)x = 0.$$

Thus, when V is irreducible an eigenspace of C is either $\{0\}$ or V. We proved

Proposition-1 *Let V be irreducible with respect to F and $C: V \to V$ a linear map commuting with every member of the family F. If C has an eigenvalue μ (in the field of V) then C is a multiple of the identity $C = \mu I$.*

What can be said for C, when the field **K** of V is not algebraicaly closed? (f.e. when **K**=**R**) In this case the minimum polynomial of C is still irreducible (but no more linear). In fact, if it where

$$f(x) = f_1(x)f_2(x),$$

then there should exist a decomposition

$$V = V_1 \oplus V_2 = \{x \in V \mid f_1(C)x = 0\} \oplus \{x \in V \mid f_2(C)x = 0\},$$

in two F-invariant subspaces (because of the commutativity). Then, one of these subspaces would be $\{0\}$, which gives a contradiction. With these preliminary remarks, we can prove the analogon of proposition-1, for real vector spaces.

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3. K. Nomizu, Linear Algebra, p. 209

Proposition-2 *Let V be a real vector space, irreducible with respect to the family F. Let also C:V→ V be a linear map commuting with every $A \in F$. Then,*

$$C = \mu I, \qquad \textit{for } \dim V = 2k+1, \textit{ and}$$
$$C = aI + bJ, \ \textit{with } J^2 = -I, \qquad \textit{for } \dim V = 2k.$$

When dimV=2k+1, there is a <u>real</u> eigenvalue of C and we can repeat the arguments of Prop-1.

In the second case the minimum polynomial of C is at most quadratic, hence C satisfies an equation of the form

$$(C-aI)^2 + b^2 I = 0 . \Leftrightarrow [(C-aI)/b]^2 = -I.$$

Take then, $J = (C-aI)/b \Leftrightarrow C = aI+bJ$.

The Schur's lemma has important applications in the context of bilinear forms. In fact, suppose that $<...,...>$ and q(...,...) are two bilinear forms and the first is non-degenerate. Then there is a unique operator C, such that

$$q(x,y) = <Cx,y>. \tag{2}$$

Suppose now that $<...,...>$ and q(...,...) are invariant with respect to a family F, for which V is irreducible. Then, for each $A \in F$ we'll have

$$q(Ax,Ay) = <CAx,Ay> = <Cx,y> = <ACx,Ay>.$$

Since $<Ax,Ay> = <x,y>$, the non-degeneracy of $<...,...>$ implies, AC=CA, for every $A \in F$. Thus, from Prop-2 we'll have

$C = \mu I$, if V is complex or if V is real and C has a real eigenvalue,
$C = \mu I + \mu^* J$, if V is real and C has no real eigenvalues.

We proved:

Theorem *Assume that $<...,...>$ and q(...,...) are two bilinear forms and the first is non-degenerate. Assume further that $<...,...>$ and q(...,...) are invariant with respect to a family F, for which V is irreducible. Then*

i) $q(x,y) = \mu <x,y>$, for all x,y when V is complex or real of odd dimension,
ii) $q(x,y) = \mu <x,y> + \mu^* <Jx,y>$, with real μ, μ^*, when V is real of even dimension.
In this case J is a complex structure for V.

Notice that in the case of real V and symmetric q(...,...) and $<...,...>$, holds i), since in that case C is symmetric, consequently has real eigenvalues. F is usually a group and the theorem implies, that there is (up to scalar multiples) at most one bilinear form invariant under this group. We'll have below the occasion to examine many bilinear forms and their "related" groups.

Oh that I had the art of easy writing
What should be easy reading! could I scale
Parnassus, where the Muses sit inditing
Those pretty poems never known to fail,
How quickly would I print (the world delighting)
A Grecian, Syrian, or *Assyrian* tale;
And sell you, mix'd with western sentimentalism,
Some samples of the finest Orientalism.

Byron, Beppo, 51

8. The general linear Lie-algebra gl(n;C)

The vector space M(n,n;C) of complex $n\times n$ matrices, has a natural associative-algebra structure with respect to the matrix product AB. The **commutator**

$$[AB] = AB-BA, \tag{1}$$

defines another kind of product on M(n,n;C), satisfying the properties

i) [AB] is bilinear and skew-symmetric in A, B. (2)

ii) $[[AB]C]+[[BC]A]+[[CA]B] = 0$, for all A, B, C. (3)

We call M(n,n;C) endowed with this product, the **general linear Lie algebra** acting on C^n, and denote it by gl(n;C). (3) is reffered to as Jacobi-identity.

gl(n;C) is the most important example of a **Lie algebra**, which is defined as a vector space endowed with a product [AB] with the formal properties (2) and (3). We'll see below several other examples of similar Lie algebras, which are, mainly, Lie-subalgebras of gl(n;C) i.e. vector subspaces, closed under the commutator [...].

A basis for gl(n;C) is given by the n^2 matrices

$$E_{ij} = \begin{pmatrix} & | & \\ \text{——} & 1 & \text{——} \\ & | & \end{pmatrix}$$

, 1 in the i-row, j-column
0 everywhere else.

The elements of E_{ij} are given by

$$(E_{ij})_{mn} = \delta_{mi}\, \delta_{jn}\,. \tag{4}$$

The product of these matrices is given by

$$E_{ij}\, E_{rs} = \delta_{jr}\, E_{is}\;, \tag{5}$$

$$[E_{ij}\, E_{rs}] = \delta_{jr}\, E_{is} - \delta_{si}\, E_{rj}\,. \tag{6}$$

For an arbitrary matrix A we have then,

$(AE_{ij})_{mn} = a_{mi}\, \delta_{jn}$ (only the j-column = i-column of A, all other =0), (7)

$(E_{ij}A)_{mn} = a_{jn}\, \delta_{mi}$ (only the i-row = j-row of A, all other =0). (8)

Center of gl(n;C) is called the subset of matrices A, commuting with every $B\in$ gl(n;C) ([AB]=0).

Exercise-1 Show that the center of gl(n;C) coincides with the subspace of scalar diagonal matrices $\{\mu I\}$.

Exercise-2 Show that the subspace diag(n) of diagonal matrices is an <u>Abelian</u> Lie-subalgebra of gl(n;C) i.e. $[AB]=0$, for diagonal A,B. Show also that $[A,\mathrm{diag}(n)]\subset\mathrm{diag}(n)$, implies the matrix A itself is also diagonal.

Exercise-3 Let

$$N_1=\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix},\ N_2=\begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 0 \end{pmatrix},\ \dots,\ N_{n-1}=\begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 0 & 0 & & \\ & \ddots & \ddots & \ddots & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Show that $N_k = N_1^{\,k}$.

Exercise-3 Show that $BN_1=N_1B \Leftrightarrow B=\sum a_i N_i + aI \Leftrightarrow B=f(N_1)$ = a polynomial in N_1.

[Examine $BN_i=N_iB$ using (7) and (8)]

Well-well, the world must turn upon its axis,
And all mankind turn with it, heads or tails,
And live and die, make love and pay our taxes,
And as the veering wind shifts, shift our sails;
The king commands us, and the doctor quacks us,
The priest instructs, and so our life exhales,
A little breath, love, wine, ambition, fame,
Fighting, devotion, dust, -perhaps a name.

Byron, Don Juan, Canto II, 4

9. Ad and ad of gl(n;C)

We denote by End(C^n) the associative algebra of n×n matrices, endowed with the usual product of matrices AB (more general End(V) = associative algebra of endomorphisms of a vector space V).

We denote by Aut(End(C^n)) the group of **automorphisms** of End(C^n) i.e. linear invertible maps f: End(C^n)→End(C^n) with the property $f(AB) = f(A)f(B)$. For a $g \in GL(n;C)$, the map

$$f_g(A) = gAg^{-1}, \qquad (1)$$

is an automorphism of End(C^n) and is called an **inner** automorphism. It can be proved[3] that any automorphism of End(C^n) is of that form. In analogy with the associative algebra, we define the corresponding notions for the Lie-algebra gl(n;C):

Aut(gl(n;C)) = group of automorphisms of gl(n;C) i.e. linear invertible maps

f: gl(n;C)→gl(n;C) with the property $f[AB] = [f(A)f(B)]$. (2)

In particular, the maps $f_g(A) = gAg^{-1}$, can be easily seen to be automorphisms of gl(n;C). We continue to call these automorphisms **inner** and denote them by Ad_g. Obviously Ad is an homomorphism of GL(n;C) into Aut(gl(n;C)).

A **derivation** D of an algebra W with product "·" is a linear map D:W→W, satisfying the identity

$$D(A{\cdot}B) = D(A){\cdot}B + A{\cdot}D(B). \qquad (3)$$

In the Lie-algebra gl(n;C) we have some natural derivations related to the inner automorphisms. In fact, fixing some $X \in gl(n;C)$ we define the **adjoint** transformation with respect to X:

$$adX: gl(n;C)\to gl(n;C), \text{ with } adX(Y) = [XY]. \qquad (4)$$

One verifies easily (using Jacobi identity) that this is indeed a derivation, which we call **inner**.

Theorem-1 *For each derivation D: gl(n;C)→gl(n;C), exp(tD) = ∑ (tD)ⁿ/n! defines an automorphism of gl(n;C), for all real numbers t.*

This can be proved using the uniqueness of solutions of linear differential equations. In fact, the two curves of gl(n;C), a(t) = exp(tD)[XY] and b(t) = [exp(tD)X, exp(tD)Y], for t=0, go through the same point [XY]. They satisfy also the same linear differential equation

$$z' = Dz. \qquad (5)$$

3. H. Flanders, Methods of proof in linear algebra, Amer. Math. Monthly, 1956 p. 1

This is obvious for a(t) and follows directly from (3,) for b(t). Thus, the uniqueness theorem implies that a(t)=b(t), for all t, which means that exp(tD) is an automorphism.

Theorem-2 *For each $X \in gl(n;C)$ holds the equation*

$$exp(tadX) = Ad_{exp(tX)} \cdot \qquad (6)$$

The proof may be given by an argument similar to that of Th-1, and by applying the two sides of (6) on an arbitrary matrix Y. As a corollary, we obtain the equation

$$(d/dt)\,|_0 Ad_{exp(tX)} Y = [XY]. \qquad (7)$$

Exercise-1 Show that the set Der(gl(n;C)) of derivations of gl(n;C) is a vector space closed under the operation of commutator i.e. [DD']=D·D'-D'·D is again a derivation, if D and D' are.

Problem Determine the automorphism group Aut(gl(n;C)) and the vector space Der(gl(n;C)).

II.

Lie algebras, general facts

Thus saith the Preacher; 'nought beneath the sun
Is new', yet still from change to change we run.
What varied wonders tempt us as they pass!
The Cow-pox, Tractors, Galvanism, and Gas
In turns appear to make the vulgar stare,
Till the swoln bubble bursts-and all is air!
Byron, English bards and scotch reviewers, 70

10. Lie-Algebras

Lie-Algebra is called a vector space $\mathfrak{g}$ endowed with a product [...], called Lie-bracket, and satisfying

$$[XY] \text{ is bilinear and skew-symmetric,} \tag{1}$$

$$[[AB]C]+[[BC]A]+[[CA]B] = 0, \text{ for all } A,B,C \in \mathfrak{g} . \tag{2}$$

(2) is reffered to as <u>Jacobi-identity</u>. The typical example is gl(n;C), examined in the preceding paragraph. This is a special case of the more general Lie-algebras gl(V), consisting of all endomorphisms of a vector space V with [...] defined by the commutator of two endomorphisms [AB]=A·B-B·A.

Lie-subalgebra $\mathfrak{h} \subset \mathfrak{g}$, is called a vector subspace$\mathfrak{h}$ of $\mathfrak{g}$, which is closed under the [...]. **Lie-ideal** $\mathfrak{h} \subset \mathfrak{g}$, is called a Lie-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which in addition satisfies

$$[\mathfrak{h}\mathfrak{g}] \subset \mathfrak{h}. \tag{3}$$

Obviously $\mathfrak{g}$ and {0} are ideals . We call them trivial.

Exercise-1 Show that $\mathbf{R}^3$ endowed with the usual exterior product

$$x \times y = (x_2y_3-y_2x_3, x_3y_1-y_3x_1, x_1y_2-y_1x_2), \tag{4}$$

is a Lie-algebra. Show that this Lie-algebra has no non-trivial ideals.

Exercise-2 Select a basis $e = (e_1, ..., e_n)$ of the Lie-algebra $\mathfrak{g}$. Expressing $[e_i,e_j]$ in terms of the basis e, we get the so-called **structure constants** c_{ijk} , defined by the equations

$$[e_i,e_j] = \sum c_{ijk}\, e_k . \tag{5}$$

Prove that (1), (2) are correspondingly equivalent with the equations

$$c_{ijk} = -c_{jik} , \tag{6}$$

$$\sum_r (c_{jkr}\, c_{irs} + c_{kir}\, c_{jrs} + c_{ijr}\, c_{krs}) = 0. \tag{7}$$

Exercise-3 Show that by a basis change $e' = eg$ (i.e. $e_i' = e_jg_{ji}$, summation over indices appearing twice), the corresponding structure constants are related by the equations

$$[e_i,e_j] = c_{ijk}\, e_k ,$$

$$[e_i',e_j'] = c'_{ijk} e_k' ,$$

$$c'_{ijk} = c_{nmr}\, g_{ni}g_{mj}h_{kr} , \text{ where h is the inverse matrix of g.} \tag{8}$$

The two last exercises show that the problem of classification of Lie-algebras (of finite dimension, tacitly assumed throughout the lectures) is equivalent with the determination of equivalence classes of three-dimensional arrays c_{ijk}, satisfying (6) and (7), the equivalence relation being defined by (8).

Exercise-4 Let $\mathfrak{g}$ be a Lie-algebra and V a vector-subspace of $\mathfrak{g}$ containing the set $[\mathfrak{g}\mathfrak{g}]$ = vector-subspace spanned by all [XY]. Show that V is a Lie-ideal of $\mathfrak{g}$. In particular $[\mathfrak{g}\mathfrak{g}]$ itself is an ideal of $\mathfrak{g}$.

The **center** $\mathfrak{c}$ of a Lie-algebra $\mathfrak{g}$ is defined to be the set of all elements "commuting" with every element of $\mathfrak{g}$

$$\mathfrak{c} = \{X \in \mathfrak{g}, [XY]=0, \text{ for all } Y \in \mathfrak{g}\}. \quad (9)$$

Exercise-5 Show that the center of every Lie-algebra is an ideal.

A linear mapping $f:\mathfrak{g}_1 \to \mathfrak{g}_2$ between two Lie-algebras is called an **homomorphism** of Lie-algebras, if it "respects" the corresponding products, i.e. for all X, Y in $\mathfrak{g}_1$

$$f[XY]_1 = [fX, fY]_2 . \quad (10)$$

When $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}$, we call f **endomorphism**. An invertible endomorphism is called **automorphism**. We use the notation End($\mathfrak{g}$) and Aut($\mathfrak{g}$), correspondingly, for the associative algebra of endomorphisms of $\mathfrak{g}$ and for the group of automorphisms of $\mathfrak{g}$.

Exercise-6 Show that the inverse mapping of an automorphism of $\mathfrak{g}$ is an automorphism too. Show also that the Kern$f$ of an homomorphism is an ideal in $\mathfrak{g}_1$, and that the Imf is a subalgebra in $\mathfrak{g}_2$.

For each Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the **normalizer** $\mathfrak{n}(\mathfrak{h})$ is defined to be the set

$$\mathfrak{n}(\mathfrak{h}) = \{X \in \mathfrak{g}, [X\mathfrak{h}] \subset \mathfrak{h}\}. \quad (11)$$

Exercise-7 Show that $\mathfrak{n}(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{h}$ is an ideal in $\mathfrak{n}(\mathfrak{h})$.

For each Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the **centralizer** $\mathfrak{c}(\mathfrak{h})$ is defined to be the set

$$\mathfrak{c}(\mathfrak{h}) = \{X \in \mathfrak{g}, [X\mathfrak{h}]=0\}. \quad (12)$$

Exercise-8 Show that $\mathfrak{c}(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{h}$ is an ideal in $\mathfrak{c}(\mathfrak{h})+\mathfrak{h}$. Show also that $\mathfrak{c}(\mathfrak{h}) \subset \mathfrak{n}(\mathfrak{h})$.

Exercise-9 Show that subalgebras are mapped, by homomorphisms f, onto subalgebras, but ideals are mapped, in general, in subalgebras (and not ideals). Show that the inverse image $f^{-1}(\mathfrak{h})$ of a subalgebra (ideal) is again a subalgebra (ideal).

For each ideal $\mathfrak{h} \subset \mathfrak{g}$ of a Lie-algebra $\mathfrak{g}$, the quotient vectorspace $\mathfrak{g}/\mathfrak{h}$ admits the structure of a Lie-algebra by defining the product on $\mathfrak{g}/\mathfrak{h}$ via the natural projection

$$p:\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}, \quad p(X) = X + \mathfrak{h},$$
$$[p(X), p(Y)] = p[XY]. \quad (13)$$

With this structure, p becomes an homomorphism of Lie-alge- bras. We call $\mathfrak{g}/\mathfrak{h}$, endowed with this structure the **quotient** Lie-algebra of $\mathfrak{g}$ by $\mathfrak{h}$.

Exercise-10 Show that for an homomorphism $f:\mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie-algebras, there is a naturally defined "induced" homomor- phism $f':\mathfrak{g}_1/\text{Kern}f \to \mathfrak{g}_2$, which is 1-1 and makes the nearby diagram commutative (p the projection onto the quotient).

Exercise-11 Show that for ideals $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{g}$, there is a natural isomorphism

f:g/b → (g/a)/(b/a) given by

$$f(X+b) = p(X)+b/a, \text{ where } p(X) = X+a. \tag{14}$$

Exercise-12 Let a, b be ideals of the Lie-algebra g. Show that a+b and [ab] are ideals too. When a, b are only subalgebras, then, in general, a+b and [ab] are not Lie-subalgebras. If however one of them, say b, is an ideal, then [ab] is a subalgebra, b∩a is an ideal in a and there is a natural homomorphism

$$f: a/(b\cap a) \to (a+b)/b, \text{ given by}$$
$$f(X+b\cap a) = X+b. \tag{15}$$

All these formal things are boring but necessary, like the rules of grammar. Except the subalgebras and quotients one can construct other examples of Lie-algebras as follows:

The **direct product** of two Lie-algebras a⊕b with bracket defined on decomposable elements by

$$[X\oplus Y, X'\oplus Y'] = [XX']\oplus[YY'], \tag{16}$$

and by bilinear extension for the other elements.

Exercise-13 Show that a, b are naturally embedded as ideals in a⊕b.

A **semi-direct-product** $a\oplus_\sigma b$, of two Lie-algebras is defined by means of a **representation** of Lie-algebras i.e. using an homomorphism

$$\sigma: a \to gl(b),$$

where gl(b) denotes the Lie-algebra of endomorphisms of b, with bracket equal to the commutator of linear operators on b. One verifies easily that the subspace Der(b) of derivations of b is a Lie-subalgebra of gl(b). For the representation σ, used in the definition of semi-direct-product, we postulate that it takes values into Der(b). Then, as a vector-space, we define $a\oplus_\sigma b$ to be identical with the vector-space-direct-sum a⊕b. But the product on $a\oplus_\sigma b$ is defined by the rules

i) [...] restricted on $a\subset a\oplus_\sigma b$ and on $b\subset a\oplus_\sigma b$, coincides with the old brackets,

ii) for "mixed" products, $[XY] = \sigma(X)Y = -[YX]$, for all $X\in a$, $Y\in b$.

Exercise-14 Show, using the characteristic property of derivations ((3) in §9), that $a\oplus_\sigma b$ is indeed a Lie-algebra, which coincides with the direct product, when σ= 0. Show further that, in general, only b is an ideal of $a\oplus_\sigma b$.

Abelian are called the Lie-algebras, which have a trivial product: [XY]=0, for all X, Y. Any vector-space V can be considered as an Abelian Lie-algebra. Considering V as Abelian Lie-algebra and taking any representation σ:a → gl(V), we get a new Lie-algebra $a\oplus_\sigma V$.

Exercise-15 Show that every two-dimensional non-abelian Lie-algebra g has one-dimensional [gg] and there is a basis e_1, e_2 of g, such that $[e_1,e_2]=e_2$. Conclude that all these Lie algebras (over a fixed field) are isomorphic. Thus, we are legitimate to say the two-dimensional non-abelian Lie-algebra . Show that the Lie-subalgebra

$$aff(1,C) = \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, x, y\in C\right\},$$

of gl(2;C), is such an example and find e_1, e_2 in this case.

Exercise-16 Show that the center of an n-dimensional Lie-algebra, cannot have n-1 dimensions.

Exercise-17 Show that for every Lie-algebra $\mathfrak{g}$, $[\mathfrak{g}\mathfrak{g}]$ is an ideal and $\mathfrak{g}/[\mathfrak{g}\mathfrak{g}]$ is Abelian.

Exercise-18 Let $\mathfrak{g}$ be a Lie-algebra with one dimensional $[\mathfrak{g}\mathfrak{g}] = \langle e \rangle$. Show that there is a skew-symmetric $f\colon \mathfrak{g}\times\mathfrak{g}\to \mathbf{K}$ ($\mathbf{K}$ is the field of $\mathfrak{g}$) such that $[XY]=f(X,Y)e$, for all X, Y and f and $\mathfrak{g}$ have the properties :

i) $f(X,Z)f(X,e)+f(Z,X)f(Y,e)+f(X,Y)f(Z,e) = 0$, for all X, Y, Z and

ii) $[e\mathfrak{g}] = 0$, or

iii) there is a basis of $e_1, \ldots, e_n$ of $\mathfrak{g}$, such that f has corresponding representation matrix

$$\begin{pmatrix} 0 & -\lambda & & & \\ \lambda & 0 & & & \\ 0 & 0 & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \text{ with } \lambda \neq 0.$$

In the last case, the Lie-subalgebra $\mathfrak{h}$, spanned by $\{e_3, \ldots, e_n\}$, is an Abelian ideal in $\mathfrak{g}$. Hence in the case $[e\mathfrak{g}]\neq 0$, $e \notin \mathfrak{g} \Rightarrow e=\mu_1 e_1+\ldots+\mu_n e_n$, with μ_1, μ_2 not both $= 0$, say $\mu_1 \neq 0$. Then, the Lie-subalgebra $\mathfrak{h}'$, spanned by $\{e_2, e\}$, is an ideal of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}'$.

Exercise-19 (Continuation of 18) In the case ii) above, there is a basis, such that f has the representation matrix

$$\begin{pmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & -\lambda_k & & & \\ & & & \lambda_k & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}, \text{ with } \lambda_i \neq 0,$$

and $\mathfrak{g}$ is the direct sum of an Abelian ideal $\mathfrak{h}$ and of an odd-dimensional Lie-subalgebra $\mathfrak{h}'$, spanned by vectors $\{e_1, f_1, e_2, f_1, \ldots, e_k, f_k\}$, such that $[e_i, f_i]=e$, for all $i=1,\ldots,k$, all other products being zero.

Reader! I have kept my word,-at least so far
As the first Canto promised. You have now
Had sketches of love, tempest, travel, war-
All very accurate, you must allow,
And Epic, if plain truth should prove no bar;
For I have drawn much less with a long bow
Than my forerunners. Carelessly I sing,
But Phoebus lends me now and then a string,

Byron, Don Juan, Canto VIII, 138

11. Adjoint representation and Killing form

A **representation** of the Lie-algebra $\mathfrak{g}$ on the vector-space V, is an homomorphism $f:\mathfrak{g}\to gl(V)$ of Lie-algebras. Using a representation and some basis of V, we "see" the Lie-algebra $\mathfrak{g}$ as a set of matrices with the commutator as product. A famous theorem of Ado says, that every Lie-algebra (over the field of complex numbers C) has a **faithful** (i.e. 1-1) representation on some vector-space V.

Given a Lie-algebra $\mathfrak{g}$, there is a special representation of $\mathfrak{g}$ on itself (viewed as a vector-space) which generalizes the adjoint representation of gl(n;C) :

$$ad:\mathfrak{g} \to gl(\mathfrak{g}),$$
$$adX(Y) = [XY], \text{ for all } X, Y \in \mathfrak{g}. \qquad (1)$$

This is called the **adjoint** representation of $\mathfrak{g}$. The Jacobi identity is easily seen to be equivalent with the

$$ad[XY] = adX\cdot adY - adY\cdot adX = [adX, adY], \text{ for all } X, Y \in \mathfrak{g}, \qquad (2)$$

which shows that ad is indeed a representation.

Despite the fact that this representation is not faithful, and

$$Kern(ad) = \text{center of } \mathfrak{g} = \mathfrak{c}, \qquad (3)$$

it is very useful for the study of the structure of Lie-algebras. Besides, from the Jacobi identity follows that adX is a derivation of $\mathfrak{g}$:

$$adX([YZ]) = [adX(Y),Z]+[Y,adX(Z)], \text{ for all } X, Y, Z \in \mathfrak{g}. \qquad (4)$$

Derivations of this kind (D=adZ, for some $Z \in \mathfrak{g}$) are called **inner**. We will see many examples of Lie-algebras (Semi-simple) having all their derivations inner.

Exercise-1 Show that all the derivations of the two-dimensional non-abelian Lie-algebra (with a basis $\{e_1,e_2\}$ s.t. $[e_1,e_2]=e_2$, see §10) are inner.

[If $De_1=ae_2$, $De_2=be_2$, show $D=ad(be_1-ae_2)$]

The **Killing** form of a Lie-algebra $\mathfrak{g}$ is a very useful symmetric bilinear form on the Lie-algebra, defined through the adjoint representation by

$$K(X,Y) = tr(adX\cdot adY), \text{ for all } X, Y, Z \in \mathfrak{g}. \qquad (5)$$

Theorem-1 *For every automorphism $f: \mathfrak{g} \to \mathfrak{g}$, we have*

$$K(fX,fY) = K(X,Y). \qquad (6)$$

Theorem-2 *Each derivation $D: \mathfrak{g} \to \mathfrak{g}$, is skew-symmetric, with respect to K:*

$$K(DX,Y)+K(X,DY) = 0. \qquad (7)$$

The first theorem is a consequence of the easily proved formula

$$\mathrm{ad}(fX) = f\cdot \mathrm{ad}X\cdot f^{-1} \quad , \quad \text{for every automorphism } f \in \mathrm{Aut}(\mathfrak{g}), \quad (8)$$

and the invariance of tr(ABCD), under a cyclic permutation of the letters. The second theorem follows from Th-1, by differetiating K(exp(tD)X,exp(tD)), and the following theorem, which is proved by applying verbatim the argument in Th-1, §9:

Theorem-3 *For each derivation D: $\mathfrak{g} \to \mathfrak{g}$, $\exp(tD) = \sum (tD)^n/n!$ defines an automorphism of $\mathfrak{g}$, for all real numbers t.*

These propositions show that K is strongly related to Aut($\mathfrak{g}$). Unfortunately K is in general degenerate. f.e. the center $\mathfrak{c}$ of $\mathfrak{g}$ is allways contained in the nullspace of K. As we'll see below, non-degeneracy of K is a drastic restriction on the Lie-algebra, and defines the so-called semi-simple Lie-algebras, which are our main object of study. In this context, the Killing form becomes an indispensable tool in the investigation of the structure theory.

Exercise-2 Show that in every Lie-algebra $\mathfrak{g}$ the nullspace $\mathfrak{r}$ of K is an ideal of $\mathfrak{g}$. $\mathfrak{r}$ is called the **radical** of $\mathfrak{g}$.

One may ask, wether there is a complementary space of the radical, which is a subalgebra. The famous theorem of Levi answers this question affirmatively.

Exercise-3 Show that for each ideal $\mathfrak{h}$ of the Lie-algebra $\mathfrak{g}$, the orthogonal complement $\mathfrak{h}^{\perp}$ with respect to K, is again an ideal of $\mathfrak{g}$.

Exercise-4 Show that for each ideal $\mathfrak{h}$ of the Lie-algebra $\mathfrak{g}$, the Killing form $K_{\mathfrak{h}}$ of $\mathfrak{h}$ coincides with the restriction of K on $\mathfrak{g}$. [Consider a basis $\{e_1,...,e_k\}$ of $\mathfrak{h}$ and extend it to a basis of $\mathfrak{g}$. Examine then the matrix representing adX, for $X \in \mathfrak{h}$.]

For every representation f: $\mathfrak{g} \to \mathrm{gl}(V)$, one can define the **traceform**

$$\mathrm{tr}_f(X,Y) = \mathrm{trace}(f(X)\cdot f(Y)), \qquad (9)$$

which specializes to the Killing form, in the case f=ad.

Exercise-5 Show that for every representation f: $\mathfrak{g} \to \mathrm{gl}(V)$, and every X, Y, Z $\in \mathfrak{g}$:

$$\mathrm{tr}_f([ZX],Y) + \mathrm{tr}_f(X,[ZY]) = 0. \qquad (10)$$

Formulate and prove the analogous of exercises -2, -3 and -4, for tr_f instead of K.

The representation f is called **irreducible**, when there is no (non-trivial i.e. different from {0} and V) subspace W of V, invariant under all f(X) (we say short "$\mathfrak{g}$-invariant subspace W"), for $X \in \mathfrak{g}$. The representation is called **semisimple** or **completely reducible**, when every $\mathfrak{g}$-invariant subspace W, admits a $\mathfrak{g}$-invariant complement. When W is a $\mathfrak{g}$-invariant subspace for the representation f, then one can define in a natural way the **induced representation**

$$\bar{f}: \mathfrak{g} \to \mathrm{gl}(V/W),$$
$$\bar{f}(X)(v+W) = f(X)v+W, \text{ for all } v \in V. \qquad (11)$$

Exercise-6 Show that the representation of the one-dimensional Abelian Lie-algebra C on C^2, given by

$$x \to \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \; ,$$

is not completely reducible. [There is no $\mathfrak{g}$-invariant complement of $\{(0,z), z \in \mathbb{C}\}$.]

Given two representations $f_i : \mathfrak{g} \to \mathrm{gl}(V_i)$, for i=1, 2, the **direct sum** is the representation defined (on decomposable elements) through

$$f_1 \oplus f_2 : \mathfrak{g} \to \mathrm{gl}(V_1 \oplus V_2),$$
$$f_1 \oplus f_2 (X)(v_1 \oplus v_2) = f_1 (X)(v_1) \oplus f_2 (X)(v_2). \qquad (12)$$

The **tensor product** is defined to be the representation defined (on decomposable elements)

$$f_1 \otimes f_2 : \mathfrak{g} \to \mathrm{gl}(V_1 \otimes V_2),$$
$$f_1 \otimes f_2 (X)(v_1 \otimes v_2) = f_1 (X)(v_1) \otimes v_2 + v_1 \otimes f_2 (X)(v_2). \qquad (13)$$

But now I will begin my poem.-'Tis
  Perhaps a little strange, if not quite new,
That from the first of Cantos up to this
  I've not begun what we have to go through.
These first twelve books are merely flourishes,
  Preloudios, trying just a string or two
Upon my lyre, or making the pegs sure;
And when so, you shall have the overture.

Byron, Don Juan, Canto XII, 54

12. sl(n;C)

This is the most important example of "semi-simple" Lie-algebra. It is the Lie-subalgebra of gl(n;C), consisting of matrices A, which have tr(A)=0. Obviously, we have

$$gl(n;C) = sl(n;C) \oplus \langle I \rangle , \tag{1}$$

where <I> is the center of gl(n;C). This is a direct sum of Lie-algebras, which contains the two summands as ideals.

Theorem-1 *sl(n;C) is simple i.e. it has no non-trivial ideals.*

In fact, a basis of sl(n;C) is given by the matrices E_{ij}, for $i \neq j$, defined in §8, together with the diagonal matrices $H_i = E_{ii} - E_{nn}$, for i=1,...,n-1. Notice the condition for diagonal matrices,

$$\sum_{i=1}^{n} \lambda_i E_{ii} \in sl(n;C) \Leftrightarrow \sum_{i=1}^{n} \lambda_i = 0 .$$

From the formulas of §8 we get

$$[E_{ij}\, E_{rs}] = \delta_{jr}\, E_{is} - \delta_{si}\, E_{rj} , \tag{2}$$

which for diagonal matrices M gives

$$[E_{ii}\, E_{rs}] = \delta_{jr}\, E_{is} - \delta_{si}\, E_{ri} ,$$
$$[ME_{rs}] = (\mu_r - \mu_s)E_{rs} . \tag{3}$$

From these formulas we can see, by simple calculations, that if an ideal h of sl(n;C) contains an element

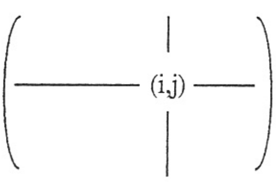
$$x = \sum_{i=1}^{n} \mu_i E_{ii} + \sum_{i \neq j} \lambda_{ij} E_{ij} \neq 0 ,$$

then, it contains all the elements of sl(n;C). In fact, if some $\lambda_{ij} \neq 0$, for $i \neq j$, then

$$[E_{ii}-E_{jj}, x] = \sum_{r \neq s} \lambda_{rs}(\delta_{ir}-\delta_{js})E_{rs} = \sum_{s \neq i} \lambda_{is}E_{is} + \sum_{r \neq j} \lambda_{rj}E_{rj} ,$$

$$x - [E_{ii}-E_{jj}, x] = -\lambda_{ij}E_{ij} + \sum_{r \neq i, s \neq j} \lambda_{rs}E_{rs} ,$$

$$[E_{ii}-E_{jj}, x - [E_{ii}-E_{jj}, x]] = -2\, \lambda_{ij}E_{ij} .$$



Thus, $E_{rs} \in h$. Then, it follows that all E_{rs}, with $r \neq s$, belong to h. In fact, from (2) we have

$[E_{ki}\, E_{ij}] = E_{kj} \in h$, for $k \neq j$, and $[E_{ij}\, E_{jk}] = E_{ik} \in h$, for $k \neq i$. Formaly, this means that if for some pair (i,j) the corresponding $E_{ij} \in h$, then we have also $E_{rs} \in h$, for all pairs (r,s) belonging to the same row or column with (i,j).

This implies also $[E_{ki}\, E_{ik}] = E_{ii}-E_{jj} \in h$.

Exercise-1 Complete the preceding proof and examine what happens when $\lambda_{ij} = 0$, for all $i \neq j$.

We turn now to the computation of the Killing form of gl(n;C) and sl(n;C).

Exercise-2 Show that for a, b > 0, the bilinear form on gl(n;C) defined by

$$t(AB) = a\,tr(AB) + b(trA)(trB), \qquad (4)$$

is symmetric, Ad-invariant and satisfies also (ad-invariance, by "differentiating" the Ad-invariance relation)

$$t([XA],B) + t(A,[XB]) = 0. \qquad (5)$$

Furthermore show that t is non-degenerate (for a,b>0). [Compute $t(AA^*)$, where $A^*=\overline{A}^t$]

Exercise-3 Show that the ideals sl(n;C), <I> of gl(n;C) are orthogonal with respect to t and that the restriction of t on sl(n;C) is a non-degenerate symmetric bilinear form, for which (5) holds.

From these exercises we can conclude a formula for the easy computation of the Killing forms of gl(n;C) and sl(n;C). In fact, sl(n;C) and <I> are also orthogonal with repect to the Killing form K of gl(n;C). Hence decomposing an element X of gl(n;C), we get

$$X = X' + X'' = (X-(1/n)tr(X)I) + (1/n)tr(X)I,$$
$$K(X,Y) = K(X',Y'). \qquad (6)$$

X', Y' are elements of sl(n;C), and in the right side of (6) K is the Killing form of sl(n;C).

Lemma *There is no subspace h of sl(n;C), invariant under all Ad_g, $g \in GL(n;C)$.*

In fact, if there were such a subspace, differentiating $Ad_{\exp tX}(Y)$, for $Y \in GL(n;C)$, we would get ((7) in §9),

$$adX(h) \subset h, \text{ for all } X \in GL(n;C).$$

In other words h would be an ideal of sl(n;C). This proves the lemma.

Now, Schur's lemma implies that the bilinear form K is a constant multiple of t on sl(n;C), $K = c\cdot t$. In order to compute c, we choose special matrices f.e. for $X = \sum x_i E_{ii}$ with $\sum x_i=0$, we have $tr(XX) = \sum x_i^2$. On the other side, computing directly the Killing form,

$$[XE_{rs}] = (x_r - x_s)E_{rs},$$
$$[XH] = 0, \text{ for diagonal } H.$$

Thus,

$$tr(adX\cdot adX) = \sum_{i\neq j}(x_i-x_j)^2.$$

Exercise-4 Prove the following identity between elementary symmetric functions:

$$\sum_{i\neq j,\, i,j=1}^{n} (x_i-x_j)^2 = 2n\sum_{i=1}^{n} x_i^2 - 2\left(\sum_{1}^{n} x_i\right)^2.$$

Conclude that the Killing form of sl(n;C) is given by

$$K(X,Y) = 2n\, tr(XY). \qquad (7)$$

Exercise-5 Prove that the Killing form of sl(2;C) is given by

$$K(X,X) = -8\, detX. \qquad (8)$$

Exercise-6 Show that the Killing form of gl(n;C) is given by

$$K(X,Y) = 2n\, tr(XY) - 2\, tr(X)tr(Y). \qquad (9)$$

We notice the role played by the set Δ of diagonal matrices. Δ is an abelian (n-1)-dimensional subalgebra of sl(n;C). The linear operators adX, for X in Δ, commute with each other and they are completely diagonalizable, the E_{kr} being their common eigenvectors.

Exercise-7 Show that Δ is self-normalizing i.e. $[X\Delta] \subset \Delta \Rightarrow X \in \Delta$. Show that Δ is maximal abelian i.e. there is no other abelian subalgebra $\Delta' \neq \Delta$, containing Δ.

In the following sections we'll see that all these facts generalize for "semi-simple" Lie-algebras i.e. algebras whose Killing form is non degenerate.

Exercise-8 Show that the Killing form of $\mathfrak{g} = sl(n;C)$ is non-degenerate and $[\mathfrak{g}\mathfrak{g}]=\mathfrak{g}$.

My Muses do not care a pinch of rosin
  About what's called success, or not succeeding:
Such thoughts are quite below the strain they have chosen;
  'Tis a "great moral lesson" they are reading.
I thought, at setting off, about two dozen
  Cantos would do; but at Apollo's pleading,
If that my Pegasus should not be foundered,
I think to canter gently through a hundred.

Byron, Don Juan, Canto XII, 55

13. Irreducible representations of sl(2;C)

$\mathrm{sl}(2;C) = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, x, y, z \in C \right\}$, has a basis consisting of the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We compute easily the multiplication table of this basis:

$$[HX_+] = 2X_+, \; [HX_-] = -2X_-, \; [X_+X_-] = H. \qquad (1)$$

Exercise-1 Show that, with respect to this basis, we have the matrix representations:

$$\mathrm{ad}H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathrm{ad}X_+ = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}X_- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Let now f: sl(2;C) $\to$ gl(V) be a representation. We write Xv, instead of $f(X)v$, for all X in sl(2;C) and all v in gl(V). The following lemma is very simple and important.

Lemma *Let the vector v of V be an eigenvector of H (we mean $f(H)$), with respect to the eigenvalue μ. Then, the vectors of V, X_+v and X_-v are either zero or eigenvectors of H, with respect to the eigenvalues $\mu+2$ and $\mu-2$ respectively.*

In fact,

$$Hv = \mu v, \text{ implies}$$
$$H(X_+v) = ([HX_+] + X_+H)v = (2+\mu)X_+v, \text{ and}$$
$$H(X_-v) = ([HX_-] + X_-H)v = (-2+\mu)X_+v.$$

The lemma leads to the complete classification of irreducible representations of sl(2;C) as follows.

We start with an eigenvector of H (exists, since V is complex),

$$Hv = \mu v,$$

and consider the eigenvectors of H:

$$v, \; X_+v, \; X_+^2v, \; X_+^3v, \; X_+^4v, \; \dots .$$

Since these vectors correspond to different eigenvalues, they are independent (the non-zero of them). Thus, since V has finite dimension, there must be some

$$v_0 \neq 0, \text{ with } Hv_0 = av_0 \text{ and } X_+v_0 = 0. \qquad (2)$$

We consider the eigenvectors of H:

$$v_1 = X_-v_0,$$

$$v_2 = X_- v_1 ,$$
$$v_3 = X_- v_2 ,$$
$$\dots\dots\dots\dots\dots$$
$$v_r = X_- v_{r-1} \neq 0, \text{ with } v_{r+1} = X_- v_r = 0. \qquad (3)$$

Such an r exists, since V is finite dimensional. Applying the lemma, we get:

$$Hv_k = (a - 2k)v_k , \text{ for } k = 0, 1, \dots, r . \qquad (4)$$

Defining also

$$v_{-1} = 0,$$

we get inductively

$$X_+ v_k = b_k v_{k-1} , \text{ for certain constants } b_k,\ k = 0, 1, \dots, r . \qquad (5)$$

We compute the constants easily:

$$\begin{aligned} X_+ v_i &= X_+ X_- v_{i-1} = ([X_+ X_-] + X_- X_+) v_{i-1} \\ &= (H + X_- X_+) v_{i-1} \\ &= ((a-2(i-1)) v_{i-1} + b_{i-1} X_- v_{i-2} \\ &= (a-2(i-1)+b_{i-1}) v_{i-1} . \qquad \Rightarrow \end{aligned}$$

$$b_k = (a - 2(k-1) + b_{k-1}), \text{ and } b_0 = 0 . \qquad (6)$$

Exercise-2 Solve the inductive relation (6) and show that, for all positive integers k:

$$b_k = k(a - k+1) . \qquad (7)$$

[Write $b_k = (a+2)-2k+ b_{k-1}$, and $b_0 = 0$.]

For the special case $k = r+1$, $0 = X_+ v_{r+1} = b_{r+1} v_r$, implies $a = r$ (positive integer!). Thus,

$$b_k = k(r - k+1) , \text{ for } k = 0, 1, \dots, r . \qquad (8)$$

and the eigenvalues of H are:

$$r ,\ r-2 ,\ r-4 , \dots ,\ r-2k , \dots ,\ -r. \qquad (9)$$

Thus, the subspace W_{r+1} of V, spanned by $\{v_0, v_1 , \dots, v_r\}$ is sl(2;C)-invariant, and, with respect to this basis of W_{r+1}, we have the matrix representation (we mean $f(H)$, $f(X_+)$, ...):

$$H = \begin{pmatrix} r & 0 & \dots & & 0 \\ 0 & r-2 & 0 & \dots & 0 \\ 0 & 0 & r-4 & 0 \dots & 0 \\ \dots & & & & \\ 0 & \dots & & & -r \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & b_1 & \dots & & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ 0 & 0 & 0 & b_3 \dots & 0 \\ \dots & & & & \\ 0 & \dots & & & b_r \\ 0 & \dots & & & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 & \dots & & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 \dots & 0 \\ \dots & & & & \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} . \qquad (*)$$

When the representation is irreducible, then $W_{r+1} = V$.

Exercise-3 Show that the preceding matrices satisfy (1) and define a representation of sl(2;C). Show for this representation, that one can find a vector v, s.t. $X_+ v = \mu v_0 \neq 0$. Conclude that this representation is irreducible.

Exercise-4 Show that for r=1, the preceding procedure gives the <u>standard</u> representation of sl(2;C), described in the first line of this paragraph. Show also that for r=2, the preceding procedure gives the adjoint representation, described in Ex-1.

From this short investigation, we notice that for every positive integer r, there is an (r+1)-dimensional irreducible representation, for which $f(H)$ has r+1 different eigenvalues, which are given by (9). The dimension of the representation is related to the highest eigenvalue of $f(H)$. Changing the lengths of $\{v_0, v_1, ..., v_r\}$, we can find for $f(X_+)$ and $f(X_-)$ more symmetric matrix-representations ($f(H)$ remains the same, since its eigenvalues don't depend on the basis). In the following exercises we construct such bases, for which $f(X_+)$ and $f(X_-)$ are transpose to each other.

Exercise-5 Change from $\{v_0, v_1, ..., v_r\}$ to $\{\mu_0 v_0, \mu_1 v_1, ..., \mu_r v_r\}$. Show that the corresponding matrices for $f(X_+)$ and $f(X_-)$ are transpose to each other, if and only if,

$$b_i \mu_{i-1}^2 = \mu_i^2, \text{ for } i = 1, ..., r, \qquad (10)$$

and for these values of μ_i the corresponding matrices to $f(X_+)$ and $f(X_-)$ have the form:

$$X_+ = \begin{pmatrix} 0 & c_1 & \cdots & & 0 \\ 0 & 0 & c_2 & \cdots & 0 \\ 0 & 0 & 0 & c_3 \cdots & 0 \\ \vdots & & \cdots & & \\ 0 & \cdots & & & c_r \\ 0 & \cdots & & & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & \cdots & & 0 \\ c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ & \cdots & & \\ 0 & \cdots & c_r & 0 \end{pmatrix},$$

with

$$c_i = \mu_i/\mu_{i-1} = \sqrt{b_i} = \sqrt{i\cdot(r+1-i)}\,.$$

Physicists use often a different notation, writing $r = 2J$, with J integer or half-integer, J=0, 1/2, 1, 3/2, 2, 5/2, ... , and denoting the representation W_{2J+1} by D_J. Using the basis of the previous exercise and the indices $i = J-k$, we get the easy to memorize

$$d_k = c_i = (J(J+1) - k(k+1))^{1/2}, \text{ for } k = J-1, J-2, ..., -J, \qquad (11)$$

for which

$$X_+ w_k = d_k w_{k+1},$$
$$X_- w_k = d_k w_{k-1},$$

where

$$w_k = \mu_{J-i} v_{J-i}\,.$$

Exercise-6 Show that a 1-dimensional representation of a Lie-algebra $\mathfrak{g}$ is a linear form on $\mathfrak{g}$ which vanishes on $[\mathfrak{g}\mathfrak{g}]$. Conclude that a 1-dimensional representation of sl(2;C) vanishes identically [Ex-8, §12].

Our brief discussion gives the proof of the following

Theorem *For every integer $r \geq 0$, there is, up to isomorphism, a unique (r+1)-dimensional representation on a space W_{r+1} with a basis $\{v_0, v_1, ..., v_r\}$, with respect to which, the matrices $f(H)$, $f(X_+)$ and $f(X_-)$ have the form (*). The integer r is the heighest eigenvalue of $f(H)$.*

But I am apt to grow too metaphysical:
"The time is out of joint,"-and so am I;
I quite forget this poem's merely quizzical,
And deviate into matters rather dry.
I ne'er decide what I shall say, and this I call
Much too poetical. Men should know why
They write, and for what end; but, note or text,
I never know the word which will come next.

Byron, Don Juan, Canto IX, 41

14. Solvable Lie-algebras

For every Lie algebra g one can define the **derived series** of ideals

$$g' = [gg],\ g'' = [g'g'],\ g^{(3)} = [g''g''], \dots , g^{(n+1)} = [g^{(n)}g^{(n)}], \dots . \qquad (1)$$

Exercise-1 Show that all $g^{(n)}$ are **characteristic** ideals i.e. they are invariant under endomorphisms and derivations of the Lie algebra g.

Exercise-2 Show that an ideal h' of an ideal h of g, is also an ideal of g.

g is called **solvable**, when $g^{(n)} = \{0\}$ for some n. Notice that $g^{(n+1)} \subset g^{(n)}$, hence, assuming finite dimensions, we'll have, for an arbitrary Lie algebra, either $g^{(n)} = \{0\}$ or $g^{(n)} = [g^{(n)}g^{(n)}]$ for some n. The most important example of solvable Lie algebra is the set of **upper triangular** matrices of gl(n;C)

$$t(n;C) = \{X \in gl(n;C), \text{ with } x_{ij} = 0, \text{ for all } i > j\}. \qquad (2)$$

Exercise-3 Show that $[t(n;C),t(n;C)] = \{X \in gl(n;C), \text{ with } x_{ij} = 0, \text{ for all } i \geq j\}$. Compute the derived series of t(n; C) and show that it is solvable.

In every Lie algebra g there is a maximal solvable ideal r, called the radical of g. A famous **theorem of Malcev** says that there is a complementary semi-simple Lie-subalgebra h in g, s.t. g is a semi direct product of r and h.

Exercise-4 Show that solvable Lie algebras g contain always abelian ideals. [If $g^{(n)} = \{0\}$ with minimal n, then $g^{(n-1)}$ is an abelian ideal of g.]

Exercise-5 Show that g is solvable, if and only if, there is a sequence of ideals

$$g_0 = g \supset g_1 \supset g_2 \supset \dots \supset g_k = \{0\}, \text{ s.t. } g_i/g_{i+1} \text{ is abelian.}$$

Exercise-6 Show that a subalgebra $h \subset g$ of a solvable Lie algebra g, is also solvable. Show that the quotient g/h of a solvable algebra with some ideal h is also solvable.

Exercise-7 Let $0 \to a \to b \to c \to 0$ be an exact sequence of Lie algebra homomorphisms. Show that b is solvable, if and only if a and c are solvable. [$b^{(k)}$ is mapped in $c^{(k)}$, which for big k vanishes, hence $b^{(k)}$ is contained in the image of $a^{(k)}$ etc....]

From the preceding exercise we conclude that the sum of two solvable ideals a, c of a Lie algebra g, is a solvable ideal too. In fact, it suffices to use the exact sequence

$$0 \to a \to a+c \to (a+c)/c \to 0,$$

and the isomorphism $(a+c)/c = a/a \cap c$. We "proved" the

Proposition *In every Lie algebra g there is a maximal solvable ideal r, called the radical of g. r is simply the sum of all solvable ideals of g.*

Exercise-8 Show that the radical of gl(n;C) coincides with its center. Show, more general, hat the center of a Lie-algebra is contained in its radical.

The following theorem, due to **Dynkin**, leads to a conveniant description of solvable subalgebras of gl(n;C).

Theorem *Let g be a solvable subalgebra of gl(V). Then there is a non-zero vector $v \in V$ and a linear form $\mu: g \to C$, with the property (common eigenvector to all $X \in g$)*

$$Xv = \mu(X)v, \text{ for every } X \in g. \tag{3}$$

We use induction on the dimension of g. For dimg=1 it is obvious. Suppose the theorem is true for dimg=k. Let dimg=k+1. There is an ideal $h \subset g$ of codimension 1([gg]≠g and take a hyperplane h containing [gg]). Let now Y be a complementary vector to h, s.t.

$$g = h \oplus <Y>, \text{ as a vector space.} \tag{4}$$

Then, h is also solvable, and, from the induction hypothesis, there is a $\mu: h \to C$, and $v \in V$ s.t.

$$Xv = \mu(X)v, \text{ for every } X \in h. \tag{5}$$

Consider now the vectors of V:

$$v_0 = v,\ v_1 = Yv,\ v_2 = Y^2v, \dots,\ v_p = Y^pv,$$

where p is the minimal integer, s.t. $\{v_0, v_1, \dots, v_p\}$ are independent. The subspace spanned:

$$V' = <v_0, v_1, \dots, v_p>,$$

is, obviously, Y-invariant. We show that it is also h-invariant. More precisely :

$$Xv_q = \mu(X)v_q \bmod <v_0, v_1, \dots, v_{q-1}>, \text{ for every } X \in h. \tag{6}$$

(6) is proved by induction. For q=0, this is the definition of v. Suppose (6) is true for some q. Then $[XY] \in h$, for all $X \in h$ and

$$Xv_{q+1} = XYv_q = ([XY] + YX)v_q =$$
$$= \mu([XY])v_q \bmod <v_0, v_1, \dots, v_{q-1}> + Y(\mu(X)v_q \bmod <v_0, v_1, \dots, v_{q-1}>) =$$
$$= \mu(X)v_{q+1} \bmod (v_0, v_1, \dots, v_q).$$

This proves (6). Thus V' is g-invariant and from the preceding formulas we get

$$tr(X|V') = \mu(X)\dim V' \quad \Rightarrow \quad \mu([XY]) = 0, \text{ for every } X, Y \in g, \tag{*}$$

since $\dim V' \neq 0$.

We improve inductively (6) to the

$$Xv_q = \mu(X)v_q, \text{ for every } X \in h. \tag{7}$$

In fact, $Xv = \mu(X)v$ by assumption. Let $Xv_q = \mu(X)v_q$, for every $X \in h$. Then from (*), (7)

$$Xv_{q+1} = XYv_q = ([XY] + YX)v_q = \mu(X)Yv_q = \mu(X)v_{q+1}.$$

To prove the theorem, take now some eigenvector v of Y|V'. Then, from (7) we'll have

$$Xv = \mu(X)v, \text{ for every } X \in h, \text{ and } Yv = \mu_0 v.$$

Hence the linear function μ^* defined on g by its restrictions on h and <Y> :

$$\mu^*, \text{ with } \mu^*|h = \mu, \text{ and } \mu^*(Y) = \mu_0,$$

satisfies the requirements of the theorem.

Notice that the solvability hypothesis is used in the very beginning of the proof, when we consider an ideal of codimension 1.

Corollary-1 *For every solvable subalgebra $\mathfrak{g} \subset gl(V)$, there is a basis of V, with respect to which all the elements of $\mathfrak{g}$ are represented by upper triangular matrices. In other words the subalgebra is isomorphic to a subalgebra of $t(n; C)$.*

To prove the corollary take $v_1 \in V$, $\mu_1 : \mathfrak{g} \to C$, as in the theorem. In $V/\langle v_1 \rangle$ use the induced representation, which defines a solvable subalgebra $\overline{\mathfrak{g}}$ of $gl(V/\langle v_1 \rangle)$. Take again $\overline{v}_2 \in V/\langle v_1 \rangle$, $\mu_2 : \overline{\mathfrak{g}} \to C$, as in the theorem. Choose $v_2 \in V$, which projects under the canonical projection onto $\overline{v}_2$. Repeat the procedure with the induced representation on $V/\langle v_1, v_2 \rangle$. One gets inductively a basis $\{v_1, v_1, \ldots, v_n\}$ of V, satisfying the requirements. The matrices representing the elements of $\mathfrak{g}$, with respect to this basis, have the form (using somewhat liberally the notation "$\mu_i(X)$"):

$$\begin{pmatrix} \mu_1(X) \ldots\ldots\ldots\ldots & * \\ 0 \;\; \mu_2(X) \ldots\ldots\ldots & * \\ 0 \;\; 0 \;\; \mu_3(X) \ldots & * \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 \;\ldots\ldots\ldots\ldots & \mu_n(X) \end{pmatrix} . \tag{8}$$

Corollary-2 *For every representation $f : \mathfrak{g} \to gl(V)$, of a solvable Lie-algebra $\mathfrak{g}$, there is a basis of V, with respect to which all $f(X)$ take the form (8).*

Corollary-3 *Every irreducible representation of a solvable Lie-algebra is either one-dimensional or trivial (zero).*

Corollary-4 *A complex Lie-algebra $\mathfrak{g}$ is solvable, if and only if there is a sequence of ideals*

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g}, \text{ with } \dim \mathfrak{g}_k = k.$$

The condition holds for solvable Lie-algebras. This is proved by applying the preceding corollaries to the adjoint representation $ad : \mathfrak{g} \to gl(\mathfrak{g})$. For the inverse we notice that $[\mathfrak{g}_k \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}$, which is a consequence of the hypothesis that $\mathfrak{g}_{k-1}$ is of codimension 1 in $\mathfrak{g}_k$.

Corollary-5 *A Lie-algebra $\mathfrak{g}$ is solvable, if and only if there is a basis of $\mathfrak{g}$, with respect to which, all adX, $X \in \mathfrak{g}$, are represented by upper triangular matrices.*

> Die Mathematik ist eine gar herrliche Wissenschaft, aber die Mathematiker taugen oft den Henker nicht. Es ist fast mit der Mathematik, wie mit der Theologie. So wie die der letztern Beflissenen, zumal wenn sie in Ämtern stehen, Anspruch auf einen besondern Kredit von Heiligkeit und eine nähere Verwandschaft mit Gott machen, obgleich sehr viele darunter wahre Taugenichtse sind, so verlangt sehr oft der so genannte Mathematiker für einen tiefen Denker gehalten zu werden, ob es gleich darunter die größten Plunderköpfe gibt, die man nur finden kann, untauglich zu irgendeinem Geschäft, das Nachdenken erfordert, wenn es nicht unmittelbar durch jene leichte Verbindung von Zeichen geschehen kann, die mehr das Werk der Routine, als des Denkens sind.
>
> Lichtenberg, Sudelbücher p. 471

15. Nilpotent Lie-algebras

For every Lie-algebra $\mathfrak{g}$ one can define the **lower central series** of ideals of $\mathfrak{g}$:

$$\mathfrak{g}^1 = \mathfrak{g},\ \mathfrak{g}^2 = [\mathfrak{g}\mathfrak{g}],\ \mathfrak{g}^3 = [\mathfrak{g}\mathfrak{g}^2],\ \dots,\ \mathfrak{g}^{k+1} = [\mathfrak{g}\mathfrak{g}^k]. \qquad (1)$$

Obviously the series, either will have $\mathfrak{g}^{k+1} = \mathfrak{g}^k$, for some k, or will terminate with $\{0\}$. In the later case we say that $\mathfrak{g}$ is **nilpotent**.

Exercise-1 Show that the $\mathfrak{g}^k$, are characteristic ideals of $\mathfrak{g}$ (i.e. invariant under endomorphisms and derivations of $\mathfrak{g}$).

Exercise-2 Show that $\mathfrak{g}^{(k)} \subset \mathfrak{g}^k$, where $\mathfrak{g}^{(k)}$ the ideals of the derived series of §14.

Corollary *Every nilpotent Lie-algebra is also solvable.*

Exercise-3 Show that the Lie-algebra of upper triangular matrices t(n;C) is solvable, but not nilpotent.

Exercise-4 Show that t'(n;C) = [t(n;C),t(n;C)] = {X gl(n;C), with $x_{ij} = 0$, for $i \geq j$} is a nilpotent Lie-algebra. Notice that all elements of t'(n;C) are nilpotent matrices.

We'll see below that the last property is quite characteristic for nilpotent Lie-subalgebras of gl(n;C) i.e. every nilpotent subalgebra of gl(n;C) consists of nilpotent matrices.

Exercise-5 Show that every Lie-subalgebra of a nilpotent Lia-algebra is nilpotent too. Show also that the quotient $\mathfrak{g}/\mathfrak{h}$ of a nilpotent by an ideal is nilpotent too.

Exercise-6 Show that for nilpotent ideals $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{g}$, $\mathfrak{a}+\mathfrak{b}$ is nilpotent ideal too.

The last exercise shows that the sum of all nilpotent ideals of a Lie-algebra $\mathfrak{g}$ is a maximal nilpotent ideal of $\mathfrak{g}$, which we call the **nil radical** of $\mathfrak{g}$. The nil radical of $\mathfrak{g}$ is, obviously, contained in the radical of $\mathfrak{g}$. The example of t(n;C) which has t'(n;C) as nilradical and itself as radical, shows that the two radicals can be different.

The next exercise relates the operator $X \in gl(V)$ with $adX \in gl(gl(V))$.

Exercise-7 Show by induction, that for all X, $Y \in gl(V)$, holds the formula

$$(\mathrm{ad}X)^n Y = \sum_{p=0}^{n} (-1)^{n-p}\binom{n}{p} X^p Y X^{n-p}. \qquad (2)$$

Conclude that if X is a nilpotent operator of V, then adX is a nilpotent operator of gl(V) .

Proposition-1 *Let* $\mathfrak{g}$ *be a Lie-subalgebra of gl(V), which consists of nilpotent operators* ($X^n=0$, $n=\dim V$) *of V. Then, there is a subspace* $W\neq\{0\}$ *of V annihilated by all X in* $\mathfrak{g}$.

We use induction on the dimension n of $\mathfrak{g}$. For $\dim\mathfrak{g}=1$, this is obvious, since all operators are then multiples of a single nilpotent operator. Let the theorem be true for $\dim\mathfrak{g}=k$. Let $\dim\mathfrak{g}=k+1$ and take a maximal subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ which is different from $\mathfrak{g}$. Such subalgebras exist. Start f.e. by a 1-dimensional (abelian) <Z> and consider the maximal dimensional subalgebra containing <Z>.

Under the hypothesis of the proposition, we can prove that $\mathfrak{m}$ is an ideal of codimension 1. In fact, for $X\in\mathfrak{m}$, $\mathrm{ad}X\,\mathfrak{m}\subset\mathfrak{m}$, hence one has the induced operator on the quotient

$\overline{\mathrm{ad}X}: \mathfrak{g}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{m}$, considering $\mathfrak{g}/\mathfrak{m}$ as a vector space.

According to Ex-7, adX is nilpotent and the same will be true for $\overline{\mathrm{ad}X}$. Hence, by the inductive hypothesis, there is a non-zero vector $\overline{U}_0$ of $\mathfrak{g}/\mathfrak{m}$, annihilated by all $\overline{\mathrm{ad}X}$, $X\in\mathfrak{m}$. This means that $[\mathfrak{m}, U_0]\subset\mathfrak{m}$, hence $\mathfrak{m}\oplus\langle U_0\rangle$, is a subalgebra and by maximality $\mathfrak{m}\oplus\langle U_0\rangle = \mathfrak{g}$. This completes the proof that $\mathrm{codim}\,\mathfrak{m}=1$. To prove the proposition, consider the subspace W of V, which is annihilated by all $X\in\mathfrak{m}$ (inductive hypothesis). For all $X\in\mathfrak{m}$, we have

$$X(U_0W) = (U_0X + [XU_0])W = 0 \quad\Rightarrow\quad U_0W \subset W.$$

Then $U_0|W$ is a nilpotent operator, hence there is some non-zero vector v in W, with $U_0(v)=0$. Then Xv=0, for every X in $\mathfrak{g}$. This proves the proposition.

Exercise-8 Let f: $\mathfrak{g} \to$ gl(V) be a representation, s.t. f(X) is nilpotent operator of V, for every X in $\mathfrak{g}$. Show that there exist a subspace $W \neq \{0\}$ annihilated by all f(X).

Proposition-2 *For every Lie-subalgebra* $\mathfrak{g}\subset gl(V)$ *consisting of nilpotent operators of V, there is some basis of V, with respect to which, all ellements of* $\mathfrak{g}$ *are represented by upper triangular matrices (which are nilpotent, hence elements of t'(n;C)).*

This is obvious. In fact, take v_1 in V s.t. $Xv_1 = 0$, for all X in $\mathfrak{g}$. Take then v_2 in $V/\langle v_1\rangle$ s.t. $Xv_2 = 0$, for all X in $\mathfrak{g}$. Take v_3 in $V/\langle v_1,v_2\rangle$ s.t. $Xv_3 = 0$, etc (the somewhat liberal use of of the same symbol, for operators and vectors in V and quotient spaces of V, can be made easily precise). In this way we construct a basis of V, with the desired properties. Using such a basis, we see that $\mathfrak{g}$ is isomorphic to a subalgebra of t'(n;C), so we get the proofs of the theorems :

Theorem-1 *Every Lie-subalgebra of gl(V) consisting of nilpotent operators of V, is nilpotent.*

Theorem-2 (Engel) *Let* $\mathfrak{g}$ *be a Lie-algebra, for which all operators adX are nilpotent, then* $\mathfrak{g}$ *is nilpotent.*

For the proof of The-2 consider the exact sequence

$$0 \to \mathfrak{c} \to \mathfrak{g} \to \mathrm{ad}\mathfrak{g} \to 0, \text{ where } \mathrm{ad}\mathfrak{g} \subset gl(\mathfrak{g}),\ \mathfrak{c} \text{ the center of } \mathfrak{g}.$$

By The-1, ad$\mathfrak{g}$ is a nilpotent subalgebra of gl($\mathfrak{g}$), hence $\mathfrak{g}^k \subset \mathfrak{c}$, for some k. Then $\mathfrak{g}^{k+1} = 0$.

Exercise-9 Show that the Lie-subalgebras of gl(n;C) consisting of matrices of the form :

$$\begin{pmatrix} \boxed{\begin{matrix}\mu_1 & & * \\ & \diagdown & \\ 0 & & \mu_1\end{matrix}} & & & \\ & \boxed{\begin{matrix}\mu_2 & & * \\ & \diagdown & \\ 0 & & \mu_2\end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix}\mu_k & & * \\ & \diagdown & \\ 0 & & \mu_k\end{matrix}} \end{pmatrix},$$

are nilpotent, but they do not consist of nilpotent matrices only.

The following theorem shows that the subalgebras of this form are, essentially, all nilpotent subalgebras of gl(n;C). We need an identity analogous to (2), expressing X^nY in terms of adX. We use induction for X, Y in gl(n;C):

$$\begin{aligned} XY &= \mathrm{ad}X(Y) + YX, \\ X^2Y &= (\mathrm{ad}X)^2(Y) + 2\,\mathrm{ad}X(Y)X + YX^2, \\ X^3Y &= X((\mathrm{ad}X)^2(Y) + 2\,\mathrm{ad}X(Y)X + YX^2) = \\ &= (\mathrm{ad}X)^2(Y) + (\mathrm{ad}X)^2(Y)X \\ &\quad + 2((\mathrm{ad}X)^3(Y) + (\mathrm{ad}X)^2(Y)X) \\ &\quad + (\mathrm{ad}XY)X^2 + YX^3 = \\ &= (\mathrm{ad}X)^3(Y) + 3(\mathrm{ad}X)^2(Y)X + 3(\mathrm{ad}XY)X^2 + YX^3 . \end{aligned}$$

Exercise-10 Show that for every X, Y $\in$ gl(n: C), holds the identity

$$X^nY = \sum_{k=0}^{n} \binom{n}{k} ((\mathrm{ad}X)^{n-k}Y)\cdot X^k, \quad \text{with } (\mathrm{ad}X)^0Y = Y. \qquad (3)$$

Proposition-3 *Let $\mathfrak{g} \subset gl(V)$ be a nilpotent Lie-subalgebra and fix an X in $\mathfrak{g}$. Let V_μ be the generalized eigenspace of X, with respect to some eigenvalue μ of X i.e.*

$$V_\mu = \{v \in V \mid (X-\mu I)^p v = 0, \text{ for some } p\}.$$

Then V_μ is a $\mathfrak{g}$-invariant subspace of V.

In fact, $\mathfrak{g}$ nilpotent $\Rightarrow$ $\mathfrak{g}$+<I> nilpotent $\Rightarrow$ $(\mathrm{ad}(Z-xI))^k = 0$, for sufficiently big k. Then, for arbitrary $Y \in \mathfrak{g}$ and $v \in V_\mu$, we have, using (3),

$$(X-\mu I)^n Yv = \left(\sum_{k=0}^{n} \binom{n}{k} ((\mathrm{ad}(X-\mu I))^{n-k} Y)\cdot(X-\mu I)^k \right) v,$$

and for suffciently big n, k either $(\mathrm{ad}(X-\mu I))^{n-k} = 0$, or $(X-\mu I)^k v = 0$, hence $Yv \in V_\mu$. q.e.d.

Fix now $X \in \mathfrak{g}$ as in the proposition. Consider the direct sum decomposition of V into the generalized eigenspaces of X.

$$V = \oplus_\mu V_\mu .$$

Each V_μ is $\mathfrak{g}$-invariant subspace of V, hence fixing some V_μ and taking an $X' \in \mathfrak{g}$, linearly independent from X, we can decompose V_μ in the eigenspaces of X'

$$V_\mu = \oplus V_{\mu,\mu'}.$$

Continuing in this way with some X'', X''' etc. we obtain (by finiteness of dimension) a decomposition of V into $\mathfrak{g}$-invariant subspaces V_i , s.t. each $X \in \mathfrak{g}$ has exactly one generalized eigenvalue $\mu_i(X)$ on V_i:

$$(X-\mu_i(X)I)^k v = 0, \quad \text{for sufficiently big k and all v in } V_i .$$

From the Jordan-form of X in V_i, we see that

$$tr(X|V_i) = \dim(V_i)\mu_i(X).$$

Thus $\mu_i(X)$ is linear in X and vanishes on $[\mathfrak{g}\mathfrak{g}]$. Thus it defines a 1-dimensional representation of $\mathfrak{g}$ into C. Besides, each $(X-\mu_i(X)I)|V_i$ is a nilpotent operator, hence, according to Prop-2, there is a basis in V_i s.t. $(X-\mu_i(X)I)|V_i$ is represented by nilpotent upper triangular matrices and consequently , with respect to this basis :

$$X|V_i = \begin{pmatrix} \mu_i(X) & & * \\ & \ddots & \\ 0 & & \mu_i(X) \end{pmatrix}.$$

We proved the

Theorem-3 *For each nilpotent Lie-subalgebra $\mathfrak{g}$ of gl(V) there is a basis of V and 1-dimensional representations of $\mathfrak{g}$, μ_i: $\mathfrak{g} \to C$, $i = 1, 2, \ldots , k$, such that every X in $\mathfrak{g}$ is represented, with respect to this basis, by an upper triangular matrix of the form:*

$$X = \begin{pmatrix} \boxed{\begin{matrix} \mu_1(X) & & * \\ & \ddots & \\ 0 & & \mu_1(X) \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \mu_k(X) & & * \\ & \ddots & \\ 0 & & \mu_k(X) \end{matrix}} \end{pmatrix}.$$

Puisque'on ne peut être universel en sachant tout ce qui se peut savoir sur tout, il faut savoir peu (*peu*, c'est-à-dire un peu, et non trop peu.) de tout. Car il est bien plus beau de savoir quelque chose de tout que de savoir tout d'une chose; cette universalité est la plus belle. Si on pouvait avoir les deux, encore mieux, mais s'il faut choisir, il faut choisir celle-là, et le monde le sent et le fait, car le monde est un bon juge souvent.

Pascal, Pensées, 37

16. Cartan's first criterion

This is a criterion to judge wether a given Lie-algebra is solvable. The method is suggested by the solvable Lie-subalgebras of gl(V), wich are represented by upper triangular matrices (§14). Then, the elements of $[\mathfrak{gg}]$ are represented by proper upper triangular matrices, hence, for X,Y in $[\mathfrak{gg}]$, we have tr(XY) = 0, which by (7) §12 gives for the Killing form K(X,Y)=0. Cartan's first criterion says that K(X,Y)=0, for all X, Y in $[\mathfrak{gg}]$, is a necessary and sufficient condition, for $\mathfrak{g}$ to be solvable. For the proof we need some basic facts about the (Jordan-Chevalley) splitting of an operator in semi-simple and nilpotent parts.

Lemma-1 *Let S, $N \in gl(V)$ be respectively a semi-simple and a nilpotent operator, then adS and adN are respectively semi-simple and nilpotent.*

The statement for the nilpotent part is identical with Ex-7, §15. For the semisimple part consider a basis $\{v_1, \dots, v_n\}$ of eigenvectors of S (since we work in C, semi-simple = diagonalizable). Suppose $Sv_i = \mu_i v_i$ and define the operators E_{ij}, through

$$E_{ij} v_k = \delta_{jk} v_i . \tag{1}$$

The E_{ij} build a basis of gl(V) and we have

$$[SE_{ij}]v = SE_{ij} v_k - E_{ij} Sv_k = \delta_{jk} \mu_i v_i - E_{ij} (\mu_k v_k)$$
$$= (\mu_i - \mu_j) E_{ij} v_i .$$

Thus,

$$adS \cdot E_{ij} = (\mu_i - \mu_j) E_{ij} ,$$

which means that E_{ij} are eigenvectors of adS. q.e.d.

Lemma-2 *If X = S + N is the Jordan-Chevalley decomposition of $X \in gl(V)$ in semi-simple and nilpotent part, then adX = adS + adN is the Jordan-Chevalley decomposition of adX.*

The Jordan-Chevalley decomposition X = S + N, in a semi-simple S and a nilpotent operator N, with [SN] = SN-NS = 0, is unique. By Lemma-1, adS, adN are respectively semi-simple and nilpotent. Besides [adS, adN] = ad[SN] = 0. Thus, by uniqueness of the decomposition, adX = adS + adN is the J-C-decomposition of adX.

Lemma-3 *If $\mathfrak{g} \subset gl(V)$ is a Lie-subalgebra, for which tr(XY) = 0, for all X, Y in $\mathfrak{g}$, then the derived subalgebra $\mathfrak{g}' = [\mathfrak{gg}]$ is nilpotent.*

We start with the Jordan-Chevalley-decomposition of some $X \in gl(V)$. It is well-known that the semi-simple and nilpotent parts may be expressed by polynomials in X:

$$S = p(X), N = q(X).$$

Then, the complex conjugate S^* of S, which is defined in the basis of eigenvectors of S, is a polynomial in S (take the interpolation-polynomial, which maps the eigenvalues μ_i of S onto $\bar{\mu}_i$) hence also in X. Thus S^* commutes with X, N and we have

$$tr(S^*X) = tr(S^*S) = \sum \mu_i\bar{\mu}_i,$$

since, by commutativity, $(S^*N)^k = S^{*k}N^k = 0$, for large k. Similarly adS^* is a polynomial with respect to adX, hence $[S^*\mathfrak{g}]\subset\mathfrak{g}$. We consider now an $X\in[\mathfrak{gg}]$, which will be of the form

$$X = \sum [A_rB_r].$$

For each summand we'll have

$$\begin{aligned} tr(S^*[AB]) &= tr(S^*AB-S^*BA) \\ &= tr(S^*AB-AS^*B-[S^*B,A]) \\ &= tr(S^*AB-AS^*B) \\ &= tr([S^*A]B) = 0, \end{aligned}$$

since $[S^*A] \in \mathfrak{g}$ and by hypothesis $tr(XY)=0$, for all X, Y in $\mathfrak{g}$. Thus, for $X\in[\mathfrak{gg}]$ we'll have

$$\sum \mu_i\bar{\mu}_i = tr(S^*S) = tr(S^*(S+N)) = tr(S^*X) = tr(S^*\sum[A_rB_r]) = 0,$$

hence X will be nilpotent and, by The-1,§15, $[\mathfrak{gg}]$ will be nilpotent. q.e.d.

Lemma-4 *The Lie-algebra $\mathfrak{g}$ is solvable, if and only if $[\mathfrak{gg}]$ is nilpotent.*

In fact, if $\mathfrak{g}$ is solvable, then there is a basis of $\mathfrak{g}$, with respect to which all adX are upper triangular. Thus ad[XY]=[adX, adY] are proper upper triangular, hence nilpotent. Thus $[\mathfrak{gg}]$ is nilpotent. Inversely, if $[\mathfrak{gg}]$ is nilpotent, then it is also solvable, hence the definition of solvability is satisfied for $\mathfrak{g}$ too.

Theorem (Cartan's first criterion) *The Lie-algebra $\mathfrak{g}$ is solvable, if and only if its Killing form is identically zero on $[\mathfrak{gg}]$.*

In fact, $\mathfrak{g}$ solvable, implies the existence of a basis in $\mathfrak{g}$, s.t. all adX are represented by upper triangular matrices. Thus ad[XY]=[adX, adY] are proper upper triangular, hence nilpotent and consequently $tr(adZ\cdot adW) = 0$, for $Z, W\in[\mathfrak{gg}]$.

For the inverse, we apply lemma-3 on $\mathfrak{q} = ad([\mathfrak{gg}])\subset gl(\mathfrak{g})$. The hypothesis implies that $[\mathfrak{qq}]$ is nilpotent, hence $\mathfrak{q}$ is solvable. Now consider the exact sequence

$$0 \to \mathfrak{c}([\mathfrak{gg}]) \to [\mathfrak{gg}] \to \mathfrak{q} = ad([\mathfrak{gg}]) \to 0.$$

Applying Ex-7, §14, we see that $[\mathfrak{gg}]$ is solvable, hence $\mathfrak{g}$ is solvable too.

Le quelque chose qui est là et qui me parle, me dit: Rameau, tu voudrais bien avoir fait ces deux morceaux-là; si tu avais fait ces deux morceaux-là, tu en ferais bien deux autres; et quand tu en aurais fait un certain nombre, on te jouerait, on te chanterait partout; quand tu marcherais, tu aurais la tête droite; la conscience te rendrait témoignage à toi-même de ton propre mérite; les autres te désigneraient de doigt. On dirait: C'est lui qui a fait les jolies gavottes (et il chantait les gavottes; ...)

Diderot, Le Neveu de Rameau, p. 34

17. Semi-simplicity, Cartan's second criterion

Semi-simple is called a Lie-algebra $\mathfrak{g}$, when it contains no solvable ideals; consequently its radical is $\{0\}$.

Exercise-1 Show that the quotient $\mathfrak{g}/\mathfrak{r}$ of a Lie-algebra by its radical $\mathfrak{r}$, is a semi-simple Lie-algebra. [Use Ex-7, §14, taking a solvable ideal in $\mathfrak{g}/\mathfrak{r}$.]

Exercise-2 Show that $\mathfrak{g}$ is semi-simple, if and only if it contains no abelian ideals. [Ex-4, §14]

Exercise-3 Show that the radical of gl(n;C) is its center <I>. Show further that the quotient gl(n;C)/<I> is isomorphic with sl(n;C), through the isomorphism of Lie-algebras

$$f(X) = X - (1/n)(\mathrm{tr}X)I. \qquad (1)$$

Cartan's second criterion is a criterion of solvability and its proof relies on his first criterion.

Theorem (Cartan's second criterion) *A Lie-algebra $\mathfrak{g}$ is semi-simple, if and only if its Killing form K is non-degenerate.*

In fact, if $\mathfrak{g}$ is not semi-simple, then it contains some abelian ideal $\mathfrak{a}$. Then for $X \in \mathfrak{g}$ and $A \in \mathfrak{a}$, we'll have

$$\mathrm{ad}A \cdot \mathrm{ad}X \cdot \mathrm{ad}A = 0.$$

Hence $\mathrm{ad}A \cdot \mathrm{ad}X$ is nilpotent (of order 2) and consequently $K(A,X) = 0$; which shows that K is degenerate. Inversely, if K is degenerate, then

$$\mathfrak{g}^{\perp} = \{X \in \mathfrak{g},\ K(X,Y)=0, \text{ for every } Y \in \mathfrak{g}\},$$

is easily seen to be an ideal of $\mathfrak{g}$ (the null-space of K) and $K|\mathfrak{g}^{\perp} = 0$. By Cartan's first criterion, $\mathfrak{g}^{\perp}$ is solvable, hence $\mathfrak{g}$ is not semi-simple. q.e.d.

A semi-simple Lie-algebra which contains no ideals is called **simple.**

Exercise-4 Show that $\mathfrak{g}$ is simple $\Leftrightarrow$ $\mathfrak{g}$ has no non-trivial ideals and its dimension is >1.

Exercise-5 Show that for every ideal $\mathfrak{h} \subset \mathfrak{g}$ of a semisimple Lie-algebra $\mathfrak{g}$, the restriction of the Killing form of $\mathfrak{g}$ on $\mathfrak{h}$ is the Killing form of $\mathfrak{h}$ and is non-degenerate there. Conclude (Ex-8, §4) that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp},$$

where $\mathfrak{h}^{\perp}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, with respect to the Killing form K.

Exercise-6 Show that a Lie-algebra is semi-simple, if and only if it is the direct sum of simple ideals.

Exercise-7 Show that for semi-simple Lie-algebras $\mathfrak{g}$, $[\mathfrak{g}\mathfrak{g}] = \mathfrak{g}$.

[$[\mathfrak{g}\mathfrak{g}]^{\perp}$ is an abelian ideal]

Exercise-8 Show that if $\mathfrak{h}\subset\mathfrak{g}$ is an ideal s.t. the restriction $K|\mathfrak{h}$ is non-degenerate, then (Ex-5) $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{h}^{\perp}$ and $\mathfrak{h}$ is a semi-simple Lie-algebra.

Using the last exercise, we can study the derivations of a semi-simple Lie-algebra $\mathfrak{g}$. Remember that a derivation is a linear endomorphism $D: \mathfrak{g} \to \mathfrak{g}$ with the property $D[XY]=[DX,Y]+[X,DY]$. The set of all derivations $Der(\mathfrak{g})$ of a Lie-algebra is a Lie-subalgebra of $gl(\mathfrak{g})$. The endomorphisms $adX : \mathfrak{g} \to \mathfrak{g}$, are easily seen to be derivations called inner. The kernel of ad is the center of $\mathfrak{g}$. Hence, when $\mathfrak{g}$ is semi-simple, then the center is $\{0\}$ and ad is an isomorphism. We can easily see that $ad(\mathfrak{g})$ is an ideal of $Der(\mathfrak{g})$:

$$[adX,D]Y = adX\cdot DY - D[XY] = [X,DY]-([DX,Y]+[X,DY])$$
$$= -[DX,Y] = -\,ad(DX)\cdot Y.$$

Hence $ad(\mathfrak{g})$ is a semi-simple ideal of $Der(\mathfrak{g})$, consequently there will be an orthogonal complement $\mathfrak{h}$ of $ad(\mathfrak{g})$, with respect to the Killing form of $Der(\mathfrak{g})$:

$$Der(\mathfrak{g}) = ad(\mathfrak{g})\oplus\mathfrak{h}.$$

Then, for an $X\in\mathfrak{g}$, $D\in\mathfrak{h}$ we'll have

$$[adX,D] = -\,ad(DX) \in ad(\mathfrak{g})\cap\mathfrak{h} = \{0\}.$$

Hence $DX=0$, for every $X\in\mathfrak{g}$, hence $D= 0$ and consequently $\mathfrak{h} =\{0\}$. We proved the

Theorem-2 *All derivations of a semi-simple Lie-algebra are inner.*

Exercise-9 Let $\mathfrak{g}= \oplus_i\mathfrak{g}_i$ be the decomposition of a semi-simple Lie-algebra in simple ideals. Assume further that $f: \mathfrak{g} \to gl(V)$ is a representation of Lie-algebras with corresponding trace-form $t_f(X,Y)=tr(f(X)\cdot f(Y))$. The relation $t_f(X,Y)=K(A_f(X),Y)$ defines linear operator $A_f: \mathfrak{g} \to \mathfrak{g}$. Show that $A_f\cdot adX = adX\cdot A_f$, for every $X\in\mathfrak{g}$. Prove then, that $A_f(\mathfrak{g}_i)\subset\mathfrak{g}_i$ for every simple ideal $\mathfrak{g}_i$. Conclude that $A_f = \sum \mu_i\,(Id|\mathfrak{g}_i)$.

Exercise-10 Show that there is a non semi-simple Lie-algebra $\mathfrak{g}$, s.t. $[\mathfrak{g}\mathfrak{g}]=\mathfrak{g}$.

[Take the semi-direct-product $sl(2;C)\times_f C^2$, where f is the standard representation fo $sl(2;C)$ and the product satisfies $[Ax] = -[xA] = Ax$. Verify that C^2 is an abelian ideal and $[\mathfrak{g}\mathfrak{g}]=\mathfrak{g}$]

The heart is like the sky, a part of heaven,
But changes night and day too, like the sky;
Now o'er it clouds and thunder must be driven,
And darkness and destruction as on high:
But when it hath been scorch'd, and pierced, and riven,
Its storms expire in water-drops; the eye
Pours forth at last the heart's-blood turn'd to tears,
Which make the English climate of our years.

Byron, Don Juan, Canto II, 214

18. The Casimir element

Let $f\colon \mathfrak{g} \to gl(V)$ be a faithful ($\ker f=\{0\}$) representation and $t_f(X,Y)=tr(f(X)\circ f(Y))$ be non-degenerate (the trace form of f). For each basis of $\mathfrak{g}$, $\{e_1, \dots, e_n\}$ one can define the <u>dual basis</u> by the equations $\{e'_1, \dots, e'_n\}$

$$t_f(e_i,e'_j) = \delta_{ij}. \tag{1}$$

The **Casimir element** of the representation f is the element of gl(V), defined by

$$C = \sum f(e_i)f(e'_i). \tag{2}$$

Exercise-1 Let $f = e\cdot g$ (i.e. $f_j = e_i\cdot g_{ij}$) be a change of basis, by the invertible matrix g. Show that the corresponding dual bases are related by

$$f' = e'\cdot(g^t)^{-1}. \tag{3}$$

Conclude that the definition of C is independent of the basis.

Exercise-2 Consider the bases of $\mathfrak{g}$, $e = \{e_1, \dots, e_n\}$, $e' = \{e'_1, \dots, e'_n\}$, as before. Let adX have matrix representations with respect to these bases, correspondingly $A= (a_{ij})$, $A' = (a'_{ij})$:

$$[Xe_i] = a_{ji}\cdot e_j,\quad [Xe'_i] = a'_{ji}\cdot e'_j \quad \text{(summation)}.$$

Show that $a_{ji} + a'_{ij} = 0$. $[t_f([Xe_i],e'_j)+t_f(e_i,[Xe'_j]) = 0$, Ex-5, §11]

Proposition-1 *The Casimir element C commutes with every endomorphism f(X).*

$$\begin{aligned}\text{In fact, } [f(X),C] &= \sum [f(X),f(e_i)f(e'_i)] = \sum [f(X)\cdot f(e_i)\cdot f(e'_i) - f(e_i)\cdot f(e'_i)\cdot f(X)]\\ &= \sum ([f(X),f(e_i)]\cdot f(e'_i) + f(e_i)\cdot[f(X),f(e'_i)])\\ &= \sum (f([X,e_i])\cdot f(e'_i) + f(e_i)\cdot f([X,e'_i]))\\ &= \sum a_{ji}f(e_j)\cdot f(e'_i) + f(e_i)a'_{ji}f(e'_j) = 0,\end{aligned}$$

according to the preceding exercise.

Proposition-2 *trace(C) = n = dim* $\mathfrak{g}$.

In fact, $trace(C) = tr(\sum f(e_i)f(e'_i)) = \sum t_f(e_i,e'_i) = n$.

The proposition shows that C is never zero. The kernel of C is $\mathfrak{g}$-invariant. In fact, $v \in \ker C$ implies $C\cdot f(X)v = f(X)\cdot Cv = 0$. Thus, we proved

Proposition-3 *When the representation $f\colon \mathfrak{g} \to gl(V)$ is irreducible, then C is an automorphism of V.*

More is true. When f is irreducible, then $C = \mu I$. In fact, take an eigenspace of C. By $\mathfrak{g}$-invariance, this must be the whole V, thus $trace(C)=\mu \dim V$. Thus, combining with Pro-2:

Proposition-4 *For the Casimir element C of an irreducible representation f: $\mathfrak{g} \to gl(V)$ we have $C = \mu I$, with $\mu = \dim\mathfrak{g}/\dim V$.*

Exercise-3 Let f: $\mathfrak{g} \to \mathrm{gl}(V)$ be a faithful representation of a semi-simple Lie-algebra. Let also $\mathfrak{g} = \oplus\mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ in simple ideals. Show that the corresponding trace form of f, satisfies $t_f = \sum_i \mu_i K_i$, where K_i the Killing forms of the $\mathfrak{g}_i$. [Ex-9, §17]

Exercise-4 With the same assumptions as in Ex-3, prove that the Casimir operator C of f, and the Casimir operators C_i of $f|\mathfrak{g}_i$, satisfy the relation $C = \sum_i C_i$.

Wenn man die richtigen Schuhe hat, so vergißt man seine Füße; wenn man den richtigen Gürtel hat, vergißt man die Hüften. Wenn man in seiner Erkenntnis alles Für und Wider vergißt, dann hat man das richtige Herz; wenn man in seinem Innern nicht mehr schwankt und sich nicht nach andern richtet, dann hat man die Fähigkeit, richtig mit den Dingen umzugehen. Wenn man erst einmal so weit ist, daß man das Richtige trifft und niemals das Richtige verfehlt, dann hat man das richtige Vergessen dessen, was richtig ist.

Dschuang Dsi, p. 205

19. Complete reducibility and semi simplicity

We will prove here that every representation f: $\mathfrak{g} \to gl(V)$ of a semi-simple Lie-algebra is **completely reducible** i.e. for every $\mathfrak{g}$-invariant subspace W of V, there is a $\mathfrak{g}$-invariant complement W'. We do this in three steps:

a) When W is a $\mathfrak{g}$-invariant and irreducible hyperplane of V (lemma-1).

b) When W is a $\mathfrak{g}$-invariant hyperplane of V (not irreducible, lemma-2).

c) The general case, for arbitrary $\mathfrak{g}$-invariant subspace W (theorem-1).

f can be assumed to be faithful. Otherwise kern(f) is an ideal of $\mathfrak{g}$, and f restricted on the orthogonal complement $\mathfrak{h}$ of kern(f), with respect to the Killing form, is faithful. The proof of a), b), c) is divided in three lemmata.

Lemma-1 *Let f: $\mathfrak{g} \to gl(V)$ be a faithful representation of a semisimple Lie-algebra and $W \subset V$ a $\mathfrak{g}$-invariant and irreducible hyperplane of V. Then W has a $\mathfrak{g}$-invariant complement.*

Notice first that the trace form is non-degenerate, since in the contrary $f(\mathfrak{g})$ would be solvable (Lemmata-3, 4, §16). The one dimensional complement, we are seeking for, is the kernel of the Casimir element C of the representation. From Pro-4, §18, we see that the intersection $W \cap \mathrm{kern}(C) = \{0\}$. $\dim(\mathrm{kern}(C)) = 1$ follows from the triviallity of the induced representation $\bar{f}$: $\mathfrak{g} \to gl(V/W)$ (see Ex-1), hence $f(X)V \subset W$ and $C(V) \subset W$, which by the identity $\dim(\mathrm{Im}C) + \dim(\mathrm{kern}C) = \dim V$ complete the proof of the lemma.

Exercise-1 For each representation f: $\mathfrak{g} \to gl(V)$ of a semi-simple Lie-algebra show that $\mathrm{tr} f(X)=0$, for all X in $\mathfrak{g}$. In particular, wenn $\dim V=1$, then f is the trivial (zero) representation. [$\mathfrak{g}=[\mathfrak{g}\mathfrak{g}]$, for semi-simple $\mathfrak{g}$]

Lemma-2 *Let f: $\mathfrak{g} \to gl(V)$ be a faithful representation of a semisimple Lie-algebra and $W \subset V$ a $\mathfrak{g}$-invariant hyperplane of V. Then W has a $\mathfrak{g}$-invariant complement.*

We use induction on dimW. For dimW=1, f is the trivial representation and the induced on V/W is also trivial, thus $X(V) \subset W$, for all X in $\mathfrak{g}$. Thus $X^2 = 0$, and $f(\mathfrak{g})$ is a nilpotent subalgebra of gl(V), hence $f(\mathfrak{g}) = \{0\}$ and every complement of W does the work.

Assume now that the lemma holds for dimW<k and that dimW = k. If W is irreducible subspace, we apply lemma-1, if it isn't then there is some proper $\mathfrak{g}$-invariant subspace $W' \subset W$ and in the induced representation on V/W' we have the $\mathfrak{g}$-invariant hyperplane W/W', with dim(W/W')<k. By the induction hypothesis, there will exist some $\mathfrak{g}$-invariant

complement W''/W' s.t. $V/W' = W/W' \oplus W''/W'$. In the induced representation on W'', W' is a $\mathfrak{g}$-invariant subspace and $\dim W'' < k$, hence by induction, there will exist a $\mathfrak{g}$-invariant 1-dimensional complement of W' in W'', say U, $W'' = W' \oplus U$. It follows immediately that $V = W \oplus U$, since $W \cap U = \{0\}$. Thus U is the complement we are seeking for. q.e.d.

Theorem-1 *Let $f: \mathfrak{g} \to gl(V)$ be a faithful representation of a semisimple Lie-algebra and $W \subset V$ a $\mathfrak{g}$-invariant subspace of V. Then W has a $\mathfrak{g}$-invariant complement.*

Let $W \subset V$ be $\mathfrak{g}$-invariant subspace of V. Using W, we define another representation on the vector space

$$V' = \{A \in End(V),\ A(V) \subset W \text{ and } A|W = \mu Id|W, \text{ with } \mu \in C\},$$

through $X' = \mathrm{ad}f(X)$. For $A \in V'$, $w \in W$, we have

$$X'(A) = [f(X),A](w) = f(X)Aw - Af(X)w = \mu f(X)w - \mu f(X)w = 0,$$

by the $\mathfrak{g}$-invariance of W. We deduce that

$$X'(V') \subset W' = \{A \in End(V),\ A(V) \subset W \text{ and } A|W = 0\} \subset V'.$$

Obviously $\dim(V'/W') = 1$. Thus, using X' we have a representation $f' : \mathfrak{g} \to gl(V')$ and W' is a $\mathfrak{g}$-invariant hyperplane of V'. By lemma-2, there is a $\mathfrak{g}$-invariant complement W" of W', which is 1-dimensional, hence of the form $\langle F \rangle \subset V'$. Multiplying by a constant, we can assume that $F|W = Id|W$. From this follows that $V = W \oplus \mathrm{kern}F$. $\langle F \rangle$ is $\mathfrak{g}$-invariant, thus

$$0 = [f(X),F] = f(X)\cdot F - F\cdot f(X),$$

which implies that the kernF is a $\mathfrak{g}$-invariant complement of W. q.e.d.

From this theorem we can deduce a characterization of semi simple Lie algebras.

Theorem-2 (Weyl) *A Lie algebra is semi simple, if and only if every representation of it is completely reducible.*

When $\mathfrak{g}$ is semi simple the result follows from the preceding theorem. Inversely, if every representation is completely reducible, then the same will be true for ad: $\mathfrak{g} \to gl(\mathfrak{g})$. If $\mathfrak{g}$ is not semi simple there will exist an abelian ideal $\mathfrak{h}$ (Ex-4, §14). By assumption, there will exist also a complementary ideal $\mathfrak{h}'$ and the first natural projection

$$p: \mathfrak{g} \to \mathfrak{h},$$

will be a representation of $\mathfrak{g}$ onto the abelian $\mathfrak{h} = \langle e_1, \dots, e_k \rangle$. Then the projection onto the i-th coordinate and the composition with the representation

$$C \ni x \to \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix},$$

will define a representation of $\mathfrak{g}$ in gl(2;C). In this representation $\{0\} \times C \subset C^2$ is a $\mathfrak{g}$-invariant subspace (in a trivial way) but it has no $\mathfrak{g}$-invariant complement. In fact, if this were true, then such a complement should be of the form $W = \langle ae_1 + be_2 \rangle$, with $a \neq 0$ and $e_1 = (1,0)$, $e_2 = (0,1)$. But then, we should have $XW = \langle ae_2 \rangle \neq W$. This contradiction shows that $\mathfrak{g}$ must be semi simple.

We turn now to applications concerning representations of sl(2;C) (§13). Consider a representation f: sl(2;C) $\to$ gl(V). From Th-2, we know that

$$f = \sum n_i D_i,$$

i.e. f is direct sum of the irreducible representations D_i, n_i (the multiplicity of D_i) showing how many copies of D_i are contained in f. We notice that for each D_i the corresponding operator H (diagonal) either has the eigenvalue 0 when i is integer and dimD = 2i+1, or H has the eigenvalue 1, when i is half-integer and dimD = 2i+1 is even. Since the eigenvalues of the operators H are simple, we have:

Proposition-1 *Let f: $sl(2;C) \to gl(V)$ be a representation and $f = \sum n_i D_i$ its decomposition in irreducible factors. Then $\sum n_i = \dim V_0 + \dim V_1$, where V_j the eigenspace of $f(H)$, with respect to the eigenvalue $j = 0, 1$.*

Proposition-2 *Let f: $sl(2;C) \to gl(V)$ be a representation, s.t. $f(H)$ has pairwise different eigenvalues whose differences are even numbers. Then f is irreducible.*

Obviously if we assume only the simplicity of eigenvalues of f(H), then f can be either irreducible or of the form $D_i \oplus D_j$ with one of the {i,j} integer and the other half-integer.

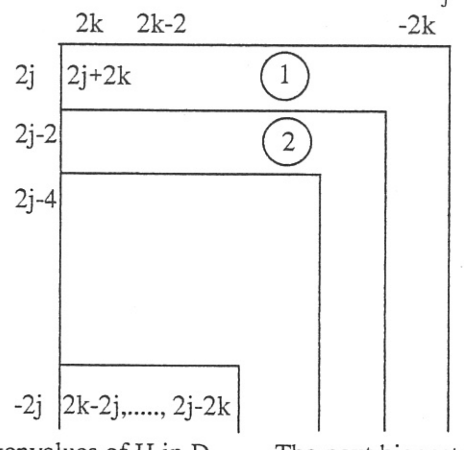
Proposition-3 *For the tensor product $D_j \otimes D_k$ of two irreducible representations of $sl(2;C)$ holds the formula*

$$D_j \otimes D_k = D_{j+k} \oplus D_{j+k-1} \oplus \dots \oplus D_{|j-k|}\,.$$

In fact, the eigenvectors v_i of H in D_j build a basis of D_j. Analogously, the eigenvectors w_i of H in D_k. Thus, the vectors $v_i \otimes w_{i'}$ which build a basis of $D_j \otimes D_k$ are also eigenvectors of H :

$$H(v_i \otimes w_{i'}) = H\, v_i \otimes w_{i'} + v_i \otimes H w_{i'} = (\mu_i + \mu_{i'}) v_i \otimes w_{i'}\,.$$

The biggest eigenvalue is 2j+2k and shows that D_{j+k} is among the irreducible components of $D_j \otimes D_k$. Assuming $j \leq k$, we can arrange the eigenvalues of $D_j \otimes D_k$ in a matrix:



Band 1 contains the eigenvalues of H in D_{j+k} . The next biggest remaining eigenvalue, outside band 1, is 2j+2k-2 and shows that D_{j+k-1} is among the irreducible components of $D_j \otimes D_k$. Band 2 of the matrix contains all the eigenvalues of D_{j+k-1}. Continuing with the other bands of the matrix we can construct a proof by induction.

Another application of the complete reducibility is the decomposition of a Lie algebra g, with respect to a semi simple subalgebra h. In fact, the restriction of ad: g $\to$ gl(g) to h, is a representation of h, for which h is h-invariant. Thus, there will exist some complementary $\mathfrak{h}$-invariant subspace $\mathfrak{m}$ (vector subspace). We proved the

Proposition-4 *For each semi simple Lie subalgebra $\mathfrak{h}\subset\mathfrak{g}$, there is a vector space complement $\mathfrak{m}$ of $\mathfrak{h}$, s.t. $[\mathfrak{h}\ \mathfrak{m}]\subset\mathfrak{m}$.*

We call such a complement $\mathfrak{m}$ of a semi simple subalgebra $\mathfrak{h}$, a **reductive complement** of $\mathfrak{h}$ in $\mathfrak{g}$. As a concrete example we can consider the Lie subalgebra o(n;C) of skew-symmetric matrices in gl(n;C). Later we'll prove that o(n;C) is semi-simple (actually, simple) and a reductive complement of it consists of the vector subspace sym(n;C) of symmetric matrices.

In the first two lemmata in §16 we saw that the adjoint representation for matrices respects the Jordan-Chevalley decomposition in semi simple and nilpotent parts. For semi simple Lie subalgebras of matrices, we can see that the components belong to the Lie subalgebra too.

Proposition-5 *Let $\mathfrak{g}\subset gl(V)$ be a semi simple subalgebra and for each $X\in\mathfrak{g}$, $X = S + N$, the Jordan-Chevalley decomposition of X. Then $S, N\in\mathfrak{g}$.*

In fact, by pro-4, there will exist a reductive complement $\mathfrak{m}\subset gl(V)$,

$$gl(V) = \mathfrak{g}\oplus\mathfrak{m},\quad [\mathfrak{g}\mathfrak{m}]\subset\mathfrak{m}. \qquad (*)$$

Thus, for $X\in\mathfrak{g}$, $adX(\mathfrak{g})\subset\mathfrak{g}$, $adX(\mathfrak{m})\subset\mathfrak{m}$ and adS is polynomial in adX, hence $adS(\mathfrak{m})\subset\mathfrak{m}$, $adS(\mathfrak{g})\subset\mathfrak{g}$. Consider now the decomposition

$$S = S_{\mathfrak{g}} + S_{\mathfrak{m}},\ \text{with } S_{\mathfrak{g}}\in\mathfrak{g} \text{ and } S_{\mathfrak{m}}\in\mathfrak{m}.$$

For each $\mathfrak{g}$-invariant subspace W, since $\mathfrak{g}=[\mathfrak{g}\mathfrak{g}]$, we have $tr(S|W) = tr(S_{\mathfrak{g}}|W) = 0$, hence also $tr(S_{\mathfrak{m}}|W)=0$. Notice that W is also S-invariant, since S is polynomial in X and $S_{\mathfrak{m}}$-invariant, since $S_{\mathfrak{m}}=S-S_{\mathfrak{g}}$.

Now, for each $Y\in\mathfrak{g}$, $[S_{\mathfrak{m}}Y]\in\mathfrak{g}$ (using $S_{\mathfrak{m}}=S-S_{\mathfrak{g}}$) and $[S_{\mathfrak{m}}Y]\in\mathfrak{m}$ (using (*)), hence $[S_{\mathfrak{m}}Y]=0$ and $S_{\mathfrak{m}}$ commutes with every $Y\in\mathfrak{g}$. Let W be a $\mathfrak{g}$-invariant and irreducible subspace of V. Then, because of the commutativity $[S_{\mathfrak{m}}Y]=0$, $S_{\mathfrak{m}}|W$ will be a multiple of the Id|W (§7) and since $tr(S_{\mathfrak{m}}|W)=0$ we'll have $S_{\mathfrak{m}}|W=0$. Since V can be decomposed in such $\mathfrak{g}$-invariant irreducible subspaces W on which $S_{\mathfrak{m}}|W=0$, we'll have $S_{\mathfrak{m}}=0$. This means $S=S_{\mathfrak{g}}\in\mathfrak{g}$ and consequently $N=X-S\in\mathfrak{g}$. q.e.d.

There is only one slight difference between
  Me and my epic brethren gone before,
And here the advantage is my own, I ween;
  (Not that I have not several merits more,
But this will more peculiarly be seen)
  They so embellish, that'tis quite a bore
Their labyrinth of fables to thread through,
Whereas this story's actually true.

Byron, Don Juan, Canto I, 202

20. Reductive Lie algebras

So is called a Lie algebra $\mathfrak{g}$, whose radical $\mathfrak{r} = \mathfrak{c}$ = center of $\mathfrak{g}$. Examples of such algebras are the abelian Lie algebras, the semi simple and all gl(V). We saw in the preceding § that semi simple Lie algebras have all their representations completely reducible. We'll see here that a Lie algebra which has <u>some</u> completely reducible representation is reductive. First we shall take delight in some easy exercises.

Exercise-1 $X \in gl(V)$ is a nilpotent operator, if and only if $tr(X) = tr(X^2) = ... = tr(X^n) = 0$.

[$p(X) = a_nX^n + ... + a_1X + a_0I = 0$, if $p(x)$ is the characteristic polynomial of X. $tr(p(X))=0$ implies $\det(X)=a_0=0$, hence there is some $e_n \in V$ with $Xe_n=0$. Complete e_n to a basis of V, write X with respect to this basis

$$X \approx \begin{pmatrix} X_1 & 0 \\ * & 0 \end{pmatrix}, \quad tr(X_1) = tr(X_1^2) = ... = tr(X_1^{n-1}) = 0,$$

and proceed by induction.]

Exercise-2 Let $Z = \sum_i [A_iB_i] \in gl(V)$ and $[ZA_i]=0$, for all these A_i. Then Z is a nilpotent operator of V.

[$Z[A_iB_i] = ZA_iB_i - ZB_iA_i = A_iZB_i - ZB_iA_i$, hence $tr(Z[A_iB_i]) = 0 \Rightarrow tr(Z^2) = 0$.
Analogously $tr(Z^3) = 0$, etc... Ex-1]

Exercise-3 Let f: $\mathfrak{g} \to gl(V)$ be a representation and $\mathfrak{h}$ an ideal of $\mathfrak{g}$, then $\mathfrak{h}V$ is a $\mathfrak{g}$-invariant subspace of V.

Exercise-4 Let f: $\mathfrak{g} \to gl(V)$ a faithful and irreducible representation and $\mathfrak{h}$ an ideal of $\mathfrak{g}$ s.t. all $f(X)$ are nilpotent operators. Then $\mathfrak{h} = \{0\}$.

[$\mathfrak{h}V$ is a proper $\mathfrak{g}$-invariant subspace (Th-2,§15), apply Ex-3.]

Exercise-5 Let V be a complex vector space irreducible, with respect to a family F of mutually commuting operators. Then V is one dimensional.

[All X in F operate on V as multiples of the identity (consider some eigenspace of X)]

Exercise-6 Let $A \in gl(V)$ and $W \subset V$ be an A-invariant subspace s.t. $A|W$ is nilpotent and the induced $\overline{A}$: $V/W \to V/W$ is also nilpotent. Then A is nilpotent too.

Let now $\mathfrak{g}$ be a reductive Lie algebra with radical $\mathfrak{r} = \mathfrak{c}$ = the center of $\mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{c}$ is semi simple (Ex-1, §17) and ad induces a faithful representation

$$\overline{ad}: \mathfrak{g}/\mathfrak{c} \to gl(\mathfrak{g}).$$

Since $\mathfrak{g}/\mathfrak{c}$ is semi simple (hence completely reducible) there is a $\mathfrak{g}/\mathfrak{c}$-invariant complement $\mathfrak{h}$ of $\mathfrak{c}$ s.t. $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{h}$. $\mathfrak{h}$ is isomorphic with $\mathfrak{g}/\mathfrak{c}$ under the natural projection, hence $\mathfrak{h}$ is semi simple, $[hh]=h$ and $[hh]=[gg]$. Thus

$$g = c \oplus [gg] \text{ and } [gg] \text{ is semi simple.} \qquad (1)$$

Inversely, if (1) holds, then the radical of g coincides with c. We prove more general:

Proposition-1 *If g is a Lie algebra with center c, then the following conditions are equivalent:* *1) $c \cap [gg] = \{0\}$ and every abelian ideal of g is contained in c.*
2) $g = c \oplus [gg]$ and $[gg]$ is semi simple.

Given 2), every abelian ideal h of g is projected in g/c, onto $\{0\}$, hence $h \subset c$.

Given 1), complete $[gg]$ to a subspace $h \supseteq [gg]$ s.t. $c \oplus h = g$. h is an ideal. An abelian ideal of h will be also abelian ideal of g, hence (hypothesis) will be contained in c, hence it will be the $\{0\}$. Thus h is semi simple and $[gg] = [hh] = h$. Combining with the remarks preceding formula (1), we see that

Proposition-2 *g is a reductive Lie algebra, if and only if for its center c holds*

$$g = c \oplus [gg] \text{ and } [gg] \text{ is semi simple.}$$

Lemma-1 *Let $\{h_i\}$ be a family of ideals of the Lie algebra g, s.t. $\cap h_i = \{0\}$ and $g_i = g/h_i$ are reductive. Then g is a reductive Lie algebra.*

We assume that 1) of Pro-1 holds for every h_i. We'll prove that it holds for g too. In fact, an $X \in c \cap [gg]$ is projected by the canonical projections onto $\bar{c}_i \cap [g_i g_i] = \{0\}$, hence $X \in h_i$ for each i, hence $X = 0$. We check the other condition of 1) Pro-1 in a similar way. If a is an abelian ideal in g, then it is projected into an abelian ideal in g_i, thus $[ag] \subset h_i$ for each i, hence $[ag] = \{0\}$ and $a \subset c$.

Proposition-3 *Let $f: g \to gl(V)$ be a faithful, completely reducible representation of g. Then g is reductive and for $X \in c$, $f(X)$ are semi simple operators of V.*

We work again with 1) Pro-1. Let us assume that f: $g \to gl(V)$ is a faithful, completely reducible representation of g and $X = \sum [A_i B_i] \in c \cap [gg]$. Then $[f(X), f(A_i)]=0$, hence (Ex-2) $f(X)$ is nilpotent. If V is g-irreducible, then (Ex-4) $c \cap [gg] = 0$. On the other side, for an abelian ideal a of g and an $X \in [ag]$ we'll have $X=\sum[A_i B_i]$ with $A_i \in a$. For the same reason, as before, we see that $f(X)$ is nilpotent and $[ag] = 0$, hence $a \subset c$.

If the representation is not irreducible, then by assumption, V is decomposable in g-irreducible subspaces $V = \oplus W_i$. Then $h_i = \text{kern}(f|W_i)$ are ideals with $\cap h_i = \{0\}$ and the induced representation f: $g/h_i \to gl(W_i)$ is faithful and irreducible. The result follows, in this case, from the lemma.

In every g-irreducible component W of V the operators $f(X)$ are multiples of the identity (Schur), since they commute with each other and with the elements of g. q.e.d.

Proposition-4 *A Lie subalgebra g of $gl(V)$ is completely reducible in V, if and only if it is reductive and the elements of the center are semi simple operators of V.*

One part of the proposition is a consequence of the preceding one. For the remaining inverse, let $g=c \oplus [gg]$ and X be semi simple whenever $X \in c$ = center of g. Then V can be decomposed in the common eigenspaces W of all $X \in c$. Since the elements of c commute with each other and with the elements of g, the spaces W are g-invariant, hence also $[gg]$-invariant. Since [gg] is semi simple, W will decompose into [gg]-invariant subspaces V_{ij}. These are also g-invariant and irreducible and give the complete reducibility of V, with respect to g.

Exercise-7 Let f: g → gl(V) be a faithful, completely reducible representation of a solvable Lie algebra g. Then g is abelian.

Exercise-8 Let g be a solvable Lie subalgebra of gl(V), A, B∈g and A be nilpotent. Then BA is nilpotent.

The last exercise is used below to give a characterization of solvable Lie algebras.

Lemma-2 *Let r be the radical of the Lie algebra g. Then in every representation f: g → gl(V) of g, [rg] is represented by nilpotent matrices.*

For the proof, we consider first the case of irreducible V. Let h = kern$f$, $\bar{g}$ = g/h, $\bar{r}$ theradical of $\bar{g}$ and $\bar{f}$: $\bar{g}$ → gl(V) the induced faithful representation. By Pro-4, $\bar{g}$ is reductive $\bar{g} = \bar{c}\oplus[\bar{g}\bar{g}]$ and $\bar{r}\subset\bar{c}$. By the natural projection r is mapped in $\bar{r}\subset\bar{c}$, hence [rg]⊂h and f([rg]) = 0. For the general case we use induction, with respect to dimV. Let W⊂V be a g-invariant subspace. By the inductive assumption, [rg] will be represented (with respect to the induced representations) with nilpotent operators in W and in V/W. From Ex-6, follows that [rg] will be represented in V by nilpotent operators too.

Applying this lemma to ad and using Engel's theorem §15 we have the

Corollary *Let r be the radical of the Lie algebra g, then [rg] is a nilpotent ideal of g.*

Lemma-3 *Let f: g → gl(V) be a representation, r the radical of g and t the trace form of the representation. Then t([gg],r) = 0.*

For X, Y∈g, A∈r we must show that t([XY],A) = 0 or t(X,[YA]) = 0. For the subspace r' = r + <X> holds [r',r']⊂r, hence r' is a solvable subalgebra. By lemma-2, f([XA]) will be nilpotent. Applying then Ex-8 on f(r'), f(X), f([XA]), we conclude that f(X)·f([YA]) is nilpotent hence t(X,[YA]) = trace(f(X)·f([YA]) = 0.

Theorem *The radical of a Lie algebra g is the orthogonal complement of [gg], with respect to the Killing form K. g is solvable, if and only if K([XY],Z) = 0, for all X, Y, Z in g.*

Applying the lemma to f = ad, we have K([gg],r) = 0, hence r⊂$[gg]^{\perp}$. Inversely, let h = $[gg]^{\perp}$. We have K([gg],h) = 0 ⇔ K([hg],g) = 0. Hence, by Cartan's first criterion, [hg] is a solvable ideal, hence [hh] and h are solvable too, hence h⊂r. The last statement of the theorem is a cosequence of the previous.

But let me change this theme, which grows too sad,
And lay this sheet of sorrows on the shelf;
I don't much like describing people mad,
For fear of seeming rather touch'd myself-
Besides I've no more on this head to add;
And as my Muse is a capricious elf,
We'll put about, and try another tack
With Juan, left half-kill'd some stanzas back.

Byron, Don Juan, Canto IV, 74

21. The classical Lie algebras

For every invertible matrix $J \in GL(n;C)$ one can define a Lie algebra

$$L_J = \{A \in gl(n;C),\ A^tJ + JA = 0\}. \tag{1}$$

Exercise-1 Show that L_J is indeed a Lie subalgebra of $gl(n;C)$.

The **classical** Lie algebras (except $sl(n;C)$, §12) are examples of Lie algebras defined by special J as in (1). There are four different types, denoted by Cartan with A_n, B_n, C_n and D_n respectively.

A_n denotes the isomorphism class of $sl(n+1;C)$, for $n = 1, 2, 3, \ldots$.

B_n denotes the isomorphism class of $o(2n+1;C)$, for $n = 2, 3, \ldots$.

C_n denotes the isomorphism class of $sp(n;C)$, for $n = 3, 4, \ldots$.

D_n denotes the isomorphism class of $o(2n;C)$, for $n = 4, 5, \ldots$.

$o(m;C)$ denotes the Lie subalgebra of $gl(m;C)$, consisting of skew-symmetric matrices. They are defined by (1), for $J = I$, and they comprise two isomorphism classes B_n and D_n. $o(m;C)$ is called the **orthogonal** Lie algebra in m parameters. $sp(n;C)$ is called the **symplectic** Lie algebra in n parameters and is defined by (1), for

$$J = \begin{pmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{pmatrix}.$$

Exercise-2 Show that all four types of Lie algebras are subalgebras of some $sl(m;C)$.

Exercise-3 For an invertible $J \in GL(n;C)$ symmetric or skew-symmetric matrix, show that the mapping

$$f\colon sl(n;C) \to sl(n;C)$$
$$f(A) = -(J^{-1})A^tJ, \tag{2}$$

is an automorphism of Lie algebras of order 2 (involution) and has two eigenvalues ± 1. Show also that L_J coincides with the $+1$-eigenspace of f, and that the -1-eigenspace,

$$m_J = \{A \in sl(n;C),\ A^tJ - JA = 0\}, \tag{3}$$

is a reductive complement (see remarks following Pro-4, §19) of L_J in sl(n;C). Show finally that

$$[m_J m_J] \subset L_J . \qquad (4)$$

The last property of the exercise shows that (L,M) is a symmetric pair of sl(n;C).

More generally, we call **symmetric pair** of a Lie algebra $\mathfrak{g}$, a pair $(\mathfrak{h},\mathfrak{m})$ s.t.

$\mathfrak{h}\subset\mathfrak{g}$ is a Lie subalgebra,

$\mathfrak{m}\subset\mathfrak{g}$ is a reductive complement of $\mathfrak{h}$ ($[\mathfrak{hm}]\subset \mathfrak{m}$), and

$[\mathfrak{mm}]\subset\mathfrak{h}$. (5)

Symmetric pairs are heavily used in the classification of the symmetric spaces of Cartan.

Exercise-4 For every $J \in GL(n;C)$ (not only symmetric or skew-symmetric as in Ex-3) formula (2) defines an automorphism of sl(n;C) and the corresponding L_J coincides with the +1-eigenspace of f. In this more general case f is not of order 2 (involution).

Exercise-5 Show that for the invertible $2n \times 2n$ matrix

$$g = \left(\begin{array}{c|c} 1\ 0\ 0 \ldots\ldots 0 & 0\ 0 \ldots\ldots 0 \\ 0\ 0 \ldots\ldots 0 & 1\ 0\ 0 \ldots\ldots 0 \\ 0\ 1\ 0\ 0 \ldots 0 & 0\ 0 \ldots\ldots 0 \\ 0\ 0 \ldots\ldots 0 & 0\ 1\ 0\ 0 \ldots 0 \\ 0\ 0\ 1 \ldots\ldots 0 & 0\ 0 \ldots\ldots 0 \\ \ldots\ldots & \ldots\ldots \\ 0\ 0 \ldots\ldots 0 & 0\ 0\ 0 \ldots\ldots 1 \end{array}\right),$$

holds

$$g^{-1} J g = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = J', \text{ where J is the matrix on p. 21-1.}$$

Conclude that sp(n;C) is isomorphic with the Lie algebra

$$L_{J'} = \{A \in gl(n;C),\ A^t J' + J'A = 0\}.$$

Splitting A in blocks, we can write the last relation $A^t J' + J'A = 0$:

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \ \Leftrightarrow$$

$$\begin{pmatrix} C^t & -A^t \\ D^t & -B^t \end{pmatrix} + \begin{pmatrix} -C & -D \\ A & B \end{pmatrix} = 0 \ \Leftrightarrow\ \{C^t = C,\ D = -A^t,\ B^t = B\}.$$

Thus, we can identify

$$sp(n;C) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where B and C are symmetric matrices} \right\}.$$

We have

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}\right] = \begin{pmatrix} HA-AH, & HB+BH \\ -HC-CH, & HA^t-A^tH \end{pmatrix},$$

which for diagonal matrices H (adopting the notations of §12, for the $n \times n$ matrices A, B, ...)

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}\right] = \begin{pmatrix} (h_i-h_j)E_{ij} & 0 \\ 0 & (h_j-h_i)E_{ji} \end{pmatrix},$$

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & (h_i+h_j)(E_{ij}+E_{ji}) \\ 0 & 0 \end{pmatrix},$$

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ -(h_i+h_j)(E_{ij}+E_{ji}) & 0 \end{pmatrix},$$

which follow from

$$HE_{ij} = h_i E_{ij}, \quad E_{ij} H = h_j E_{ij}.$$

Exercise-6 Show that the matrices

$$H' = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \quad A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix}, \quad C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix},$$

build a basis of sp(n;C) (more precisely of $L_{J'}$, which is isomorphic to sp(n;C)) whose dimension is $n^2+n(n+1)$.

Exercise-7 Show that

$$[A_{ij}, A_{kr}] = \delta_{jk} A_{ir} - \delta_{ri} A_{kj}, \quad [B_{ij}, C_{rk}] = \delta_{jr} A_{ik} + \delta_{ir} A_{jk} + \delta_{jk} A_{ir} + \delta_{ik} A_{jr},$$

$$[A_{ij}, B_{rk}] = \delta_{jr} B_{ik} + \delta_{jk} B_{ir},$$

$$[A_{ij}, C_{rk}] = - \delta_{ir} C_{jk} - \delta_{ik} C_{jr}, \quad [B_{ij}, B_{rk}] = [C_{ij}, C_{rk}] = 0.$$

Exercise-8 The preceding model for $\mathfrak{g}$ = sp(n;C) shows that

a) The set of diagonal matrices is an abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

b) This subalgebra is self-normalizing: $[X\ \mathfrak{h}] \subset \mathfrak{h} \Rightarrow X \in \mathfrak{h}$.

c) Every element X of $\mathfrak{h}$ has adX semi simple.

d) There is a basis of $\mathfrak{g}$ s.t. adX are simultaneous diagonal, for all X in $\mathfrak{h}$.

These results for sp(n;C) will be generalized below for arbitrary semi simple Lie algebras. The basic ingredient will be the existence of a subalgebra $\mathfrak{h}$, as the subalgebra of diagonal matrices above. Notice that for every abelian subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, the operators adX, for X in $\mathfrak{h}$, are similtaneous diagonalizable, since they commute [adX, adY] = ad[XY] = 0. The other properties, beyond commutativity, have to do with the bracketing of elements of the different eigenspaces.

For the orthogonal matrices we compute separately the cases n = even and n = odd.

In the even case, o(2n;C) is isomorphic to L_J, where

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad J^2 = I_{2n}.$$

In fact, the invertible 2n ×2n matrix

$$g = \frac{1}{\sqrt{2}}\begin{pmatrix} I & i\cdot I \\ I & -i\cdot I \end{pmatrix}, \quad \text{satisfies} \quad g^t J g = I_{2n}.$$

We have then,

$$A^t(g^tJg)+(g^tJg)\,A = 0 \;\Leftrightarrow\; (gAg^{-1})^tJ + J(gAg^{-1}) = 0 \;\Leftrightarrow\; B^tJ + JB = 0, \text{ for } B = gAg^{-1}.$$

Thus, L_J and o(2n;C) are conjugate. Splitting the matrices into blocks, we see that

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \cdot \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \quad \Leftrightarrow \quad \{ C^t + C = 0 , \;\; A^t + D = 0 , \;\; B^t + B = 0 \} .$$

Hence o(2n;C) may be identified with the set of matrices

$$\left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ B, C skew-symmetric, A arbitrary} \right\}.$$

For the subalgebra of diagonal matrices we have

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right] = \begin{pmatrix} HA\text{-}AH , & HB\text{+}BH \\ \text{-}HC\text{-}CH , & HA^t\text{-}A^tH \end{pmatrix} .$$

Exercise-9 Compute as in Ex-6, for

$$H' = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix} \text{ (diagonal)}, \quad A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & E_{ij}\text{-}E_{ji} \\ 0 & 0 \end{pmatrix}, \quad C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij}\text{-}E_{ji} & 0 \end{pmatrix},$$

and prove the relations

$$[H', A_{ij}] = (h_i - h_j)\, A_{ij} , \qquad [A_{ij}, A_{kr}] = \delta_{jk} A_{ir} - \delta_{ri} A_{kj} ,$$

$$[H', B_{ij}] = (h_i + h_j)\, B_{ij} , \qquad [A_{ij}, B_{rk}] = \delta_{jr} B_{ik} - \delta_{jk} B_{ir} ,$$

$$[H', C_{ij}] = -(h_i + h_j)\, C_{ij} , \qquad [A_{ij}, C_{rk}] = \delta_{ik} C_{jr} - \delta_{ir} C_{jk} ,$$

$$[B_{rk}, C_{ij}] = \delta_{ki} A_{rj} - \delta_{ri} A_{kj} - \delta_{kj} A_{ri} + \delta_{rj} A_{ki} ,$$

$$[B_{ij}, B_{rk}] = 0,$$

$$[C_{ij}, C_{rk}] = 0.$$

The Lie algebras o(n2+1;C) can be identified with L_J, for $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$.

Exercise-10 Show for the preceding J and the invertible matrix

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I , & \frac{i}{\sqrt{2}} I \\ 0 & \frac{1}{\sqrt{2}} I , & \frac{-i}{\sqrt{2}} I \end{pmatrix},$$

that $g^t J g = I_{2n+1}$, hence o(2n+1;C) and the corresponding L_J are conjugate.

L_J can be described by matrices of blocks

$$\begin{pmatrix} a & b & c \\ d & A , & B \\ e & C , & D \end{pmatrix}, \text{ which satisfy } \begin{pmatrix} a & d^t & e^t \\ b^t & A^t , & C^t \\ c^t & B^t , & D^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & A , & B \\ e & C , & D \end{pmatrix} = 0.$$

$$\Leftrightarrow \quad L_J = \left\{ \begin{pmatrix} a & b & c \\ -c^t & A, & B \\ -b^t & C, & -A^t \end{pmatrix}, \text{ where C, B skew-symmetric, A arbitrary} \right\}.$$

Exercise-11 Show that o(2n;C) is a subalgebra of o(2n+1;C). The basis of o(2n;C), considered in Ex-9, extends in a natural way to an independent set in o(2n+1;C), which can be completed to a basis of the later by the addition of the 2n matrices:

$$A_i = \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ -e_i^t & 0 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 0 & e_i \\ -e_i^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_i = (0,\dots,1,\dots,0) = \text{i-th vector of the standard basis.}$$

Denoting by the same letters the extension of the basis of o(2n;C), the relations of Ex-9 hold also in o(2n+1;C). In addition we have the relations

$$\begin{aligned}
&[H', A_i] = -h_i A_i, && [A_i, A_{kr}] = \delta_{ik} A_r, \\
&[H', B_i] = h_i B_i, && [B_i, B_{kr}] = 0, \\
&[A_i, A_j] = -C_{ij}, && [A_i, B_{kr}] = \delta_{ik} B_r, \\
&[A_i, B_j] = -A_{ji}, && [A_i, C_{kr}] = 0, \\
&[B_i, B_j] = B_{ij}, && [B_i, C_{kr}] = \delta_{ik} A_r, \\
& && [B_i, A_{kr}] = -\delta_{ir} B_k.
\end{aligned}$$

Exercise-12 Show that the orthogonal Lie algebras o(2n;C) and o(2n+1;C) contain abelian subalgebras of dimension n(n-1)/2. [Generated by the B_{ij}. The same is true for sp(n;C)]

Notice that the orthogonal Lie algebras are not self-normalizing in the appropriate gl(m;C). For the diagonal (abelian) subalgebras however we have:

Exercise-13 Show that the diagonal abelian subalgebras of sp(n;C) and o(n;C) are self-normalizing. Conclude that the center of these algebras is the {0}.

Exercise-14 Show that the matrices of sp(n;C) and <I> generate the associative algebra End(C^{2n}). Analogously the matrices of o(n;C) and <I> generate End(C^n). [Start with $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}$, then $\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix} \in \text{End}(C^{2n})$. Analogously $\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix} \in \text{End}(C^{2n})$. Thus, all diagonal matrices are generated by sp(n;C) and <I>, ...]

Exercise-15 Let $\mathfrak{g} \subset gl(V)$ be a Lie subalgebra and $W \subset V$ a $\mathfrak{g}$-invariant subspace. Show that W is also invariant with respect to the associative algebra $A(\mathfrak{g}) \subset End(V)$, generated by $\mathfrak{g}$ and <I>. Conclude that when $A(\mathfrak{g}) = End(V)$, then V is $\mathfrak{g}$-irreducible. [No (non-trivial) subspace of V is End(V)-invariant.]

Exercise-16 Show (using Ex-15) that C^{2n} is sp(n;C)-irreducible. Analogously C^n is o(n;C)-irreducible. Conclude (Pro-4, §20) that these Lie algebras are reductive and, by Ex-13, even semi simple.

Later we'll prove that the classical Lie algebras are simple. Anticipating this fact, we shall produce now some formulas for the easy computation of the Killing form of the classical Lie algebras. As in the case of sl(n;C) (§12), the Killing form and tr(AB) are both symmetric non-degenerate bilinear forms $f(X, Y)$ on the Lie algebra $\mathfrak{g}$, satisfying

$$f([XY], Z) + f(Y, [XZ]) = 0. \tag{6}$$

The equation

$$K(X, Y) = tr(AX, Y), \tag{7}$$

defines a linear operator $A: \mathfrak{g} \rightarrow \mathfrak{g}$, which, using (6) for K and tr, is seen to satisfy

$$A[XY] = [X, AY], \text{ for all } X, Y \text{ in } \mathfrak{g}. \tag{8}$$

From this, follows easily that an eigenspace of A is also an ideal of $\mathfrak{g}$. Thus, when $\mathfrak{g}$ is simple, $A = \mu I$. We compute μ in each case separately.

a) In sp(n;C) we consider the matrix $A = diag(1, 0, ..., 0, -1, 0, ..., 0)$ which has $tr(AA) = 2$. On the other side $K(A, A) = tr(adA \cdot adA)$ may be computed by Ex-9. In fact, using the formulas there, we see that $adA \cdot adA$ is diagonal with eigenvalue 1, and 1-eigenpace of dimension $2(n-1)+2(n-1)+ 8$ (8 comming from B_{11}, C_{11}). Thus, in this case:

$$K(X, Y) = (2n+2)\ tr(XY). \tag{9}$$

b) In o(2n;C) we consider the same matrix A and analogous calculations to obtain

$$K(X, Y) = 2(n-1)\ tr(XY). \tag{10}$$

c) In o(2n+1;C), using the same A we obtain analogously

$$K(X, Y) = (2n-1)\ tr(XY). \tag{11}$$

The table below summarizes the results.

Cartan's notation	Model	Dimension	Killing form
A_n, n = 1, 2, ...	sl(n+1;C)	$n^2+ 2n$	2(n+1) tr(XY)
B_n, n = 2, 3, ...	o(2n+1;C)	$2n^2+ n$	(2n-1) tr(XY)
C_n, n = 3, 4, ...	sp(n;C)	$2n^2+ n$	(2n+2) tr(XY)
D_n, n = 4, 5, ...	o(2n;C)	$2n^2- n$	(2n-2) tr(XY)

n is the dimension of the abelian diagonal subalgebra and is called the **rank** of the Lie algebra. The restrictions on n are posed in order to avoid coincidences in the different classes.

Exercise-17 Show that D_1 is abelian (1-dimensional).

Exercise-18 Looking at the bases of A_1 , B_1 , C_1 , constructed in this §, prove that these Lie algebras (3-dimensional) are isomorphic.

Remark For other coincidences between (real or complex) Lie algebras see in §54.

III.

Structure of Complex Semi Simple Lie Algebras

Das Ergebniß derselben ist, daß der Werth der Mathematik nur ein mittelbarer sei, nämlich in der Anwendung zu Zwecken, welche allein durch sie erreichbar sind, liege; an sich aber lasse die Mathematik den Geist da, wo sie ihn gefunden hat, und sei der allgemeinen Ausbildung und Entwickelung desselben keineswegs förderlich, ja sogar entschieden hinderlich.

A. Schopenhauer, Die Welt als ... , II p.154

22. Cartan subalgebras and roots

Our prototype in this § is the abelian subalgebra of diagonal matrices of the classical Lie algebras. The operators adX, for X in this subalgebra commute with each other and are simultaneously diagonalizable. A basis of this subalgebra together with the common eigenvectors of adX build a basis of the whole Lie algebra, with respect to which the structure constants are relatively simple and allow the comparison and the classification of the Lie algebras. The analogon of the diagonal subalgebra in the general case is the Cartan subalgebra.

We call **Cartan subalgebra** h, of a Lie algebra g, a nilpotent Lie subalgebra of g, which is self normalizing (i.e. $[X h] \subset h \;\Rightarrow\; X \in h$).

In general, for a nilpotent Lie subalgebra h of g, we can apply The-3, §15 to the nilpotent subalgebra $ad(h) \subset gl(g)$. According to that theorem, there is a basis of g, s.t.

$$\mathrm{ad}X = \begin{pmatrix} \boxed{\begin{matrix} \alpha(X) & & * \\ & \ddots & \\ 0 & & \alpha(X) \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \omega(X) & & * \\ & \ddots & \\ 0 & & \omega(X) \end{matrix}} \end{pmatrix}, \text{ for all } X \text{ in } h. \qquad (*)$$

The linear functions $\alpha(X), \ldots, \omega(X)$ on h are called **roots** of the nilpotent subalgebra h and their set is denoted by Δ. g decomposes in a direct sum of generalized eigenspaces g_i called **root spaces** of the nilpotent subalgebra h :

$$g = g_0 \oplus g_\alpha \oplus \ldots \oplus g_\omega , \qquad (1)$$

where

$$g_\mu = \{X \in g, (\mathrm{ad}Y - \mu(Y)I)^p (X) = 0, \text{ for all } Y \in h \text{ and sufficient large } p\}. \quad (2)$$

The algebraic structure of g implies the nice property of the root spaces g_μ :

$$[g_\mu , g_\pi] \subset g_{\mu+\pi} . \qquad (3)$$

In this relation we understand that $g_{\mu+\pi} = \{0\}$, when $\mu+\pi$ is not a root of g. The proof of this relation relies on a simple consequence of the Jacobi identity:

Exercise-1 Show inductively, that in every Lie algebra and for arbitrary constants a, b:

$$(adX - (a+b)I)^n [Y Z] = \sum_{i=0}^{n} \binom{n}{i} [(adX - bI)^i Y, (adX - aI)^{n-i} Z] .$$

Obviously 0 is a root of h and we denote the corresponding root space by g_0. Because of the nilpotency of h, we see immediately that

$$h \subset g_0 . \qquad (4)$$

Besides, (3) implies that g_0 is a subalgebra of g. The additional condition of self-normalization, for Cartan subalgebras, implies $h = g_0$. In fact, by the definition of the root space g_0, and considering arbitrary $H_1, \dots, H_n$ in h, dim $g_0 = n$ and X in g_0 we'll have (since $adX|g_0$ is nilpotent for all X in h, see Pro-2, Ex-11 §15):

$[H_1 [H_2 [\dots [H_n, X] \dots] = 0$, for every $H_1, \dots, H_n$ in h $\Rightarrow$

$[H_2 [H_3 [\dots [H_n, X] \dots] \in h$, for every $H_2, \dots, H_n$ in h $\Rightarrow$

....

$[H_n, X] \in h$, for every H_n in h $\Rightarrow$

$X \in h$.

Inversely, if $g_0 = h$, then h is self normalizing. In fact, if $Y \in g$ and $[HY] \in h$, for all $H \in h$, then decomposing Y in the root spaces, we have

$$Y = Y_0 + Y_\alpha + \dots + Y_\omega ,$$
$$[HY] = [HY_0] + [HY_\alpha] + \dots + [HY_\omega] .$$

By assumption, $[HY] \in h = g_0$, $[HY_0] \in g_0$. Thus, $adHY_\mu = 0$, for every $H \in h$ and $\mu \neq 0$ $\Rightarrow$ $Y_\mu = 0$, for $\mu \neq 0$, hence $Y = Y_0 \in h$. We proved the

Proposition-1 *A nilpotent Lie subalgebra h of the Lie algebra g is self-normalizing (i.e. a Cartan subalgebra), if and only if its zero root space $g_0 = h$.*

Proposition-2 *For every nilpotent Lie subalgebra h of the Lie algebra g, with Killing form K, we have*

$$K(g_\mu, g_\pi) = 0, \text{ when the corresponding roots } \mu + \pi \neq 0. \qquad (5)$$

In fact, $[g_\mu, [g_\pi, g_\alpha]] \subset g_{\mu+\pi+\alpha}$. Hence for $X \in g_\mu$, $Y \in g_\pi$, $Z \in g_\alpha$, we have,

$$(adX \cdot adY)^n (Z) \in g_{n(\mu+\pi)+\alpha} = \{0\}, \text{ for sufficiently large n,}$$

since $\mu + \pi \neq 0$ and since there are finitely many roots. This shows that $(adX \cdot adY)$ is nilpotent, hence $tr(adX \cdot adY) = 0$. q.e.d. We get also immediately the corollary :

Proposition-3 *For every nilpotent Lie subalgebra of a Lie algebra g, the restriction of the Killing form K on the root spaces g_μ, with $\mu \neq 0$, vanishes identically.*

For the restriction of K on the nilpotent subalgebra h we get, considering the triangular form of adX, for X in h :

$$K(X,Y) = \textstyle\sum_{\mu\neq 0} n_\mu \, \mu(X)\mu(Y), \; n_\mu = \dim(g_\mu). \qquad (6)$$

Some easy corollaries for Cartan subalgebras of a semi simple Lie algebra are the following:

Proposition-4 *Let h be a Cartan subalgebra of the semi simple Lie algebra g and $X \in h$. If $\mu(X) = 0$, for all roots of h, then $X = 0$.*

Proposition-5 *Let g, h be as before. Then the set Δ of roots of h contains a basis of the dual space h^*.*

Proposition-6 *Every Cartan subalgebra of a semi simple Lie algebra is abelian.*

Proposition-7 *The restriction of the Killing form on a Cartan subalgebra h, of a semi simple Lie algebra g, is non-degenerate.*

Proposition-8 *For every $X\in h$, where h a Cartan subalgebra of the semi simple Lie algebra g, the operator adX is semi simple and $adX|g_\mu = \mu(X)Id|g_\mu$, hence*

$$\mathrm{ad}X = \begin{pmatrix} \boxed{\begin{matrix}\alpha(X) & & 0\\ & \ddots & \\ 0 & & \alpha(X)\end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix}\omega(X) & & 0\\ & \ddots & \\ 0 & & \omega(X)\end{matrix}} \end{pmatrix} \quad \text{(diagonal), for all } X \text{ in } h. \qquad (7)$$

Proposition-9 *For every root μ of the Cartan subalgebra h of the semi simple Lie algebra g the linear form $-\mu$ is also a root of h.*

Here are the proofs :

Pro-4 : follows from (6), the orthogonality $K(h, g_i) = 0$, and the semi simplicity of g.

Pro-5 : immediate corollary of Pro-4.

Pro-6 : follows from Pro-4 and the fact $\mu([hh]) = 0$, for every root μ of h.

Pro-7 : again the non-degeneracy of the Killing form and the fact $g_0 \perp g_\mu$, for $\mu\neq 0$.

Pro-8 : this important proposition is a consequence of Pro-5, §19. According to that proposition and since $ad(g)\subset gl(g)$ is semi simple, the semi simple part of the Jordan-Chevalley decomposition of $adX = S + N$ will be $S \in ad(g)$, hence $S = adY$, for some Y in g. Then, for arbitrary $H\in h$, (and since $X\in h$) the equation $adX(H) = 0 \Rightarrow S(H) = 0 \Leftrightarrow [YH] = 0 \Rightarrow Y\in h$. Finally, for every root μ of h we'll have $\mu(X\text{-}Y)$ and we conclude the result by Pro-4.

Pro-9 : this is certainly true, since in the contrary case $g_\mu \perp g_\pi$, for $\mu + \pi \neq 0$, would imply that K is degenerate.

Before to proceed to the analysis of the structure by means of a Cartan subalgebra, we should perhaps spend some time on the existence of such subalgebras and their relations. These questions however will be considered later in §36. Among other things, we'll prove there, that all Cartan subalgebras of a complex semisimple Lie algebra have the same dimension, which we call the **rank** of the Lie algebra. Here we content ourselves with another aspect of the Cartan subalgebras of semi simple Lie algebras.

Proposition-10 *A subalgebra h, of a semi simple Liealgebra g, is a Cartan subalgebra, if and only if the following conditions are true:*

a) h is a maximal abelian subalgebra, and

b) for every $X\in h$, the operator adX is semi simple.

Notice that condition a) has a relative meaning. It means that there is no other abelian subalgebra, containing h as a proper subspace. Note also that in the case of classical Lie algebras we encountered abelian subalgebras with more dimensions than their Cartan (diagonal) subalgebras. Now to the proof. The preceding propositions show that a), b) are true for such Cartan subalgebras.

Inversely, if $\mathfrak{h}$ satisfies a), b), then $\mathfrak{h}$ is in particular nilpotent, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \ldots \oplus \mathfrak{g}_\omega$ will be the splitting in the root spaces of $\mathfrak{h}$, and $[XH] \in \mathfrak{h}$, for all $H \in \mathfrak{h}$, will imply, by the semi simplicity of adH, $X \in \mathfrak{h}$. In fact,

$$X = X_0 + X_\alpha + \ldots + X_\omega \Rightarrow [HX] = [HX_0] + \alpha(H)X_\alpha + \ldots + \omega(H)X_\omega \Rightarrow$$
$$\alpha(H)X_1 + \ldots + \omega(H)X_k = 0, \text{ for every } H \text{ etc. } \ldots$$

Exercise-2 Let $X = S + N$ be the Jordan Chevalley decomposition of $X \in gl(V)$ and assume that $Xv = 0$. Show that $Sv = 0$. [v belongs to the generalized eigenspace of the eigenvalue 0, where $S = 0$]

Here my chaste Muse a liberty must take-
Start not! still chaster reader-she'll be nice hence-
Forward, and there is no great cause to quake;
This liberty is a poetic licence,
Which some irregularity may make
In the design, and as I have a high sense
Of Aristotle and the Rules, 'tis fit
To beg his pardon when I err a bit.

Byron, Don Juan, Canto I, 120

23. Strings of roots, coroots

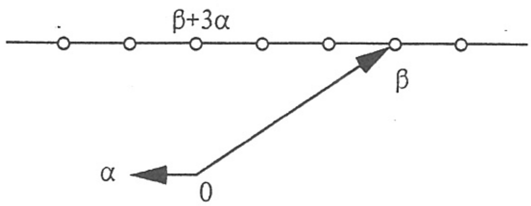
We continue here the analysis of structure of a semi simple Lie algebra g, based on the roots of a Cartan subalgebra h. For two such roots α, $\beta : h \to C$, we define the **α-string of the root β**, to be the (vector) subspace of g

$$g_\beta{}^\alpha = \oplus_{\nu \in Z}\ g_{\beta+\nu\alpha} \ . \qquad (1)$$

We use the same name also for the sequence of the roots

$$\{\beta+\nu\alpha \mid \nu \in \mathbf{Z}\}.$$

In the dual space h*, the α-string consists of a finite number of points on the line $\beta+t\alpha$ of this space.



We'll see below that the characteristics of all possible such strings describe the structure of the whole Lie algebra.

Proposition-1 *For every root $\mu \neq 0$ of h consider the hyperplane $\mu^\perp = \{X \in h \mid \mu(X) = 0\}$.*

a) *$[g_\mu, g_{-\mu}] \subset h$, has dimension one.*

b) *$[g_\mu, g_{-\mu}] \cap \mu^\perp = \{0\}$.*

To prove a), we note that each $X \in g_\mu$ has nilpotent adX. In fact for every other root π we'll have $adX(g_\pi) \subset g_{\pi+\mu}$ and more generally, $(adX)^n(g_\pi) \subset g_{\pi+n\mu} = \{0\}$, for sufficiently large n. Consider now two vectors $X \in g_\mu$, $Y \in g_{-\mu}$ with $[XY] = 0$. This implies $[adX, adY] = 0$, hence adX, adY are nilpotent and commuting operators. Thus, $adX \cdot adY$ is nilpotent too and consequently $K(X,Y) = 0$. Since K is non-degenerate for semi simple g, there must be some $X \in g_\mu$, $Y \in g_{-\mu}$ with $[XY] \neq 0$ (see Pro-2, §22). The statement on the dimension follows now from b).

To prove b), we consider the operation of adX, $X \in g_\mu$, adY, $Y \in g_{-\mu}$ and ad[XY], $[XY] \in h$ on the string $g_\pi{}^\mu$, which is invariant under these operators. We compute the traces of the restrictions on $g_\pi{}^\mu$ of these operators, in two different ways.

$$tr(ad[XY]) = tr(adX \cdot adY - adY \cdot adX) = 0.$$

$$tr(ad[XY]) = \sum_\nu (\dim g_{\pi+\nu\mu})(\pi+\nu\mu)([XY]) =$$
$$\sum_\nu (\dim g_{\pi+\nu\mu})\pi([XY]) + \sum_\nu (\dim g_{\pi+\nu\mu})\nu\mu([XY]).$$

Thus, if $\mu([XY]) = 0$, then we'll have also $\pi([XY]) = 0$, for every root π, hence (Pro-4, §22) $[XY] = 0$. This completes the proof of the proposition.

Because of the non-degeneracy of K on h, each root $\mu \in h^*$ defines a corresponding vector of $h_\mu \in h$ through the equation

$$\mu(X) = K(h_\mu, X), \text{ for all } X \in h . \qquad (2)$$

Proposition-2 *For $X \in g_\mu$, $Y \in g_{-\mu}$ such that $K(X,Y) = 1$, follows $[XY] = h_\mu$.*

In fact, for $Z \in h$ we have $K([XY],Z) = - K(Y,[XZ]) = \mu(Z)K(X,Y)$. Note also that $[g_\mu, g_{-\mu}] \cap \mu^\perp = \{0\}$ implies

$$K(h_\mu, h_\mu) = \mu(h_\mu) \neq 0 . \qquad (3)$$

The **coroots** of the Cartan subalgebra h are the vectors of h

$$H_\mu = \frac{2}{\mu(h_\mu)} h_\mu, \text{ for every root } \mu \in h. \qquad (4)$$

Note that

$$\mu(H_\mu) = 2. \qquad (5)$$

Note also that each root $\mu \in h^*$ generates a copy of sl(2;C) (as subalgebra of g). In fact, choosing $X \in g_\mu$, $Y \in g_{-\mu}$ such that $[XY] = H_\mu$, we have

$$[H_\mu X] = 2X, \quad [H_\mu Y] = -2Y, \quad [XY] = H_\mu . \qquad (6)$$

Proposition-3 *For each root $\mu \in h^*$, $\dim g_\mu = 1$ and $2\mu, 3\mu, \ldots$, are not roots.*

To prove this, we choose X, Y as in (6) and compute again in two different ways the trace of ad[XY], restricted on the ad[XY]-invariant subspace

$$CY \oplus CH_\mu \oplus_{t \in N} g_{t\mu} .$$
$$0 = tr(ad[XY]) = tr(adH_\mu)$$
$$= -2 + \sum_{t \in N} t\, 2\, (\dim g_{t\mu})$$
$$= 2(-1 + \dim g_\mu + 2\dim g_{2\mu} + 3\dim g_{3\mu} + \ldots) .$$

The only possibility for that is

$$\dim g_\mu = 1, \dim g_{2\mu} = \dim g_{3\mu} = \ldots = 0.$$

Choosing for each root $\mu \in h^*$ the **root-vectors** $X_\mu = X$ and $X_{-\mu} = Y$, as in (6) and denoting by Δ the set of roots of h, we get the decomposition of the Lie algebra

$$g = h \oplus_{\mu \in \Delta} CX_\mu . \qquad (7)$$

Exercise-1 Adopting the notation of §12, prove the following facts for $g = sl(n;C)$:

a) The subalgebra of diagonal matrices is a Cartan subalgebra h of g.

b) $\mu_{ij}(H) = h_i - h_j$, for $i \neq j$, and diagonal $H = diag(h_1, \ldots, h_n)$, are the roots of h.

c) $[HE_{ij}] = \mu_{ij}(H)E_{ij}$, shows $\mu_{ij} = -\mu_{ji}$ and $\langle E_{ij} \rangle$ is the root space of μ_{ij} (1-dimensional).

d) $H_{ij} = [E_{ij}E_{ji}] = E_{ii} - E_{jj}$, hence E_{ij}, E_{ji} and H_{ij} are as in (6).

All these things will be specified in time,
  With strict regard to Aristotle's rules,
The *vade mecum* of the true sublime,
  Which makes so many poets, and some fools,
Prose poets like blank-verse, I'm fond of rhyme,
  Good workmen never quarrel with their tools,
I've got new mythological machinery,
and very handsome supernatural scenery.

Byron, Don Juan, Canto I, 201

24. Cartan integers and Weyl group

We continue here the analysis of the structure of a semi simple Lie algebra $\mathfrak{g}$ in terms of a Cartan subalgebra $\mathfrak{h}$ with corresponding set of roots Δ root-vectors X_μ and coroots H_μ, as in (6) in the preceding §. Cartan's integers are connected with the "length" of strings of roots and contain all the information needed for the reconstruction of the Lie algebra.

We denote in the sequel by $\mathfrak{g}^\mu$ the copy of sl(2;C) corresponding to the root μ:

$$\mathfrak{g}^\mu = CH_\mu \oplus CX_\mu \oplus CX_{-\mu} . \qquad (1)$$

This subalgebra of $\mathfrak{g}$ operates via ad on the string $\mathfrak{g}_\pi{}^\mu$ and leaves it invariant. Thus, we have a representation of sl(2;C) for which we know that

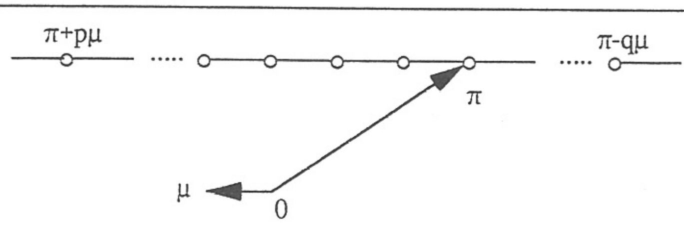
a) the root-vectors $X_{\pi+t\mu}$ are eigenvectors of adH_μ with corresponding eigenvalues:

$$(\pi+t\mu)(H_\mu) = \pi(H_\mu) + 2t, \qquad (2)$$

b) the multiplicities of the eigenvalues are 1 and their pairwise differences are even numbers.

We conclude, that this representation of $\mathfrak{g}^\mu$ is irreducible (§13) and coincides with some D_J, whose eigenvalues are known to run from 2J to -2J, with step 2. Applying this to (2) we get:

c) the integers $t \in Z$, for which $\pi+t\mu$ is a root, cover an interval [-q,p] without gaps,



the string $\mathfrak{g}_\pi{}^\mu$ has no gaps

d) the extremal eigenvalues are opposite ($\pm 2J$) hence

$$\pi(H_\mu) + 2p = -(\pi(H_\mu) - 2q)$$

$$\pi(H_\mu) \text{ are integers, } \pi(H_\mu) = q-p = c_{\pi\mu} . \qquad (3)$$

The integers $c_{\pi\mu}$ are called **Cartan integers**. Using the duals h_μ of the roots instead of the coroots $H_\mu (=2h_\mu/K(h_\mu,h_\mu))$ we obtain

$$c_{\pi\mu} = \pi(H_\mu) = 2K(h_\pi,h_\mu)/K(h_\mu,h_\mu). \qquad (4)$$

Theorem-1 *For every non zero root $\pi \in \Delta$ the only multiples of π, which are again roots are $\pm\pi$.*

In fact, if $\mu = d\pi$ were a root, then we should have $\mu(H_\pi) = d\pi(H_\pi) = 2d$ is integer and $\pi(H_\mu) = (1/d)\mu(H_\mu) = 2/d$ is integer. The only possibilities for d are $\pm 1, \pm 2, \pm 1/2$, from which the two last are ruled out by Pro-3 of §23.

Using the non-degeneracy of the Killing form K on h (Pro-7, §22), we can define a bilinear form on the dual h* of the Cartan subalgebra. In fact, each linear form (the roots are special cases) $\alpha \in h^*$ defines a dual vector $h_\alpha \in h$, by the equation

$$\alpha(X) = K(h_\alpha, X), \text{ for all } X \in h. \tag{5}$$

We define the inner product on h* by duality i.e. so that $\alpha \to h_\alpha$ becomes an isomorphism.

$$<\alpha,\beta> = K(h_\alpha, h_\beta). \tag{6}$$

Theorem-2 *For every non zero root $\alpha \in \Delta$, the linear map $S_\alpha: h^* \to h^*$,*

$$S_\alpha(x) = x - 2\frac{<\alpha,x>}{<\alpha,\alpha>}\alpha, \tag{7}$$

is a reflexion on the hyperplane $<\alpha,x> = 0$, and has the property $S_\alpha(\Delta) \subset \Delta$.

In fact, we have a reflexion since $S_\alpha(\alpha) = -\alpha$ and $<\alpha,x> = 0 \Rightarrow S_\alpha(x) = x$. The second property is a consequence of (4) since $\beta \in \Delta \Rightarrow$

$$S_\alpha(\beta) = \beta - 2(<\alpha,\beta>/<\alpha,\alpha>)\alpha = \beta - c_{\beta\alpha}\alpha \text{ and}$$

$$-q \leq -(q-p) \leq p.$$

The last inequality means that

$$S_\alpha(\beta) = \beta - c_{\beta\alpha}\alpha \tag{8}$$

belongs to the string $g_\beta^{\ \alpha}$. q.e.d.

Theorem-3 *Let $h^*_0 = <\Delta>_R \subset h^*$ be the <u>real</u> linear span of the set of roots Δ. Then the restriction of <...,...> on h^*_0 defines an inner product on it. Thus, h^*_0 becomes in a natural way a Euclidean space.*

In fact, by (6) of §22, the restriction of the Killing form on h is given by

$$K(X,Y) = \sum_{\alpha \in \Delta} \alpha(X)\alpha(Y), \tag{9}$$

which for $X = H_\alpha$ and $Y = H_\beta$ implies that

$$K(H_\alpha, H_\beta) = \sum_{\alpha \in \Delta} \alpha(H_\alpha)\alpha(H_\beta) \text{ is an integer } \Rightarrow$$

$$K(H_\alpha, H_\alpha) = 4K(h_\alpha, h_\alpha)/K(h_\alpha, h_\alpha)^2 = 4/K(h_\alpha, h_\alpha) \text{ is an integer } \Rightarrow$$

$$K(h_\alpha, h_\beta) \text{ is a rational } \Rightarrow \alpha(h_\beta) \text{ is rational } \Rightarrow$$

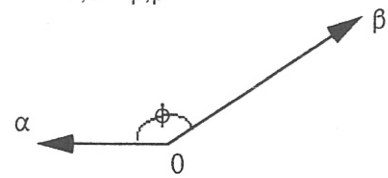
$\alpha(X)$ is real for every real combination of the h_α's.

The last assertion together with (9) and the fact that $\{h_\alpha \mid \alpha \in \Delta\}$ contains a basis of h (Pro-5 of §22), shows that <.......> is positiv definit on h^*_0.

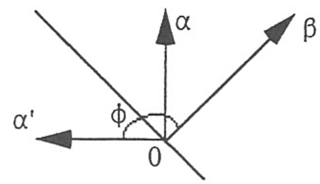
Having the inner product at our disposal, we can use the geometric language. The group of isometries of h^*_0 generated by the reflexions $\{S_\alpha \mid \alpha \in \Delta\}$ is called the **Weyl group** of the Lie algebra. Also, we can see that the angles and relative lengths of the roots are strongly restricted by the integrality of the $c_{\alpha\beta}$. In fact,

$$c_{\beta\alpha} = \beta(H_\alpha) = q - p = 2<\beta,\alpha>/<\alpha,\alpha> \Rightarrow$$

$$c_{\beta\alpha}\cdot c_{\alpha\beta} = 4\frac{{<\beta,\alpha>}^2}{<\alpha,\alpha><\beta,\beta>} = 4\cos^2\phi \quad \text{is an integer.} \tag{10}$$



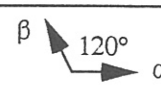
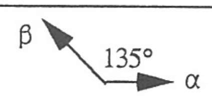
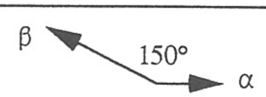
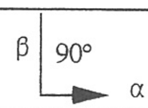
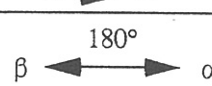
The angle ϕ between α,β may be assumed to be $\geq \pi/2$, since in the contrary case, we can reflect α by S_β and obtain $\alpha' = S_\beta(\alpha)$, which forms then an obtuse angle with β.



Condition (10) means that the only possibilities are

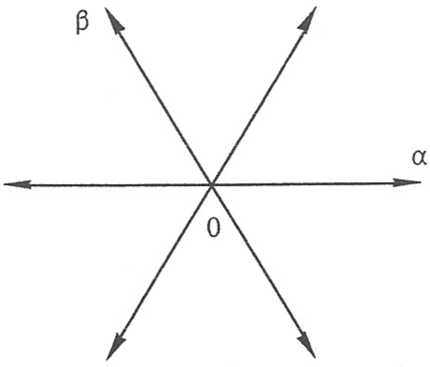
$$c_{\beta\alpha}\cdot c_{\alpha\beta} = 4\cos^2\phi = 0,\ 1,\ 2,\ 3 \text{ and } 4, \tag{11}$$

which produce the following table:

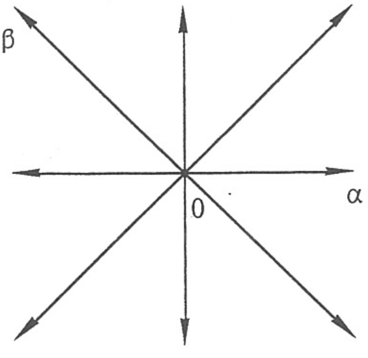
$c_{\alpha\beta}\cdot c_{\beta\alpha}$	$c_{\beta\alpha}$	$c_{\alpha\beta}$	ϕ	$\lvert\beta\rvert^2/\lvert\alpha\rvert^2 = c_{\beta\alpha}/c_{\alpha\beta}$	**vector configuration**
1	-1	-1	$2\pi/3$	1	
2	-2	-1	$3\pi/4$	2	
3	-3	-1	$5\pi/6$	3	
0	?	?	$\pi/2$	?	
4	-2	-2	π	1	

Let us see the systems of rank two i.e. the root systems for Cartan subalgebras of dimension two:

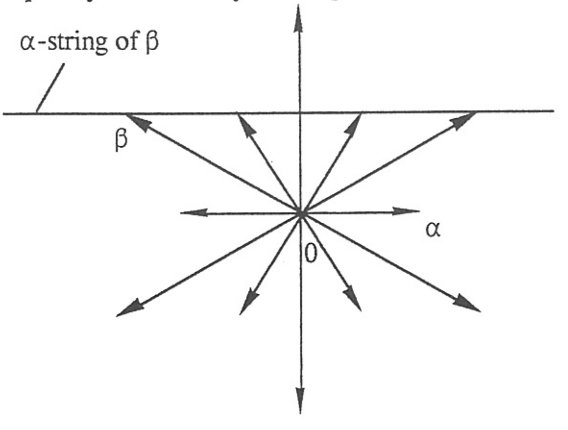
A_2: All the roots are equal and their pairwise angles are 60°. The Cartan integers are $c_{\beta\alpha} = q-p = -1$, hence $q = 0$, $p = 1$. The Weyl group coinsides with the symmetric group of three elements and has 6 elements.



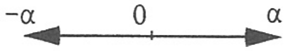
B_2 : $q\text{-}p = -2$, $q = 0$, $p = 2$ i.e. the α-string of β contains three elements. The Weyl group coincides with the dihedral group D_4 and has 8 elements.



G_2 : $q\text{-}p = -3$, $q = 0$, $p = 3$. One determines the α-string of β and gets all other roots by reflexions and symmetries. The Weyl group in this case is the dihedral group D_6 of 12 elements. Note that the corresponding Lie algebra has 2+12 = 14 dimensions and, as we'll see, its structure is completely described by the diagram below.

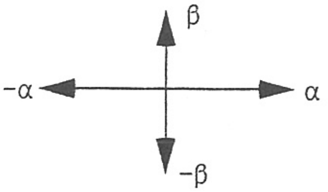


There is only one system of rank one, this of $A_1 = sl(2;C) = \langle H, X_+, X_- \rangle$ (in the notation of §13) with Cartan subalgebra CH and root diagram



$\alpha(cH) = c\alpha(H) = 2c.$

From this results one more diagram of rank 2, which corresponds to a direct sum of two copies of sl(2;C) : $A_1 \oplus A_1$



Note that in a Cartan subalgebra $\mathfrak{h}$ of arbitrary dimension, the plane spanned by two non zero roots $\alpha, \beta \in \Delta$ will define a subset of Δ (the roots contained in that plane), which will have one of the four configurations A_2 , B_2,G_2 and $A_1 \oplus A_1$.

This happens because the restrictions which imply these four possibilities, hold for any system of roots contained in some two dimensional plane.

Exercise-1 Let $\mathfrak{h}_0$ be the real subspace of $\mathfrak{h}$ generated by $\{h_\alpha | \alpha \in \Delta\}$. Show that the restriction of the Killing form on $\mathfrak{h}_0$ defines an inner product $<\ldots,\ldots>$ s.t. the map $\alpha \to h_\alpha$ becomes an isometry of euclidean spaces. The set of roots Δ, may be identified through this isometry, with the set of vectors $\{h_\alpha | \alpha \in \Delta\}$, and the reflexions S_α may by identified with the reflexions (of $\mathfrak{h}_0$) on the hyperplanes $\alpha^\perp$, defined by the equations $\alpha(X) = 0$.

Exercise-2 Show that, with the preceding identifications, the Weyl group becomes a group of isometries of $\mathfrak{h}_0$ and its elements f have the property $f(\Delta) \subset \Delta$.

Nicht die bewiesenen Urtheile, noch ihre Beweise; sondern jene aus der Anschauung unmittelbar geschöpften und auf sie, statt alles Beweises, gegründeten Urtheile sind in der Wissenschaft das, was die Sonne im Weltgebäude: denn von ihnen geht alles Licht aus, von welchem erleuchtet die andern wieder leuchten.

A. Schopenhauer, Die Welt als ... I, p. 103

25. Coxeter - Dynkin diagrams

The classification of the configurations of the root systems Δ (considered as finite subsets of vectors of some euclidean space) or their dual vectors $\{h_\alpha | \alpha \in \Delta\}$ we obtained for the case of rank two in the preceding §, may be generalized to arbitrary rank. Our analysis so far suggests the following formal definition:

We call **Root system** a pair (V, Δ) consisting of a euclidean vector space V, with inner product denoted by $<...,...>$ and a finite subset Δ of V which generates V and whose elements are called "roots" and have the properties :

a) $c_{\beta\alpha} = 2<\beta,\alpha>/<\alpha,\alpha>$ is an integer for all α,β in Δ,

b) $\alpha, \beta \in \Delta \Rightarrow \beta - c_{\beta\alpha}\,\alpha \in \Delta$, for all α, β in Δ,

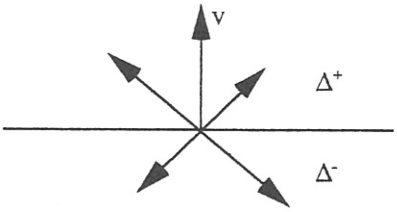
c) $\beta \in \Delta$ and $\mu\beta \in \Delta \Rightarrow \mu = \pm 1$.

The **direct sum** of two root systems (V', Δ'), $(V", \Delta")$ is the root system $(V'\oplus V", \Delta'\oplus\Delta")$, which trivially verifies the above conditions, with respect to the metric $<...,...>'\oplus<...,...>"$ of $V'\oplus V"$. Note that $<\alpha,\beta> = 0$, with respect to this metric, when $\alpha\in\Delta'$ and $\beta\in\Delta"$. A root system is called **simple**, when it is not the direct sum of two other. We'll see that simple root systems correspond to simple Lie algebras, whereas semi simple Lie algebras define root systems which are direct sums of other simple systems.

Exercise-1 Let (V, Δ) be a root system and W a vector subspace of V. Show that $(W, \Delta\cap W)$ is again a root system.

Exercise-2 Show that the root system (h^*_0, Δ) of a simple Lie algebra g (with respect to some Cartan subalgebra h of g) is a simple root system in the preceding sense.

We pass now to the classification of simple root systems. Condition c) implies that root systems are symmetric with respect to the origin 0. The following trick reduces to the "half" of the root system. For this, we consider a vector $v \in V$ not contained in any hyperplane $\alpha^\perp$ orthogonal to a root $\alpha\in\Delta$ i.e. $<v,\alpha> \neq 0$, for every $\alpha\in\Delta$. The hyperplane $v^\perp$ separates Δ in two subsets, $\Delta^+ = \{\alpha\in\Delta \,|<\alpha,v> > 0\}$ and $\Delta^- = \{\alpha\in\Delta\,|<\alpha,v> < 0\}$.



Obviously Δ^+, Δ^- are disjoint and their union is Δ. The elements of Δ^+ are called **positive** roots (write $\alpha>0$), those of Δ^- **negative** (write $\alpha<0$). A root $\alpha\in\Delta^+$ is called **simple** when it cannot be written as a sum $\alpha = \alpha'+\alpha"$, with $\alpha', \alpha"\in\Delta^+$. Obviously there are simple roots

(Start f.e. with some positive root and decompose successively in other positive roots).

Proposition-1 *For two simple roots* $\alpha, \beta \in \Delta^+$ *we have* $<\alpha,\beta> \leq 0$.

The proof of this follows from the table of two dimensional root systems of the preceding §. This table is valid also in the present abstract setting, since it uses only the integrality of the $c_{\alpha\beta}$. Looking at this table and supposing $<\alpha,\beta> > 0$ and $\alpha \neq \pm\beta$, we see that $c_{\alpha\beta} = 1$ or $c_{\beta\alpha} = 1$. In the first case we have (by b)) $\alpha-\beta \in \Delta^+$ and $\alpha = (\alpha-\beta) + \beta$ i.e. α is not simple, contradiction. Similarly in the second case, $\alpha-\beta \in \Delta^+ \Rightarrow \beta-\alpha \in \Delta^+$ and $\beta = (\beta-\alpha) + \alpha$, hence β is not simple, contradiction. q.e.d.

Proposition-2 *The set of simple roots* $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ *is independent and every root can be uniquely written as a linear combination with integer coefficients :*

$$\alpha = n_1\alpha_1 + n_2\alpha_2 + \ldots + n_d\alpha_d, \text{ with all } n_i \geq 0, \text{ when } \alpha \in \Delta^+, \text{ and all } n_i \leq 0, \text{ when } \alpha \in \Delta^-.$$

Decomposing successively a positive root, we see immediately that it can be written as a linear combination of simple roots. For a negative root, we work similarly with $-\alpha$. Uniqueness follows from independence and independence from the following:

Let $\sum r_i \alpha_i = 0$. Separate the summands in positive and negative r_i:

$$0 = \sum r_i \alpha_i = \sum r'_i \alpha_i - \sum r''_j \alpha_j \Rightarrow \sum r'_i \alpha_i = \sum r''_j \alpha_j = \gamma \ (r'_i, r''_j \geq 0) \Rightarrow$$

$$0 \leq <\gamma,\gamma> = \sum_{ij} r'_i r''_j <\alpha_i,\alpha_j> \leq 0 \Rightarrow$$

$$\gamma = 0 \Rightarrow$$

$$0 = <v,\gamma> = \sum r'_i <v,\alpha_i>, \text{ which by } <v,\alpha_i> > 0 \Rightarrow r'_i = 0.$$

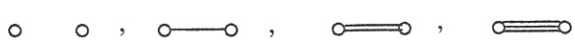
Similarly $r''_i = 0$. This completes the proof of the proposition.

A set $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$, as in the preceding proposition is called a **fundamental system of roots.** The number d is called the **rank** of the root system. It corresponds (in the present abstract setting) to the dimension of the Cartan subalgebra. We proceed now to the classification of the fundamental systems by means of their corresponding **Dynkin diagrams**, which are defined from Π in the following way :

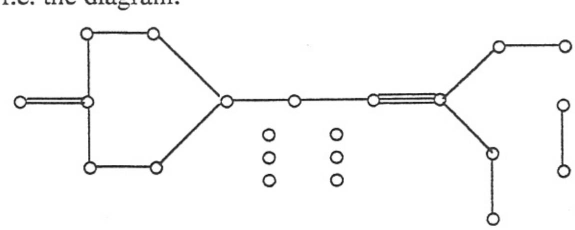
A) To each vector μ of Π we correpond a vertex.

B) We connect two vertices corresponding to the roots α, β by $c_{\beta\alpha}c_{\alpha\beta}$ lines.

Thus, there are the following four possibilities for connections between two points of the graph:



One could ask if f.e. the diagram:

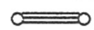


is a possible Dynkin diagram, corresponding to some fundamental system of roots. We'll see that this is impossible. In fact there are severe restrictions, which allow only a few configurations. These restrictions result from the fact that Π is a basis of the vector space V and the number of lines connecting two vertices α, β is related to the angle ϕ between these vectors ($c_{\beta\alpha}c_{\alpha\beta} = 4\cos^2\phi$), thus considering unit vectors in the direction of the roots we get a matrix representing the inner product of V whose entries are directly related to the diagram. This matrix must define an inner product (positive definite). This is the restriction!

Exercise-3 Show that a root system is simple if and only if, its Dynkin diagram is connected.

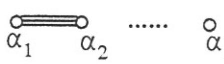
We proceed now to the classification of the fundamental systems of roots by assuming that their corresponding Dynkin diagrams are connected. The method is to consider the corresponding basis of unit vectors, rule out the impossibilities which lead to negative lengths and check the remaining cases.

1) When the diagram contains the subdiagram



then it contains nothing else.

In fact, in the contrary case the diagram should contain a subdiagram of the form



which for the corresponding unit vectors of the basis would give the inner products:

$$<\alpha_1,\alpha_2> = -\sqrt{3}/2,$$
$$<\alpha_2,\alpha_3> \leq -1/2\ ,$$
$$<\alpha_1,\alpha_3> \leq 0\ .$$

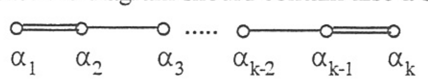
Then the vector $\alpha = \sqrt{3}\alpha_1+2\alpha_2+\alpha_3$ would have square length

$$<\alpha,\alpha> = 3|\alpha_1|^2+4|\alpha_2|^2+|\alpha_1|^2+ 4\sqrt{3}<\alpha_1,\alpha_2>+2\sqrt{3}<\alpha_1,\alpha_3>+4<\alpha_2,\alpha_3>$$
$$\leq 3+4+1-6+0-2 = 0,$$

a contradiction.

2) Each diagram contains at most one subdiagram of the form

In fact, in the contrary case the diagram should contain also a subdiagram of the form



which gives the inner products for the corresponding unit vectors:

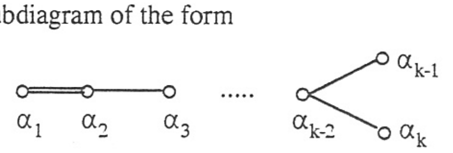
$$<\alpha_1,\alpha_2> = <\alpha_{k-1},\alpha_k> = -1/\sqrt{2}\ ,$$
$$<\alpha_r,\alpha_{r+1}> = -1/2\ , \text{ for } r = 2, \ldots, k\text{-}2.$$

Then the vector $\alpha = 1/\sqrt{2}\alpha_1+\alpha_2+\ldots+\alpha_{k-1}+1/\sqrt{2}\alpha_k$ has square-length

$$<\alpha,\alpha> = 1/2 + 1 + \ldots + 1 + 1/2 + (-2(1/2) - 2(1/2) - \ldots - 2(1/2) - 2(1/2)) = 0$$

a contradiction.

3) There is no subdiagram of the form



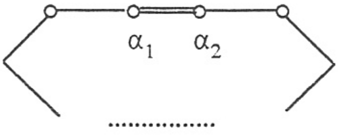
In fact, in the contrary case the vector

$$\alpha = 1/\sqrt{2}\alpha_1+\alpha_2+...+ \alpha_{k-2}+ 1/2\alpha_{k-1} + 1/2\alpha_k$$

would have square length $<\alpha,\alpha> \leq 0$.

4) For the diagrams containing  remain the possibilities:

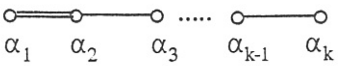
4.1)



(closed)

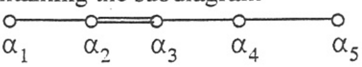
which is impossible, since in the contrary case the root vectors α_1, α_2 should have equal lengths (since they are joined by simple lines, see the table of §24).

4.2)



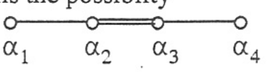
which exists and is the Dynkin diagram of the classical Lie algebras of type B_k and C_k.

4.3) Diagrams containing the subdiagram



which is impossible since the vector $\alpha = \sqrt{2}\alpha_1+ 2\sqrt{2}\alpha_2+ 3\alpha_3+2\alpha_4 + \alpha_5$ has $<\alpha,\alpha> \leq 0$.

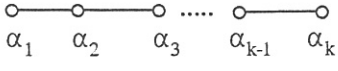
4.4) There remains the possiblity



which actually occurs and corresponds to the exceptional Lie algebra of type F_4.

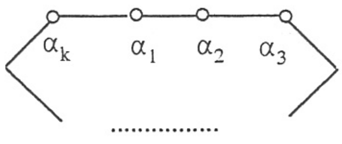
Connected Dynkin diagrams comprising only simple lines are the following:

5)



which corresponds to the Lie algebra of type A_k.

6)



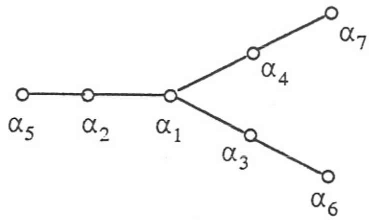
(closed)

which is impossible since the vector $\alpha = \alpha_1+\alpha_2+...+ \alpha_k$ has $<\alpha,\alpha> \leq 0$.

7) A subdiagram of the form

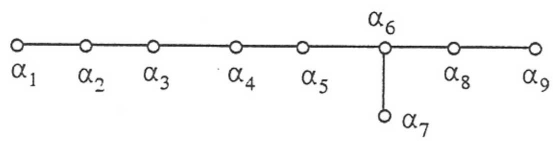
is impossible since the vector $\alpha = (\alpha_1+\alpha_2)/2+\alpha_3+...+ \alpha_{k-2} + (\alpha_{k-1} + \alpha_k)/2$ has $<\alpha,\alpha> \leq 0$.

8) A subdiagram of the form



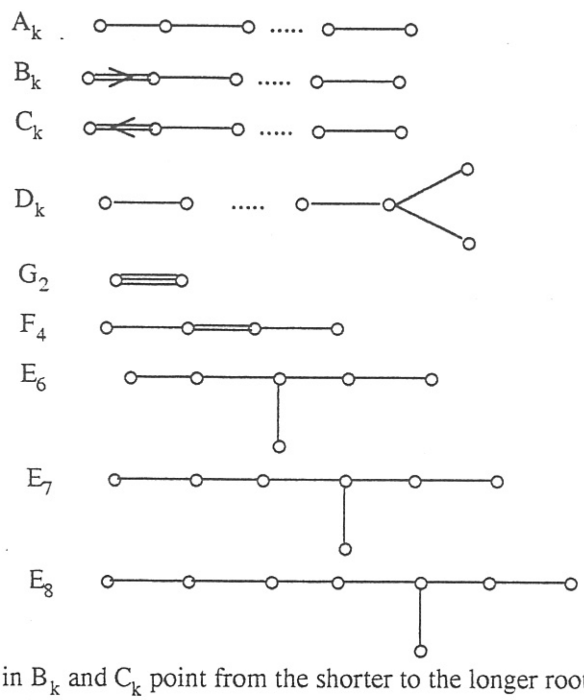
is impossible since the vector $\alpha = 3\alpha_1+2(\alpha_2+\alpha_3+\alpha_4)+\alpha_5+\alpha_6+\alpha_7$ has $<\alpha,\alpha> \leq 0$.

9) Finally, there is no subdiagram of the form



since the vector $\alpha = (\alpha_1+\alpha_2+\alpha_4+\alpha_9)/2+(\alpha_5+\alpha_8+\alpha_7)+3/2\alpha_6$ has $<\alpha,\alpha> \leq 0$.

Thus, all possible diagrams are the following:



The arrows in B_k and C_k point from the shorter to the longer roots. As we'll see in the following, the first four types correspond to the classical Lie algebras whereas the remaining five types correspond to the exeptional Lie algebras of respective dimensions 14, 52, 78, 133 and 248. Instead of verifying that the matrix $(<\alpha_i,\alpha_j>)_{i,j}$ corresponding to each of the 9 preceding diagrams, defines actually a positive definite inner product on V, we'll do much more in the next paragraphs, namely we'll construct explicit models of Lie algebras, in which the preceding 9 configurations appear as the Dynkin diagrams of their root systems.

The following exercise formulates in the abstract setting of root systems a property already proved in the case of the root system of a semi simple Lie algebra (see c), p. 24-1).

Exercise-4 Let (V,Δ) be a root system and $\alpha,\beta\in\Delta$, $\beta\neq\pm\alpha$, then $<\alpha,\beta> > 0 \Rightarrow \alpha-\beta\in\Delta$, and $<\alpha,\beta> < 0 \Rightarrow \alpha+\beta\in\Delta$. [at least one of $c_{\beta\alpha}$ or $c_{\alpha\beta}$ is +1 (see p. 243)]

Exercise-5 Let (V,Δ) be a root system, $\alpha,\beta\in\Delta$, $\beta\neq\pm\alpha$ and p, q the largest non negative integers such that $\beta-q\alpha$, $\beta+p\alpha\in\Delta$. Show that

a) $\beta+k\alpha\in\Delta$, for all k with $-q\leq k\leq p$,

b) $c_{\beta\alpha} = 2<\alpha,\beta>/<\alpha,\alpha> = q-p$.

[Apply the preceding ex. to the extremal roots of a maximal gap lying inside the string. To prove b) notice that S_α leaves the string invariant and that $S_\alpha(\beta+p\alpha) = \beta-q\alpha$.]

Exercise-6 Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be a subset of the root system Δ. Show that this is a fundamental system of roots if and only if

1) $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ are independent, and

2) every root can be written as a linear combination with integer coefficients :

$\alpha = n_1\alpha_1+n_2\,\alpha_2+ \dots +n_d\,\alpha_d$, with all $n_i \geq 0$, or all $n_i \leq 0$.

[Consider the "duals" of the α_i's, $<\beta_i, \alpha_j> = \delta_{ij}$. Define $v=\sum\beta_i$, $<v,\alpha_i>=1$, for all i.]

Exercise-7 Show that a set of roots $\{\alpha, \beta, \gamma, \dots\}$ lying strictly on one side of a hyperplane of V, and having all pairwise products $<a,b> \leq 0$, is independent. [imitate the proof of Pro-2]

The classification of fundamental systems implies the classification of simple root systems, and this in turn implies the classification of simple complex Lie algebras. To prove this, we must still show that the different fundamental systems are isometric and independent of the particular Cartan subalgebra. The question of the congruence of the different fundamental systems of the same root system, will be handled, in the abstract setting of root systems, in the next paragraph. The question of the independence from the different Cartan subalgebras will be tackled in § 38.

La nature a mis toutes ses vérités chacune en soi-même, notre art les renferme les unes dans les autres, mais cela n'est pas naturel, chacune tient sa place.

Pascal, Pensées, 21

26. Weyl group and Weyl chambers

We continue here the analysis of abstract root systems initiated in the preceding §. Our main concern is to find out all the fundamental systems of a given simple root system and their relations. The corner-stone of the subject is proved to be the **Weyl group** of the abstract root system (V,Δ), which is defined (as for a root system of a Lie algebra, §24) to be the subgroup of isometries of V, generated by the reflections

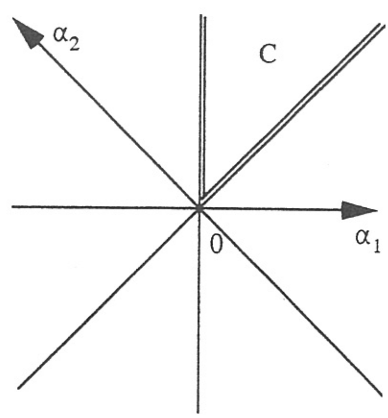
$$S_\alpha(x) = x - 2\frac{<\alpha,x>}{<\alpha,\alpha>}\alpha, \quad \alpha\in\Delta. \tag{1}$$

By the axioms of root systems (see p. 25-1) we know that for any pair of roots α, $\beta\in\Delta$ the numbers $c_{\beta\alpha} = 2<\alpha,\beta>/<\alpha,\alpha>$ are integers and $S_\alpha(\Delta) = \Delta$. Consequently the Weyl group is a subgroup of permutations of the elements of Δ, hence it is finite.

Proposition-1 *Any vector $v\in V$ not contained in any hyperplane $\alpha^\perp$ orthogonal to a root $\alpha\in\Delta$ i.e. $<v,\alpha> \neq 0$, for every $\alpha\in\Delta$, defines a fundamental system $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ uniquely, up to a permutation of its elements.*

In fact, the first two propositions of §25 give the way to construct Π out of v. We prove here the uniqueness. If there were two such systems, Π and Π', then the matrices A, B which change from one basis to the other, should have both only non-negative elements and satisfy AB=I (inverses). The rows $\mathbf{a}_i$ of A and the columns $\mathbf{b}_j$ of B should satisfy $<\mathbf{a}_i,\mathbf{b}_j> = \delta_{ij}$, and consequently, for $i\neq j$ the $\mathbf{b}_j$'s should have zeroes at the places where $\mathbf{a}_i$ has non-zero coordinates. By independence, $\mathbf{a}_i$ has one non-zero coordinate, and in fact exactly one, since in the contrary case the $\mathbf{b}_j$'s with $i\neq j$ would spann a subspace of dimension $\leq n-2$. The non-zero coordinate must be 1, since simple roots are not multiples of other roots. The argument shows that A is a permutation matrix and completes the proof.

Given a root system (V,Δ), the union of planes $\{\alpha^\perp | \alpha\in\Delta\}$ separates V into connected components which are called **Weyl chambers**.



Exercise-1 Show that the Weyl chambers of a root system are connected, open and convex subsets of V. Show also that two points **x**,**y** $\in V$, belong to the same chamber C, if and only if the product

$<\alpha,x><\alpha,y> > 0$, for every $\alpha\in\Delta$.

Exercise-2 Show that a fundamental system $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ defines a chamber

$C = \{x\in V | <\alpha_i,x> > 0, \text{ for } i=1,\dots,d\}$.

This is called **fundamental chamber for Π**.

The **maximal dimension faces** of the Weyl chamber C are defined to be the subsets of the closure cl(C), having $<\alpha,x> = 0$, for exactly

one root $\alpha \in \Delta$.
The hyperplane $\alpha^\perp$ containing such a face of C is called a **wall** of the Weyl chamber C. Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be a fundamental system and denote by S_i the reflections on the hyperplanes $\alpha_i^\perp$, which are the walls of the fundamental chamber C for Π. The following is a technical fact, used repeatedly in the arguments below.

Proposition-2 *For each positive (negative) root $\alpha \neq \alpha_i$ ($\alpha \neq -\alpha_i$), $S_i(\alpha)$ is again a positive (negative) root.*

In fact, $S_i(\alpha)$ is again a root and (expressing it in terms of the basis Π) contains some $\alpha_j \neq \alpha_i$ with positive coefficient, hence all the coefficients in $\alpha = \sum n_k \alpha_k$ will be positive (see Pro-2, §25).

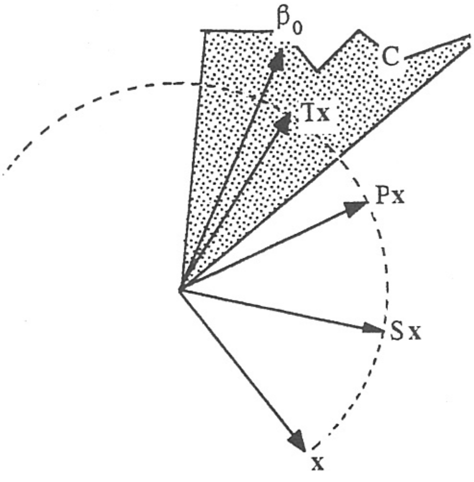
Proposition-3 *Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be a fundamental system and C_0 the corresponding fundamental chamber. For each other chamber C, there is an element S of the Weyl group W such that $S(C) = C_0$ Thus, the Weyl group operates transitively on the set of Weyl chambers.*

In fact, inside C_0 there is a special element:

$$\beta_0 = (1/2) \sum_{\alpha \in \Delta^+} \alpha .$$

To prove this, notice that for each i = 1,...,d we have (using Pro-2):

$$S_i(\beta_0) = (1/2) \sum_{\alpha \in \Delta^+ - \alpha_i} S_i(\alpha) + (1/2) S_i(\alpha_i) = (1/2) \sum_{\alpha \in \Delta^+ - \alpha_i} \alpha - (1/2)\alpha_i = \beta_0 - \alpha_i .$$



On the other side $S_i(\beta_0) = \beta_0 - 2<\beta_0,\alpha_i>/<\alpha_i,\alpha_i>\alpha_i$, hence $2<\beta_0,\alpha_i>/<\alpha_i,\alpha_i> = 1 > 0$. This proves that β_0 is indeed in C_0. Let now $x \in V$ and consider the orbit $Wx = \{S(x) | S \in W\}$. By the finiteness of W, there is some point Tx of the orbit nearest to β_0, for which $<\beta_0,Tx>$ is a maximum (angle $\angle(\beta_0,Tx)$ is a minimum). Then $<\beta_0,Tx> \geq <\beta_0,S_iTx> = <S_i\beta_0,Tx> = <\beta_0-\alpha_i,Tx> = <\beta_0,Tx> - <\alpha_i,Tx>$. Hence

$$<\alpha_i,Tx> \geq 0.$$

Thus, for each $x \in V$ there is some T in W, such that $Tx \in cl(C_0)$. Given a chamber C take then some x in C and as before find a T in W, such that $Tx \in cl(C_0)$. Since $T(\Delta)=\Delta$ we'll have $T(C) \subset C_0$ which implies $T(C) = C_0$. q.e.d.

Theorem-1 *The fundamental systems Π of a root system (V,Δ) are isometric by elements of the Weyl group W. Each of them consists of the roots which are orthogonal to the walls of some Weyl chamber C and point to the side of the wall containing C. C is then the fundamental chamber for this system.*

In fact, each fundamental system is defined by some $v \in V$, such that $<v,\alpha> \neq 0$ for all $\alpha \in \Delta$. Then v defines a Weyl chamber C and the fundamental system consisting of the roots orthogonal to the walls of C. The rest follows from the preceding proposition.

Exercise-3 For each $\alpha \in \Delta$ let S_α be the corresponding reflexion defined as in (1). Then for T in the Weyl group and the root $\beta = T(\alpha)$ we have $S_\beta = T \circ S_\alpha \circ T^{-1}$. [examine the operation of $T \circ S_\alpha \circ T^{-1}$ on β and on $\beta^\perp$]

Exercise-4 Let $\alpha \in \Delta$ and $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ a fundamental system of Δ. Then there is a T in the Weyl group W, such that $T(\alpha) \in \Pi$. [consider a Weyl chamber having $\alpha^\perp$ as a wall]

Exercise-5 With the previous notations, show that for every $\alpha \in \Delta^+$ there is some $\alpha_i \in \Pi$, with $<\alpha,\alpha_i> > 0$. [if $<\alpha,\alpha_i> \leq 0$ for all i, then $<\alpha,\alpha> = \sum n_i <\alpha,\alpha_i> \leq 0$]

Proposition-4 *Let $\alpha \in \Delta^+$ and $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be a fundamental system of roots. Let also S_i be the reflection corresponding to α_i. Then there are indices $i_1, \dots, i_k$ such that*

$$(S_{i_k} \circ \dots \circ S_{i_1})(\alpha) \in \Pi .$$

In fact, if $\alpha \in \Pi$ then we are done. Let $\alpha \notin \Pi$ and $v \in C_0$. Then by Ex-5, there is some i such that $<\alpha,\alpha_i> > 0$. Hence by Pro-2, $S_i(\alpha) \in \Delta^+$ and $<S_i(\alpha),v> < <\alpha,v>$. If $S_i(\alpha) \in \Pi$, then we are done, else by the same reasoning there will be some j such that $S_j(S_i(\alpha)) \in \Delta^+$ $<S_j(S_i(\alpha)),v> < <S_i(\alpha),v> < <\alpha,v>$. Continuing in this way, by the finiteness of Δ^+, we find $i_1=i$, $i_2=j$, ... , as required.

Theorem-2 *Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be a fundamental system of roots. Then the Weyl group W is generated by the corresponding "fundamental" reflections $\{S_1, \dots, S_d\}$.*

The proof is contained in the previous proposition and the exercises. Since $S_{-\alpha} = S_\alpha$ we consider only $\alpha \in \Delta^+$. By Pro-4, there is a $T = S_{i_k} \circ \dots \circ S_{i_1}$, such that $T(\alpha) = \alpha_j \in \Pi$. Then, by Ex-3, $S_\alpha = T^{-1} \circ S_j \circ T$. q.e.d.

Proposition-3 shows that the Weyl group operates transitively on the set of Weyl chambers and on the set of fundamental systems. These two sets are intimatelly connected, as is proved by theorem-1. The next theorem shows that this operation of the Weyl group is "simple transitive".

Theorem-3 *The Weyl group W operates simply transitively on the set of Weyl chambers i.e. if $T \in W$ and $T(C)=C$ for a Weyl chamber C, then $T= Id$.*

To prove the theorem we consider a fundamental system of roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ and denote the fundamental Weyl chamber for Π by C. We assume further that $T(C)=C$. By the previous theorem T can be written $T = S_{i_k} \circ \dots \circ S_{i_1}$, hence $T(\Pi) = \Pi$. We show that in the preceding decomposition of T, we can reduce the factors from k to k-2. To do this we observe the orbit of the root α_{i_1}, under the successive reflections:

$$\alpha_{i_1} \xrightarrow[S_{i_1}]{} -\alpha_{i_1} \xrightarrow[S_{i_2}]{} \dots \xrightarrow[S_{i_k}]{} T(\alpha_{i_1}) \in \Pi .$$

Let S_{i_r} be the first reflexion, after which the root $S_{i_r} \circ \dots \circ S_{i_1}(\alpha_{i_1})$ becomes again positive. We have the formula:

$$S_{i_{r-1}} \circ \dots \circ S_{i_2}(-\alpha_{i_1}) = -\alpha_{i_r}. \qquad (*)$$

In fact, by the hypothesis for r, all the roots

$$S_{i_2}(-\alpha_{i_1}),\ \ S_{i_3}\circ S_{i_2}(-\alpha_{i_1})\ ,\dots,\ \ S_{i_{r-1}}\circ\dots\circ S_{i_2}(-\alpha_{i_1}) = \beta\ ,$$

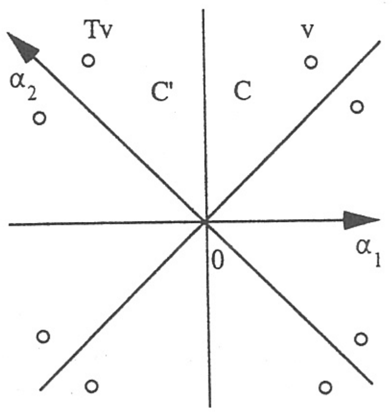
are negative. If $\beta \neq -\alpha_{i_r}$, then by Pro-2, $S_{i_r}(\beta)$ should be again negative. Thus (*) is true. Then, by Ex-3 we'll have for

$$P = S_{i_{r-1}}\circ\dots\circ S_{i_2}\ ,\quad S_{i_r} = P\circ S_{i_1}\circ P^{-1} \quad \Rightarrow \quad S_{i_r}\circ P = P\circ S_{i_1} \quad \Rightarrow$$

$$T = S_{i_k}\circ\dots\circ(S_{i_r}\circ P)\circ S_{i_1} = S_{i_k}\circ\dots\circ(P\circ S_{i_1})\circ S_{i_1} = S_{i_k}\circ\dots\circ P\ ,\ \left((S_{i_r})^2 = \mathrm{Id}\right).$$

i.e. we reduced the factors of T by two. Repeating the procedure we land at T = Id, which gives the proof or at $T=S_i$ for some i. But the last case gives the contradiction $T(C)\neq C$.

q.e.d.



The theorem implies that (fixing some fundamental Weyl chamber) the group elements of W are in one-to-one correspondance with the Weyl chambers. It follows also that for every point v of a Weyl chamber, the points of the orbit W(v) lie, each in a different Weyl chamber.

Exercise-6 If the roots $\alpha_1, \alpha_2, \dots, \alpha_t$ are simple (but not necessary different) and $S_1, \dots, S_t$ are the corresponding reflexions, show that if

$$(S_1\circ\dots\circ S_{t-1})(\alpha_t) \text{ is negative,}$$

then for some k, $1\leq k\leq t$, we'll have

$$S_1\circ\dots\circ S_t = S_1\circ\dots\circ S_{k-1}\circ S_{k+1}\circ\dots\circ S_{t-1}\ .$$

[see the proof of theorem-3]

Fixing a fundamental system of roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$, every element T of the Weyl group can be written as a product of "fundamental reflections"

$$T = S_{i_k}\circ\dots\circ S_{i_1}\ .$$

The smallest k with this property is called the **length** of the element T (with respect to Π). Obviously, if in the preceding expression, k is the length of T, then every "partial" product

$$T_s = S_{i_k}\circ\dots\circ S_{i_s}$$

will have length k-s+1 and, according to Ex-6 we'll have

$$T_s(\alpha_{i_s}) = (S_{i_k}\circ\dots\circ S_{i_s})(\alpha_{i_s}) > 0.$$

Thus, if v is in the closure $cl(C_0)$ of the fundamental Weyl chamber, then

$$\langle v, \alpha\rangle \geq 0, \text{ for every positive root } \alpha, \text{ and}$$

$$\langle v, T_s(\alpha_{i_s})\rangle = \langle T_s^{-1}(v), \alpha_{i_s}\rangle \geq 0. \qquad \Rightarrow$$

$$S_{i_s}T_s^{-1}(v) = T_s^{-1}(v) - 2\frac{\langle T_s^{-1}(v), \alpha_{i_s}\rangle}{\langle \alpha_{i_s}, \alpha_{i_s}\rangle}\alpha_{i_s}, \text{ with positive coefficient } \langle T_s^{-1}(v), \alpha_{i_s}\rangle.$$

But

$$S_{i_s}T_s^{-1}(v) = T_{s-1}^{-1}(v), \quad \text{thus } T_s^{-1}(v) = T_{s-1}^{-1}(v) + \mu_s\alpha_{i_s} \text{ and inductively}$$

$$T^{-1}(v) = v + \sum \mu_s \alpha_{i_s} \text{ with } \mu_s \geq 0.$$

This equation holds for every element T of the Weyl group. We conclude that for $T(v) \neq v$, $T(v)$ cannot belong again to the closure $cl(C_0)$, since in that case we'll have

$$v = T^{-1}T(v) = (v + \sum \mu_s \alpha_{i_s}) + \sum \lambda_s \alpha_{i_s} \text{ with } \mu_s, \lambda_s \geq 0.$$

Hence, $\mu_s = \lambda_s = 0$ and $T(v) = v$, a contradiction. We proved the theorem :

Theorem-4 *For every $v \in V$, the orbit of v under the Weyl group, intersects the closure $cl(C_0)$ of a Weyl chamber in exactly one point.*

Exercise-7 Show that the roots of a simple fundamental system Π have at most two different lenghts. [apply Ex-4 and inspect the Dynkin diagrams in p. 25-6]

Exercise-8 Show that every root of a simple root system Δ belongs to some fundamental system of Δ.

A very importand object, associated to each simple fundamental system, is the so-called **highest** or **maximal** root. This is characterized by the following theorem :

Theorem-5 *Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be a fundamental system of the simple root system Δ. Then there is a unique positive root α with the properties :*

1) $\alpha = n_1\alpha_1 + \dots + n_d\alpha_d$, with all $n_i > 0$ and

2) for every other positive root $\beta = m_1\alpha_1 + \dots + m_d\alpha_d$, $m_i \leq n_i$ for all i.

In fact, call $h(\beta) = \sum m_i$ the **height** of the positive root $\beta \in \Delta^+$. Choose then α to be of maximal height. This is the desired highest root. To see this, divide Π in two different subsets $\Pi' = \{\alpha_i \mid \alpha \text{ has } \alpha_i\text{-component}\}$ and $\Pi'' = \{\alpha_i \mid \alpha \text{ has not } \alpha_i\text{-component}\}$. By simplicity of Π, there is some root $\beta \in <\Pi''>$ non-orthogonal to $<\Pi'>$. i.e. there is some $\alpha' \in \Pi'$ such that $<\alpha',\beta> \neq 0$. Then $<\alpha',\beta> < 0$, by Pro-1, p.25-2, and consequently $<\alpha,\beta> < 0$ too. But then (by Ex-6, §25) $\alpha+\beta \in \Delta$ would be higher than α, a contradiction. Thus $\Pi'' = \emptyset$, and we proved 1).

To show uniqueness of α, notice first that $<\alpha,\beta> \geq 0$, for every root $\alpha \in \Delta^+$, since in the contrary case $\alpha+\beta \in \Delta^+$ would contradict the maximality of α. Also $<\alpha_i,\alpha> > 0$, for some i, since Π spans V. If β were another root of maximal hight, then $<\beta,\alpha> \geq 0$ and since $<\alpha_i,\alpha> > 0$, for some i, we would have $<\beta,\alpha> > 0$, which implies that α-β is a root and either $\alpha = (\alpha-\beta)+\beta$ or $\beta = (\beta-\alpha)+\alpha$ would be higher, a contradiction unless $\alpha=\beta$. Thus, there is a unique positive root of maximal hight.

2) is immediate, since every positive root β different from the heighest α, leads to α by adding α_i's until we reach α.

Exercise-9 Show that the maximal root, with respect to the fundamental system Π, lies in the closure of the corresponding fundamental Weyl chamber.

Exercise-10 Show that the integers $\{n_1, \dots, n_d\}$ entering as coeficients of the highest root α, with respect to the fundamental system $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$, are, up to permutation, invariants of the simple root system (Δ, V), and do not depend on the particular fundamental system Π. [all Weyl chambers are isometric under the Weyl group, which permutes the roots]

Exercise-11 Prove that the maximal root is a long root. [suffices to show that $<\alpha,\alpha> \geq <\beta,\beta>$, for every root in the $cl(C_0)$ (The-4). For such a β, $<\gamma,\alpha-\beta>\geq 0$, for every γ in $cl(C_0)$. Apply this for $\gamma=\alpha$ and $\gamma=\beta$.]

Exercise-12 Show that every positive root α, with respect to some fundamental system $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$, can be written as a sum $\alpha = \alpha_k + \alpha_m + \dots + \alpha_r$ in such a way that every "partial sum" α_k, $\alpha_k + \alpha_m$, ..., $\alpha_k + \alpha_m + \dots + \alpha_r$ is again a root. [use Ex-5, and (Ex-6, §25)]

Exercise-13 Prove 2) in The-5, by showing that, for every root β, different from the maximal one α, there is some fundamental root γ, such that $\beta+\gamma$ is also a root.

[If $<\beta,\gamma>\geq 0$, for every root γ, then, show by contradiction, that $\Pi''(\beta)$, constructed as in the proof of The-5, must be empty. Then, both α and β belong to $cl(C_0)$, and a reasoning like that of the proof of The-5, would show that $\alpha-\beta$ is a root etc. ...]

Eine Regel beim Lesen ist die Absicht des Verfassers, und den Hauptgedanken sich auf wenig Worte zu bringen und sich unter dieser Gestalt eigen zu machen. Wer so liest ist beschäftigt, und gewinnt, es gibt eine Art von Lektüre wobei der Geist gar nichts gewinnt, und viel mehr verliert, es ist das Lesen ohne Vergleichung mit seinem eigenen Vorrat und ohne Vereinigung mit seinem Meinungs-System.

Lichtenberg, Sudelbücher p. 321

27. The structure of sl(n+1;C) (A_n)

To prove the properties of the concrete models of Lie algebras that follow, we need the lemma:

Lemma *Let g be a complex Lie algebra, $h \subset g$ be a subalgebra, $\Delta \subset h^* - \{0\}$ be finite and such that, for every $\alpha \in \Delta$ there is a subspace g_α of g with the property*

$$g_\alpha = \{X \in g \mid adH(X) = \alpha(H)X, \text{for every } H \text{ in } h\}.$$

Suppose moreover that the following conditions hold:

a) $<\Delta> = h^*$,

b) $\Delta = -\Delta$ *and* $[g_\alpha, g_\alpha] \neq 0$, *for every* $\alpha \in \Delta$,

c) $g = h \dotplus \sum_{\alpha \in \Delta} g_\alpha$.

Then g is semisimple, h is a Cartan subalgebra and Δ is the set of roots of g, with resp. to h.

Obviously in c) we have a <u>direct</u> sum, $h = g_0$ hence, by Pro-1 §22, h is a Cartan subalgebra and $\alpha \in \Delta$ are its roots. It remains to show that g is semisimple. Obviously $[g_\alpha, g_\alpha] \subset g_{\alpha+\beta}$ and we can choose $X_\alpha \in g_\alpha$, $X_{-\alpha} \in g_{-\alpha}$, $H_\alpha = [X_\alpha, X_{-\alpha}]$ which generate a copy of sl(2;C). Then we can repeat the proofs of §§23, 24 and show that $\dim g_\alpha = 1$, $\beta(H_\alpha) = q-p$, etc.

Let now r be the radical of g. Since this is adh-invariant ideal it must satisfy

$$r = r \cap h \oplus \sum_\alpha (r \cap g_\alpha).$$

But $r \cap g_\alpha \neq \{0\} \Rightarrow X_\alpha \in r \Rightarrow$ the copy of sl(2;C) $<X_\alpha, X_{-\alpha}, H_\alpha> \subset r$, a contradiction, hence $r \subset h$. If now for some $X \in r$, $\alpha(X) \neq 0$, then $X_\alpha = \alpha(X)^{-1}[X\ X_\alpha] \in h$, again a contradiction. Thus $\alpha(X) = 0$, for every $\alpha \in \Delta$, hence X=0. q.e.d.

The rest of this § is a continuation of §12. We use the notation of that § and denote by h the abelian subalgebra of diagonal matrices $H = \mathrm{diag}(h_1, \dots, h_{n+1})$. The n linear forms defined on h by

$$\alpha_i(H) = h_i - h_{i+1}, \text{ for } i = 1, \dots, n \tag{1}$$

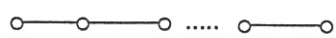
build a basis of h^* and equation (3) of §12 becomes

$$[H, E_{rs}] = (h_r - h_s)E_{rs}, \tag{2}$$

which shows (applying the lemma) that the linear forms

$$\beta_{rs} = \alpha_r + \dots + \alpha_{s-1}, \text{ for } s>r, \tag{3}$$

and their negatives are the roots of the Lie algebra. $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a fundamental system of roots and the Dynkin diagram is that of A_n.

A_n 

We compute immediately that $[E_{rs},E_{rs}] = E_{rr} - E_{ss} = H_{rs}$ and since $\beta_{rs}(H_{rs}) = 2$

$$H_{rs} = E_{rr} - E_{ss}\ , \quad \text{for } r\neq s, \text{ are the coroots.} \tag{4}$$

For the reflection S_{rs} with respect to the root β_{rs}, we compute explicitly for a $H \in h$:

$$S_{rs}(H) = H - 2(<H,H_{rs}>/<H_{rs},H_{rs}>)H_{rs} = H - 2(tr(HH_{rs})/tr(H_{rs}^2))H_{rs} = H-(h_r-h_s)H_{rs}.$$

Which means that S_{rs} operates on H by interchanging the r-th and s-th coordinates of H and leaving all other coordinates fixed. Thus the Weyl group of sl(n+1;C) is isomorphic to the permutation group S_{n+1} of n+1 objects.

So wie gewisse Schriftsteller nachdem sie ihrer Materie erst einen derben Hieb versetzt haben hernach sagen sie zerfalle von selbst in zwei Teile.

Lichtenberg, Sudelbücher p. 149

28. The structure of sp(n;C) (C_n)

In §21 we saw a model for the complex symplectic Lie algebra sp(n;C), consisting of 2n×2n complex matrices :

$$sp(n;C) = \left\{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B, C \text{ symmetric matrices}\right\}.$$

In the same § we saw that the matrices

$$H' = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix},\ A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix},\ B_{ij} = \begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix},\ C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix}$$

build a basis of sp(n;C), and it is obvious that the set $\mathfrak{h}$ of diagonal matrices H' is an abelian subalgebra. The relations

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}\right] = \begin{pmatrix} (h_i-h_j)E_{ij} & 0 \\ 0 & -(h_i-h_j)E_{ji} \end{pmatrix},$$

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}\begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & (h_i+h_j)(E_{ij}+E_{ji}) \\ 0 & 0 \end{pmatrix},$$

$$\left[\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}\begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ -(h_i+h_j)(E_{ij}+E_{ji}) & 0 \end{pmatrix}.$$

suggest that the roots are expressible by the linear forms on $\mathfrak{h}$:

$$\lambda_i(H') = h_i,\quad \text{for } i = 1, \dots, n. \tag{1}$$

In fact, we see easily that

$$\alpha_1 = \lambda_1-\lambda_2, \dots, \alpha_{n-1} = \lambda_{n-1}-\lambda_n,\ \alpha_n = 2\lambda_n \tag{2}$$

are simple roots of $\mathfrak{h}$ and build a fundamental system of roots. The positive roots are

$$\lambda_i-\lambda_j = \alpha_i + \dots + \alpha_i,\ \text{ for } i<j, \text{ and}$$
$$\lambda_i+\lambda_j = (\alpha_i+\dots+\alpha_{n-1})+(\alpha_j+\dots+\alpha_n),\ \text{for } i<j. \tag{3}$$

Applying the lemma of the preceding § and the preceding calcutations, we see that sp(n;C) is semisimple. In order to find the Dynkin diagram and the corresponding Weyl group, we compute the positive coroots :

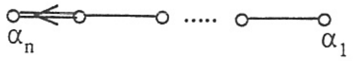
$$H'_{ij} = A_{ii} - A_{jj},\ \text{corresponding to } \lambda_i-\lambda_j,\ \text{ for } i<j,$$
$$H''_{ij} = A_{ii} + A_{jj},\ \text{corresponding to } \lambda_i+\lambda_j,\ \text{ for } i\neq j, \text{ and}$$
$$H'''_{ii} = A_{ii},\ \text{corresponding to } 2\lambda_i. \tag{4}$$

Using this we compute the Cartan Integers :

$$2<\alpha_i,\alpha_{i+1}>/<\alpha_i,\alpha_i> = 2\mathrm{tr}((A_{ii}-A_{i+1,i+1})(A_{i+1,i+1}-A_{i+2,i+2}))/\mathrm{tr}((A_{ii}-A_{i+1,i+1})^2)$$
$$= 2<\alpha_i,\alpha_i>/<\alpha_{i+1},\alpha_{i+1}> = -1, \text{ for } i=1,\dots,n-2.$$
$$2<\alpha_{n-1},\alpha_n>/<\alpha_{n-1},\alpha_{n-1}> = 2\mathrm{tr}((A_{n-1,n-1}-A_{nn})A_{nn})/\mathrm{tr}((A_{n-1,n-1}-A_{nn})^2) = -1.$$
$$2<\alpha_{n-1},\alpha_n>/<\alpha_n,\alpha_n> = 2\mathrm{tr}((A_{n-1,n-1}-A_{nn})A_{nn})/\mathrm{tr}((A_{nn})^2) = -2.$$

By $\alpha(H_\alpha)=2$, we know the lengths of roots are inverse proportional to the lengths of coroots. A_{nn} has length shorter than $A_{ii}-A_{i+1,i+1}$, hence α_n is longer, in the corresponding Dynkin diagram. Thus the Dynkin diagram is that of C_n.

C_n



We compute now the reflections, with respect to the different roots. For the case of $\lambda_i-\lambda_j$, for i<j, the corresponding reflection is

$$S_{ij}(H') = H' - 2\mathrm{tr}(H'(A_{ii}-A_{jj}))/\mathrm{tr}((A_{ii}-A_{jj})^2)(A_{ii}-A_{jj})$$
$$= H' - (h_i-h_j)(A_{ii}-A_{jj}). \qquad (5)$$

This operates on H' by interchanging i-th and j-th coordinates and leaving all other coordi nates of H' fixed. Thus, these S_{ij} generate a group isomorphic to the permutation group S_n of n objects. Analogously, we have for the reflection corresponding to the root $\lambda_i+\lambda_j$, for i<j

$$S'_{ij}(H') = H' - (h_i+h_j)(A_{ii}+A_{jj}), \qquad (6)$$

and for the root corresponding to $2\lambda_i$

$$S'_{ii}(H') = H' - 2h_iA_{ii}. \qquad (7)$$

The first of them operates on H' by interchanging the i-th and j-th coordinates and simultaneously myltiplying these coordinates by (-1). The second operates by changing only the sign of the i-th coordinate of H'. We conclude that (6) can be written as a composition of (5) and (7). The group generated by (7) has 2^n elements and the Weyl group is the product of this and the permutation group S_n. Thus the Weyl group of sp(n;C) has $n!2^n$ elements.

Exercise-1 Show that the subgroup K_n of the Weyl group W, generated by the transformations (7), is abelian and isomorphic to $(Z_2)^n$.

Exercise-2 Show that the Weyl group is a semi direct product of K_n and S_n by proving that:

a) Every subset $J \subset \{e_1, \dots, e_n\}$ defines an element $f_J \in K_n$, and inversely.
b) For two subsets J, J' as before, $f_J o f_{J'} = f_{J\#J'}$, where $J\#J' = J \cup J' - J \cap J'$.
c) Every $S \in W$ is written uniquely as a product $S = f_J oT$, with $f_J \in K_n$ and $T \in S_n$.
d) $(f_J oT)o(f_{J'} oT') = f_{J\#T(J')} o(ToT')$.

Wenn es der Himmel für nötig und nützlich finden sollte mich und mein Leben nocheinmal neu aufzulegen, so wollte ich ihm einige nicht unnütze Bemerkungen zur neuen Auflage mitteilen, die hauptsächlich die Zeichnung des Porträts und den Plan des Ganzen angehen.

Lichtenberg, Sudelbücher p. 397

29. The structure of o(2n;C) (D_n)

In §21 we saw that the complex orthogonal Lie algebra o(2n; C) may be described by the set of $2n \times 2n$ complex matrices:

$$\left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ B, C skew-symmetric, A arbitrary} \right\}.$$

A "natural" basis of this Lie algebra consists of the matrices

$$H' = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix} \text{(diagonal)},\ A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix},\ B_{ij} = \begin{pmatrix} 0 & E_{ij}-E_{ji} \\ 0 & 0 \end{pmatrix},\ C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij}-E_{ji} & 0 \end{pmatrix},$$

for which the following relations hold:

$$[H', A_{ij}] = (h_i - h_j)\, A_{ij}, \text{ for } i \neq j,$$
$$[H', B_{ij}] = (h_i + h_j)\, B_{ij}, \text{ for } i<j,$$
$$[H', C_{ij}] = -(h_i + h_j)\, C_{ij}, \text{ for } i<j. \qquad (1)$$

It follows that the linear forms on the subalgebra h of diagonal matrices:

$$\lambda_1(H') = h_1, \dots, \lambda_n(H') = h_n,$$

build a basis of h* and the roots of the Lie algebra are expressible through them. In fact, we see easily that

$$\alpha_1 = \lambda_1 - \lambda_2,\ \alpha_2 = \lambda_2 - \lambda_3,\ \dots,\ \alpha_{n-1} = \lambda_{n-1} - \lambda_n,\ \alpha_n = \lambda_{n-1} + \lambda_n, \qquad (2)$$

is a fundamental system of roots, the corresponding positive roots being:

$$\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}, \text{ for } i<j, \text{ and}$$
$$\lambda_i + \lambda_j = (\alpha_i + \alpha_{i+1} + \dots + \alpha_{n-1}) + (\alpha_j + \alpha_{j+1} + \dots + \alpha_{n-2}), \text{ for } i<j. \qquad (3)$$

Applying the lemma of §27 and the relations of (1) we see easily that o(2n; C) is semi simple. We compute the Dynkin diagram via the coroots which correspond to (3):

$$\lambda_i - \lambda_j \iff H'_{ij} = A_{ii} - A_{jj}, \text{ for } i<j, \text{ and}$$
$$\lambda_i + \lambda_j \iff H''_{ij} = A_{ii} + A_{jj}, \text{ for } i<j. \qquad (4)$$

The Cartan integers are easily computed by

$$2\langle \alpha_i, \alpha_{i+1} \rangle / \langle \alpha_i, \alpha_i \rangle = 2\mathrm{tr}((A_{ii} - A_{i+1,i+1})(A_{i+1,i+1} - A_{i+2,i+2}))/\mathrm{tr}((A_{ii} - A_{i+1,i+1})^2) = -1,$$

and analogously

$$2\langle \alpha_i, \alpha_{i+1} \rangle / \langle \alpha_{i+1}, \alpha_{i+1} \rangle = -1, \text{ for } i = 1, \dots, n-2.$$

We have also

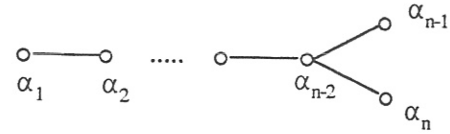
$$2\langle \alpha_{n-1}, \alpha_n \rangle / \langle \alpha_{n-1}, \alpha_{n-1} \rangle = 2\mathrm{tr}((A_{n-1,n-1} - A_{nn})(A_{n-1,n-1} + A_{nn}))/\mathrm{tr}((A_{n-1,n-1} - A_{nn})^2) = 0,$$
$$2\langle \alpha_{n-2}, \alpha_n \rangle / \langle \alpha_n, \alpha_n \rangle = 2\mathrm{tr}((A_{n-2,n-2} - A_{n-1,n-1})(A_{n-1,n-1} + A_{nn}))/\mathrm{tr}((A_{n-1,n-1} + A_{nn})^2) = -1,$$

and for all other cases

$$<\alpha_i,\alpha_j> = 0.$$

The result of these calculation is the Dynkin diagram D_n.

D_n



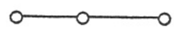
It is here important to notice some coincidences with other types of Lie algebras, in the case of n = 2 and n = 3. For n = 2, we have

$$\alpha_1 = \lambda_1 - \lambda_2 \,,\ \alpha_2 = \lambda_1 + \lambda_2 \,,\ <\alpha_1,\alpha_2> = 0,$$

and the Dynkin diagram is not connected and consists of two separate points.

$A_1 \oplus A_1$ 

One copy of A_1 is generated by $\{A_{12}, A_{21}, A_{11}-A_{22}\}$ and the other by $\{B_{12}, C_{12}, A_{11}+A_{22}\}$. For n = 3 the Dynkin diagram becomes isomorphic to A_3.

$A_3 \approx D_3$ 

The corresponding isomorphisms between the Lie algebras will be discussed later.

The reflexions corresponding to the different roots and generating the Weyl group are computed easily by using the coroots. The reflexion S_{ij} corresponding to the root $\lambda_i-\lambda_j$ is

$$S_{ij}(H') = H' - 2tr(H'(A_{ii}-A_{jj}))/tr((A_{ii}-A_{jj})^2)(A_{ii}-A_{jj}) = H' - (h_i-h_j)(A_{ii}-A_{jj}). \qquad (5)$$

This operates on H' by interchanging i-th and j-th coordinates and leaving all other fixed. For the reflection S'_{ij} corresponding to the root $\lambda_i+\lambda_j$ we have analogously

$$S'_{ij}(H') = H' - (h_i+h_j)(A_{ii}+A_{jj}). \qquad (6)$$

This operates on H' by interchanging i-th and j-th coordinates, changing also simultaneously their signs, and leaving all other coordinates of H' fixed. Thus the simultaneous change of the sign of two coordinates of H' is an isometry of h contained in the Weyl group. The totality of these transformations generates an abelian subgroup G of the Weyl group with 2^{n-1} elements. The Weyl group is the semi direct product of this group and the permutation group S_n, hence it has $2^{n-1}n!$ elements.

Exercise-1 Show that the elements of G can be described by the group of diagonal matrices $diag(x_1, \ldots , x_n)$, with $x_i = \pm 1$, and the number of (-1)'s is even. Conclude that $|G| = 2^{n-1}$.

Exercise-2 Show that the Weyl group W is the semi direct product of S and G.

[Use arguments analogous to those of Ex-2, §28]

L' hommme est visiblement fait pour penser; c' est toute sa dignité et tout son mérite; et tout son devoir est de penser comme il faut. Or l' ordre de la pensée est de commencer par soi, et par son auteur et sa fin.

Or à quoi pense le monde? Jamais à cela; mais à danser, à jouer du luth, à chanter, à faire des vers, à courir la bague, etc., à se battre, à se faire roi, sans penser à ce que c' est qu' être roi, et qu' être homme.

Pascal, Pensées, 146.

30. The structure of o(2n+1;C) (B_n)

In §21 we saw that the complex orthogonal Lie algebra o(2n+1;C) may be described by the set of 2n+1)×(2n+1) complex matrices:

$$o(2n+1;C) = \left\{ \begin{pmatrix} 0 & b & c \\ -c^t & A, & B \\ -b^t & C, & -A^t \end{pmatrix}, \text{ where C, B skew-symmetric, A arbitrary} \right\}.$$

A "natural" basis of this Lie algebra consists of the matrices

$$A_i = \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ -e_i^t & 0 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 0 & e_i \\ -e_i^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_i = (0,...,1,...,0) = \text{i-th vector of the standard basis},$$

together with the matrices

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ij} & 0 \\ 0 & 0 & -E_{ji} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E_{ij}-E_{ji} \\ 0 & 0 & 0 \end{pmatrix}, \quad C_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{ij}-E_{ji} & 0 \end{pmatrix}.$$

Denoting by H' the diagonal matrices, which build an abelian subalgebra h, we easily see (§21) that the following relations hold:

$$[H', A_i] = -h_i A_i,$$
$$[H', B_i] = h_i B_i,$$
$$[H', A_{ij}] = (h_i - h_j) A_{ij}, \text{ for } i\neq j,$$
$$[H', B_{ij}] = (h_i + h_j) B_{ij}, \text{ for } i<j,$$
$$[H', C_{ij}] = -(h_i + h_j) C_{ij}, \text{ for } i<j. \tag{1}$$

It follows that the linear forms on the subalgebra h of diagonal matrices:

$$\lambda_1(H') = h_1, \dots, \lambda_n(H') = h_n,$$

build a basis of h* and the roots of the Lie algebra are expressible through them. In fact, we see easily that

$$\alpha_1 = \lambda_1-\lambda_2,\ \alpha_2 = \lambda_2-\lambda_3, \dots, \alpha_{n-1} = \lambda_{n-1}-\lambda_n,\ \alpha_n = \lambda_n, \tag{2}$$

is a fundamental system of roots, the corresponding positive roots being:

$$\lambda_i = \alpha_i + \alpha_{i+1} + \dots + \alpha_n,$$

$$\lambda_i-\lambda_j = \alpha_i+\alpha_{i+1}+\ldots+\alpha_{j-1},\ \text{for } i<j, \text{ and}$$
$$\lambda_i+\lambda_j = (\alpha_i+\alpha_{i+1}+\ldots+\alpha_n)+(\alpha_j+\alpha_{j+1}+\ldots+\alpha_n),\ \text{for } i<j. \tag{3}$$

Applying the lemma of §27 and the relations of (1) we see easily that o(2n+1;C) is semi simple. We compute the Dynkin diagram via the coroots which correspond to (3):

$$\lambda_i \Leftrightarrow H'_i = 2A_{ii},$$
$$\lambda_i-\lambda_j \Leftrightarrow H'_{ij} = A_{ii}-A_{jj},\ \text{for } i<j, \text{ and}$$
$$\lambda_i+\lambda_j \Leftrightarrow H''_{ij} = A_{ii}+A_{jj},\ \text{for } i<j. \tag{4}$$

The Cartan integers are easily computed as in the preceding §:

$$2<\alpha_i,\alpha_{i+1}>/<\alpha_i,\alpha_i> = -1, \text{ for } i = 1, \ldots, n-1,$$
$$2<\alpha_i,\alpha_{i+1}>/<\alpha_{i+1},\alpha_{i+1}> = -1, \text{ for } i = 1, \ldots, n-1.$$

We have also

$$2<\alpha_{n-1},\alpha_n>/<\alpha_{n-1},\alpha_{n-1}> = 2\mathrm{tr}((A_{n-1,n-1}-A_{nn})A_{nn})/\mathrm{tr}(A_{nn}^2) = -2,$$
$$2<\alpha_{n-1},\alpha_n>/<\alpha_{n-1},\alpha_{n-1}> = -1,$$

and for all other cases

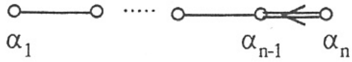
$$<\alpha_i,\alpha_j> = 0.$$

Comparison of the coroot lengths gives

$$<H'_n, H'_n>/<H'_{i,i+1}, H'_{i,i+1}> = \mathrm{tr}(4A_{ii}^2)/\ \mathrm{tr}((A_{ii}-A_{i+1,i+1})^2) = 2.$$

Thus H'_n is longer than $H'_{i,i+1}$, hence, by $\alpha(H_\alpha)=2$, the root α_n is shorter than the roots α_i. The result of these calculation is the Dynkin diagram B_n.

B_n



The reflexions corresponding to the different positive roots are again easily computed:

$$\lambda_i \Leftrightarrow S_{ij}(H') = H' - 2\mathrm{tr}(H'A_{ii})/\mathrm{tr}(4A_{ii}^2)2A_{ii} = H' - 2h_iA_{ii},$$
$$\lambda_i-\lambda_j \Leftrightarrow S'_{ij}(H') = H' - (h_i-h_j)(A_{ii}-A_{jj}),$$
$$\lambda_i+\lambda_j \Leftrightarrow S''_{ij}(H') = H' - (h_i+h_j)(A_{ii}+A_{jj}).$$

The first of them operates on H' by changing the sign of the i-th coordinate and leaving all other coordinates of H' fixed. The other two operate as the corresponding reflections of o(2n;C). We conclude that the Weyl group of B_n is isomorphic with that of C_n and has $n!2^n$ elements.

Einige *kommen* auf einen Gedanken, andere *stoßen* darauf, andere *fallen* darauf, andere *verfallen* darauf (hier fehlt noch das zerfallen), auch *gerät* man darauf. Man sagt nicht, ich habe mich nach dem Gedanken *hin*begeben. Das wäre via regia.

Lichtenberg, Sudelbücher p. 403

31. Freudenthal's construction

This is a method to construct Lie algebras, which gives as special cases models of the exceptional Lie algebras G_2 and E_8. The other three simple complex exceptional Lie algebras, corresponding to the Dynkin diagrams E_7, E_6 and F_4 can be found as subalgebras in E_8. Freudenthal's construction proceeds as follows :

Given are the following data:

$\mathfrak{g}$: a simple Lie algebra,

V : a complex vector space,

V^*: the dual of V,

f : $\mathfrak{g} \to gl(V)$ a representation of $\mathfrak{g}$,

f^* : $\mathfrak{g} \to gl(V^*)$ the dual representation of f, defined by

$$(f^*(X)(\alpha))(v) = -\alpha(f(X)v), \text{ for all } v \in V \text{ and } \alpha \in V^*. \qquad (1)$$

f being fixed we write often Xv and $X\alpha$ instead of the longer notation $f(X)v$ and $f^*(X)(\alpha)$. The right side of (1) defines by the equation

$$\langle \alpha\cdot v, X\rangle = \alpha(f(X)v), \text{ for all } X \in \mathfrak{g}, \qquad (2)$$

an element $\alpha\cdot v \in \mathfrak{g}$. $\langle ...,...\rangle$ denotes here the Killing form of $\mathfrak{g}$, which by assumption is non-degenerate.

Exercise-1 Show that the map $V^* \times V \to \mathfrak{g}$ defined by $(\alpha,v) \to \alpha\cdot v$ (and depending on f) is bilinear.

A further fundamental assumption for this construction is that our spaces V and V^* are endowed with fixed trilinear antisymmetric forms

$$[\dots, \dots, \dots] : V \times V \times V \to \mathbb{C},$$

$$[\dots, \dots, \dots]' : V^* \times V^* \times V^* \to \mathbb{C},$$

which are invariant with respect to all automorphisms of V (respectively of V^*) of the form $\exp(f(X))$ (resp. $\exp(f^*(X))$). In other words equations

$$[\exp(f(X))v_1, \exp(f(X))v_2, \exp(f(X))v_3] = [v_1, v_2, v_3],$$

$$[\exp(f^*(X))\alpha_1, \exp(f^*(X))\alpha_2, \exp(f^*(X))\alpha_3]' = [\alpha_1, \alpha_2, \alpha_3]', \qquad (3)$$

hold for all $X \in \mathfrak{g}$, $v_i \in V$ and $\alpha_i \in V^*$.

Exercise-2 Show by differenciating (3) that the following "infinitesimal" invariance of the trilinear forms is true:

$$[Xv_1, v_2, v_3] + [v_1, Xv_2, v_3] + [v_1, v_2, Xv_3] = 0,$$

$$[X\alpha_1, \alpha_2, \alpha_3]' + [\alpha_1, X\alpha_2, \alpha_3]' + [\alpha_1, \alpha_2, X\alpha_3]' = 0, \qquad (4)$$

for all $X \in \mathfrak{g}$, $v_i \in V$ and $\alpha_i \in V^*$.

Using these trilinear forms we can define in V(resp. V^*) something like the exterior product of $\mathbf{R}^3$:

$V \times V \to V^*$, with
$(v_1,v_2) \to v_1 \times v_2 \in V^*$, defined by the property
$(v_1 \times v_2)(v_3) = [v_1, v_2, v_3]$, for every $v_3 \in V$. (5)

Analogously we define
$V^* \times V^* \to V$, with
$(\alpha_1,\alpha_2) \to \alpha_1 \times \alpha_2 \in V$, defined by the property
$(\alpha_1 \times \alpha_2)(\alpha_3) = [\alpha_1, \alpha_2, \alpha_3]'$, for every $\alpha_3 \in V$ (6)

In (6) we use the natural isomorphism $V^{**} \approx V$.

Finally we assume that, in addition to $\mathfrak{g}$-invariance, the trilinear products are related to the structure of the Lie algebra through the relation

$$(v_1 \times v_2)(\alpha_1 \times \alpha_2) = <\alpha_1 \cdot v_2, \alpha_2 \cdot v_1> - <\alpha_1 \cdot v_1, \alpha_2 \cdot v_2>. \quad (7)$$

With all these beautiful assumptions and constructions we can define a bracket on the vector space $\mathfrak{g} \oplus V \oplus V^*$ which turns it into a Lie algebra!

> Gradso wie manche Menschen das für göttlich halten was keinen vernünftigen Sinn hat. Das Vergnügen an dem Anblick unnützer algebraischer Rechnungen, die man selbst gemacht hat, gehört mit in diese Klasse.
>
> Lichtenberg, Sudelbücher p. 403

In fact, we define the bracket on $\mathfrak{g} \oplus V \oplus V^*$ to be a bilinear map

$$[\ldots, \ldots] : (\mathfrak{g} \oplus V \oplus V^*) \times (\mathfrak{g} \oplus V \oplus V^*) \to \mathfrak{g} \oplus V \oplus V^*,$$

which satisfies (under the natural identification of $X \in \mathfrak{g}$, with $X \oplus 0 \oplus 0$ in $\mathfrak{g} \oplus V \oplus V^*$ etc.) :

i) $[XY] = [XY]$, as in $\mathfrak{g}$, for all $X, Y \in \mathfrak{g}$,
ii) $[Xv] = -[vX] = Xv$, for all $X \in \mathfrak{g}$ and $v \in V$,
iii) $[X\alpha] = -[\alpha X] = X\alpha$ $(= f^*(X)\alpha)$, for all $X \in \mathfrak{g}$, $\alpha \in V^*$,
iv) $[vw] = -[wv] = v \times w \in V^*$, for all $v, w \in V$,
v) $[\alpha\beta] = -[\beta\alpha] = \alpha \times \beta \in V$, for all $\alpha, \beta \in V^*$,
vi) $[\alpha v] = -[v\alpha] = \alpha \cdot v \in \mathfrak{g}$, for all $\alpha \in V^*$, $v \in V$.

	$\mathfrak{g}$	V	V*
$\mathfrak{g}$	$\mathfrak{g}$	V	V*
V	V	V*	$\mathfrak{g}$
V*	V*	$\mathfrak{g}$	V

Obviously the skew-symmetry of [...,...] is involved in the definition. The truble is with the Jacobi identity, which we verify in 10 different cases. For brevity I use the notation

$$@[X[YZ]] = [X[YZ]] + [Y[ZX]] + [Z[XY]].$$

The 10 different cases are :

1) $@[X[YZ]] = 0$, for all $X, Y, Z \in \mathfrak{g}$, which is obvious.
2) $@[X[Yv]] = [X[Yv]] + [Y[vX]] + [v[XY]] = XYv - YXv - [XY]v = 0$. $(v \in V)$.
3) $@[X[Y\alpha]] = 0$, analogous to the preceding $(\alpha \in V^*)$.
4) $@[X[v_1 v_2]] = [X[v_1 v_2]] + [v_1[v_2 X]] + [v_2[Xv_1]]$
$= X(v_1 \times v_2) - v_1 \times (Xv_2) - (Xv_1) \times v_2 = 0$, because of the $\mathfrak{g}$-invariance of "$\times$".
5) $@[X[\alpha_1 \alpha_2]] = 0$, analogous to the preceding.
6) $@[X[v\alpha]] = [X[v\alpha]] + [v[\alpha X]] + [\alpha[Xv]] = -[X, \alpha \cdot v] + (X\alpha) \cdot v + \alpha \cdot Xv = Z$.
To show that $Z = 0$, take now an arbitrary $Y \in \mathfrak{g}$ and compute using the Killing form:
$<Y, Z> = <\alpha \cdot v, [XY]> + <(X\alpha) \cdot v, Y> + <\alpha \cdot (Xv), Y>$

$= \alpha([XY]v) + (X\alpha)\bullet(Yv) + \alpha(YXv) = \alpha([XY]v - XYv + YXv) = 0.$

7) $@[v_1[v_2v_3]] = [v_1[v_2v_3]] + [v_2[v_3v_1]] + [v_3[v_1v_2]]$

$= [v_1,v_2\times v_3] + [v_2,v_3\times v_1] + [v_3,v_1\times v_2] = Z\in g.$

To show that $Z = 0$, take again an arbitrary $Y\in g$ and compute using the Killing form:

$- <Z,Y> = <(v_2\times v_3)\bullet v_1,Y> + <(v_3\times v_1)\bullet v_2,Y> + <(v_1\times v_2)\bullet v_3,Y>$

$= (v_2\times v_3)\bullet(Zv_1) + (v_3\times v_1)\bullet(Zv_2) + (v_1\times v_2)\bullet(Zv_3)$

$= [Zv_1,v_2,v_3] + [v_1, Zv_2,v_3] + [v_1,v_2, Zv_3] = 0.$

8) $@[\alpha_1[\alpha_2\alpha_3]] = 0$, analogous to the preceding.

9) $@[v[\alpha_1\alpha_2]] = [v[\alpha_1\alpha_2]] + [\alpha_1[\alpha_2 v]] + [\alpha_2[v\alpha_1]] = [v,\alpha_1\times\alpha_2] + [\alpha_1,\alpha_2\bullet v] - [\alpha_2,\alpha_1\bullet v] =$

$= v\times(\alpha_1\times\alpha_2) - (\alpha_2\bullet v)\alpha_1 + (\alpha_1\bullet v)\alpha_2 = \beta\in V^*.$

To see that $\beta = 0$, we apply this on a $w\in V$.

$\beta(w) = [v,\alpha_1\times\alpha_2,w] + \alpha_1((\alpha_2\bullet v)w) - \alpha_2((\alpha_1\bullet v)(w))$

$= -(v\times w)(\alpha_1\times\alpha_2) + <\alpha_1\bullet w,\alpha_2\bullet v> - <\alpha_1\bullet v,\alpha_2\bullet w> = 0.$

10) $@[\alpha[v_1v_2]] = [\alpha[v_1v_2]] + [v_1[v_2\alpha]] + [v_2[\alpha v_1]] = \alpha\times(v_1\times v_2) + [v_1,\alpha\bullet v_2] - [v_2,\alpha\bullet v_1] =$

$= \alpha\times(v_1\times v_2) + (\alpha\bullet v_2)v_1 - (\alpha\bullet v_1)v_2 = w\in V.$

To see that $w = 0$, we apply this on a $\beta\in V^*$.

$\beta(w) = [\alpha,v_1\times v_2,w] + <\beta\bullet v_1,\alpha\bullet v_2> - <\beta\bullet v_2,\alpha\bullet v_1>$

$= -(v_1\times v_2)(\alpha\times\beta) + <\beta\bullet v_1,\alpha\bullet v_2> - <\beta\bullet v_2,\alpha\bullet v_1> = 0.$

Exercise-3 Complete the proof by examining the remaining cases.

But let me to my story: I must own,
If I have any fault, it is digression;
Leaving my people to proceed alone,
While I soliloquize beyond expression;
But these are my addresses from the throne,
Which put off business to the ensuing session:
Forgetting each omission is a loss to
The world, not quite so great as Ariosto.

Byron, Don Juan, Canto III, 96

32. The structure of G_2

We carry out Freudenthal's construction with the following data:

$\mathfrak{g} = sl(3;C)$,

$V = C^3$ (column vectors),

$V^* = (C^3)^* = C^3$ (row vectors),

$f : \mathfrak{g} \to gl(V)$ the "natural" representation of $sl(3;C)$,

Xv = matrix product of matrix X by the column vector v,

$f^* : \mathfrak{g} \to gl(V^*)$ the dual representation of f, defined by

$f^*(X)(\alpha) = -\alpha X$ = matrix product of the row vector $-\alpha$ by the matrix X.

Obviously

$$f^*(X)(\alpha)(v) = -\alpha(Xv) \quad \text{(on the right side matrix products).} \qquad (1)$$

The trilinear products are defined via the determinant of the column vectors (resp. row vectors):

$$[v_1, v_2, v_3] = \det(v_1, v_2, v_3),$$
$$[\alpha_1, \alpha_2, \alpha_3]' = \mu\det(\alpha_1, \alpha_2, \alpha_3). \qquad (2)$$

The factor μ will be determined below ($\mu=2/9$), so as to satisfy an identity of Freudenthal's construction. The product $\alpha\bullet v$ turns out to be the most natural one. By its definition it must satisfy the equation

$$\langle\alpha\bullet v, X\rangle = \alpha(f(X)v), \text{ for all } X\in\mathfrak{g},$$

$\langle\ldots,\ldots\rangle$ denoting the Killing form of $sl(3;C)$, which is (§21) $\langle X,Y\rangle = 6tr(XY)$.

Thus (2), in the present case, takes the form

$$\alpha(Xv) = 6tr((\alpha\bullet v)X) = 6(\alpha\bullet v)_{ij}X_{ji} = \alpha_i(X_{ij}v_i) \quad\Leftrightarrow\quad (\alpha_j v_i - 6(\alpha\bullet v)_{ij})X_{ji} = 0.$$

For $i\neq j$ we get $\qquad (\alpha\bullet v)_{ij} = (1/6)v_i\alpha_j$.

The diagonal elements of the matrix $\alpha\bullet v$ satisfy

$$(\alpha_i v_i - 6(\alpha\bullet v)_{ii})X_{ii} = 0, \text{ for all } X \text{ with } \sum X_{ii} = 0,$$

which implies that the $(\alpha_i v_i - 6(\alpha\bullet v)_{ii})$ are all equal, say to d. Then the condition $tr(\alpha\bullet v) = 0$ implies

$$\alpha(v) = 3d \quad\Rightarrow\quad (\alpha\bullet v)_{ii} = (1/6)\alpha_i v_i - (1/18)\alpha(v),$$
$$\alpha\bullet v = (1/6)(v\otimes\alpha - (1/3)\alpha(v)I). \qquad (3)$$

The $v_1\times v_2\in V^*$ has coordinates given by the usual exterior product. The same is true also for $\alpha_1\times\alpha_2 \in V$ up to the factor μ. The well-known formula for the exterior product gives

$$(v_1\times v_2)(\alpha_1\times\alpha_2) = \mu(\alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)). \qquad (4)$$

On the other side we compute

$$<\alpha_1 \cdot v_2, \alpha_2 \cdot v_1> = (6/6^2)\mathrm{tr}((v_2 \otimes \alpha_1 - (1/3)\,\alpha_1(v_2)I)(v_1 \otimes \alpha_2 - (1/3)\alpha_2(v_1)I))$$
$$= (1/6)\{\alpha_1(v_1)\alpha_2(v_2) - (1/3)\alpha_1(v_2)\alpha_2(v_1) - (1/3)\alpha_2(v_1)\alpha_1(v_2) + (1/9)3\alpha_1(v_2)\alpha_2(v_1)\}$$
$$= (1/6)(\alpha_1(v_1)\alpha_2(v_2) - (1/3)\alpha_1(v_2)\alpha_2(v_1)).$$

Analogously

$$<\alpha_1 \cdot v_1, \alpha_2 \cdot v_2> = (1/6)(\alpha_1(v_2)\alpha_2(v_1) - (1/3)\alpha_1(v_1)\alpha_2(v_2)).$$

If follows

$$<\alpha_1 \cdot v_2, \alpha_2 \cdot v_1> - <\alpha_1 \cdot v_1, \alpha_2 \cdot v_2> = (1/6)(1+(1/3))\{\alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)\}.$$

In view of (4), to satisfy the identity

$$(v_1 \times v_2)(\alpha_1 \times \alpha_2) = <\alpha_1 \cdot v_2, \alpha_2 \cdot v_1> - <\alpha_1 \cdot v_1, \alpha_2 \cdot v_2> \qquad (5)$$

we must have $\mu = (1/6)(1+(1/3)) = 2/9$. We are now ready to prove the

Proposition *With the preceding definitions the corresponding Freudenthal construction $sl(3;C) \oplus C^3 \oplus (C^3)^*$ is a simple Lie algebra of type G_2.*

That this is a Lie algebra, is already checked in the preceding §. To prove the rest we find a Cartan subalgebra and apply the lemma of §27, which we used also in the study of the structure of classical Lie algebras.

We denote by $\mathfrak{h}$ the abelian subalgebra of sl(3;C)(identified with the corresponding subalgebra in $sl(3;C) \oplus C^3 \oplus (C^3)^*$) which consists of the diagonal matrices. The roots of sl(3;C) are expressible in terms of the linear forms on $\mathfrak{h}$

$$\lambda_i(H) = h_i\,,\ i = 1, 2, 3,$$

and as we saw in §27 they are exactly the

$$\pm(\lambda_1-\lambda_2)\,,\ \pm(\lambda_2-\lambda_3)\,,\ \pm(\lambda_3-\lambda_1)\,.$$

From the definition of the bracket in $sl(3;C) \oplus C^3 \oplus (C^3)^*$ we see that

$$[He_1] = \lambda_1(H)e_1, \qquad [He_1^*] = -\lambda_1(H)e_1^*,$$
$$[He_2] = \lambda_1(H)e_2, \qquad [He_2^*] = -\lambda_2(H)e_2^*,$$
$$[He_3] = \lambda_3(H)e_3, \qquad [He_3^*] = -\lambda_3(H)e_3^*,$$
$$[e_i e_i^*] = -e_i^* \cdot e_i = -(1/6)(e_i \otimes e_i^* - (1/3)I) \neq 0. \qquad (6)$$

Thus applying the lemma of §27 we conclude that $sl(3;C) \oplus C^3 \oplus (C^3)^*$ is semi simple, and that $\mathfrak{h}$ is also a Cartan subalgebra of $sl(3;C) \oplus C^3 \oplus (C^3)^*$ of dimension 2, and finaly that

$$\pm(\lambda_1-\lambda_2)\,,\ \pm(\lambda_2-\lambda_3)\,,\ \pm(\lambda_3-\lambda_1)\,,\ \pm\lambda_i\,,\ i = 1, 2, 3,$$

are the roots of $\mathfrak{h}$.

Obviously a fundamental system of roots consists of the two roots

$$\alpha_1 = \lambda_1-\lambda_2\,,\quad \alpha_2 = \lambda_2.$$

The α_2-string of the root α_1 is easily computed:

$\alpha_1-\alpha_2$	is not a root,
$\alpha_1+\alpha_2 = \lambda_1$	is a root,
$\alpha_1+2\alpha_2 = \lambda_1+\lambda_2 = -\lambda_3$	is a root,

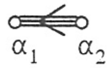
$$\alpha_1+3\alpha_2 = \lambda_1+2\lambda_2 = \lambda_2-\lambda_3 \qquad \text{is a root,}$$

$$\alpha_1+4\alpha_2 \qquad \text{is not a root.}$$

From the preceding list we see that the Cartan integers is

$$c_{12} = 2\langle\alpha_1,\alpha_2\rangle/\langle\alpha_2,\alpha_2\rangle = -3,\ c_{21} = -1.$$

Thus $\alpha_1 = \lambda_1-\lambda_2$ is the long root, the Dynkin diagram is connected and the Lie algebra is simple of type G_2. Notice that the 6-th root (by which we complete the explicit construction of the positive roots) $2\alpha_1+3\alpha_2 = \lambda_1-\lambda_3$ is the maximal root.

G_2 

Formulas (6) show also that the coroots corresponding to α_1, α_2 are respectively

$$H_1 = \mathrm{diag}(1,-1,0) \quad \text{and} \quad H_2 = \mathrm{diag}(-1,2,-1).$$

The Killing form on $\mathfrak{h}$ is

$$\langle X,Y\rangle = \sum_{\alpha\in\Delta}\alpha(X)\alpha(Y) = 2\sum_{\alpha>0}\alpha(X)\alpha(Y)$$
$$= 2(X_1Y_1+X_2Y_2+X_3Y_3+(X_1-X_2)(Y_1-Y_2)+(X_2-X_3)(Y_2-Y_3)+(X_3-X_1)(Y_3-Y_1))$$
$$= 8(X_1Y_1+X_2Y_2+X_3Y_3). \quad (\text{since } \textstyle\sum X_i = \sum Y_i = 0)$$

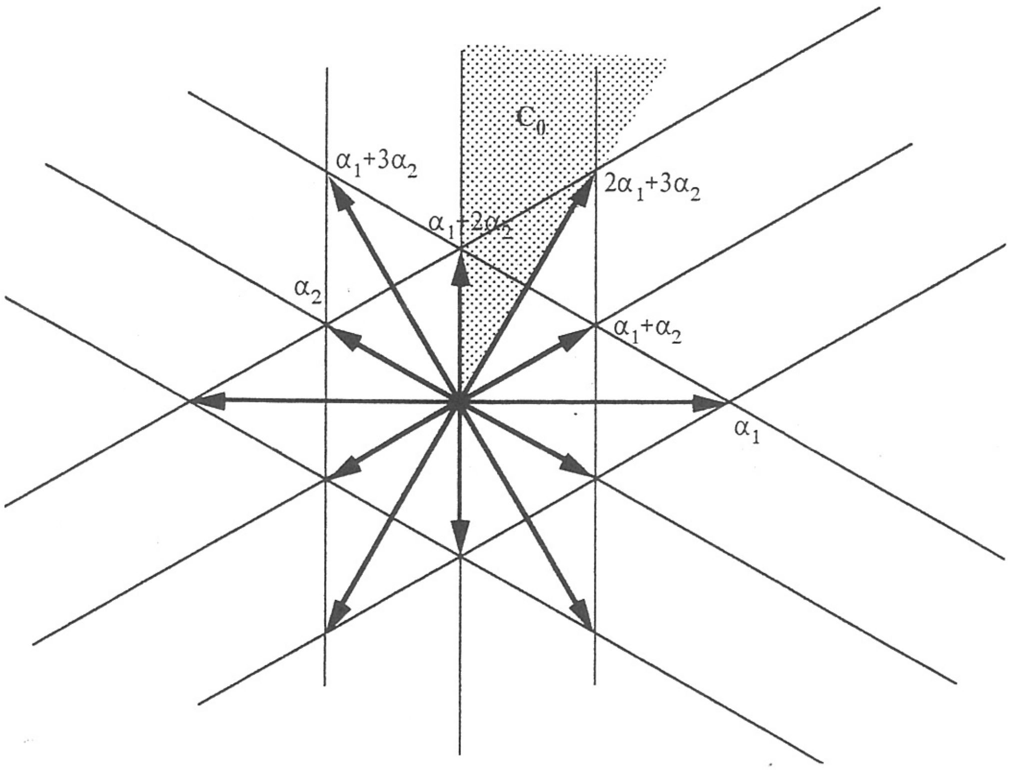
By The-2 in §26 we know that the Weyl group is generated by the reflections S_1, S_2 corresponding to the roots α_1, α_2. From this we see easily that the Weyl group is isomorphic to D_6 which has 12 elements. These are the 6 reflexions corresponding to the roots, and the 6 rotations $(k\pi/3)\mathrm{mod}(2\pi)$.

I therefore do denounce all amorous writing,
  Except in such a way as not to attract;
Plain-simple-short, and by no means inviting,
  But with a moral to each error tack'd,
Form'd rather for instructing than delighting,
  And with all passions in their turn attack'd;
Now, if my Pegasus should not be shod ill,
This poem will become a moral model

Byron, Don Juan, Canto V, 2

(turn the page)

G_2



Kroklokwafzi? Semememi!
Seiokrontro - prafriplo:
Bifzi, bafzi; hulalemi
quasti bast bo ...
Lalu lalu lalu lalu la!

Hontraruru miromente
zasku zes rü rü?
Entepente, Leiolente
klekwapufzi lü?
Lalu lalu lalu lalu la!

Simarar kos malzipempu
silzuzankunkrei(;)!
Marjimar dos: Quempu Lempu
Siri Suri Sei []
Lalu lalu lalu lalu la!

Chr. Morgenstern, Das große Lalula
Werke, p. 226

33. The structure of E_8

Here we carry out Freudenthal's construction with the following data:

$\mathfrak{g} = \mathrm{sl}(9;C)$, (A_8),

V = the vector space of skew-symmetric contravariant tensors of C^9.
= $\{v^{ijk}e_i \otimes e_j \otimes e_k \mid v^{ijk}$ skew-symmetric, (e_i) the canonical basis of $C^9\}$.
A basis of V consists of the $\binom{9}{3} = 84$ tensors $e_i \wedge e_j \wedge e_k$, for $i<j<k$.
$u = u^{ijk}e_i \otimes e_j \otimes e_k$ can be written in this basis $u = \sum_{i<j<k} \hat{u}^{ijk} e_i \wedge e_j \wedge e_k$,
which by the relation $e_i \wedge e_j \wedge e_k = \sum \mathrm{sgn}(\pi) e_{\pi(i)} \otimes e_{\pi(j)} \otimes e_{\pi(k)}$ gives
$\hat{u}^{ijk} = (1/3!) \sum \mathrm{sgn}(\pi) u^{\pi(i),\pi(j),\pi(k)}$. The last two sums run over $\pi \in S_9$ = set of permutations of 9 elements, and $\mathrm{sign}(\pi)$ is the signature of π.

V^*= the vector space of skew-symmetric covariant tensors of C^9.
= $\{v_{ijk}e^i \otimes e^j \otimes e^k \mid v_{ijk}$ skew-symmetric, (e^i) the canonical basis of $(C^9)^*\}$.
A basis of V^* consists of the $\binom{9}{3} = 84$ tensors $e^i \wedge e^j \wedge e^k$, for $i<j<k$.
$u = u_{ijk}e^i \otimes e^j \otimes e^k$ can be written in this basis $u = \sum_{i<j<k} \hat{u}_{ijk} e^i \wedge e^j \wedge e^k$,
which by the relation $e^i \wedge e^j \wedge e^k = \sum \mathrm{sgn}(\pi) e^{\pi(i)} \otimes e^{\pi(j)} \otimes e^{\pi(k)}$ gives
$\hat{u}_{ijk} = (1/3!) \sum \mathrm{sgn}(\pi) u_{\pi(i),\pi(j),\pi(k)}$.

$f : \mathfrak{g} \to \mathrm{gl}(V)$ the "natural" representation of $\mathrm{sl}(9;C)$:
$f(X)(v_1 \wedge v_2 \wedge v_3) = Xv_1 \wedge v_2 \wedge v_3 + v_1 \wedge Xv_2 \wedge v_3 + v_1 \wedge v_2 \wedge Xv_3$.

$f^* : \mathfrak{g} \to \mathrm{gl}(V^*)$ the dual representation of f, defined by
$f^*(X)(v^1 \wedge v^2 \wedge v^3) = -v^1 X \wedge v^2 \wedge v^3 - v^1 \wedge v^2 X \wedge v^3 - v^1 \wedge v^2 \wedge v^3 X$.

The trilinear products are defined using the determinant of the column vectors (resp. row vectors) for decomposable elements (of the form $v_1 \wedge v_2 \wedge v_3$) and extending linearly to the whole tensor space V:

$$[u_1\wedge u_2\wedge u_3, v_1\wedge v_2\wedge v_3, w_1\wedge w_2\wedge w_3] = \det(u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3),$$
$$[u^1\wedge u^2\wedge u^3, v^1\wedge v^2\wedge v^3, w^1\wedge w^2\wedge w^3] = \mu\det(u^1, u^2, u^3, v^1, v^2, v^3, w^1, w^2, w^3).$$

The factor μ will be determined below (μ=432), so as to satisfy an identity of Freudenthal's construction. The duality between V and V* is expressed in the two kinds of components by the relation

$$a(u) = a_{ijk}u^{ijk} = 3!\sum_{i<j<k} \hat{a}_{ijk}\hat{u}^{ijk}, \text{ for every } a\in V^* \text{ and } u\in V.$$

The product a•u, by its definition, must satisfy the equation

$$\langle a\bullet u, X\rangle = a(f(X)v), \text{ for all } X\in \mathfrak{g},$$

$\langle \ldots,\ldots\rangle$ denoting the Killing form of sl(9;C),which is (§21), $\langle X,Y\rangle = 18\mathrm{tr}(XY)$.

Thus

$$a(Xu) = 18\mathrm{tr}((a\bullet u)X) = 18(a\bullet u)_{ij}X_{ji} = a_{ijk}(Xu)^{ijk}. \quad (1)$$

$$\begin{aligned}Xu &= u^{ijk}(Xe_i\otimes e_j\otimes e_k + e_i\otimes Xe_j\otimes e_k + e_i\otimes e_j\otimes Xe_k)\\ &= u^{ijk}(X_{mi}e_m\otimes e_j\otimes e_k + X_{mj}e_i\otimes e_m\otimes e_k + X_{mk}e_i\otimes e_j\otimes e_m) \quad \Rightarrow\end{aligned}$$

$$a(Xu) = u^{ijk}(X_{mi}a_{mjk} + X_{mj}a_{imk} + X_{mk}a_{ijm}) = 3u^{ijk}X_{mi}a_{mjk} = 18(a\bullet u)_{ij}X_{ji}.$$

$$\Leftrightarrow \quad (u^{ijk}a_{mjk} - 6(a\bullet u)_{im})X_{mi} = 0.$$

For $i\neq m$ we get

$$(a\bullet u)_{im} = (1/6)u^{ijk}a_{mjk}.$$

The diagonal elements of the matrix a•u satisfy

$$(u^{ijk}a_{ijk} - 6(a\bullet u)_{ii})X_{ii} = 0, \quad \text{for all X with } \sum X_{ii} = 0, \quad (2)$$

which implies that the $(u^{ijk}a_{ijk} - 6(a\bullet u)_{ii})$ are all equal, say to d. Then, condition $\mathrm{tr}(a\bullet u) = 0$ implies

$$a(u) = 9d \quad \Rightarrow \quad (a\bullet u)_{ii} = (1/6)(u^{ijk}a_{ijk} - (1/9)a(u)) \quad \Rightarrow$$

$$(a\bullet u)_{im} = (1/6)(u^{ijk}a_{mjk} - (1/9)a(u)\delta_{im}). \quad (3)$$

We have also

$$\langle a\bullet u,b\bullet v\rangle = 18\{(1/6)(u^{ijk}a_{mjk} - (1/9)a(u)\delta_{im})(1/6)(v^{mrs}b_{irs} - (1/9)b(v)\delta_{mi})\},$$

$$\langle a\bullet v,b\bullet u\rangle = 18\{(1/6)(v^{ijk}a_{mjk} - (1/9)a(v)\delta_{im})(1/6)(u^{mrs}b_{irs} - (1/9)b(u)\delta_{mi})\},$$

$$\langle a\bullet u,b\bullet v\rangle - \langle a\bullet v,b\bullet u\rangle$$
$$= (1/2)\{u^{ijk}a_{mjk}v^{mrs}b_{irs} - v^{ijk}a_{mjk}u^{mrs}b_{irs} - (1/9)(a(u)b(v)-a(v)b(u))\}. \quad (4)$$

And

$$(u\times v) = \sum_{i<j<k} c_{ijk}e^i\wedge e^j\wedge e^k,$$

$$(u\times v)(e_i\wedge e_j\wedge e_k) = 3!\, c_{ijk} = [u, v, e_i\wedge e_j\wedge e_k]$$

$$= \hat{u}^{mrs}v^{\wedge pqt}[e_m\wedge e_r\wedge e_s, e_p\wedge e_q\wedge e_t, e_i\wedge e_j\wedge e_k] = \hat{u}^{mrs}v^{\wedge pqt}(mrs,pqt,ijk), \quad (5)$$

where (mrs,pqt,ijk) denotes the signature of the permutation. Analogously we compute the coordinates of $a\times b\in V$,

$$(a\times b) = \sum_{i<j<k} d^{ijk}e_i\wedge e_j\wedge e_k,$$

$$d^{ijk} = (\mu/3!)\hat{a}_{mrs}b^{\wedge}{}_{pqt}(mrs,pqt,ijk). \quad (6)$$

From (5) and (6) we get

$$(u\times v)(a\times b) = 3!\sum_{i<j<k} c_{ijk}d^{ijk}$$

$$= 3!\sum_{i<j<k}\{(1/3!)\hat{u}^{mrs}v^{\wedge pqt}(mrs,pqt,ijk)(\mu/3!)\hat{a}_{m'r's'}b^{\wedge}{}_{p'q't'}(m'r's',p'q't',ijk)\}.$$

Using the other coordinates ($\hat{u}_{ijk} = (1/3!)\sum \mathrm{sgn}(\pi)u_{\pi(i),\pi(j),\pi(k)}$) and absorbing the sign of the permutation π into(mrs,pqt,ijk), we get

$$(u\times v)(a\times b) = \mu/(3!)^5\sum_{i<j<k}\{u^{mrs}v^{pqt}(mrs,pqt,ijk)a_{m'r's'}b_{p'q't'}(m'r's',p'q't',ijk)\}.$$

We notice that the ordering of each triple is inessencial (complete skew-symmetry) and we distinguish the following cases:

a) {mrs} = {m'r's'} which implies {pqt} = {p'q't'}. The corresponding summands are

$$u^{mrs}v^{pqt}a_{mrs}b_{pqt},\ \text{and their sum is}\ \ a(u)b(v).$$

b) {mrs} = {p'q't'} which implies {pqt} = {m'r's'}. The corresponding summands are

$$u^{mrs}v^{pqt}a_{pqt}b_{mrs},\ \text{and their sum is}\ -a(v)b(u).$$

c) Two elements of {mrs} are equal to two elements of {m'r's'} which implies that also two elements of {pqt} will be equal to two elements of {r's't'}. f.e. we have the case

$$u^{mrs}v^{pqt}(mrs,pqt,ijk)a_{mrs'}b_{pqt'}(mrs',pqt',ijk),$$

and 8 similar cases, which give equal summands, when we bring the equal indices in the two first places. We bring mr and pq in the first places in u, v respectively. To put the indices at the two first places in a, b, there are $3\times3=9$ possibilities. For all these s=t', t =s' and the signature of the permutation is -1. Thus the sum of all these summands is

$$-9u^{mrs}v^{pqt}a_{mrt}b_{pqs}.$$

d) Two elements of {mrs} are equal with two of {p'q't'}, consequently two of {pqt} are equal with two of {m'r's'}. f.e. the case

$$u^{mrs}v^{pqt}(mrs,pqt,ijk)a_{pqs'}b_{mrt'}(pqs',mrt',ijk).$$

The signature, since s=s', t=t', is +1. As before, we get the summands

$$9u^{mrs}v^{pqt}a_{pqs}b_{mrt}.$$

Thus, the sum takes the form

$$(u\times v)(a\times b) = \mu/(3!)^5\{a(u)b(v) - a(v)b(u) + 9(u^{mrs}v^{pqt}a_{pqs}b_{mrt} - u^{mrs}v^{pqt}a_{mrt}b_{pqs})\}.$$

In Freudenthal's construction $(u\times v)(a\times b) = \langle a\bullet v,b\bullet u\rangle - \langle a\bullet u,b\bullet v\rangle$, which using (4) gives

$$9\mu/(3!)^5 = 1/2 \quad\Rightarrow\quad \mu = 2(3!)^3 = 432.$$

> Junges frisches Gehirn auf solche Art zu desorganisieren ist wahrlich eine Sünde, die weder Verzeihung noch Schonung verdient.
> Schopenhauer, Über die Universitäts-Philosophie, p. 194

Proposition *With the preceding definitions the corresponding Freudenthal construction $sl(9;C)\oplus V\oplus V^*$ is a simple Lie algebra of type E_8.*

That this is a Lie algebra, is already checked in the §30. To prove the rest we find a Cartan subalgebra and apply the lemma of §27.

We denote by $\mathfrak{h}$ the abelian subalgebra of sl(9;C)(identified with the corresponding subalgebra in $sl(9;C)\oplus V\oplus V^*$) which consists of the diagonal matrices. The roots of sl(9;C) are expressible in terms of the linear forms on $\mathfrak{h}$

$$\lambda_i(H) = h_i,\ i = 1, ..., 9,$$

and as we saw in §27 they are exactly the

$$\pm(\lambda_i-\lambda_j),\ i<j.$$

From the definition of the bracket in sl(9;C)⊕V⊕V* we see that

$$\pm(\lambda_i+\lambda_j+\lambda_k)\ ,\ i<j<k\ ,$$

are also roots of sl(9;C)⊕V⊕V*, since

$$[H, e_i\wedge e_j\wedge e_k\ ] = (\lambda_i+\lambda_j+\lambda_k)(H)e_i\wedge e_j\wedge e_k\ ,$$
$$[H, e^i\wedge e^j\wedge e^k] = -(\lambda_i+\lambda_j+\lambda_k)(H)e^i\wedge e^j\wedge e^k\ ,$$

and using (3)

$$[e_i\wedge e_j\wedge e_k, e^i\wedge e^j\wedge e^k] = (1/6)\{2(E_{ii}+E_{jj}+E_{kk}\ ) - (3!/9)I\} \neq 0.$$

Applying the lemma of §27 we see that sl(9;C)⊕V⊕V* is semi simple and h is a Cartan subalgebra with roots

$$\pm(\lambda_i-\lambda_j)\ \text{, for } i<j \text{ and } \pm(\lambda_i+\lambda_j+\lambda_k)\ ,\ \text{ for } i<j<k\,. \tag{7}$$

The Killing form of the Lie algebra can be easily computed

$$<H,H> = 2\ \Sigma_{i<j}\,(h_i-h_j)^2 + 2\Sigma_{i<j<k}\ (h_i+h_j+h_k)^2.$$

The two sums are symmetric functions in the h_i's and are expressible in terms of elementary symmetric functions. The first was computed in Ex-4, §12,

$$2\ \Sigma_{i<j}\,(h_i-h_j)^2 = 18\Sigma\ h_i^2\ \ (\text{since } \Sigma\ h_i = 0).$$

For the second we compute the coefficient of h_1^2 , which is $\binom{8}{2}$ and put

$$\Sigma_{i<j<k}\ (h_i+h_j+h_k)^2 = \binom{8}{2}(\Sigma\ h_i)^2 + A\Sigma_{i<j}\,(h_ih_j).$$

Taking h_1=1, h_2=1, h_i=0 for i>2, we find A = -42. Using this and $\Sigma\ h_i = 0$, we find

$$\Sigma_{i<j<k}\ (h_i+h_j+h_k)^2 = 21\Sigma\ h_i^2\ ,$$
$$<H,H> = 60\ \Sigma\ h_i^2\ . \tag{8}$$

A fundamental system of roots is given by

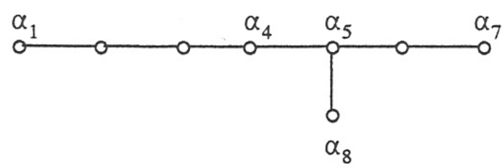
$$\alpha_1 = \lambda_1-\lambda_2, \ldots, \alpha_7 = \lambda_7-\lambda_8, \text{ and } \alpha_8 = \lambda_6+\lambda_7+\lambda_8. \tag{9}$$

In fact, $\lambda_8-\lambda_9 = \lambda_8+(\lambda_1+\ldots+\lambda_8) = (\lambda_1+\lambda_2+\lambda_3)+(\lambda_4+\lambda_5+\lambda_8)+(\lambda_6+\lambda_7+\lambda_8)$. This implies that all the roots of the form $(\lambda_i-\lambda_j)$, for i<j are positive. For i<j<k<8

$$(\lambda_i+\lambda_j+\lambda_k) = \lambda_i-\lambda_6+\lambda_j-\lambda_7+\lambda_k-\lambda_8+(\lambda_6+\lambda_7+\lambda_8),$$

as well as all $(\lambda_i+\lambda_j+\lambda_8)$ are positive, whereas all $(\lambda_i+\lambda_j+\lambda_9) = -(\lambda_{i'}+\lambda_{j'}+\lambda_{k'})- (\lambda_r+\lambda_s+\lambda_8)$ are negative. Finaly computing the Cartan integers, we find that the Lie algebra is simple and its Dynkin diagram is the E_8.

E_8



The Weyl group of this and the other exceptional Lie algebras we'll compute elsewhere.

Exercise-1 Identifying $(\lambda_1, \ldots ,\lambda_8)$ with the canonical basis of $\mathbf{R}^8$, {e, ... , e}, show that the coefficients of the other (non-simple) roots $(\lambda_i-\lambda_j)$, for i<j and $(\lambda_i+\lambda_j+\lambda_k)$, for i<j<k with

respect to the fundamental system of roots ($\alpha_1, \dots, \alpha_8$) are given from the solutions of the linear systems $(\lambda_i - \lambda_j) = Ax$, and $(\lambda_i + \lambda_j + \lambda_k) = Ax$, where A is the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \text{ with } A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & & & & & & \\ 3 & 3 & 0 & & & & & \\ 3 & 3 & 3 & 0 & & & & \\ 3 & 3 & 3 & 3 & 0 & & & \\ 3 & 3 & 3 & 3 & 3 & 0 & & \\ 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Exercise-2 Solve the preceding linear systems and show that the following list is true.

Exercise-3 Prove that E_8 has 248 dimensions.

The list, on the next page, gives the half of the whole set of roots of E_8, namely these which can be written

$$(\lambda_i - \lambda_j) \text{ or } (\lambda_i + \lambda_j + \lambda_k) = n_1\alpha_1 + \dots + n_8\alpha_8, \quad \text{for } i<j<k.$$

The other roots are the negatives of those in the list.

$n_1 = 0$ characterizes the roots of E_7,

$n_1 = n_2 = 0$ characterizes the roots of E_6.

The underlined roots are correspondingly the highest (or negatives of them) of E_8, E_6, E_7.

The roots of E_8

$\lambda_i - \lambda_j$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_8$	$\lambda_i + \lambda_j + \lambda_k$	$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_8$	$\lambda_i + \lambda_j + \lambda_k$	$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_8$
1 2	1 0 0 0 0 0 0 0	1 2 3	1 2 3 3 3 2 1 1	2 5 9	-1 -1 -2 -3 -3 -2 -1 -2
1 3	1 1 0 0 0 0 0 0	1 2 4	1 2 2 3 3 2 1 1	2 6 7	0 1 1 1 1 1 1 1
1 4	1 1 1 0 0 0 0 0	1 2 5	1 2 2 2 3 2 1 1	2 6 8	0 1 1 1 1 1 0 1
1 5	1 1 1 1 0 0 0 0	1 2 6	1 2 2 2 2 2 1 1	2 6 9	-1 -1 -2 -3 -4 -2 -1 -2
1 6	1 1 1 1 1 0 0 0	1 2 7	1 2 2 2 2 1 1 1	2 7 8	0 1 1 1 1 0 0 1
1 7	1 1 1 1 1 1 0 0	1 2 8	1 2 2 2 2 1 0 1	2 7 9	-1 -1 -2 -3 -4 -3 -1 -2
1 8	1 1 1 1 1 1 1 0	1 2 9	0 0 -1 -2 -3 -2 -1 -2	2 8 9	-1 -1 -2 -3 -4 -3 -2 -2
1 9	2 3 4 5 6 4 2 3	1 3 4	1 1 2 3 3 2 1 1	3 4 5	0 0 1 2 3 2 1 1
2 3	0 1 0 0 0 0 0 0	1 3 5	1 1 2 2 3 2 1 1	3 4 6	0 0 1 2 2 2 1 1
2 4	0 1 1 0 0 0 0 0	1 3 6	1 1 2 2 2 2 1 1	3 4 7	0 0 1 2 2 1 1 1
2 5	0 1 1 1 0 0 0 0	1 3 7	1 1 2 2 2 1 1 1	3 4 8	0 0 1 2 2 1 0 1
2 6	0 1 1 1 1 0 0 0	1 3 8	1 1 2 2 2 1 0 1	3 4 9	-1 -2 -2 -2 -3 -2 -1 -2
2 7	0 1 1 1 1 1 0 0	1 3 9	0 -1 -1 -2 -3 -2 -1 -2	3 5 6	0 0 1 1 2 2 1 1
2 8	0 1 1 1 1 1 1 0	1 4 5	1 1 1 2 3 2 1 1	3 5 7	0 0 1 1 2 1 1 1
2 9	1 3 4 5 6 4 2 3	1 4 6	1 1 1 2 2 2 1 1	3 5 8	0 0 1 1 2 1 0 1
3 4	0 0 1 0 0 0 0 0	1 4 7	1 1 1 2 2 1 1 1	3 5 9	-1 -2 -2 -3 -3 -2 -1 -2
3 5	0 0 1 1 0 0 0 0	1 4 8	1 1 1 2 2 1 0 1	3 6 7	0 0 1 1 1 1 1 1
3 6	0 0 1 1 1 0 0 0	1 4 9	0 -1 -2 -2 -3 -2 -1 -2	3 6 8	0 0 1 1 1 1 0 1
3 7	0 0 1 1 1 1 0 0	1 5 6	1 1 1 1 2 2 1 1	3 6 9	-1 -2 -2 -3 -4 -2 -1 -2
3 8	0 0 1 1 1 1 1 0	1 5 7	1 1 1 1 2 1 1 1	3 7 8	0 0 1 1 1 0 0 1
3 9	1 2 4 5 6 4 2 3	1 5 8	1 1 1 1 2 1 0 1	3 7 9	-1 -2 -2 -3 -4 -3 -1 -2
4 5	0 0 0 1 0 0 0 0	1 5 9	0 -1 -2 -3 -3 -2 -1 -2	3 8 9	-1 -2 -2 -3 -4 -3 -2 -2
4 6	0 0 0 1 1 0 0 0	1 6 7	1 1 1 1 1 1 1 1	4 5 6	0 0 0 1 2 2 1 1
4 7	0 0 0 1 1 1 0 0	1 6 8	1 1 1 1 1 1 0 1	4 5 7	0 0 0 1 2 1 1 1
4 8	0 0 0 1 1 1 1 0	1 6 9	0 -1 -2 -3 -4 -2 -1 -2	4 5 8	0 0 0 1 2 1 0 1
4 9	1 2 3 5 6 4 2 3	1 7 8	1 1 1 1 1 0 0 1	4 5 9	-1 -2 -3 -3 -3 -2 -1 -2
5 6	0 0 0 0 1 0 0 0	1 7 9	0 -1 -2 -3 -4 -3 -1 -2	4 6 7	0 0 0 1 1 1 1 1
5 7	0 0 0 0 1 1 0 0	1 8 9	0 -1 -2 -3 -4 -3 -2 -2	4 6 8	0 0 0 1 1 1 0 1
5 8	0 0 0 0 1 1 1 0	2 3 4	0 1 2 3 3 2 1 1	4 6 9	-1 -2 -3 -3 -4 -2 -1 -2
5 9	1 2 3 4 6 4 2 3	2 3 5	0 1 2 2 3 2 1 1	4 7 8	0 0 0 1 1 0 0 1
6 7	0 0 0 0 0 1 0 0	2 3 6	0 1 2 2 2 2 1 1	4 7 9	-1 -2 -3 -3 -4 -3 -1 -2
6 8	0 0 0 0 0 1 1 0	2 3 7	0 1 2 2 2 1 1 1	4 8 9	-1 -2 -3 -3 -4 -3 -2 -2
6 9	1 2 3 4 5 4 2 3	2 3 8	0 1 2 2 2 1 0 1	5 6 7	0 0 0 0 1 1 1 1
7 8	0 0 0 0 0 0 1 0	2 3 9	-1 -1 -1 -2 -3 -2 -1 -2	5 6 8	0 0 0 0 1 1 0 1
7 9	1 2 3 4 5 3 2 3	2 4 5	0 1 1 2 3 2 1 1	5 6 9	-1 -2 -3 -4 -4 -2 -1 -2
8 9	1 2 3 4 5 3 1 3	2 4 6	0 1 1 2 2 2 1 1	5 7 8	0 0 0 0 1 0 0 1
		2 4 7	0 1 1 2 2 1 1 1	5 7 9	-1 -2 -3 -4 -4 -3 -1 -2
		2 4 8	0 1 1 2 2 1 0 1	5 8 9	-1 -2 -3 -4 -4 -3 -2 -2
		2 4 9	-1 -1 -2 -2 -3 -2 -1 -2	6 7 8	0 0 0 0 0 0 0 1
		2 5 6	0 1 1 1 2 2 1 1	6 7 9	-1 -2 -3 -4 -5 -3 -1 -2
		2 5 7	0 1 1 1 2 1 1 1	6 8 9	-1 -2 -3 -4 -5 -3 -2 -2
		2 5 8	0 1 1 1 2 1 0 1	7 8 9	-1 -2 -3 -4 -5 -4 -2 -2

Außerdem wird noch die logische Behandlung der Mathematik dem Genius widerstehn, da diese, die eigentliche Einsicht verschließend, nich befriedigt, sondern eine bloße Verkettung von Schlüssen, nach dem Satz des Erkenntnißgrundes darbietend, von allen Geisteskräften am meisten das Gedächtniß in Anspruch nimmt, um nämlich immer alle die früheren Sätze, darauf man sich beruft, gegenwärtig zu haben. Auch hat die Erfahrung bestätigt, daß Große Genien in der Kunst zur Mathematik keine Fähigkeit haben: nie war ein Mensch zugleich in Beiden sehr ausgezeichnet.

Schopenhauer, Welt als Vorstellung p. 244

34. The structure of E_7 and E_6

Consider the roots of the preceding table characterized by the condition $n_1 = 0$.

$$n_2\alpha_2 + \dots + n_8\alpha_8 . \qquad (1)$$

This subset Δ' of the roots of E_8 consists of those elements, which can be written

$\pm(\lambda_i-\lambda_j)$, for $2\le i<j\le 8$ (42 in number),

$\pm(\lambda_i+\lambda_j+\lambda_k)$, for $2\le i<j<k\le 8$ (70 in number), and the 14 roots

$$\pm(\lambda_1+\lambda_i+\lambda_9) = \pm(\lambda_i-\lambda_2-\lambda_3- \dots -\lambda_8),\ i = 2, \dots, 8. \qquad (2)$$

We check easily that the subset of E_8,

$$\mathfrak{g} = \langle H_\alpha | \alpha\in\Delta'\rangle \oplus_{\alpha\in\Delta'} \mathbb{C}X_\alpha , \qquad (3)$$

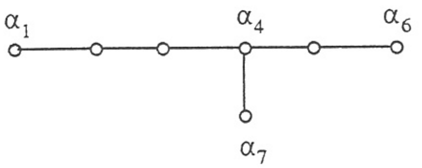
is a subalgebra, and applying the lemma of §27, that this is a semi simple Lie algebra with Cartan subalgebra $\mathfrak{h}' = \langle H_\alpha | \alpha\in\Delta'\rangle$ of dimension 7. Δ' (when restricted on $\mathfrak{h}'$) is the set of roots of this Lie algebra, and

$$\alpha'_1 = \alpha_2|\mathfrak{h}' ,\ \alpha'_2 = \alpha_3|\mathfrak{h}' , \dots ,\ \alpha'_7 = \alpha_8|\mathfrak{h}' ,$$

is a fundamental system of roots, whose Dynkin diagram coincides with that of E_7. This is most easily seen by noticing that the Cartan integers coincide with the corresponding in E_8:

$$c_{\alpha'\beta'} = \alpha'(H_{\beta'}) = \alpha(H_\beta) = c_{\alpha\beta} .$$

E_7



Exercise-1 Show the identities

$$\Sigma_{1\le i<j<k\le n} (h_i+h_j+h_k)^2 = \binom{n-1}{2}(\Sigma\, h_i)^2 + (n-2)(n-3)\Sigma_{i<j} (h_ih_j)$$
$$= (n-2)(\Sigma\, h_i)^2 + ((n-2)(n-3)/2)\Sigma\, h_i^2 .$$

Exercise-2 Prove that the Killing form of E_7, restricted on $\mathfrak{h}'$ is given by

$$\langle H,H\rangle = 36\Sigma_{2\le i\le 8}\, h_i^2 + 18(h_1+h_9)^2 .$$

Exercise-3 Find the fundamental coroots H_i, for $i = 1, \dots, 7$ of E_7.

Exercise-4 Show that the subalgebra $\mathfrak{g}$ in (3) coincides with the direct sum

$$g^{19} \oplus V^{19} \oplus (V^{19})^*,$$

where g^{19} is the subalgebra of sl(9;C) consisting of matrices of the form

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & \boxed{\quad * \quad} & & 0 \\ | & & & | \\ 0 & \cdots & 0 & a \end{pmatrix}.$$

[All the coroots H_α are contained in this subspace, as well as the 42 E_{ij}, for $2 \leq i \neq j \leq 8$. Coincidence follows for dimension (=48) reasons.]

Show also that

$$V^{19} = \langle e_i \wedge e_j \wedge e_k \mid \text{with } i \neq 1 \text{ and } k \neq 9 \text{ or } (i,k)=(1,9) \rangle,$$
$$(V^{19})^* = \langle e^i \wedge e^j \wedge e^k \mid \text{with } i \neq 1 \text{ and } k \neq 9 \text{ or } (i,k)=(1,9) \rangle .$$

The construction of a model of E_6 is analogous to the previous one. We consider again the model of E_8, constructed in the previous § and in this, the roots not containing α_1, α_2 : $n_3\alpha_3 + \ldots + n_8\alpha_8$. This subset Δ'' of roots of E_8 contains the following roots :

$\pm(\lambda_i - \lambda_j)$, for $3 \leq i < j \leq 8$ (30 in number),

$\pm(\lambda_i + \lambda_j + \lambda_k)$, for $3 \leq i < j < k \leq 8$ (40 in number), and the 2 roots

$\pm(\lambda_1 + \lambda_2 + \lambda_9)$. (4)

Exercise-5 Show that Δ'' coincides also with the subset of the roots of E_7, which do not contain the root $\alpha'_1 = \alpha_2|h'$.

Applying again the lemma of §27, we see that

$$g'' = \langle H_\alpha | \alpha \in \Delta'' \rangle \oplus_{\alpha \in \Delta''} CX_\alpha , \qquad (5)$$

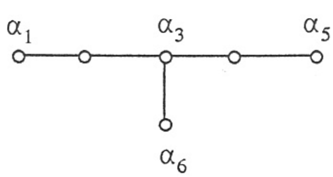
is a semi simple Lie algebra with Cartan subalgebra $h'' = \langle H_\alpha | \alpha \in \Delta'' \rangle$ of dimension 6. Δ'' (when restricted on h'') is the set of roots of this Lie algebra, and

$$\alpha''_1 = \alpha_3|h'' , \alpha''_2 = \alpha_4|h'' , \ldots , \alpha''_7 = \alpha_8|h'' , \qquad (6)$$

is a fundamental system of roots, whose Dynkin diagram coincides with that of E_6. This is most easily seen by noticing that the Cartan integers coincide with the corresponding in E_8:

$$c_{\alpha''\beta''} = \alpha''(H_{\beta''}) = \alpha(H_\beta) = c_{\alpha\beta} .$$

E_6



Exercise-6 Show that the subalgebra g'' in (5) coincides with the direct sum

$$g^{129} \oplus V^{129} \oplus (V^{129})^*,$$

where g^{129} is the subalgebra of sl(9;C) consisting of matrices of the form

$$\begin{pmatrix} a & 0 & \cdots & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ | & | & \boxed{\quad * \quad} & & | \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}.$$

Show also that

$$V^{129} = \langle e_i \wedge e_j \wedge e_k \mid \text{with } i,j,k \neq 1,2,9 \rangle + \langle e_1 \wedge e_2 \wedge e_9 \rangle,$$
$$(V^{129})^* = \langle e^i \wedge e^j \wedge e^k \mid \text{with } i,j,k \neq 1,2,9 \rangle + \langle e^1 \wedge e^2 \wedge e^9 \rangle.$$

Exercise-7 Show that the Killing form of E_6, when restricted on $\mathfrak{h}''$ is given by

$$\langle H,H \rangle = 24\sum_{3\leq i\leq 8} h_i^2 + 8(h_1+h_2+h_9)^2 .$$

Exercise-8 Show that E_6 and E_7 have respectively the dimensions 78 and 133.

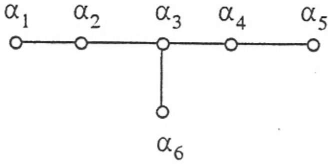
But these are foolish things to all the wise,
  And I love Wisdom more than she loves me;
My tendency is to philosophize
  On most things, from a tyrant to a tree;
But still the spouseless Virgin *Knowledge* flies.
  What are we? and whence came we? what shall be
Our *ultimate* existence? what's our present?
Are questions answerless, and yet incessant.

Byron, Don Juan, Canto VI, 63

35. The structure of F_4

The Dynkin diagram of E_6 has an obvious symmetry :

E_6



Call f the permutation of the fundamental system $\Pi = \{\alpha_1, \dots, \alpha_6\}$ suggested by the figure (interchange $\alpha_2 \leftrightarrow \alpha_4$, $\alpha_1 \leftrightarrow \alpha_5$, fixing α_3, α_6). Since Π is a basis of $\mathfrak{h}^*$ (dual of a Cartan subalgebra) we can Extend f linearly on $\mathfrak{h}^*$ and on the root system Δ of E_6 and denote by

$$\alpha' = f(\alpha), \text{ for every } \alpha \in \mathfrak{h}^*. \qquad (1)$$

We extend also f on $\mathfrak{h}$, so as to preserve duality :

$$\alpha'(f(H)) = \alpha(H), \text{ for every } \alpha \in \mathfrak{h}^* \text{ and } H \in \mathfrak{h}. \qquad (2)$$

We have then

$$f(H_\alpha) = H_{\alpha'}, \text{ for every } \alpha \in \Delta. \qquad (3)$$

In fact, because of the symmetry of the diagram, f is an isometry on $\mathfrak{h}$ and we have

$$\beta'(f(H_\alpha)) = \beta(H_\alpha) = 2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle = 2\langle\beta',\alpha'\rangle/\langle\alpha',\alpha'\rangle = \beta'(H_{\alpha'}).$$

We extend f on the whole Lie algebra by defining it on the root vectors :

$$f(X_\alpha) = X_{\alpha'}, \text{ for all } \alpha \in \Delta. \qquad (4)$$

It will be seen later in §37 (or by calculation in the model of E_6 of the preceding §) that f is a well defined involutive ($f^2 = I$) automorphism of the Lie algebra, which leaves $\mathfrak{h}$ invariant and permutes the root vectors X_α, as well as the coroots H_α. Since f is a linear involution, it decomposes the Lie algebra into a direct sum of its ± 1-eigenspaces, the $+1$-eigenspace being a Lie subalgebra $\mathfrak{g}$ of E. $\mathfrak{g}$ is a model for F_4. The following table gives the roots of F_4 and relates them to those of E_6 and their representation in terms of $\lambda_i - \lambda_j$, $\lambda_i + \lambda_j + \lambda_k$, in §33. The Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ is characterized by the equation

$$\alpha'(H) = \alpha(H), \text{ for all } \alpha \in \Delta, \quad \Rightarrow$$

$$\mathfrak{h}' = \langle H_3, H_6, H_1 + H_5, H_2 + H_4 \rangle. \qquad (5)$$

A fundamental system of roots consists of the restrictions of the roots α_1, α_2, α_3, α_6 of E_6 on $\mathfrak{h}'$:

$$\gamma_1 = \alpha_1|\mathfrak{h}', \; \gamma_2 = \alpha_2|\mathfrak{h}', \; \gamma_3 = \alpha_3|\mathfrak{h}', \; \gamma_4 = \alpha_6|\mathfrak{h}'. \qquad (6)$$

The roots of F_4

$m_1\gamma_1+\ldots+m_4\gamma_4$	$n_1\alpha_1+\ldots+n_6\alpha_6$	(i,j) or (i,j,k)	α'	in terms of $\bar{\omega}_i$
1 0 0 0	1 0 0 0 0 0	3 4	0 0 0 0 1 0	1/2 -1 -1 -1 -1
0 1 0 0	0 1 0 0 0 0	4 5	0 0 0 1 0 0	0 0 1 0
0 0 1 0	0 0 1 0 0 0	5 6	fix	0 1 -1 0
0 0 0 1	0 0 0 0 0 1	6 7 8	fix	1 -1 0 0
1 1 0 0	1 1 0 0 0 0	3 5	0 0 0 1 1 0	1/2 -1 -1 1 -1
1 1 1 0	1 1 1 0 0 0	3 6	0 0 1 1 1 0	1/2 -1 1 -1 -1
1 2 1 0	1 2 1 1 0 0	3 7	0 1 1 1 1 0	1/2 -1 1 1 -1
2 2 1 0	1 1 1 1 1 0	3 8	fix	-1 0 0 -1
0 1 1 0	0 1 1 0 0 0	4 6	0 0 1 1 0 0	0 1 0 0
0 2 1 0	0 1 1 1 0 0	4 7	fix	0 1 1 0
0 0 1 1	0 0 1 0 0 1	5 7 8	fix	1 0 -1 0
0 1 1 1	0 0 1 1 0 1	5 6 8	0 1 1 0 0 1	1 0 0 0
1 1 1 1	0 0 1 1 1 1	5 6 7	1 1 1 0 0 1	1/2 1 -1 -1 -1
0 2 1 1	0 1 1 1 0 1	4 6 8	fix	1 0 1 0
1 2 1 1	0 1 1 1 1 1	4 6 7	1 1 1 1 0 1	1/2 1 -1 1 -1
0 2 2 1	0 1 2 1 0 1	4 5 8	fix	1 1 0 0
1 2 2 1	0 1 2 1 1 1	4 5 7	1 1 2 1 0 1	1/2 1 1 -1 -1
1 3 2 1	0 1 2 2 1 1	4 5 6	1 2 2 1 0 1	1/2 1 1 1 -1
2 2 1 1	1 1 1 1 1 1	3 6 7	fix	0 -1 0 -1
2 2 2 1	1 1 2 1 1 1	3 5 7	fix	0 0 -1 -1
2 3 2 1	1 1 2 2 1 1	3 5 6	1 2 2 1 1 1	0 0 0 -1
2 4 2 1	1 2 2 2 1 1	3 4 6	fix	0 0 1 -1
2 4 3 1	1 2 3 2 1 1	3 4 5	fix	0 1 0 -1
2 4 3 2	1 2 3 2 1 2	-(1 2 9)	fix	1 0 0 -1

The table contains 24 roots. 12 of them are restrictions of 12 roots of E_6 satisfying $\alpha' = \alpha$. We denote them by $\sigma_1, \ldots, \sigma_{12}$. The other 12 satisfy $\alpha' \neq \alpha$ and we denote them by $\beta_1, \ldots, \beta_{12}$. The Lie algebra may be represented by the direct sum

$$<H_3, H_6, H_1+H_5, H_2+H_4> \bigoplus_1^{12} CX_{\pm\sigma_i} \bigoplus_1^{12} C(X_{\beta_i}+X_{\beta'_i}) \bigoplus_1^{12} C(X_{-\beta_i}+X_{-\beta'_i}) .$$

Also the following relations hold :

$$[HX_{\pm\sigma}] = \pm\sigma_i(H)X_{\pm\sigma_i},$$

$$[H(X_{\beta_i}+X_{\beta'_i})] = \beta_i(H)(X_{\beta_i}+X_{\beta'_i}),$$

$$[H(X_{-\beta_i}+X_{-\beta'_i})] = -\beta_i(H)(X_{-\beta_i}+X_{-\beta'_i}),$$

$$[X_{\sigma_i}, X_{-\sigma}] = H_{\sigma_i},$$

$$[(X_{\beta_i}+X_{\beta'_i}), (X_{-\beta_i}+X_{-\beta'_i})] = (H_{\beta_i}+H_{\beta'_i}) \neq 0 .$$

The last relation holds because $\beta'_i - \beta_i$ is not a root. This follows from the fact that f preserves the sign of the roots and $f(\beta'_i - \beta_i) = -(\beta'_i - \beta_i)$. Also the last term is non-zero because

$$\beta_i(H_{\beta_i} + H_{\beta'_i}) \neq 0.$$

In fact, in the contrary case we should have $2+2<\beta_i,\beta'_i>/<\beta_i,\beta_i> = 0$, which ($f$ preserving the sign) gives the contradiction $<\beta_i+\beta'_i,\beta_i+\beta'_i> = 2<\beta_i,\beta_i>+2<\beta_i,\beta'_i> = 0$.

Applying the lemma of §27, we conclude that $\mathfrak{g}$ is a semi simple Lie algebra, with Cartan subalgebra $\mathfrak{h}'$ and roots the $\pm\sigma_i$ and $\pm\beta_i$.

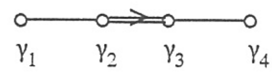
Computing the coroots and the Cartan integers we find

$$H_{\gamma_1} = H_{\alpha_1} + H_{\alpha_5},\ H_{\gamma_2} = H_{\alpha_2} + H_{\alpha_4},\ H_{\gamma_3} = H_{\alpha_3},\ H_{\gamma_4} = H_{\alpha_6},$$

$$\gamma_1(H_{\gamma_2}) = \gamma_2(H_{\gamma_3}) = -1,\ \gamma_3(H_{\gamma_2}) = -2,\ \gamma_3(H_{\gamma_4}) = -1,\ \text{all other integers being 0.}$$

Thus the Dynkin diagram of $\mathfrak{g}$ coincides with the F_4.

F_4



Exercise-1 Show that $\mathfrak{h}'$ is described in the model of E_6, used in the previous §, by the equations

$$\lambda_1 + \dots + \lambda_9 = 0,$$
$$\lambda_1 = \lambda_2 = \lambda_9,$$
$$\lambda_3 - \lambda_4 = \lambda_7 - \lambda_8,$$
$$\lambda_4 - \lambda_5 = \lambda_6 - \lambda_7.$$

Define then the linear forms on $\mathfrak{h}'$

$$\bar\omega_1 = (\lambda_4 + \lambda_7 + \lambda_8)|\mathfrak{h}' = (\lambda_5 + \lambda_6 + \lambda_8)|\mathfrak{h}',$$
$$\bar\omega_2 = (\lambda_4 - \lambda_6)|\mathfrak{h}' = (\lambda_5 - \lambda_7)|\mathfrak{h}',$$
$$\bar\omega_3 = (\lambda_4 - \lambda_5)|\mathfrak{h}' = (\lambda_6 - \lambda_7)|\mathfrak{h}',$$
$$\bar\omega_4 = -(\lambda_3 + \lambda_5 + \lambda_6)|\mathfrak{h}' = -(\lambda_3 + \lambda_4 + \lambda_7)|\mathfrak{h}'.$$

Show that all the roots of F_4 can be written in the form (verify the last column of the table)

$$\pm\bar\omega_i,\ \pm\bar\omega_i \pm\bar\omega_j,\ (1/2)(\pm\bar\omega_1 \pm\bar\omega_2 \pm\bar\omega_3 \pm\bar\omega_4).$$

> Demnach bedarf ein schönes Werk eines empfindenden Geistes, ein gedachtes Werk eines denkenden Geistes, um wirklich dazuseyn und zu leben. Allein, nur gar zu oft kann Dem, der ein solches Werk in die Welt schickt, nachher zu Muthe werden, wie einem Feuerwerker, der sein lange und mühsam vorbereitetes Erzeugniß endlich mit Enthusiasmus abgebrannt hat und dann erfährt, daß er damit an den unrechten Ort gekommen, und sämmtliche Zuschauer die Zöglinge der Blindenanstalt gewesen seien.
>
> Schopenhauer, Parerga p. 504

36. The order of the Weyl group

The Weyl group of the classical Lie algebras and of G_2 has been determined in §§27-30. The order of the Weyl group of the Lie algebras F_4, E_6, E_7 and E_8 is found using a theorem of Witt, stated below.

Let $\mathfrak{g}$ be a semi simpole Lie algebra and $\Pi = \{\alpha_1, \dots, \alpha_d\}$ be a fundamental system of roots. Consider the "dual basis" $\{\beta_1, \dots, \beta_d\}$ defined via the relations

$$<\alpha_i,\beta_j> = \delta_{ij} . \tag{1}$$

Notice that, in general, the β_j's are not roots themselves.

Exercise-1 Show that $\{t\beta_j|t>0\}$ are the 1-dimensional faces of the fundamental Weyl chamber, corresponding to Π. Analogously $\{t\beta_i+t\beta_j|t,t>0\}$ are the 2-dimensional faces, etc. Show finally that the fundamental Weyl chamber C_0 is described by

$$C_0 = \{\textstyle\sum_i t_i\beta_i|t_i>0\}.$$

After these preliminary remarks, consider now the set of roots $\alpha \in \Delta$, which are orthogonal to the last dual root β_d. All these roots are contained in the (d-1)-dimensional plane $\beta_d^{\perp}$ and form a root system which has $\Pi'=\{\alpha_1, \dots, \alpha_{d-1}\}$ as a fundamental system. Let W' be the corresponding Weyl group generated by the fundamental reflections $S_1, \dots, S_{d-1}$. Let also C'_0, C'_1, ... denote the Weyl chambers of W'. W' may be considered as a subroup of W (Weyl group of Δ) and all its elements fix β_d. This is characteristic for W' :

Theorem *W' coincides with the (isotropy) subgroup W_β of W, which fixes β_d.*

In fact, suppose $g \in W_\beta$-W'. g induces an isometry on the plane $\beta_d^{\perp}$, hence $g(C'_0)\cap C'_k \neq \emptyset$, for some C'_k. Let $T \in W'$ be the element of W' which sends C'_k to C'_0 and assume that $y=g(x) \in g(C'_0)\cap C'_k$ with $x \in C'_0$. Then $T(g(x)) \in C'_0$ and we can choose x so that

$$\beta_d+x \in C_0 \text{ and } T(g(x) \in C_0, (C_0 \text{ fundamental corr. to } \Pi). \tag{2}$$

By simple transitivity of W, $T\circ g=I$, hence $g = T^{-1} \in W'$, a contradiction.

That (2) holds, is due to the fact that C'_k and C_m are "cones" i.e. with each x belonging in there, the whole postive line $\{tx|t>0\}$ is contained in the same set. Also ty = g(tx) $\in g(C'_0)\cap C'_k$, since the intersection is again a cone (g being isometry). For i=1, .. , d-1 and t>0 we have,

$$<\alpha_i,\beta_d+tx> = <\alpha_i,tx> > 0,$$
$$<\alpha_i,T(g(\beta_d+tx))> = <\alpha_i,\beta_d+T(g(tx))> >0,$$
$$<\alpha_i,\beta_d+tx> = 1+t<\alpha_i,x>,$$
$$<\alpha_i,T(g(\beta_d+tx))> = 1+t<\alpha_i,T(g(x))>. \qquad (3)$$

Since x, $T(g(x)) \in C'_0$, (hence positive linear combinations of the elements of Π') we have

$$<\alpha_d,x> \leq 0,\ <\alpha_d,T(g(x))>\leq 0.$$

Thus we obtain (2) by taking t sufficiently small, so that the right sides of (3) be positive.

q.e.d.

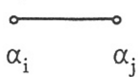
The choice of β_d is insignificant. An analogous result holds also for the subgroup of W leaving any other fundamental root fixed. Thus, for a given root system Δ, we can consider a) the orbit $W(\beta_d)$ and b) the isotropy group W_β. It is elementary that

$$|W| = |W(\beta_d)||W_\beta|,$$

and this allows in certain cases the determination of the order |W|. The two next propositions assist the subsequent calculations.

Exercise-2 Let $\Pi = \{\alpha_1, \ldots, \alpha_d\}$ and α be a root orthogonal to all α_i but α_d. Then $\beta_d = t\alpha$.

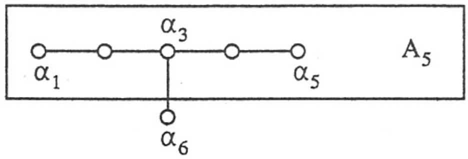
Proposition *Let the Dynkin diagram of the root system Δ be connected and have only simple connections. Then the Weyl group W operates transitively on Δ and all the roots have the same length.*



In fact, in this case, for two adjacent roots α_i, α_j of the fundamental system, we have $c_{ij}=-1$, hence $S_iS_j(\alpha) = S_i(\alpha_i+\alpha_j) = -\alpha_i+S_i(\alpha_j) = -\alpha_i+(\alpha_i+\alpha_j) = \alpha_j$. Hence the Weyl group is transitive on the simple roots. Since the Weyl group is transitive on the Weyl chambers, every root of Δ is conjugate, under W, to some fundamental root (Ex-4,§26). Thus W is transitive on the whole Δ and consequently all roots must have the same length.

q.e.d.

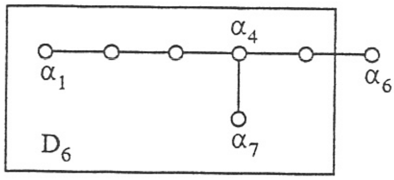
We examine now the Dynkin diagrams of E_6, E_7, E_8 and F_4.



E_6 The maximal root (negative, see the table in p. 33-6) $\alpha = \lambda_1+\lambda_2+\lambda_9$ of E_6 is orthogonal to $\alpha_1, \ldots, \alpha_5$ ($\alpha+(\lambda_i-\lambda_j)$, $\alpha-(\lambda_i-\lambda_j)$ are not roots). Thus, $\beta_d = t\alpha$, and in $\beta_d^\perp$ we have a root system of type A_5, whose Weyl group has the order 6!, hence $|W_\beta|=6!$. Since E_6 has simple connectins, we have also $|W(\beta_d)|=|W(\alpha)| = 72$ = number of roots of E.

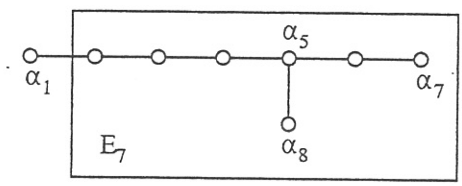
Thus,

$$|W| = |W(\beta_d)|\cdot|W_\beta| = 72\cdot 6! = 2^7\cdot 3^4\cdot 5 = 51{,}840.$$



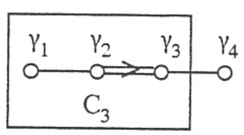
E$_7$ The maximal root (negative, see the table in p. 33-6) $\alpha = \lambda_1+\lambda_8+\lambda_9$ of E_7 is orthogonal to $\alpha_1, \ldots, \alpha_5$ and α_7 ($\alpha\pm(\lambda_2-\lambda_3), \ldots, \alpha\pm(\lambda_6-\lambda_7), \alpha+(\lambda_7-\lambda_8), \alpha\pm(\lambda_6+\lambda_7+\lambda_8)$ are not roots). Thus, $\beta_7=t\alpha$, and in $\beta_7^\perp$ we have a root system of type D_6, whose Weyl group has the order $6!\cdot 2^5$, hence $|W_\beta|=6!\cdot 2^5$. Since E_7 has simple connections, we have also $|W(\beta_7)| = |W(\alpha)| = 126$ = number of roots of E_7. Thus,

$$|W| = |W(\beta_d)|\cdot|W_\beta| = 126\cdot 6!\cdot 2^5 = 2^{10}\cdot 3^4\cdot 5\cdot 7 = 2{,}903{,}040.$$



E$_8$ The maximal root (see the table in p. 33-6) $\alpha=\lambda_1-\lambda_9$ of E_8 is orthogonal to $\alpha_2, \ldots, \alpha_8$ ($\alpha\pm\alpha_2, \ldots, \alpha\pm\alpha_8$ are not roots). Thus, $\beta_8 = t\alpha$, and in $\beta_8^\perp$ we have a root system of type E_7, whose Weyl group has the order $126\cdot 6!\cdot 2^5$, hence $|W_\beta| = 126\cdot 6!\cdot 2^5$. Since E_8 has simple connections, we have also $|W(\beta_8)| = |W(\alpha)| = 240$ = number of roots of E_8. Thus

$$|W| = |W(\beta_8)||W_\beta| = 126\cdot 6!\cdot 2^5\cdot 240 = 2^{14}\cdot 3^5\cdot 5^2\cdot 7 = 696{,}729{,}600.$$



F$_4$ The maximal root (see the table in p. 35-2) $\alpha = \bar\omega_1-\bar\omega_4$ of F_4 is orthogonal to $\gamma_1, \gamma_2, \gamma_3$ ($\alpha\pm\gamma_1, \alpha\pm\gamma_2, \alpha\pm\gamma_3$ are not roots). Thus $\beta_4 = t\alpha$, and in $\beta_4^\perp$ we have a root system of type C_3, whose Weyl group has the order $3!\cdot 2^3$, hence $|W_\beta| = 3!2^3$.

Now we need an exercise, which is proved with the arguments used in the previous proposition (transitivity on Weyl chambers + transitivity between fundamental roots of the same length).

Exercise-3 Show that the Weyl group W of F_4 has two orbits in the root system Δ. One is $\{\pm\bar\omega_i\pm\bar\omega_j\}$ (long roots) and the other (short) $\{\pm\bar\omega_i\}\cup\{\pm\bar\omega_1\pm\bar\omega_2\pm\bar\omega_3\pm\bar\omega_4\}$, each containing 24 elements.

Using this result, we see that $|W(\beta_4)| = |W(\alpha)| = 24$ = number of roots in the orbit of α. Thus,

$$|W| = |W(\beta_8)||W_\beta| = 24 \cdot 3! \cdot 2^3 = 2^7 \cdot 3^2 = 1,152.$$

The portion of this world which I at present
Have taken up to fill the following sermon,
Is one of which there's no description recent:
The reason why, is easy to determine:
Although it seems both prominent and pleasant,
There is a sameness in its gems and ermine,
A dull and family likeness through all ages,
On no great promise for poetic pages.

Byron, Don Juan, Canto XIV, 15

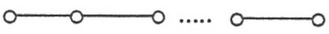
37. Weyl-Chevalley normal form

We show in this § that the Dynkin diagram completely describes the structure of the corresponding semi simple Lie algebra. Let $\Pi = \{\alpha_1, \ldots, \alpha_d\}$ be a fundamental system of roots (notation as in the preceding §§). The Cartan matrix of Π is the integer matrix whose elements are the Cartan integers of the fundamental system:

$$c_{ij} = 2\langle\alpha_i,\alpha_j\rangle/\langle\alpha_j,\alpha_j\rangle\ ,\ \ i, j = 1, \ldots, d. \tag{1}$$

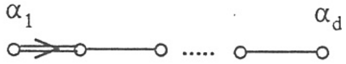
The next list gives the Cartan matrices corresponding to the diagrams of the list p. 25-5.

A_d



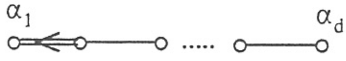
$$\begin{pmatrix} 2 & -1 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & 0 & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

B_d



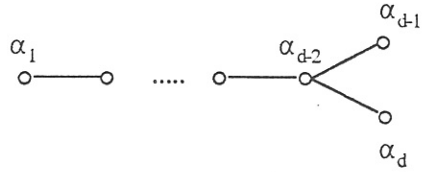
$$\begin{pmatrix} 2 & -1 & 0 & 0 & & \\ -2 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & 0 & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

C_d



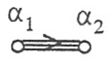
$$\begin{pmatrix} 2 & -2 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & 0 & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

D_d



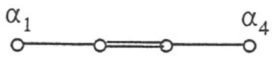
$$\begin{pmatrix} 2 & -1 & 0 & 0 & & & \\ -1 & 2 & -1 & 0 & & & \\ 0 & -1 & 2 & 0 & & & \\ & & & \ddots & & & \\ & & & & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}$$

G_2



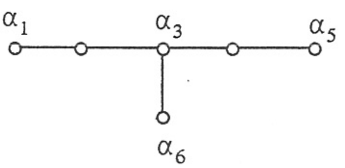
$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

F_4



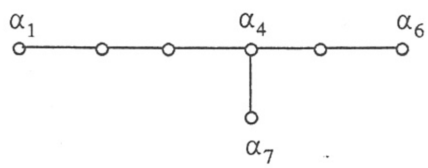
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

E_6



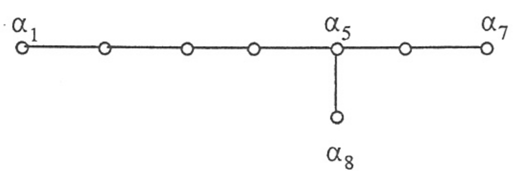
$$\begin{pmatrix} 2 & -1 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & -1 & 0 & -1 \\ & & -1 & 2 & -1 & 0 \\ & & 0 & -1 & 2 & 0 \\ & & -1 & 0 & 0 & 2 \end{pmatrix}$$

E_7



$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & 0 & 0 & & \\ & -1 & 2 & -1 & 0 & & \\ & 0 & -1 & 2 & -1 & 0 & -1 \\ & & & -1 & 2 & -1 & 0 \\ & & & 0 & -1 & 2 & 0 \\ & & & -1 & 0 & 0 & 2 \end{pmatrix}$$

E_8



$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & 0 & 0 & & \\ & & -1 & 2 & -1 & 0 & & \\ & & 0 & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & 0 \\ & & & & -1 & 0 & 0 & 2 \end{pmatrix}$$

Exercise-1 Prove that reversing the arrow in the Dynkin diagram gives transposed corresponding Cartan matrices (f.e. B_d and C_d).

Exercise-2 Show that the determinant of the corresponding Cartan matrix is as in the list below.

A_d	B_d	C_d	D_d	E_6	E_7	E_8	F_4	G_2
d+1	2	2	4	3	2	1	1	1

Proposition-1 *The Cartan matrix (or the Dynkin diagram) of a fundamental system Π, completely determines the system of roots Δ.*

In fact, Δ can be constructed inductively, using the hight of a root (p. 26-5). Suppose we know all the roots of hight less than the fixed hight k. By Ex-10 §26, all the roots of hight k+1 will have the form $\alpha+\alpha_i$, where α a root of hight k. Thus it suffices to know which of the $\alpha+\alpha_i$ are roots, when α runs into the set of roots of hight k and α_i runs into Π. But, by induction, for a root α of hight k, we know which of $\alpha-\alpha_i$, $\alpha-2\alpha_i$, ... are roots (of hight < k). In other words, in the formula

$$2<\alpha,\alpha_i>/<\alpha_i,\alpha_i> = q\text{-}p,$$

we know the q. But then

$$p = q - 2<\alpha,\alpha_i>/<\alpha_i,\alpha_i>, \tag{2}$$

and if $\alpha = \sum_{1\le j\le d} n_i\alpha_i$, then (2) gives

$$p = q - \sum_{1\le j\le d} n_j c_{ji}. \tag{3}$$

Hence, knowing also n_j and c_{ji}, if p>0, then $\alpha+\alpha_i$ will be a root, otherwise not. Thus, we can construct all the roots of hight k+1, and the induction is completed. q.e.d.

The next step is to show that the Cartan matrix not only determines Δ, but also a suitable basis of the Lie algebra $\mathfrak{g}$. In fact, we show that the basis of the Lie algebra can be choosen so that the corresponding structure constants are all integer numbers. This will be called a **Weyl-Chevalley** basis (or normal form). We start with the copy of sl(2;**R**) corresponding to each positive root $\alpha\in\Delta^+$ (see (6), §23):

$$\mathfrak{g}^{(\alpha)} = <H_\alpha, X_\alpha, X_{-\alpha}> \approx sl(2;\mathbf{R}),$$
$$[H_\alpha X_\alpha] = 2X_\alpha,\ [H_\alpha X_{-\alpha}] = -2X_{-\alpha},\ [X_\alpha X_{-\alpha}] = H_\alpha\ . \tag{4}$$

These are certain subalgebras of the Lie algebra $\mathfrak{g}$, and by 1-dimensionality of the root spaces $\mathfrak{g}_\alpha$ (Pro-3, §23) we know that there are (complex) constants $N_{\alpha\beta}$, such that

$$[X_\alpha X_\beta] = N_{\alpha\beta}X_{\alpha+\beta}\ . \tag{5}$$

We agree that $N_{\alpha\beta} = 0$, when $\alpha+\beta$ is not a root, or when α, or β is not a root.

The $N_{\alpha\beta}$'s satisfy some important identities which suffice to their determination via the α-strings of roots and the integers appearing in there. In fact, we saw in §24 that $\mathfrak{g}^{(\alpha)}$ operates via ad on the α-string

$$\mathfrak{g}_\beta{}^\alpha = X_{\beta-q\alpha}\oplus \dots \oplus X_\beta \oplus \dots \oplus X_{\alpha+p\beta}, \tag{6}$$

and induces there an irreducible representation D_J of sl(2;**R**), with 2J+1=p+q+1. In this representation the eigenvectors of H_α are

$$v_0 = X_{\alpha+p\beta},\ \dots,\ v_p = X_\beta,\ \dots,\ v_{2J} = X_{\beta-q\alpha},$$

2J = p+q is the maximal eigenvalue of H_α, and by the properties (§13) of these representations :

$$X_{-\alpha}X_\alpha v_i = i(p+q+1)v_i \qquad \Rightarrow$$
$$[X_{-\alpha}[X_\alpha X_\beta]] = p(q+1)X_\beta\ . \tag{7}$$

On the other side

$$[X_{-\alpha}[X_\alpha X_\beta]] = N_{\alpha\beta}[X_{-\alpha}X_{\alpha+\beta}] = N_{\alpha\beta}N_{-\alpha,\alpha+\beta}X_\beta\ . \tag{8}$$

Proposition-2 *For two roots $\alpha\neq\pm\beta$ of $\mathfrak{g}$, the following equations hold:*

a) $N_{\alpha\beta} = -N_{\beta\alpha}$, (9)

b) $\alpha+\beta+\gamma = 0 \Rightarrow N_{\alpha\beta}/<\gamma,\gamma> = N_{\beta\gamma}/<\alpha,\alpha> = N_{\alpha\gamma}/<\beta,\beta>$, (10)

c) $N_{\alpha\beta}N_{-\alpha,-\beta} = -p(q+1)<\alpha+\beta,\alpha+\beta>/<\beta,\beta>$, (11)

d) $N_{\alpha\beta}N_{-\alpha,-\beta} = -(q+1)^2$. (12)

In fact, a) is a consequence of the skew-symmetry of the Lie bracket. b) follows from Jacobi's identity. In fact, by duality, we have

$$\alpha+\beta+\gamma=0 \quad\Rightarrow\quad h_\alpha+h_\beta+h_\gamma=0 \quad\Rightarrow$$

$$<\alpha,\alpha>H_\alpha + <\beta,\beta>H_\beta + <\gamma,\gamma>H_\gamma = 0\,. \tag{*}$$

Also

$$[X_\alpha[X_\beta X_\gamma]] + [X_\beta[X_\gamma X_\alpha]] + [X_\gamma[X_\alpha X_\beta]] = 0 \quad\Rightarrow$$

$$N_{\beta\gamma}H_\alpha + N_{\gamma\alpha}H_\beta + N_{\alpha\beta}H_\gamma = 0. \tag{**}$$

The vectors H_α, H_β are independent, hence (*) and (**) are not independent $\Rightarrow$ b). c) is a consequence of (7), (8) and b) applied on the triple $(\alpha+\beta)+(-\alpha)+(-\beta)=0 \Rightarrow$

$$N_{-\alpha,\alpha+\beta} = N_{-\alpha,-\beta} <\beta,\beta>/<\alpha+\beta,\alpha+\beta>.$$

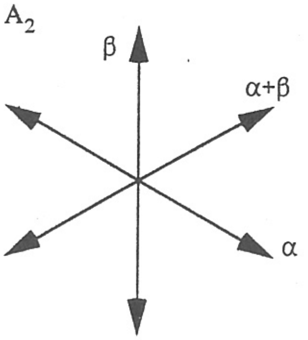
Substitution of this in (8) and comparison with (7) gives c).

Finally d) follows from c) by proving that for each pair of roots the following equation is true:

$$p<\alpha+\beta,\alpha+\beta>/<\beta,\beta> = q+1. \tag{13}$$

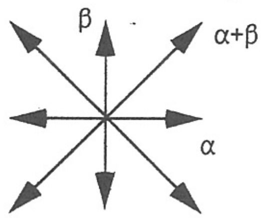
This is verified case by case, for all 2-dimensional root systems (§24) spanned by two roots α, β and such that $\alpha+\beta$ is again a root ($A_1\oplus A_1$ excluded). The different cases are:

A_2

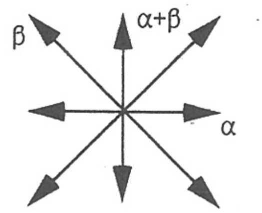


$p=1, q=0,$
$<\alpha+\beta,\alpha+\beta> = <\beta,\beta>$
and (13) holds.

B_2 (2 cases)

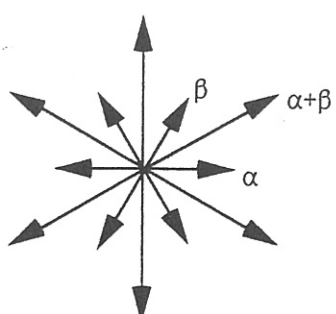


$p=1, q=1,$
$<\alpha+\beta,\alpha+\beta> = 2<\beta,\beta>$
and (13) holds.

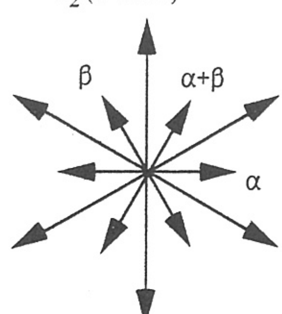


$p=2, q=0,$
$2<\alpha+\beta,\alpha+\beta> = <\beta,\beta>$
and (13) holds.

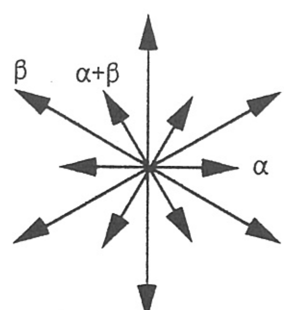
G_2 (3 cases)



i) when $p=1, q=2,$
$<\alpha+\beta,\alpha+\beta> = 3<\beta,\beta>$
and (13) is true.



ii) when $p=2, q=1,$
$<\alpha+\beta,\alpha+\beta> = <\beta,\beta>$
and (13) is true.



iii) when $p=3, q=0,$
$3<\alpha+\beta,\alpha+\beta> = <\beta,\beta>$
and (13) is true.

This completes the proof of Pro-2.

The root vectors X_α can be modified to $X'_\alpha = c_\alpha X_\alpha$, where, in order to preserve rela tions (4) the c_α's have to satisfy

$$c_\alpha c_{-\alpha} = 1. \tag{14}$$

From $[c_\alpha X_\alpha, c_\beta X_\beta] = N'_{\alpha\beta} c_{\alpha+\beta} X_{\alpha+\beta}$, we have also the relation between the N's

$$N'_{\alpha\beta} = N_{\alpha\beta}(c_\alpha c_\beta / c_{\alpha+\beta}). \tag{15}$$

Now we modify the X_α's so that the corresponding $N_{\alpha\beta}$'s become integer numbers. This is done inductively, with respect to the hight $|\alpha|=\sum|n_i|$ of the root $\alpha=\sum n_i\alpha_i$. For the roots of the fundamental system Π (hight=1) we start with arbitrary X_α. For the roots of hight 2, $\alpha+\beta$ with $\alpha,\beta\in\Pi$, we define $c_{\alpha+\beta} = N_{\alpha\beta}$, which by (15) gives $N'_{\alpha\beta} = 1$. From Pro-2(d) we have then (since q=0) $N'_{-\alpha,-\beta} = -1$. The idea is to continue inductively with respect to the hight and construct the $N_{\alpha\beta}$'s, so as to satisfy

$$N_{\alpha\beta} = -N_{-\alpha,-\beta}\ . \tag{16}$$

Pro-2(d) implies then that these $N_{\alpha\beta}$ are integers.

Proposition-3

a) If (16) holds for positive roots α, β with $|\alpha+\beta|<k$, then it holds also for negative with $|\alpha+\beta|<k$ as well as for positive α, $|\alpha|<k$ and negative β, $|\beta|<k$.

b) If (16) holds for positive roots of hight $< k$ and η is positive of hight k, and the positive roots γ, δ, ε, ζ have $\eta = \gamma+\delta = \varepsilon+\zeta$, then we have

$$N_{\gamma\delta}/N_{-\gamma,-\delta} = N_{\varepsilon\zeta}/N_{-\varepsilon,-\zeta}\,. \tag{17}$$

The statement about negative roots in a) is obvious. Suppose α positive, β negative, $\gamma= -\alpha-\beta$, with $|\gamma|<k$ and γ negative. By Pro-2(b), since $\alpha+\beta+\gamma=0$ (and β, γ are negative) implies

$$N_{\alpha\beta}/\langle\gamma,\gamma\rangle = N_{\beta\gamma}/\langle\alpha,\alpha\rangle = -N_{-\beta,-\gamma}/\langle\alpha,\alpha\rangle = N_{-\gamma,-\beta}/\langle\alpha,\alpha\rangle = N_{-\beta,-\alpha}/\langle\gamma,\gamma\rangle = -N_{-\alpha,-\beta}/\langle\gamma,\gamma\rangle\,.$$

This proves a). To prove b), apply Jacobi's identity

$$[X_\gamma[X_\delta X_{-\varepsilon}]] + [X_\delta[X_{-\varepsilon}X_\gamma]] + [X_{-\varepsilon}[X_\gamma X_\delta]] = 0 \ \Rightarrow$$

$$N_{\delta,-\varepsilon}N_{\gamma,\zeta-\gamma} + N_{-\varepsilon,\gamma}N_{\delta,\zeta-\delta} + N_{\gamma\delta}N_{-\varepsilon,\varepsilon+\zeta} = 0\,. \quad \Rightarrow$$

$$N_{\delta,-\varepsilon}N_{\gamma,\zeta-\gamma} + N_{-\varepsilon,\gamma}N_{\delta,\zeta-\delta} = N_{\gamma\delta}N_{-\varepsilon,-\zeta}\,(\langle\zeta,\zeta\rangle/\langle\eta,\eta\rangle)\,. \tag{18}$$

By a) and since the roots $\delta-\varepsilon$, γ, $\zeta-\gamma$, $-\varepsilon$ and $\zeta-\delta$ have all hight <k, we see that the sign of the left side of (18) does not change by replacing $\gamma\to-\gamma$, $\delta\to-\delta$, $\varepsilon\to-\varepsilon$, and $\zeta\to-\zeta$. Comparison of the corresponding right side with (18) completes the proof of b). q.e.d.

Proposition-3 completes the inductive modification of the X_α's so that (16) is valid for all roots. For each positive root η we choose a definite representation $\eta=\gamma+\delta$, with roots of less hight and modify X_η by c_η, so as to have (using (15)) $N_{\gamma\delta} = (q+1)$ and consequently also $N_{-\gamma,-\delta} = -(q+1)$. The whole procedure (choosing a specific representation for η etc.) is called a **normalization** and Pro-3(b) shows that for an other representation with positive roots $\varepsilon+\zeta=\eta$, (16) remains also valid. Notice also that, specifying a representation $\eta=\gamma+\delta$, and taking $N_{\gamma\delta} = (q+1)$, fixes also the $N_{\alpha\beta}$'s for all other representations of η, inductively and through formula (18). Also the "mixed" $N_{\alpha\beta}$'s are determined inductively through Pro-2(b).

Theorem-1 *In every semi simple Lie algebra* $\mathfrak{g}$ *we can choose the root vectors* X_α *in such a way as to satisfy the equations :*

$$[H_\alpha X_\alpha] = 2X_\alpha,\ [H_\alpha X_{-\alpha}] = -2X_{-\alpha},\ [X_\alpha X_{-\alpha}] = H_\alpha,$$
$$[X_\alpha X_\beta] = \pm(q+1)X_{\alpha+\beta},$$

where $\beta-q\alpha, \dots, \beta, \dots, \beta+p\alpha$ *is the* α*-string of the root* β.

Theorem-2 *Let* $\mathfrak{g}_1, \mathfrak{g}_2$ *be semi simple Lie algebras with root systems* Δ_1, Δ_2 *and suppose* f *to be a* ***weak equivalence*** *of the two root systems i.e. a bijective map such that*

$$f(-\alpha) = -f(\alpha) \text{ and } f(\alpha+\beta) = f(\alpha)+f(\beta),$$

then f *can be extended to an isomorphism between the two Lie algebras.*

In fact, extend f between the corresponding Cartan subalgebras in the obvious way. The map preserves the Cartan integers and the restriction of the Killing form on the Cartan subalgebra. We choose then a normalization for $\mathfrak{g}_1$ and its image in $\mathfrak{g}_2$. The extension of f on the whole $\mathfrak{g}_1$ (depends on the normalization choosen in $\mathfrak{g}_1$ and) is then defined so as to satisfy $f(X_\alpha) = X_{f(\alpha)}$. By its construction, f preserves α-strings and satisfies $N_{\alpha\beta} = N_{f(\alpha)f(\beta)}$. This proves that f is an isomorphism. q.e.d.

The following exercises, taken from Jacobson (pp. 123-127), show the way the Lie algebra is constructed out of a fundamental system and the corresponding $H_i = H_{\alpha_i}$, $X_i = X_{\alpha_i}$, $X_{-i} = X_{-\alpha_i}$ (satisfying (4)).

Exercise-3 Fix $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ and denote the corresponding root vectors (satisfying (4)) by $e_i = X_i$ and $f_i = X_{-i}$. After Ex-10, §26, every positive root α can be written as a sum, $\alpha = \alpha_k + \alpha_m + \dots + \alpha_r$ in such a way, that every "partial sum" α_k, $\alpha_k + \alpha_m, \dots, \alpha_k + \alpha_m + \dots + \alpha_r$ is again a root. Denote by $[e_k e_m \dots e_r] = [\dots[[e_k e_m]e_n] \dots e_r]$ and $[f_k f_m \dots f_r] = [\dots[[f_k f_m]f_n] \dots f_r]$ and show them to be equal with non zero multiples of X_α and $X_{-\alpha}$, respectively.

Exercise-4 With the preceding notation, let $\{k', m', \dots, r'\}$ denote a permutation of $\{k, m, \dots, r\}$. Show that $[e_{k'} e_{m'} \dots e_{r'}]$ (resp. $[f_{k'} f_{m'} \dots f_{r'}]$) is a rational multiple of $[e_k e_m \dots e_r]$ (resp. of $[f_k f_m \dots f_r]$), the multiplier being determined by the Cartan matrix (c_{ij}).

Exercise-5 Show that $\{H_1, \dots, H_d\} \cup \{[e_k e_m \dots e_r], [f_k f_m \dots f_r]\}$ form a basis of the Lie algebra $\mathfrak{g}$, and the structure constants with respect to this basis are determined by the Cartan matrix (c_{ij}).

Exercise-6 Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$, $\Pi' = \{\alpha'_1, \alpha_2, \dots, \alpha'_d\}$ be fundamental systems of roots for the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. Suppose also that the corresponding Cartan matrices are equal. Then there is a unique isomorphism of Lie algebras $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ which maps $X_i \to X'_i$ and $X_{-i} \to X'_{-i}$. [Use the basis of the preceding exercise.]

Remark We used this exercise in §35, in order to extend the obvious symmetry of the Dynkin diagram E_6 to an involutive automorphism of the whole Lie algebra. The same method can be used to extend any symmetry (isometry) of a Dynkin diagram to an automorphism of the Lie algebra. The corresponding extension of such a symmetry will be called henceforth **canonical extension of the symmetry**.

Es ist kein witziger Einfall sondern die lautere Wahrheit, daß vor der Revolution die Jagdhunde des Königs von Frankreich mehr Gehalt hatten, als die Akademie der Inschriften. S. Neue Bibliothek der schönen Wissenschaften. Band 44. Stück 2 p. 234. Die Hunde 40 000, die Akademisten 30 000, Hunde waren 300, Mitglieder der Akademie 30.

Lichtenberg, Sudelbücher p. 410

38 . Existence and conjugacy of Cartan algebras

In §22 we introduced the notion of a Cartan subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, and saw that for semi simple $\mathfrak{g}$, there is a set of roots $\Delta \subset \mathfrak{h}^*$, root vectors $\{X_\alpha | \alpha \in \Delta\}$ and $\mathfrak{g}$ decomposes into the direct sum :

$$\mathfrak{h} \oplus_{\alpha \in \Delta} \mathbb{C}X_\alpha \,. \tag{1}$$

Here we fill in the gap concerning the existence of such a subalgebra.

An element $X \in \mathfrak{g}$ is called **regular**, when the dimension of the vector subspace

$$\mathfrak{g}_0(X) = \{Y \in \mathfrak{g} | \mathrm{ad}X(Y) = 0\}, \tag{2}$$

is minimal. The characteristic polynomial of adX

$$\det(\mathrm{ad}X\text{-}tI) = (-1)^n t^n + \ldots + \det(\mathrm{ad}X), \tag{3}$$

has coefficients $a_i(X)$ which are polynomials with respect to X ($\det(\mathrm{ad}X)=0$, since $\mathrm{ad}XX=0$). In (3) there is a last term $a_r(X)$ which is not identically zero. X is a regular element of $\mathfrak{g}$, exactly when $a_r(X) \neq 0$. Since $a_r(X)$ is an analytic function and vanishing on an open subset would imply identical vanishing of the function, the set of regular elements is an open and dense subset of $\mathfrak{g}$.

Lemma-1 *For every regular element $X \in \mathfrak{g}$ the corresponding $\mathfrak{g}_0(X)$ is a Cartan subalgebra of $\mathfrak{g}$.*

That $\mathfrak{g}_0(X)$ is a subalgebra, was shown in §22. $\mathfrak{g}_0(X)$ is nilpotent : In fact, let $\mathfrak{g}_i(X)$ be the generalized eigenspaces of the linear endomorphism $\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}$. For $i \neq 0$, $\mathrm{ad}X | \mathfrak{g}_i(X)$ is invertible. It follows that for all Z in a neighborhood U(X) of X in $\mathfrak{g}_0(X)$, $\mathrm{ad}Z | \mathfrak{g}_i(X)$ is also invertible. Then, $\mathrm{ad}Z | \mathfrak{g}_0(X)$ must be a nilpotent operator, since in the contrary case $\mathfrak{g}_0(Z)$ would have dimension less than that of $\mathfrak{g}_0(X)$, contradicting the choice of X. Since nilpotency of an operator is a polynomial condition (hence analytic) and $\mathrm{ad}Z | \mathfrak{g}_0(X)$ is nilpotent for $Z \in U(X)$, $\mathrm{ad}Z | \mathfrak{g}_0(X)$ is nilpotent for all $Z \in \mathfrak{g}_0(X)$.

We have still to show that $\mathfrak{g}_0(X)$ is self-normalizing. In fact, $\mathfrak{g}$ decomposes into the direct sum of generalized eigenspaces of adX

$$\mathfrak{g} = \mathfrak{g}_0(X) \oplus \mathfrak{g}_1(X) \oplus \ldots \oplus \mathfrak{g}_k(X). \tag{4}$$

Let now $Y \in \mathfrak{g}$, such that $[Y\mathfrak{g}_0(X)] \subset \mathfrak{g}_0(X)$. If Y had a component in $\mathfrak{g}_i(X)$, $i \neq 0$, then $[YX]$ would have a component in $\mathfrak{g}_i(X)$ too, which contradicts $[YX] \in \mathfrak{g}_0(X)$. Thus $Y \in \mathfrak{g}_0(X)$.

Lemma-2 *Every Cartan subalgebra $\mathfrak{h}$ of a semi simple Lie algebra $\mathfrak{g}$, coincides with the $\mathfrak{g}_0(X)$, for some regular element X contained in $\mathfrak{h}$.*

In fact, for a Cartan subalgebra $\mathfrak{h}$, we have the decomposition (1) into the root spaces of h. Take then some $X \in h$, with $\alpha(X) \neq 0$, for all $\alpha \in \Delta$. Then $g_0(X) = h$. q.e.d.

Remark For every element X of a Cartan subalgebra h, for which $\alpha(X) \neq 0$, for all roots $\alpha \in \Delta$, we have $g_0(X) = h$.

We proceed now to the investigation of the different Cartan subalgebras of the same semi simple Lie algebra. Recall that inner automorphisms of g are these which can be written as compositions of automorphisms of the form

$$f = \exp(\mathrm{ad}X), \text{ for } X \in g.$$

Recall also the formula

$$(d/dt)_0 f_t(Z) = (d/dt)_0(\exp(\mathrm{ad}tX))Z = [XZ]. \quad (5)$$

From the well known formula

$$\dim\mathrm{Kern}F + \dim\mathrm{Im}F = \dim V, \quad (6)$$

for a linear map $F: V \to W$, we have the following result :

$$g = h \oplus [X,g], \text{ for every regular } X \in h. \quad (7)$$

This, combined with (5), implies that the derivative of the map

$$h \times g \ni (X,Y) \to \exp(\mathrm{ad}Y)X,$$

at $(X,0) \in h \times g$ is of maximal rank, hence there is a neighborhood V of X in g, whose all elements are of the form :

$$Y \in V \Rightarrow Y = \exp(\mathrm{ad}Z)W, \text{ with (regular)} W \in h. \quad (8)$$

By the remark above, for all W in a convenient neighborhood U of $X \in h$ we'll have $g_0(X) = g_0(W)$. Hence for $Y \in V$ or eventualy a smaller neighborhood of X in g, we'll have $g_0(Y) = \exp(\mathrm{ad}Z(g_0(W))) = \exp(\mathrm{ad}Z(g_0(X)))$. This means that for every regular X also a neighborhood V of X consists of regular elements and $g_0(X)$, $g_0(Y)$ for $Y \in V$ are conjugate with respect to an inner automorphism. If follows immediately that for every connected component S_i of the set of regular elements, the Cartan subalgebras $g_0(X)$ and $g_0(Y)$ are conjugate. That there is only one connected component, follows from the more general:

Lemma-3 *Let V be a complex vector space and $f: V \to C$ a polynomial function. Then the set $V' = \{v \in V | f(v) \neq 0\}$ is connected.*

In fact, for v, $v' \in V$ the function of t, $f(tv+(1-t)v') = g(t)$ is (non-zero) polynomial in t, hence it has finite many zeroes $t_1, \dots, t_k$, hence $C' = C - \{t_1, \dots, t_k\}$ is connected and with this $V' \supset \{tv+(1-t)v' | t \in C'\}$ is connected too.

Since the set of regular elements of g is identical with $\{X | a_r(X) \neq 0\}$, we have the proof of the:

Theorem *In every complex semi simple Lie algebra g there are Cartan subalgebras, and two such subalgebras are conjugate through an inner automorphism of g.*

IV.

Structure of Real Semi Simple Lie Algebras

Während die Wissenschaft dem rast- und bestandlosen Strohm vierfach gestalteter Gründe und Folgen nachgehend, bei jedem erreichten Ziel immer wieder weiter gewiesen wird und nie ein letztes Ziel, noch völlige Befriedigung finden kann, so wenig als man durch Laufen den Punkt erreicht, wo die Wolken den Horizont berühren, so ist dagegen die Kunst überall am Ziel.

Schopenhauer, Die Welt als ..., I, p. 239

39. Automorphisms

Automorphism of a Lie algebra $\mathfrak{g}$, is a linear isomorphism $f\colon \mathfrak{g} \to \mathfrak{g}$, with the property

$$f([XY]) = [f(X),f(Y)], \quad \text{for all } X,Y \in \mathfrak{g}. \qquad (1)$$

The set Aut($\mathfrak{g}$) of all automorphisms of $\mathfrak{g}$ is an algebraic closed Lie subgroup of GL($\mathfrak{g}$) and its Lie algebra is Der($\mathfrak{g}$), the set of derivations of $\mathfrak{g}$. In §17 we saw that when $\mathfrak{g}$ is semi simple, then all derivations are inner i.e. of the form D=adX, for some $X \in \mathfrak{g}$. In this § we assume that $\mathfrak{g}$ is <u>semi simple</u> and use the notation introduced in the preceding §§. The connected component Int($\mathfrak{g}$) of Aut($\mathfrak{g}$) is a normal subgroup of Aut($\mathfrak{g}$) and consists of automorphisms of the form

$$\exp(adX_1)\circ \dots \circ \exp(adX_k). \qquad (2)$$

In this § we show that

$$G = Aut(\mathfrak{g})/Int(\mathfrak{g}) \qquad (3)$$

is a finite group which is isomorphic to the group of automorphisms of the corresponding Dynkin diagram. A useful remark is the :

Exercise-1 Every automorphism $f\colon \mathfrak{g} \to \mathfrak{g}$, wich leaves invariant some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, induces there an isometry which leaves invariant the root system Δ. [f is an isometry with respect to the Killing form]

In §37 we saw that, inversely, every isometry of Δ comes from some automorphism of $\mathfrak{g}$ (the isometry induces isometry between fundamental systems of Δ, which can be extended choosing a normalization etc.). Two automorphisms with $f_i(\Delta)\subset\Delta$, for i=1,2, can induce the same isometry on Δ. Then the automorphism $(f_1)^{-1}\circ f_2$ will leave Δ fixed and it is useful to know that :

Lemma-1 *The following assertions are equivalent :*

1) *The automorphism $f\colon \mathfrak{g} \to \mathfrak{g}$, has $f(\mathfrak{h})\subset\mathfrak{h}$ and $f|\mathfrak{h} = id$,*
2) *$f = \exp(adH)$, for some $H\in \mathfrak{h}$.*

2) $\Rightarrow$ 1) is obvious.

1) $\Rightarrow$ 2) is also easy. In fact, with the usual notation $\Pi = \{\alpha_1, \dots, \alpha_d\}$, H_i, X_i etc. we'll have

$$f(H_i) = H_i,\ f(X_i) = a_iX_i,\ f(X_{-i}) = b_iX_{-i}, \text{ and by invariance}$$

$$f([X_i,X_{-i}]) = f(H_i) = H_i = a_ib_i H_i \ \Rightarrow\ a_ib_i = 1.$$

Take then (complex numbers) t_i with $\exp(t_i) = a_i$, and $H\in\mathfrak{h}$ such that $a(H) = t_i$. Then we verify easily, that $\exp(adH)X_i = \exp(t_i)X_i = a_iX_i$. Thus, exp(adH) and f are two automorphisms which coincide on $\mathfrak{h}$ and the X_i's. Since these generate the Lie algebra $\mathfrak{g}$, they coincide everywhere. q.e.d.

Lemma-2 *Every inner automorphism of* $\mathfrak{g}$ *has +1-eigenspace* $(=\{X\in\mathfrak{g}\mid (f-1)^kX=0$, *for some integer* $k\})$ *of dimension* $\geq d=\dim\mathfrak{h}$.

We need this lemma below and his proof uses the analyticity and connectedness of the Lie group Int($\mathfrak{g}$). From §22, we know that adX, for every $X\in\mathfrak{g}$ has 0-eigenspace

$$\mathfrak{g}_0(X)=\{Y\in\mathfrak{g}\mid (\mathrm{ad}X)^kY=0, \text{ for some integer } k\},$$

has dimension $\geq d=\dim\mathfrak{h}$. But $\mathfrak{g}_0(X)$ is a subset of the +1-eigenspace of exp(adX), hence the lemma is true for some neighborhood of $1\in$ Int($\mathfrak{g}$). The characteristic polynomial of an automorphism $f\in$ Int($\mathfrak{g}$) can be written in powers of (t-1):

$$\chi_f(t)=\sum_{0\leq i\leq r}(t-1)^i\, p_i(f), \tag{4}$$

where $p_i(f)$ are analytic functions on Int($\mathfrak{g}$). But for these functions, the preceding remark shows that

$$p_1(f)=\ldots=p_{d-1}(f)=0, \tag{*}$$

on a neighborhood of the identity. By analyticity and connectedness of Int($\mathfrak{g}$) we conclude that (*) must be true everywhere on Int($\mathfrak{g}$). This proves the lemma.

Lemma-3 *Fix a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$. *Then* $Aut(\mathfrak{g})/Int(\mathfrak{g})=G_{\mathfrak{h}}/G^o_{\mathfrak{h}}$, *where* $G_{\mathfrak{h}}$ = *the subgroup of automorphisms leaving* $\mathfrak{h}$ *invariant, and* $G^o_{\mathfrak{h}}$ = *the subgroup of inner automorphisms leaving* $\mathfrak{h}$ *invariant.*

The proof is easy. For an automorphism f: $\mathfrak{g}\to\mathfrak{g}$, $\mathfrak{h}'=f(\mathfrak{h})$ is again a Cartan subalgebra and by the conjugacy theorem (Theorem §38) there is an inner automorphism ϕ, such that $\phi\circ f(\mathfrak{h})=\mathfrak{h}$. Then also $f^{-1}\circ\phi^{-1}(\mathfrak{h})=\mathfrak{h}$ and for $\chi=f^{-1}\circ\phi^{-1}$ we have

$$f^{-1}=\chi\circ\phi.$$

Thus, each automorphism can be written as a product of a $\chi\in G_{\mathfrak{h}}$ and an inner automorphism ϕ. This means that the canonical projection restricted on $G_{\mathfrak{h}}$

$$G_{\mathfrak{h}}\to \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}), \text{ with } \chi\to[\chi],$$

is surjective. The lemma follows then from the fact that $G_{\mathfrak{h}}^o$ is the kernel of this homomor phism of groups.

After this lemma, in order to investigate the group Aut($\mathfrak{g}$)/Int($\mathfrak{g}$), we may restrict ourselves to automorphisms f leaving a definite Cartan subalgebra $\mathfrak{h}$ invariant. Such an automorphism induces then an isometry of Δ onto itself. The next step will be to show that an inner automorphism leaving $\mathfrak{h}$ invariant (i.e. $f\in G_{\mathfrak{h}}^o$), when restricted in $\mathfrak{h}$, gives some element of the Weyl group. The lemma is the easy part:

Lemma-4 *For every element* $S\in W$ *of the Weyl group, there is some* $f\in G^o_{\mathfrak{h}}$ *(inner automorphism preserving* $\mathfrak{h}$ *) such that* $S=f|\mathfrak{h}$.

Since W is generated by the reflexions S_α, with respect to the roots $\alpha\in\Delta$, it suffices to verify it for such a S_α. The proof is constructive. With the usual notation, we have

$$[X_\alpha+X_{-\alpha},H]=-\alpha(H)(X_\alpha-X_{-\alpha}),$$

$$(\mathrm{ad}(X_\alpha+X_{-\alpha}))^2H = 2\alpha(H)H_\alpha\ ,$$
$$(\mathrm{ad}(X_\alpha+X_{-\alpha}))^3H = -\,4\alpha(H)(X_\alpha-X_{-\alpha}),\quad \text{and inductively}$$
$$(\mathrm{ad}(X_\alpha+X_{-\alpha}))^{2k+1}H = -\,2^{2k}\,\alpha(H)(X_\alpha-X_{-\alpha}),$$
$$(\mathrm{ad}(X_\alpha+X_{-\alpha}))^{2k}H = 2^{2k-1}\,\alpha(H)H_\alpha\ .$$

Take then

$$X = i\pi t(X_\alpha+X_{-\alpha})\,.$$

$$\exp(\mathrm{ad}X)H = H - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(2it\pi)^{2k+1}}{(2k+1)!}\alpha(H)\,(X_\alpha-X_{-\alpha}) + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(2it\pi)^{2k+2}}{(2k+2)!}\alpha(H)\,H_\alpha.$$

For $t = 1/2$, $X = (i\pi/2)(X_\alpha+X_{-\alpha})$, $H\in h$ $\Rightarrow$

$$\exp(\mathrm{ad}X)H = H - \alpha(H)H_\alpha = S_\alpha(H)\,.$$ q.e.d.

The converse to the lemma is also valid, but to prove this we need to examine inner automorphisms permuting, not only Δ, but also the fundamental system Π.

Lemma-5 *Let f be an inner automorphism of g, with $f(h)\subset h$ and $f(\Pi)\subset\Pi$, then $f|h = Id$.*

We divide the proof in several steps:

1) f defines a permutation of Δ, $\tau:\Delta\to\Delta$, which is decomposable to cyclic permutations
$$\tau = \sigma_1\circ \dots \circ\sigma_p\ .$$
2) For each cyclic permutation $\sigma=\sigma_i$, and each root α, non-constant with respect to σ, there is an integer $q=q_i$, such that (somewhat loosely speaking)
$$\sigma = (\alpha, \sigma(\alpha), \sigma^2(\alpha), \dots , \sigma^{q-1}(\alpha)),\ \text{and}\ \sigma^q(\alpha) = \alpha\ .$$
3) The subspace of g
$$g^\sigma = CX_\alpha\oplus CX_{\sigma(\alpha)}\oplus \dots \oplus CX_{\sigma^{q-1}(\alpha)}\ ,$$

is obviously f-invariant and taking $\{X_\alpha, \dots , X_{\sigma^{q-1}(\alpha)}\}$ as a basis, the matrix representing $f|g^\sigma$ is

$$\begin{pmatrix} 0 & \dots\dots\dots & v_q \\ v_1 & 0\dots\dots\dots & 0 \\ 0 & v_2\ \ 0\ \dots\dots & 0 \\ & \dots\dots\dots\dots & \\ 0 & \dots\dots\dots v_{q-1} & 0 \end{pmatrix}.$$

4) The characteristic polynomial of $f|g^\sigma$ is $t^q - v_1\cdot v_2\cdot\dots\cdot v_q$.
5) For $H\in h$, $\exp(\mathrm{ad}H)g^\sigma\subset g^\sigma$ and $\{X_\alpha, \dots , X_{\sigma^{q-1}(\alpha)}\}$ are also eigenvectors of $\exp(\mathrm{ad}H)$
$$\exp(\mathrm{ad}H)X_{\sigma^i(\alpha)} = \exp(\sigma^i(\alpha)(H))X_{\sigma^i(\alpha)}.$$

Thus, the matrix representing $\exp(\mathrm{ad}H)\circ f|g^\sigma$ has the same form with the matrix in 3) with $v'_i = \exp(\sigma^{i-1}(\alpha))v_i$ and characteristic polynomial

$$t^q - v_1\cdot v_2\cdot\dots\cdot v_q\exp(\Sigma_{1\le i\le q}\ \sigma^{i-1}(\alpha)(H)).$$

6) Since $f(\Pi)\subset\Pi$, for positive (resp. negative) roots α, $f(\alpha)$ is again positive (negative), hence $\sum_{0\le i\le q} \sigma^i(\alpha) \neq 0$, and we can find some $H \in h$ such that

$$v_1 \cdot v_2 \cdot \ldots \cdot v_q \exp(\sum_{1\le i\le q} \sigma^{i-1}(\alpha)(H)) \neq 1. \qquad (*)$$

Moreover we can choose H so that (*) is valid for all $\sigma = \sigma_1, \ldots, \sigma_p$.

7) We have obviously

$$g = h \oplus g^{\sigma_1} \oplus, \ldots, \oplus g^{\sigma_p} \ ,$$

and the result in 6) shows that $\exp(\mathrm{ad}H)\circ f | g^{\sigma_1} \oplus, \ldots, \oplus g^{\sigma_p}$ has not the eigenvalue 1. This means that the +1-eigenspace of $\exp(\mathrm{ad}H)\circ f$ is contained in h, and from lemma-2 it must be identical with h. Thus, by lemma-1 there must be some $H' \in h$, such that

$$\exp(\mathrm{ad}H)\circ f = \exp(\mathrm{ad}H'),$$

which proves lemma-5. q.e.d.

Theorem-1 *For each element $S \in W$ of the Weyl group, there is an inner automorphism f of the Lie algebra g, leaving the Cartan subalgebra h invariant and extending S. Inversely, for every inner automorphism f, with $f(h)\subset h$ the restriction $f|h \in W$.*

The first part of the theorem is the lemma-4. The second (inverse) follows from lemma-5. In fact, if f is an automorphism with $f(h)\subset h$, and if Π is a fundamental system of roots, then $f(\Pi) = \Pi'$ is a fundamental system too. Hence, by the simple transitivity of W on the Weyl chambers, there must be some $S \in W$ such that $S(f(\Pi')) = \Pi$. According to lemma-4, S can be extended to an automorphism f' of g, with $f'|h = S$. Then $f'\circ f(\Pi) = \Pi$ and $f'\circ f$ is an inner automorphism which satisfies the hypothesis of lemma-5. Thus $f'\circ f|h = \mathrm{Id}$, and consequently $f|h = S^{-1}$. q.e.d.

Theorem-2 *Aut(g)/Int(g) = Aut(DD), where Aut(DD) is the group of symmetries of the Dynkin diagram or equivalently, the group of isometries of the corresponding fundamental system Π in itself.*

In fact, we saw that $\mathrm{Aut}(g)/\mathrm{Int}(g) = G_h/G_h^o$, and we have a natural mapping

$$F: G_h \rightarrow \mathrm{Aut}(\Delta) = \text{group of isometries of } \Delta,$$

which is defined by restricting its automorphism $f \in G_h$, on h. $W\subset\mathrm{Aut}(\Delta)$ is a normal subgroup, and by the preceding theorem $F(G_h^o)\subset W$, hence F induces an isomorphism of groups

$$F' : G_h/G_h^o \rightarrow \mathrm{Aut}(\Delta)/W.$$

It suffices to show that the map

$$f : \mathrm{Aut}(DD) \rightarrow \mathrm{Aut}(\Delta)/W, \ a \rightarrow [\hat{a}],$$

where $\hat{a}$ is the extension of a from Π to Δ, is an isomorphism. In fact, f is onto, since for every $u \in \mathrm{Aut}(\Delta)$, there is a unique $T \in W$ such that $u\circ T(\Pi)\subset\Pi$. Then $a = u\circ T|\Pi$, has $[\hat{a}] = [u]$. f is one to one, since $[\hat{a}] = W$ means $a=(\mathrm{Id}|\Pi)\mathrm{mod}W$. q.e.d.

Exercise-2 Show that $W\subset\mathrm{Aut}(\Delta)$ is a normal subgroup. [For $T \in \mathrm{Aut}(\Delta)$ and a reflection S_α, we have $T\circ S_\alpha \circ T^{-1} = S_{T(\alpha)}$]

Inspecting the Dynkin diagrams, we see that $\mathrm{Aut}(g)\neq\mathrm{Int}(g)$ (equivalently $\mathrm{Aut}(DD)\neq\{1\}$)

only in the four cases of the following table :

	A_d, $d \geq 2$	D_4	D_d	E_6
Aut(DD)	Z_2	S_3	Z_2	Z_2

»At sapiens colaphis percussus, quid faciet?« quod Cato, cum illi os percussum esset: non excanduit, non vindicavit injuriam: nec remisit quidem, sed factam negavit.

»Ja«, ruft ihr, »das waren Weise!« - Ihr aber seid Narren? Einverstanden. -

Wir sehn also, daß den Alten das ganze ritterliche Ehrenprincip durchaus unbekannt war, weil sie eben in allen Stücken der unbefangenen, natürlichen Ansicht der Dinge getreu blieben und daher solche sinistre und heillose Fratzen sich nicht einreden ließen.

Schopenhauer, Aphorismen ... p. 413

40 . Real forms, Cartan decomposition

We call **real form** of a complex Lie algebra $\mathfrak{g}$, a real subalgebra $\mathfrak{g}_0$, with the property:

$$\mathfrak{g}_0 + i\mathfrak{g}_0 = \mathfrak{g}. \tag{1}$$

$\mathfrak{g}_0$ has real dimension = complex dimension of $\mathfrak{g}$ and defines a **conjugation** in $\mathfrak{g}$ i.e. a coju-gate-linear mapping with the properties:

$$\begin{aligned} &\sigma: \mathfrak{g} \to \mathfrak{g}, \\ &\sigma(\lambda X + \mu Y) = \bar{\lambda} s(X) + \bar{\mu} s(Y), \\ &\sigma^2 = 1, \\ &\sigma[XY] = [\sigma(X), \sigma(Y)]. \end{aligned} \tag{2}$$

Such a σ is defined by $\mathfrak{g}_0$:

$$\sigma(X+iY) = X-iY, \quad \text{for all } X, Y \in \mathfrak{g}_0. \tag{3}$$

Lemma-1 *Every real form $\mathfrak{g}_0$ of a complex Lie algebra $\mathfrak{g}$ defines a conjugation σ of $\mathfrak{g}$ and inversely, every conjugation σ of $\mathfrak{g}$ defines a real form $\mathfrak{g}_0$, through the fixed points of σ:*

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \sigma(X) = X\}. \tag{4}$$

We need only to prove the inverse. But this is a consequence of the characteristic properties (2) of the conjugation σ.

We call two conjugations **equivalent**, when the corresponding real forms are isomorphic real Lie algebras.

Lemma-2 *Two conjugations σ_1, σ_2 are equivalent, if and only if, there is an automorphism f: $\mathfrak{g} \to \mathfrak{g}$, such that*

$$\sigma_2 = f \circ \sigma_1 \circ f^{-1}. \tag{5}$$

Given (5), define corresponding real subalgebras using (4):

$$\mathfrak{g}_i = \{X \in \mathfrak{g} \mid \sigma_i(X) = X\}, \quad i = 1, 2.$$

Then $f(\mathfrak{g}_1) = \mathfrak{g}_2$ and $f|\mathfrak{g}_1$ is a real isomorphism. For the inverse, extend the given $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ to a complex $f': \mathfrak{g} \to \mathfrak{g}$, through

$$f'(X+iY) = f(X) + if(Y), \quad \text{for all } X, Y \in \mathfrak{g}_1. \qquad \text{q.e.d.}$$

Exercise-1 Show that the product of two conjugations $f = \sigma_1 \circ \sigma_2$ of a Lie algebra $\mathfrak{g}$, is an automorphism of $\mathfrak{g}$.

Exercise-2 Show that every conjugation σ can be written as a product

$$\sigma = f \circ \sigma_0,$$

where σ_0 is an arbitrary fixed conjugation and f is an automorphism, such that

$$\sigma_0 \circ f \circ \sigma_0 = f^{-1}.$$

[Use Ex-1, and write $\sigma = (\sigma \circ \sigma_0) \circ \sigma_0$, $f = \sigma \circ \sigma_0$]

It is not necessary for a complex Lie algebra to possess real forms. However semi simple Lie algebras do possess real forms. In fact, using the root space decomposition of such an algebra, we can define the so-called **normal real form** corresponding to a fixed Cartan subalgebra $\mathfrak{h}$, and a definite set of root vectors X_α, $\alpha \in \Delta$:

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus_{\alpha \in \Delta} \mathbf{R} X_\alpha, \tag{6}$$

where $\mathfrak{h}_0 = \langle H_\alpha | \alpha \in \Delta \rangle_{\mathbf{R}}$ is the real subspace of $\mathfrak{h}$, spanned by the coroots H_α.

Exercise-3 Consider the classical Lie algebras realized by the matrix algebras sl(n;C) and sp(n;C). Show that, using the diagonal Cartan subalgebra and the usual root vectors (as in §§27-30), the corresponding normal real forms are sl(n;**R**) and sp(n;**R**) i.e. they coincide with the (real) subspaces of real matrices of the corresponding matrix spaces.

Besides normal real forms, semi simple Lie algebras possess also an other class of very important forms, the so-called **compact real forms**. These are characterized by the property: the Killing form $\langle ..., ... \rangle$ of $\mathfrak{g}$, restricted on $\mathfrak{g}_0$ is definite.

Exercise-4 Show that the restriction of the Killing form of $\mathfrak{g}$ on a real form $\mathfrak{g}_0$, is the Killing form of $\mathfrak{g}_0$ (and consequently takes real values there).

[Use an **R**-basis of $\mathfrak{g}_0$, which is also a C-basis of $\mathfrak{g}$].

Exercise-5 Show that when the Killing form on $\mathfrak{g}_0$ is definite, then it is negative definite. [For $X \in \mathfrak{g}_0$, adX is skew symmetric w.r. to $\langle ..., ... \rangle$ $\Rightarrow$ has purely imaginary eigenvalues $\Rightarrow$ $(\mathrm{ad}X)^2$ has negative eigenvalues].

Theorem-1 *Every semi simple Lie algebra has a compact real form, namely:*

$$\mathfrak{u} = i\mathfrak{h}_0 \oplus_{\alpha \in \Delta^+} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus_{\alpha \in \Delta^+} i\mathbf{R}(X_\alpha + X_{-\alpha}), \tag{7}$$

where $\mathfrak{h}$ is some Cartan subalgebra and $\mathfrak{h}_0$ and X_α as in (6) above.

In fact, setting $U_\alpha = X_\alpha - X_{-\alpha}$ and $V_\alpha = i(X_\alpha + X_{-\alpha})$ we compute easily:

$$[iH, U_\alpha] = \alpha(H) V_\alpha,$$

$$[iH, V_\alpha] = -\alpha(H) U_\alpha$$

$$[U_\alpha, V_\alpha] = 2iH_\alpha. \tag{8}$$

iH, U_α, V_α are orthogonal to each other (see §22, Pro-2), hence the Killing form for

$$X = iH + \sum r_\alpha U_\alpha + \sum s_\alpha V_\alpha,$$

$$\langle X,X \rangle = -\langle H,H \rangle + \sum r_\alpha^2 \langle U_\alpha, U_\alpha \rangle + \sum s_\alpha^2 \langle V_\alpha, V_\alpha \rangle, \text{ but}$$

$$\langle U_\alpha, U_\alpha \rangle = \langle X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha} \rangle = \langle X_\alpha, X_\alpha \rangle + \langle X_{-\alpha}, X_{-\alpha} \rangle - 2\langle X_\alpha, X_{-\alpha} \rangle$$

and since (Pro-3, §22)

$$\langle X_\alpha, X_\alpha \rangle = \langle X_{-\alpha}, X_{-\alpha} \rangle = 0 \text{ and } \langle X_\alpha, X_{-\alpha} \rangle = 2/\langle \alpha, \alpha \rangle,$$

we have

$$\langle U_\alpha, U_\alpha \rangle = -4/\langle \alpha,\alpha \rangle.$$

Analogously

$$\langle V_\alpha, V_\alpha \rangle = -4/\langle \alpha,\alpha \rangle, \qquad \Rightarrow$$

$$\langle X,X \rangle = -\langle H,H \rangle - (4/\langle \alpha,\alpha \rangle)\sum (r_\alpha^2 + s_\alpha^2) < 0. \qquad \text{q.e.d.}$$

Compact real forms $\mathfrak{u}$ are very important since all other real forms of the complex semi simple Lie algebra $\mathfrak{g}$, can be constructed out of involutive automorphisms of $\mathfrak{u}$. Besides all compact real forms of $\mathfrak{g}$ are isomorphic. In the next lines we discuss these facts.

Lemma-3 *Let $\mathfrak{u}$ be a compact real form of the complex semi simple Lie algebra $\mathfrak{g}$ and $f:\mathfrak{u}\to\mathfrak{u}$ an involutive automorphism ($f^2=1$). Let furthermore k, p be the (± 1) eigenspaces of f ($\mathfrak{u} = k \oplus p$):*

$$k = \{X \in \mathfrak{u} \mid fX = X\},$$
$$p = \{X \in \mathfrak{u} \mid fX = -X\}.$$

Then

$$g_0 = k \oplus ip$$

is a real form of $\mathfrak{g}$.

In fact, this follows immediately from the properties

$$[kk]\subset k,$$
$$[kp]\subset p,$$
$$[pp]\subset k, \qquad (9)$$

for the corresponding eigenspaces of the automorphism f. k is called the **characteristic** subalgebra of g_0.

Theorem-2 *Every real form g_0 of the complex semi simple Lie algebra $\mathfrak{g}$ is conjugate, with respect to some inner automorphism of $\mathfrak{g}$, to a real form $k \oplus ip$ as above.*

The proof relies on the following lemma:

Lemma-4 *Let $\mathfrak{u}$ be a compact real form and g_0 a real form of the complex semi simple Lie algebra $\mathfrak{g}$, and let τ, σ_0 be the conjugations with respect to $\mathfrak{u}$ and g_0 respectively. Assume also that*

$$\tau \circ \sigma_0 = \sigma_0 \circ \tau. \qquad (10)$$

Then the restriction $f = \sigma_0|\mathfrak{u}$ is an involutive automorphism of $\mathfrak{u}$, with the property

$$g_0 = k \oplus ip,$$

where k, p, as above, are the ± 1-eigenspaces of f.

In fact (10) and $\sigma_0^2 = 1$ implies

$$\sigma_0(\mathfrak{u})\subset\mathfrak{u},\ f^2 = 1, \qquad (11)$$

and, since we are considering only real linear combinations, $f=\sigma_0|\mathfrak{u}$ is $\mathbf{R}$-linear. Thus,

$$\sigma_0(k) = k,$$
$$\sigma_0(ip) = -i\sigma_0(p) = ip, \qquad \Rightarrow$$

$$\mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}_0.$$

Equality follows for dimension reasons. q.e.d.

To prove now the theorem, it suffices to find some conjugate $\mathfrak{g}_1$ of the real form $\mathfrak{g}_0$, whose conjugation σ_1 satisfies

$$\tau\circ\sigma_1 = \sigma_1\circ\tau. \tag{10'}$$

The construction of σ_1 out of σ_0 is done by the following computation :

1) First, since the Killing form $K(X,Y)$ of $\mathfrak{g}$ is real on $\mathfrak{u}$ and $\mathfrak{g}_0$, we have

$$K(\tau X,\tau Y) = K(\sigma_0 X, \sigma_0 Y) = \overline{K}(X,Y), \text{ (conjugate) for all } X, Y\in\mathfrak{g}.$$

2) We show that the new bilinear form H, defined by

$$H(X,Y) = K(\tau X,Y)$$

is an Hermitian form of $\mathfrak{g}$:

i) It is conjugate-linear with respect to X, and linear with respect to Y.

ii) $H(Y,X) = K(\tau Y,X) = \overline{K}(\tau^2 Y,\tau X) = \overline{K}(Y,\tau X) = \overline{H}(X,Y)$.

iii) Definite: $H(X,X) = K(\tau X,X) = K(X'-iX'',X'+iX'') = K(X',X')+K(X'',X'')<0$, for $X', X'' \in \mathfrak{u}$.

3) $P = \sigma_0\circ\tau$ is C-linear automorphism of $\mathfrak{g}$ and selfadjoint with respect to H. In fact,
$H(PX,Y) = K((\tau\circ\sigma_0\circ\tau)X,Y) = \overline{K}(\sigma_0\circ\tau X,\tau Y) = K(\tau X,\sigma_0\circ\tau Y) = H(X,PY)$.

4) It follows that $P:\mathfrak{g}\to\mathfrak{g}$ can be diagonalized, and has real non-zero eigenvalues $\lambda_1, \dots, \lambda_r$. Let

$$\mathfrak{g} = \oplus_{1\le i\le r} V_i, \quad V_i = \{X\in\mathfrak{g} \mid PX=\lambda_i X\},$$

be the corresponding decomposition into the eigenspaces of P. Notice that $[V_i V_j]\subset V_{(ij)}$ with $\lambda_{(ij)} = \lambda_i\lambda_j$.

5) We define $Q = |P|^{-1/2}$ (inner automorphism conjugating $\mathfrak{g}_0$ to $\mathfrak{g}_1$) by means of

$$Q|V_i = |\lambda_i|^{-1/2} (\mathrm{Id}|V_i).$$

From the definition and the fact $|\lambda|^2 = \lambda^2$ (holding for reals), we have the equation

$$Q^2P = P^{-1}Q^{-2}. \tag{12}$$

We note also that Q embeds into the 1-parameter-subgroup of automorphisms of $\mathfrak{g}$, given by

$$\{|P|^t \mid t\in \mathbf{R}\}.$$

Thus, it is of the form $\exp(t\,\mathrm{ad}X)$, for some $X\in\mathfrak{g}$.

6) We define the real form

$$\mathfrak{g}_1 = Q(\mathfrak{g}_0), \text{ whose conjugation is}$$

$$\sigma_1 = Q\circ\sigma_0\circ Q^{-1} \quad (\sigma_1|\mathfrak{g}_1 = \mathrm{id},\ \sigma_1|i\mathfrak{g}_1 = -\mathrm{id}).$$

We'll show that (10') holds.

i) $\sigma_0\circ P\circ\sigma_0^{-1} = P^{-1}$ because $\sigma_0\circ P\circ\sigma_0^{-1} = \sigma_0\circ\sigma_0\circ\tau\circ\sigma_0^{-1} = \tau\circ\sigma_0^{-1} = P^{-1}$.

ii) This implies $\sigma_0\circ Q\circ\sigma_0^{-1} = Q^{-1}$ because $\sigma_0\circ P\circ\sigma_0^{-1} = P^{-1} \Leftrightarrow \sigma_0\circ P = P^{-1}\circ\sigma_0 \Rightarrow$

$$\sigma_0(V_{\lambda_i})\subset V_{\lambda_i^{-1}}, \quad \text{and}$$

$$\sigma_0\circ Q\circ\sigma_0(v_i) = \sigma_0\circ Q\circ(\sigma_0 v_i) = \sigma_0(|\lambda_i|^{1/2}\sigma_0 v_i) = |\lambda_i|^{1/2} v_i .$$

iii) Finally, $\sigma_1\circ\tau = Q\circ\sigma_0\circ Q^{-1}\circ\tau = Q\circ\sigma_0\circ(\sigma_0\circ Q\circ\sigma_0^{-1})\circ\tau = Q^2\circ\sigma_0\circ\tau = Q^2P = P^{-1}Q^{-2}$

$$= \tau\circ\sigma_0\circ Q^{-1}\circ Q^{-1} = \tau\circ Q\circ\sigma_0\circ Q^{-1} = \tau\circ\sigma_1 .$$

q.e.d.

Theorem-3 *Two compact real forms of the same complex semi simple Lie algebra g, are conjugate (isomorphic) by an inner automorphism of g.*

The proof is an application of the previous theorem, for $p=\{0\}$. Thus, we can say the compact real form of g. In fact, by The-2, a second compact form of g will be conjugate to some $g_0 = k \oplus ip$, as in the lemma-3. But then $<ip,ip> > 0$, and we must have $p=\{0\}$ and $g_0=u$. q.e.d.

Absorbing i into p, we see that every real form g_0 has a **Cartan decomposition** i.e. a decomposition into subspaces $g_0 = k \oplus p$ such that

$$[kk]\subset k,$$
$$[kp]\subset p,$$
$$[pp]\subset k,$$
$$<k,k> < 0, \text{ and } <p,p> > 0. \qquad (13)$$

Notice that we pass from the Cartan decomposition to the compact form by taking again

$$u = k \oplus ip.$$

The Cartan decomposition is essentially unique. In fact if

$$g_0 = k_1 \oplus p_1 = k_2 \oplus p_2,$$

are two such decompositions then, let $u_1 = k_1 \oplus ip_1$, $u_2 = k_2 \oplus ip_2$, be the corresponding compact forms and σ_1, σ_2 the corresponding conjugations. Then,

$$Q = |\sigma_2 \circ \sigma_1|^{-1/2},$$

is, as we saw in the proof of The-2, an automorphism mapping u_2 onto u_1. Then, by lemma-4, the conjugation σ_0 of g_0 commutes with σ_1, σ_2 hence also with Q. Thus, Q maps g_0 onto itself and defines an **R**-automorphism of g_0. We have also $Q(k_2) = Q(g_0 \cap u_2) = g_0 \cap Q(u_2) = g_0 \cap u_1 = k_1$. Analogously also $Q(p_2)=p_1$. We proved the

Theorem-4 *Every real form g_0 of the complex semi simple Lie algebra g has a Cartan decomposition (as in (13)). Also two such decompositions are R-isomorphic through an inner automorphism of g_0, of the form $f= exp(adX)$, $X \in g_0$ ($f(k_2)=k_1$, and $f(p_2)=p_1$).*

The last assertion of the theorem follows from the fact that $Q= |\sigma_2 \circ \sigma_1|^{-1/2}$ embeds into a 1-parameter subgroup $\{|\sigma_2 \circ \sigma_1|^t \mid t \in \mathbf{R}\}$, of automorphisms of g_0. q.e.d.

From the discussion in this § follows that the problem of classification of all real forms of a complex semi simple Lie algebra is equivalent to the problem of classification of all involutive automorphisms of the compact real form u of g. We note first, that for conjugate automorphisms $f' = P \circ f \circ P^{-1}$ ($P: u \to u$ automorphism of u), the corresponding real forms $g_0' = k' \oplus ip'$ and $g_0 = k \oplus ip$ are isomorphic through P_C= the complex extension of P.

$$P_C(g_0)= g_0'.$$

Inversely, let $u = k \oplus p = k' \oplus p'$ and $g_0 = k \oplus ip$, $g_0' = k' \oplus ip'$ be isomorphic real forms, through an isomorphism

$$A: g_0 \rightarrow g_0'.$$

Then $A(k)+iA(p)$ is a Cartan decomposition of g_0'. Thus, there is an automorphism

$$R: g_0' \rightarrow g_0', \text{ with}$$

$$R(A(k)) = k', \; R(A(p)) = p'. \qquad (*)$$

Extending A and R on g we obtain $B=R\circ A$, which, by (*), implies $B(u) = u$ and for the conjugations σ_1, σ_2 of $g_0 = k\oplus ip$, $g_0' = k'\oplus ip'$, respectively, $\sigma_2 = B\circ\sigma_1\circ B^{-1}$. We proved the

Theorem-5 *Let g be a complex semi simple Lie algebra and u be a compact real form of g. Then, there is a bijection betweeen the set of conjugacy classes, in Aut(g), of all real forms of g and the set of the conjugacy classes, in Aut(u), of all involutive automorphisms of u.*

Dies aber weiß ich, daß wenn, wie es jetzt droht, die Erlernung der alten Sprachen ein Mal aufhören sollte, dann eine neue Litteratur kommmen wird, bestehend aus so barbarischem, plattem und nichtswürdigem Geschreibe, wie es noch gar nicht dagewesen;

Schopenhauer, Parerga, p. 610

41 . Real semi simple Lie algebras

The definition of real is the same with the corresponding for complex (§17) : the real Lie algebra g is called semi simple, when it has no solvable ideals. The properties of complex semi simple lead to properties of real semi simple Lie algebras by taking the complexification :

$$g_C = g + ig \,. \qquad (1)$$

g_C has complex dimension = real dimension of g . Multiplication by complex numbers :

$$(a+ib)(X+iY) = (aX-bY) + i(bX+aY). \qquad (2)$$

The Killing form of g_C :

$$\langle X+iY, X'+iY' \rangle = (\langle X,X' \rangle - \langle Y,Y' \rangle) + i(\langle X',Y \rangle + \langle Y',X \rangle). \qquad (3)$$

Exercise-1 Show that for every Lie subalgebra (ideal) $a \subset g$, is a subalgebra (ideal) of g_C. Show also that $(a_C)^{(n)} = (a^{(n)})_C$.

From the exercise, we see that if a is a solvable ideal of the real g, then g_C has the solvable ideal a_C. Inversely if g_C is not semi simple, then its Killing form is degenerate, hence

$$g_C^{\perp} = \{X+iY \mid \langle X+iY, U+iV \rangle = 0, \text{ for every } U+iV \in g\} \neq \{0\},$$

is a solvable ideal (Cartan's 1st criterion) of g_C. Then

$$a = \{X \in g \mid X+iY \in g_C^{\perp}, \text{ for some } Y \in g\} \neq \{0\},$$

and

$$a_C = g_C^{\perp}.$$

By Ex-1, $(a^{(n)})_C = (a_C)^{(n)}$ hence a is a solvable ideal and g is not semi simple. We proved the

Theorem-1 *The real Lie algebra g is semi simple, if and only if the complexification g_C is semi simple.*

Cartan's theorem says that g_C is semi simple if and only if its Killing form is non-degenerate. This means that the matrix representing the Killing form with respect to some basis of g_C is invertible. Choosing a basis of g (which is also a basis of g_C) we see that the corresponding matrix is real and invertible, hence

Theorem-2 *The real Lie algebra g is semi simple if and only if its Killing form is non-degenerate.*

Exercise-2 Show that for a real semi simple Lie algebra g and an ideal $h \subset g$ the orthogonal complement with respect to the Killing form $h^{\perp}$ is again an ideal. Conclude that

$\mathfrak{g}$ semi simple $\Leftrightarrow$ $\mathfrak{g} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$ is direct sum of simple ideals.

Exercise-3 Show that for a real semi simple Lie algebra $\mathfrak{g}$, $[\mathfrak{g}\mathfrak{g}] = \mathfrak{g}$.

[$[\mathfrak{g}\mathfrak{g}]^{\perp}$ is an abelian ideal]

The two exercises are proved as the corresponding ones for complex Lie algebras (§17). Analogous is also the proof of the following

Theorem-3 *All the derivations of a real semi simple Lie algebra $\mathfrak{g}$ are inner i.e. of the form $D=adX$, for some X in $\mathfrak{g}$.*

For the construction of certain Cartan subalgebras of a real Lie algebra $\mathfrak{g}$, we start with the Cartan decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $[\mathfrak{k}\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}\mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}\mathfrak{p}] \subset \mathfrak{k}$, $<\ldots,\ldots>$ negative definite on $\mathfrak{k}$, and positive definite on $\mathfrak{p}$. Then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact form of the complexification $\mathfrak{g}_C$ and for the conjugation with respect to $\mathfrak{u}$, $\tau : \mathfrak{g}_C \to \mathfrak{g}_C$, we have $\tau|\mathfrak{k} = Id|\mathfrak{k}$, $\tau|\mathfrak{p} = -Id|\mathfrak{p}$ $\Rightarrow$ $\mathfrak{g}$ is τ-invariant.

Proposition-1 *Let $\mathfrak{h}_p$ be a maximal abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{h} \subset \mathfrak{g}$ a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}_p$. Then the complexification $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$.*

First we study, how $\mathfrak{h}$ is situated in $\mathfrak{g}$.

$X \in \mathfrak{h}$, $Y \in \mathfrak{h}_p$ $\Rightarrow$ $\tau X - X \in \mathfrak{g}$ and $[\tau X - X, Y] = [\tau X, Y] = \tau[X, \tau Y] = \tau[X, -Y] = 0$. (*)

We have also $\tau(\tau X - X) = -(\tau X - X)$ $\Rightarrow$ $(\tau X - X) \in \mathfrak{p}$. Hence, by the maximality of $\mathfrak{h}_p$, and since (*) holds for every $Y \in \mathfrak{h}_p$ $(\tau X - X) \in \mathfrak{h}_p$ $\Rightarrow$ $\tau(X) \in \mathfrak{h}$. In other words, $\mathfrak{h}$ is τ-invariant and consequently

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p} \quad \Rightarrow \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}_p.$$

Let $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k}$. The complexification $\mathfrak{h}_C$ is generated by vectors $X \in \mathfrak{h}_k \oplus i\mathfrak{h}_p \subset \mathfrak{u}$. Thus for $X \in \mathfrak{u}$ and the definite hermitian form H (defined in the preceding §)

$$H(X,Y) = <\tau X, Y>,$$

we have

$$H([XU],V) + H(U,[XV]) = 0,$$

$$\Leftrightarrow \quad <\tau[XU],V> + <\tau U,[XV]> = 0,$$

$$\Leftrightarrow \quad <[X,\tau U],V> + <\tau U,[XV]> = 0.$$

Thus, for $X \in \mathfrak{u}$ and an adX-invariant subspace V of $\mathfrak{g}$, the H-orthogonal complement $V^{\perp}$ is again adX-invariant, hence adX is a semi simple operator. Besides, $\mathfrak{h}_C$ is a maximal abelian subalgebra of $\mathfrak{g}_C$ (Ex-4), hence, by Pro-10 §22, $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$.

Notice that $\mathfrak{h} \subset \mathfrak{g}$ is a real Cartan subalgebra (according to the definition of §22). In fact $\mathfrak{h}$ is nilpotent (since abelian) and $[X\mathfrak{h}] \subset \mathfrak{h}$ implies $[X\mathfrak{h}_C] \subset \mathfrak{h}_C$, hence $X \in \mathfrak{h}_C \Rightarrow X \in \mathfrak{h}$. Thus we proved the

Theorem-4 *For every real semi simple Lie algebra $\mathfrak{g}$, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and every maximal abelian subalgebra $\mathfrak{h}_p$ of $\mathfrak{p}$, there is a real Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which satisfies*

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p}, \text{ and } \mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}_p.$$

Exercise-4 Let g be a real semi simple Lie algebra and g_C its complexification. Show that the complexification h_C of a maximal abelian subalgebra of g is a maximal abelian subalgebra of g_C.

Exercise-5 Show that if h_k is a maximal abelian subalgebra of k and $h_k \subset h \subset g$ a maximal abelian subalgebra of g, then $h = h \cap k \oplus h \cap p$, and h is a Cartan subalgebra of g.

The maximal abelian subalgebra $h_k \subset k$ of the preceding exercise determines a unique maximal abelian $h_k \subset h \subset g$. In fact, if $h' = h' \cap k \oplus h' \cap p$ were an other, we would have $h' \cap k = h \cap k = h_k$. For $h' \cap p$: Let $Y \in h' \cap p$, $X \in h$. Then

$$X = (1/2)(X+\tau X)+(1/2)(X-\tau X) \in h_k \oplus h \cap p,$$
$$[YX] = [Y, (1/2)(X-\tau X)] \in [pp] \subset k, \text{ and for } Z \in h_k,$$
$$-[Z[YX]] = [Y[XZ]]+[X[ZY]] = 0,$$

hence $[YX]$ commutes with every element of h_k. Since this is a maximal abelian subalgebra, we have $[YX] \in h_k \subset h$. X was arbitrary, hence $[Yh] \subset h \Rightarrow Y \in h$. We proved the

Theorem-5 *Let $g = k \oplus p$ be a Cartan decomposition of the real semi simple Lie algebra g, and $h_k \subset k$ a maximal abelian subalgebra of k. Then there is a unique maximal abelian subalgebra h of g containing h_k. h is a Cartan subalgebra of g, and $h = h \cap k \oplus h \cap p$, $h \cap k = h_k$.*

Exercise-6 Show that every subalgebra $a \subset p$ is abelian. Conclude that h_p is a maximal subalgebra of p. [$[pp] \subset k$]

Notice that h, constructed either from $h_k \subset k$ or from $h_p \subset p$, is τ-invariant, where τ the involution of g. In both cases $h_k \oplus i h_p$ is a Cartan subalgebra of the compact form $u = k \oplus ip$. It is interesting to find the location of the roots of g_C relative to h_C. Using the standard notation, $h_0 = \oplus_{\alpha \in \Delta^+} \mathbf{R} H_\alpha$ is a real subspace of $h_C \subset g_C$ and we have the

Theorem-6 *Let $h \subset g$ be a Cartan subalgebra of the complex semi simple Lie algebra g, which is τ-invariant with respect to the conjugation τ of a compact form u of g. Then*

i) $h_0 = \oplus_{\alpha \in \Delta^+} \mathbf{R} H_\alpha \subset iu$,

ii) *The root vectors X_α, $X_{-\alpha}$ can be choosen in such a way that*

$$u = ih_0 \oplus_{\alpha \in \Delta^+} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus_{\alpha \in \Delta^+} \mathbf{R} i(X_\alpha + X_{-\alpha}).$$

To prove the theorem, we use again the hermitian form $H(X,Y) = \langle \tau X, Y \rangle$ (negative definite p. 40-4) on g. For $H \in h$, $\alpha \in \Delta$, and $X \in g_\alpha$ we have

$$[\tau H, \tau X] = \tau[HX] = \tau\alpha(H)X = \overline{\alpha(H)}\, \tau(X) = \overline{(\alpha \circ \tau)}(\tau H)\, \tau X. \qquad (*)$$

This shows that $\alpha^\tau = \overline{(\alpha \circ \tau)}$ is again a root. We show now that h_0 is τ-invariant.

$$\langle \tau h_\alpha, H \rangle = \overline{\langle h_\alpha, \tau H \rangle} = \overline{(\alpha \circ \tau)}(H) = \langle h_{\alpha^\tau}, H \rangle \quad \Rightarrow$$
$$\tau(h_\alpha) = h_{\alpha^\tau}.$$

Thus, h_0 is τ-invariant and $\tau|h_0$ is an involution, hence there is a corresponding decomposition $h_0 = h_1 \oplus h_{-1}$ of h_0 into ±1-eigenspaces. It follows $h_1=\{0\}$, since $X \in h_1 \Rightarrow \langle X,X\rangle = \langle \tau X,X\rangle < 0$, which contradicts the fact, that the Killing form is positive definite on h_0 (§24). Hence $h_0=h_{-1}$, which shows i) and in addition also the $\alpha^\tau = -\alpha$.

From (*) we have $\tau(X_\alpha) = c_\alpha X_{-\alpha} \Rightarrow \bar{c}_\alpha c_{-\alpha} = 1$, $(\tau^2 = 1)$

$$[X_\alpha, X_{-\alpha}] = H_\alpha \quad \Rightarrow \quad \tau[X_\alpha X_{-\alpha}] = H_{-\alpha} \quad \Rightarrow \quad c_\alpha c_{-\alpha}=1.$$

Thus, c_α is real and replacing X_α with $-(c_\alpha)^{-1}X_\alpha$ for all $\alpha \in \Delta^+$, we see that

$$\tau X_\alpha = -X_{-\alpha} \text{ and } \tau X_{-\alpha} = -X_\alpha,$$

This implies ii). q.e.d.

We conclude that $h_0 \subset ik \oplus p$, hence (using the previous notation $g=k\oplus p$ etc.)

$$h_0 = ih_k \oplus h_p,$$
$$ih_0 = h_k \oplus ih_p \subset u. \qquad (4)$$

Notice that ih_0 is a maximal abelian subalgebra of u. In fact $[X, ih_0] =0 \Rightarrow [Xh_C]=0 \Rightarrow X \in h_C \Rightarrow X \in ih_0$. Inversely, every maximal abelian subalgebra of u is a Cartran subalgebra of u. This happens because the Killing form of u is definite and every adX-invariant subspace $V \subset u$ defines an orthogonal complement $V^\perp$ which is adX-invariant too. Consequently adX is a semi-simple operator.

Eine Wirkung völlig zu hindern, dazu gehört eine Kraft, die der Ursache von jener gleich ist, aber ihr eine andere Richtung zu geben bedarf es öfters nur einer Kleinigkeit.
Lichtenberg, Sudelbücher p. 431

42 . Compact Lie algebras

So are called the real Lie algebras $\mathfrak{g}$, which possess some metric <...,...> such that all $f \in \mathrm{Aut}(\mathfrak{g})$ are isometries. The basic example is that of compact real forms of a complex semi simple Lie algebra, endowed with the negative Killing form (this gives a positive definite metric on the Lie algebra). For these Lie algebras $\mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$ is a closed subgroup of the orthogonal group $O(\mathfrak{g},<...,...>)$, hence a compact Lie group. The connected component $\mathrm{Int}(\mathfrak{g})$ of $\mathrm{Aut}(\mathfrak{g})$ is generated by the inner automorphisms of $\mathfrak{g}$, of the form $\exp(adX)$ and the map $\exp: \mathfrak{g} \to \mathrm{Int}(\mathfrak{g})$ may be identified with the exponential of $\mathfrak{g}$ and is onto the compact group $\mathrm{Int}(\mathfrak{g})$.

Proposition-1 *For every compact Lie algebra $\mathfrak{g}$, we have*

$$\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}\mathfrak{g}],$$

where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}\mathfrak{g}]$ is semi simple.

In fact, let <...,...> be the $\mathrm{Int}(\mathfrak{g})$-invariant metric of $\mathfrak{g}$, $\mathfrak{c}$ the center of $\mathfrak{g}$ and $\mathfrak{c}^\perp$ the orthogonal complement of $\mathfrak{c}$. For $X \in \mathfrak{c}^\perp$, $<adYX,\mathfrak{c}> = -<X,[Y\mathfrak{c}]> = 0$, hence $\mathfrak{c}^\perp$ is an ideal and $\mathfrak{g}=\mathfrak{c}\oplus\mathfrak{c}^\perp$. Besides $adX:\mathfrak{c}^\perp \to \mathfrak{c}^\perp$ is skew-symmetric with respect to <...,...>, hence $tr(adX \circ adX)<0$ and $=0 \Leftrightarrow adX=0$ i.e. $X= 0$. Thus, by Cartan's criterion $\mathfrak{c}^\perp$ is semi simple and coincides with $[\mathfrak{g}\mathfrak{g}]$.

We saw in the previous § that the Cartan subalgebras of a compact real form $\mathfrak{u}$ coincide with the maximal abelian subalgebras of $\mathfrak{u}$. This motivates our interest for the maximal abelian subalgebras of a compact Lie algebra.

Proposition-2 *Let $\mathfrak{a} \subset \mathfrak{g}$ be a maximal abelian subalgebra of the compact Lie algebra $\mathfrak{g}$. Then there is some $X \in \mathfrak{a}$ with*

$$\mathfrak{a} = \mathfrak{g}_0(X) = \{Y \in \mathfrak{g} | [YX]=0\}.$$

In fact, $\exp(ad\mathfrak{a}) \subset \mathrm{Int}(\mathfrak{g})$ is a closed abelian subgroup (a maximal toroid), hence a compact torus. In such tori there are elements $x=\exp(adX)$, such that their powers $x,x^2,x^3, ...$ build a dense subset (of the torus). $X \in \mathfrak{a}$ is what we want. We call it a **generator** of $\mathfrak{a}$. In fact, $Y \in \mathfrak{g}_0(X) \Rightarrow [YX]=0$ and consequently $\exp(adtX)Y = Y \Rightarrow \exp(ad\mathfrak{a})Y=Y \Rightarrow [\mathfrak{a}Y]=0$ $\Rightarrow Y \in \mathfrak{a}$.

Proposition-3 *For a maximal abelian subalgebra $\mathfrak{a}$ of the compact Lie algebra $\mathfrak{g}$ we have*

$$\mathfrak{g} = \bigcup_{X\in\mathfrak{g}} \exp(adX)\mathfrak{a}.$$

In other words, every element of $\mathfrak{g}$ is contained, at least, in one maximal abelian subalgebra of $\mathfrak{g}$ (conjugate to $\mathfrak{a}$).

In fact, take Y arbitrary and $X \in \mathfrak{a}$ as before. Consider then the function on $\mathrm{Int}(\mathfrak{g})$ $f(g)=<X,gY>$, $g \in \mathrm{Int}(\mathfrak{g})$. This has a minimum on some $g_0 \in \mathrm{Int}(\mathfrak{g})$. Hence, for every $Z \in \mathfrak{g}$ we'll have

$$0 = (d/dt)|_0 f(\exp(\mathrm{ad}tZ)g_o) = (d/dt)|_0 \langle X, \exp(t\mathrm{ad}Z)g_o Y\rangle$$
$$= \langle X,[Z,g_oY]\rangle = -\langle [X,g_oY],Z\rangle \;\Rightarrow$$
$$[X,g_oY]=0 \Rightarrow g_oY\in\mathfrak{a} \;\Rightarrow Y\in g_o^{-1}(\mathfrak{a}).$$

q.e.d.

Proposition-4 *Two maximal abelian subalgebras* $\mathfrak{a}$, $\mathfrak{a}'$ *of a compact Lie algebra* $\mathfrak{g}$, *are conjugate with respect to some inner automorphism* $g\in \mathrm{Int}(\mathfrak{g})$.

In fact, choose some $X\in\mathfrak{a}$, as in pro-2. Then, by Pro-3, there is some $g\in\mathrm{Int}(\mathfrak{g})$ with $gX\in\mathfrak{a}' \Rightarrow X\in g^{-1}(\mathfrak{a}') = \mathfrak{g}_0(X) \Rightarrow \dim\mathfrak{a}\geq\dim\mathfrak{a}'$. Analogously $\dim\mathfrak{a}'\geq\dim\mathfrak{a}$ q.e.d.

Corollary *All Cartan subalgebras of a compact semi simple Lie algebra are conjugate to each other.*

When $\mathfrak{g}$ is a compact semi simple Lie algebra and $\mathfrak{h}\subset\mathfrak{g}$ is a maximal abelian subalgebra, then, by Theo-6 §41, the complexification $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$ and $\mathfrak{g}$ can be written

$$\mathfrak{g} = i\mathfrak{h}_0 \oplus_{\alpha\in\Delta^+} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus_{\alpha\in\Delta^+} \mathbf{R}i(X_\alpha + X_{-\alpha}).$$

Using the formulas §40 (8), we compute the matrix representing ad(iH) with respect to the basis U_α, V_α (for $H\in\mathfrak{h}_0$):

$$\mathrm{ad}(iH) = \begin{pmatrix} 0 & & & \\ & \begin{matrix} 0 & -\alpha(H) \\ \alpha(H) & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & -\omega(H) \\ \omega(H) & 0 \end{matrix} \end{pmatrix}.$$

Proposition-5 *For every maximal abelian subalgebra* $\mathfrak{h}$ *of the compact semi simple Lie algebra* $\mathfrak{g}$, *there are linear forms* $\alpha, \ldots, \omega$ *and a basis of* $\mathfrak{g}$, *such that for all* $X\in\mathfrak{a}$:

$$\exp(\mathrm{ad}X) = \begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & & & \\ & \begin{matrix} \cos\alpha(X) & -\sin\alpha(X) \\ \sin\alpha(X) & \cos\alpha(X) \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} \cos\omega(X) & -\sin\omega(X) \\ \sin\omega(X) & \cos\omega(X) \end{matrix} \end{pmatrix}.$$

$\{i\alpha, \ldots, i\omega\} = \Delta^-$ build a system of positive roots of the Cartan subalgebra $\mathfrak{h}_C$ of the complexification $\mathfrak{g}_C$.

Instead of $\{\alpha, \ldots, \omega\}$ it is often useful to work with $\{\alpha'=\alpha/2\pi, \ldots, \omega'=\omega/2\pi\}$. We call $\{\alpha', \ldots, \omega'\}$ **roots of the compact Lie algebra** $\mathfrak{g}$. The analogon of Pro-5 holds also for more general compact Lie algebras (non-semi-simple). The only difference is in the box with 1's which will have dimension greater than the maximal abelian subalgebra $\mathfrak{a}$, whereas the vector space spanned by $\{\alpha, \ldots, \omega\} \subset \mathfrak{a}^*$ will be a (proper) subspace of dimension = rank of $[\mathfrak{g}\mathfrak{g}]$.

Proposition-5 (and its analogon for more general compact Lie algebras) completely describes the maximal toroids of Int($\mathfrak{g}$), in terms of the roots of of the Lie algebra. A crucial role in the study of compact Lie algebras plays the **diagram**, which is a set of hyperplanes inside $\mathfrak{a}$:

$$D = \{X \in \alpha' \mid \alpha'(X) \in \mathbf{Z}\}. \qquad (1)$$

By Pro-5, if $\alpha'(X) \in \mathbf{Z}$, then in exp(adX) the corresponding 2×2 matrix

$$\begin{pmatrix} \cos 2\pi\alpha'(X) & -\sin 2\pi\alpha'(X) \\ \sin 2\pi\alpha'(X) & \cos 2\pi\alpha'(X) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the normalizing subgroup of f=exp(adX) (i.e. the subgroup of the automorphisms commuting with f) has dimension bigger than the dimension of $\mathfrak{a}$.

The **central latice** Λ_Z of the Lie algebra is

$$\Lambda_Z = \{X \in \mathfrak{a} \mid \alpha'(X) \in \mathbf{Z}, \text{ for every } \alpha' \in \Delta\}.$$

Here we use the symbol Δ to denote the set of roots of $\mathfrak{g}_C$, with respect to $\mathfrak{a}_C$. This, having in mind that $2\pi i\alpha' \in \Delta$. The points $X \in \Lambda_Z$ have obviously exp(adX) = Id. The relations between roots, their angles, their ratios, positive roots, fundamental systems etc., all this can be transfered to the roots $\{\alpha', \ldots, \omega'\}$ of $\mathfrak{g}$, since $2\pi i\alpha'$ are roots of $\mathfrak{g}_C$.

The vectors $X \in D$ are called **singular**, whereas the elements of $\mathfrak{a}$-D are called **regular**. Later we'll see that for every singular $X \in D$, adX is conjugate in Int($\mathfrak{g}$) to some adY, with $Y \in D_0$: $\mathrm{ad}Y = f \circ \mathrm{ad}X \circ f^{-1}$, with $f \in$ Int($\mathfrak{g}$), where

$$D_0 = \{X \in \mathfrak{a} \mid \alpha'(X)=0, \text{ for some } \alpha \in \Delta\}.$$

The characterization of an element $X \in \mathfrak{g}$ as singular, involves the use of a maximal abelian subalgebra and its corresponding diagram but is independent of these. In fact, every $X \in \mathfrak{g}$ is contained in a maximal abelian subalgebra $\mathfrak{a}$ (Pro-3). When $\alpha'(X) \neq 0$, for every $\alpha \in \Delta$, the eigenspace $\mathfrak{g}_0(X) = \{Y \in \mathfrak{g} \mid [YX]=0\}$ has minimum dimension = dim$\mathfrak{a}$ = rank of $\mathfrak{g}$. However this property is independent of the particular abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. The preceding consideration shows that the elements $X \in D_0$ are characterized by the non-minimality of the zero-eigenspace $\mathfrak{g}_0(X)$. More general, the $X \in D$ are characterized by the same property, since they are conjugate to elements of D_0.

Proposition-6 *For a compact lie algebra $\mathfrak{g}$, $X \in \mathfrak{g}$ is regular if and only if it is contained in a unique maximal abelian subalgebra. X is singular if it is contained in more than one maximal abelian subalgebras.*

In fact, X regular, means $\mathfrak{g}_0(X)=\mathfrak{a}$=maximal abelian subalgebra of $\mathfrak{g}$. Every other maximal abelian subalgebra containing X, should be contained in $\mathfrak{g}_0(X)$ and for dimension reasons should be identical with $\mathfrak{a}$. On the other side, if X is singular then $\mathfrak{g}_0(X)$ contains some U_α, V_α and $\{X, U_\alpha\}$ generates an abelian subalgebra which lies in some maximal abelian subalgebra $\mathfrak{a}$. Similarly, $\{X,V_\alpha\}$ generates an abelian subalgebra, which lies in some maximal abelian $\mathfrak{a}'$. These facts can be easily proved using an argument like the one used in Pro-2. Obviously $\mathfrak{a}'\neq\mathfrak{a}$, since in the contrary case, we should have $[U_\alpha,V_\alpha]=0$, which contradicts $[U_\alpha,V_\alpha]=2iH_\alpha$ (§40 (8)). q.e.d.

Notice that $\dim \mathfrak{g}_0(X)=(\operatorname{rank}\mathfrak{g})\bmod 2$.

Theorem *There are as many isomorphy classes of compact semi simple Lie algebras, as the isomorphy classes of complex semi simple Lie algebras. In particular the compact simple Lie algebras are classified by their corresponding Dynkin diagrams* A_n, B_n, C_n, D_n, E_8, E_7, E_6, F_4, G_2.

The theorem is a consequence of the fact, that to every such Lie algebra corresponds bijectively a (isomorphy class of) root system Δ of $\mathfrak{g}$. The theorem combined with Pro-1, classifies also the more general compact Lie algebras $\mathfrak{g}=\mathfrak{c}\oplus\mathfrak{g}'$, where $\mathfrak{g}'$ semi simple. Notice also that models of compact simple Lie algebras can be constructed from a Cartan decomposition of a real form $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ of a simple complex Lie algebra, by taking the corresponding $\mathfrak{u}=\mathfrak{k}\oplus i\mathfrak{p}$.

Unser Daseyn nämlich ist ein wesentlich rastloses, daher wird die gänzliche Unthätigkeit uns bald unerträglich, indem sie die entsezlichste Langeweile herbeiführt. Diesen Trieb nun soll man regeln, um ihn methodisch und dadurch besser zu befriedigen.

Schopenhauer, Aphorismen ... p. 478

43 . Automorphisms of Compact Lie algebras

The work has been done in §39. If $\mathfrak{u}$ is a compact semi simple Lie algebra, then every maximal abelian subalgebra $\mathfrak{a}\subset\mathfrak{u}$ defines a Cartan subalgebra $\mathfrak{h}=\mathfrak{a}\oplus i\mathfrak{a}$, of the complexification $\mathfrak{g}_C=\mathfrak{u}\oplus i\mathfrak{u}$ and the roots of $\mathfrak{h}$ in $\mathfrak{g}_C$ define the corresponding roots of $\mathfrak{u}$ (§42).

Lemma-1 *Every automorphism $f:\mathfrak{u}\rightarrow\mathfrak{u}$ of a compact semi simple Lie algebra $\mathfrak{u}$, which leaves a maximal abelian subalgebra $\mathfrak{a}$ invariant, induces in $\Delta'\subset\mathfrak{a}^*$ an isometry, and inversely every isometry of Δ' extends to an automorphism of $\mathfrak{u}$ which leaves $\mathfrak{a}$ invariant.*

The first part is obvious, since f is an isometry with respect to the Killing form, which maps roots to roots etc.

The inverse follows from the corresponding The-2 (§37) by extending to an automorphism, using a Weyl-Chevalley normalization. In fact, f will be naturally extended to $\mathfrak{h}$ and then to $\mathfrak{u}$. Using a normalization we can define (set $\alpha'=f(\alpha)$)

$$f(X_\alpha) = c_\alpha X_{\alpha'},$$
$$\text{where, } c_\alpha c_{-\alpha}=1,\ N_{\alpha\beta} = N_{\alpha'\beta'},\ c_\alpha c_\beta=c_{\alpha+\beta}. \qquad (1)$$

Choose then a fundamental system $\Pi=\{\alpha_1, \dots, \alpha_n\}$ and take $H\in\mathfrak{h}$ with

$$c_i = \exp(-\alpha_i(H)).$$

Using (1) and writting every root as a sum of simple roots $\alpha=\sum n_i\alpha_i$ we get

$$c_\alpha= \exp(-\alpha(H)).$$

Then

$$f\circ\exp(\mathrm{ad}H)(X_{\alpha_i}) = X_{\alpha_i'},$$
$$f\circ\exp(\mathrm{ad}H)(X_\alpha) = X_{f(\alpha)},$$

and $[X_\alpha X_{-\alpha}]=H_\alpha$ implies $[X_{\alpha'}X_{-\alpha'}]=H_{\alpha'}$, hence vectors of the form iH_α, $X_\alpha-X_{-\alpha}$ and $i(X_\alpha+X_{-\alpha})$ are mapped to similar ones. But these generate $\mathfrak{u}_C$ (The-6, §41), which then remains invariant under f.

q.e.d.

Corollary *Let $f:\mathfrak{u}\rightarrow\mathfrak{u}$ be an automorphism of the compact semi simple Lie algebra $\mathfrak{u}$, leaving the maximal abelian subalgebra $\mathfrak{a}$ invariant. Then the extension of f on $\mathfrak{u}$ satisfies*

$$f(X_\alpha) = c_\alpha X_{\alpha'},\ \text{with } c_\alpha c_{-\alpha}=1 \text{ and } |c_\alpha|=1.$$

In fact, we have $c_\alpha c_{-\alpha}=1$, as before, and $f\circ\tau=\tau\circ f$, where τ is the conjugation of $\mathfrak{u}$. Since

$$X = (1/2)\{X_\alpha-X_{-\alpha}-i(i(X_\alpha+X_{-\alpha}))\} \Rightarrow \tau X_\alpha = -X_{-\alpha}, \text{ we have}$$
$$\tau\circ f(X) = \tau c_\alpha X_{\alpha'} = -\bar{c}_\alpha X_{-\alpha'},$$
$$= f\circ\tau X_\alpha = -fX_{-\alpha} = -c_{-\alpha}X_{-\alpha'}.$$

From these we have $\bar{c}_\alpha= c_{-\alpha}$. q.e.d.

Lemma-2 *An automorphism $f:\mathfrak{u}\rightarrow\mathfrak{u}$ of a compact semi simple Lie algebra leaves a maxi-*

mal abelian subalgebra $\mathfrak{a}$ *pointwise fixed, if and only if, it is of the form*

$$f = \exp(adH), \text{ with } H \in \mathfrak{a}.$$

In lemma-1 of §39 was shown that $f=\exp(adH)$, with $H \in \mathfrak{h}=\mathfrak{a}_C$, leaves the Cartan subalgebra $\mathfrak{h}$ pointwise fixed. The point is that $H \in \mathfrak{a}$. In fact,

$$f(X_\alpha) = \exp(\alpha(H))X_\alpha$$

and from the preceding corollary, we have $|\exp(\alpha(H))|=1$, hence $\alpha(H)$ is purely imaginary $\Rightarrow H \in \mathfrak{a}$. q.e.d.

Lemma-3 *Let* $\mathfrak{u}$ *be a compact semi simple Lie algebra. Then* $Aut(\mathfrak{u})/Int(\mathfrak{u}) \cong G_{\mathfrak{a}}/G_{\mathfrak{a}}^o$, *where* $G_{\mathfrak{a}}$ *is the group of automorphisms of* $\mathfrak{u}$, *leaving a maximal abelian subalgebra* $\mathfrak{a}$ *invariant, and* $G_{\mathfrak{a}}^o$ *the subgroup of* $G_{\mathfrak{a}}$, *consisting of inner automorphisms.*

The proof is the same with that of Lemma-3 of §39.

Theorem-1 *For every element* $S \in W$ *of the Weyl group* W *of a compact semi simple Lie algebra* $\mathfrak{u}$, *with respect to some abelian subalgebra* $\mathfrak{a}$, *there is an inner automorphism* f *with the properties i)* $f(\mathfrak{a}) \subset \mathfrak{a}$ *and ii)* $(f|\mathfrak{a})=S$. *Inversely, every inner automorphism* f *leaving* $\mathfrak{a}$ *invariant, has* $(f|\mathfrak{a}) \in W$.

The proof of the theorem is that of The-1 in §39. Notice there, that the element

$$X = (i\pi/2)(X_\alpha + X_{-\alpha}),$$

whose adX is the reflexion S_α, belongs to $\mathfrak{u}$. For the inverse, apply The-1(§39) to $\mathfrak{h}=\mathfrak{a}_C$. The rest is consequence of the definitions.

Theorem-2 *For every compact semi simple Lie algebra* $\mathfrak{u}$, $Aut(\mathfrak{u})/Int(\mathfrak{u}) = Aut(DD)$, *where* $Aut(DD)$ *is the group of symmetries of the Dynkin diagram.*

The proof is again the same with that of the corresponding theorem in §39.

Remark-1 The automorphisms of compact semi simple Lie algebras play a crucial role in the classification of real semi simple Lie algebras. According to The-5 §40, the real forms of a semi simple complex Lie algebra are defined through (conjugacy classes of) involutions of a compact real form $\mathfrak{u}$. From the analysis we made here and the description of Aut(DD) (p.39-5), we know that, but for 4 cases, all automorphisms of compact semi simple Lie algebras are inner. For inner involutive automorphisms we have the following

Theorem-3 *Let* $f: \mathfrak{u} \rightarrow \mathfrak{u}$ *be an involutive automorphism of a compact semi simple Lie algebra* $\mathfrak{u}$. *Let also* $\mathfrak{k}$ *be the (fixed by f)* $+1$*-eigenspace of* f. *Then the following statements are equivalent :*

1) $f \in Int(\mathfrak{u})$,
2) $rank(\mathfrak{u}) = rank(\mathfrak{k})$,

where, as usual, $rank(\mathfrak{u})$*=dimension of a maximal abelian subalgebra of* $\mathfrak{u}$.

In fact, if $f \in Int(\mathfrak{u})$, then by compactness, $f=\exp(adX)$, for some $X \in \mathfrak{u}$. By Pro-3 §42, X will belong to some maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{u}$, which by lemma-2 will be invariant with respect to f, hence $\mathfrak{a} \subset \mathfrak{k}$.

Inversely, if $\mathfrak{a} \subset \mathfrak{k}$ is also maximal abelian in $\mathfrak{k}$ and $\mathfrak{u}$, then by lemma-2, the automorphism f will be of the form f=exp(adX), with X∈a. q.e.d.

Remark-2 Notice that k is compact (not necessarily semi simple) subalgebra of u and the orthogonal complement $p=k^{\perp}$, u=k⊕p is the -1-eigenspace of f. Hence, all these f are determined if we know all possible k i.e. all the compact Lie subalgebras of u, which have maximal rank (=rank(u)). As we noticed earlier, k is called the characteristic subalgebra of the corresponding real (form) Lie algebra k⊕ip.

denn jeder Irrthum trägt ein Gift in seinem Innern. Ist es der Geist, ist es die Erkenntniß, welche den Menschen zum Herrn der Erde macht, so giebt es keine unschädlichen Irrtümer, noch weniger ehrwürdige, heilige Irrthümer.

Schopenhauer, Die Welt als ... I. p. 67

44 . Diagram and latices

The definitions of roots and diagram of a semi simple compact Lie algebra $\mathfrak{g}$ were given in §42. Recall from there that the diagram, with respect to some maximal abelian subalgebra $\mathfrak{a}$, and the corresponding root system Δ is

$$D = \{X \in \mathfrak{a} | \alpha(X) \in \mathbf{Z}, \text{ for some root } \alpha \in \Delta\}, \qquad (1)$$

the central latice being

$$\Lambda_Z = \{X \in \mathfrak{a} | \alpha(X) \in \mathbf{Z}, \text{ for every root } \alpha \in \Delta\}. \qquad (2)$$

Δ denotes here the set of roots of the compact Lie algebra $\mathfrak{g}$, related to the roots of the corresponding complexification $\mathfrak{g}_C$, with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{a}_C$, through

$$\Delta \ni \alpha \Leftrightarrow 2\pi i\, \alpha \text{ is a root of } \mathfrak{g}_C \text{ (with respect to } \mathfrak{a}_C). \qquad (3)$$

Exercise-1 If $\Pi = \{\alpha_1, \dots, \alpha_d\}$ is a fundamental system of roots for Δ and $\{Y_1, \dots, Y_d\}$ are the duals of $\{\alpha_1, \dots, \alpha_d\}$, defined by the relations $\alpha_i(Y_j) = \delta_{ij}$, show that

$$\Lambda_Z = \mathbf{Z}Y_1 + \dots + \mathbf{Z}Y_d . \qquad (4)$$

Lemma-1 Λ_Z *operates simply transitively on itself (by translations) as a normal subgroup of the group G of isometries of* $\mathfrak{a}$, *which leave the latice* Λ_Z *invariant.*

Only the normality needs some comment, $v \in \Lambda_Z$ and $f = (A,w) \in G$ have

$$(A,w)^{-1} \circ (I,v) \circ (A,w) = (A^{-1}, -A^{-1}w) \circ (A, v+w) = (I, A^{-1} v),$$

where the composition is that of the semi direct product $O(d) \times_\sigma \mathbf{R}^d$. q.e.d.

Lemma-2 *With the previous notation,* $G/\Lambda_Z \cong G_0$, *where* G_0 *is the isotropy group at 0. We have also* $G = G_0 \times_\sigma \Lambda_Z$.

In fact, the map $(A,v) \to A$ defines an homorphism of G onto G_0, whose kernel is Λ_Z.

Exercise-2 Show that if $f = (A,v) \in G$, then $v \in \Lambda_Z$ and $A(\Lambda_Z) \subset \Lambda_Z$.

[$v = f(0)$, $(A,0) = (I,-v) \circ (A,v)$]

Using the basis $\{Y_1, \dots, Y_d\}$ of Λ_Z, we can describe G_0 by a matrix representation. This basis is not orthogonal (in general) and consequently the matrices representing $f \in G_0$ are not orthogonal. We have (α' denoting the roots of $\mathfrak{g}$ and α the roots of $\mathfrak{g}_C$)

$$f(Y_j) = \sum a_{ij} Y_i, \quad f(Y_j) \in \Lambda_Z$$

$$\alpha'_i(f(Y_j)) = a_{ij} \in \mathbf{Z}.$$

$$<f(Y_j), f(Y_k)> = <Y_j, Y_k> \quad \Rightarrow$$

$$<a_{rj} Y_r, a_{sk} Y_s> = <Y_j, Y_k> \quad \Leftrightarrow$$

$$A^t(<Y_r, Y_s>)A = (<Y_r, Y_s>). \qquad (*)$$

The matrix $(<Y_r, Y_s>)$ can be expressed through the Cartan matrix of the Lie algebra $\mathfrak{g}_C$.

$$iY_j = b_{rj} h_r \ (h_r = h_{\alpha_r} \in \mathfrak{h}_0, \text{ the dual of } \alpha_r).$$

$$2\pi\delta_{kj} = 2\pi\alpha'_k(Y_j) = -2\pi i\alpha'_k(iY_j) = -\alpha_k(iY_j) = -b_{rj}<h_i,h_k>$$
$$(b_{ij}) = 2\pi(<h_i,h_j>)^{-1}.$$
$$<Y_r,Y_s> = <b_{ir}h_i,b_{js}h_j> = b_{ir}<h_i,h_j>b_{js}$$
$$(<Y_r,Y_s>) = B^t(<a_i,a_j>)B = -2\pi B^t = -2\pi B.$$

Taking the inverse matrices in (*) we see that G is described by the matrices A, with integer coefficients, which leave invariant the quadratic form defined by the matrix $(<a_i,a_j>)$:

$$A^t(<\alpha_i,\alpha_j>)A = (<\alpha_i,\alpha_j>). \tag{5}$$

Exercise-3 Using the definition of the Cartan matrix (§37), show that

$$<\alpha_i,\alpha_j>=c_{ij}<\alpha_j,\alpha_j>/2.$$

Inspecting the Dynkin diagrams for the different simple complex Lie algebras, we notice that A_d, D_d, E_6, E_7, E_8 have simple links, hence all vectors of the corresponding fundamental system Π have equal lengths. Thus, in these cases (5) is equivalent with

$$A^tCA = C, \tag{6}$$

C being the corresponding Cartan matrix.

Exercise-4 For the remaining cases of simple complex Lie algebras show that the matrix $(<\alpha_i,\alpha_j>)$ is a constant multiple of the matrices :

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \Leftrightarrow B_d \qquad \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & -1 & \\ & & -1 & 2 & -2 \\ & & & -2 & 2 \end{pmatrix} \Leftrightarrow C_d \qquad \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 4 \end{pmatrix} \Leftrightarrow F_4$$

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \Leftrightarrow G_2$$

Problem For the symmetric matrices $(<\alpha_i,\alpha_j>)$ of the two last exercises, find the group of invertible integer matrices A, satisfying the equation

$$A^t(<\alpha_i,\alpha_j>)A = (<\alpha_i,\alpha_j>).$$

The correspondence established by (3) $a^* \leftrightarrow h_0^*$, defines a similarity between a and h_0

$$h_0 \ni H \rightarrow 2\pi i H = H' \in a, \tag{7}$$

such that

$$\alpha'(H') = \alpha(2\pi i H) = \alpha(H). \tag{8}$$

Through this similarity h_α is mapped into $h_{\alpha'} = 2\pi i h_\alpha$ and the corrots H_α to $2\pi i 2h_\alpha/<\alpha,\alpha>$.

Exercise-5 Show that

$$H_{\alpha'} = 2\pi i H_\alpha \in \Lambda_Z.$$

Exercise-6 For every $\alpha' \in a^*$ there is a "dual" $t_{\alpha'} \in a$, defined by

$$<t_{\alpha'},X> = \alpha'(X), \text{ for every } X \in a. \tag{9}$$

Show that

$$h_{\alpha'} = -4\pi^2 t_{\alpha'} . \tag{10}$$

[for $X \in a$, $<h_{\alpha'},X> = <2\pi i h_\alpha,X> = <h_\alpha,2\pi i X> = \alpha(2\pi i X) = \alpha'(-4\pi X) = <-4\pi t_{\alpha'},X>$]

Exercise-7 Show that

$$H_{\alpha'} = 2t_{\alpha'} / <\alpha', \alpha',>,$$

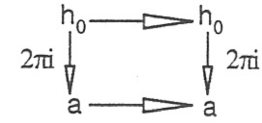
and from this conclude that the vectors $H_{\alpha'}$ are orthogonal to the planes

$$(\alpha',0) = \{X \in \mathsf{a} | \alpha'(X) = 0\},$$

and their length is twice the distance of the parallel planes $(\alpha',0)$ and $(\alpha',1)$, where

$$(\alpha',1) = \{X \in \mathsf{a} | \alpha'(X) = 1\}.$$

Exercise-8 Every isometry of h_0 defines an isometry of a, so that the nearby diagram is commutative. Show that the reflexion with respect to the plane $(\alpha,0)$ of h_0, corresponds to the reflexion with respect to the plane $(\alpha,0)$ of a.



Exercise-9 Show that the reflexions with respect to the planes $(\alpha',0)$ leave invariant the sets $\{h_{\alpha'}\}$, $\{t_{\alpha'}\}$, the latice Λ_Z and the diagram of the compact semi simple Lie algebra g. Conclude that in G_0 is contained a subgroup W', which through the mapping of Ex-8, corresponds to the Weyl group of g_C.

Of fundamental importance for the structure of semi simple compact Lie algebras is the subgroup Γ of the isometry group of the latice Λ_Z, which is generated by the reflexions on the planes

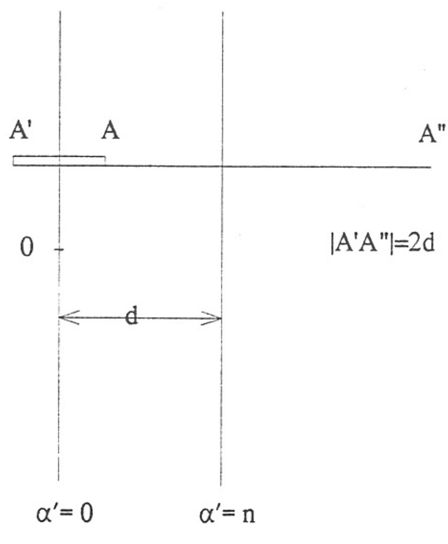
$$(\alpha',n) = \{X \in \mathsf{a} | \alpha'(X) = n\}, n \in Z. \qquad (11)$$

Proposition-1 *$\Gamma = W' \times_\sigma \Lambda_*$, where $\Lambda_* = ZH'_1 + \dots + ZH'_d$, and H'_i are the coroots of a fundamental system $\Pi = \{\alpha_1, \dots, \alpha_d\}$ of g_C.*

First, the reflexions on the planes (11) leave the latice Λ_Z invariant. In fact, the reflection on the plane $\alpha'(X)=n$, decomposes to the Weyl reflection $S_{\alpha'}$ and the translation: twice the distance of the planes $\alpha'(X)=n$ and $\alpha'(X)=0$. As noticed in Ex-7, this distance is a multiple of $H_{\alpha'}$. This shows that

$$\Gamma \subset W' \times_\sigma \Lambda_*.$$

The inverse inclusion is obvious.



The connected components of the complement a-D of the diagram are called **cells** of the diagram. They are open, convex and bounded polyeders of a. Convex, since they are intersections of (convex) "strips"

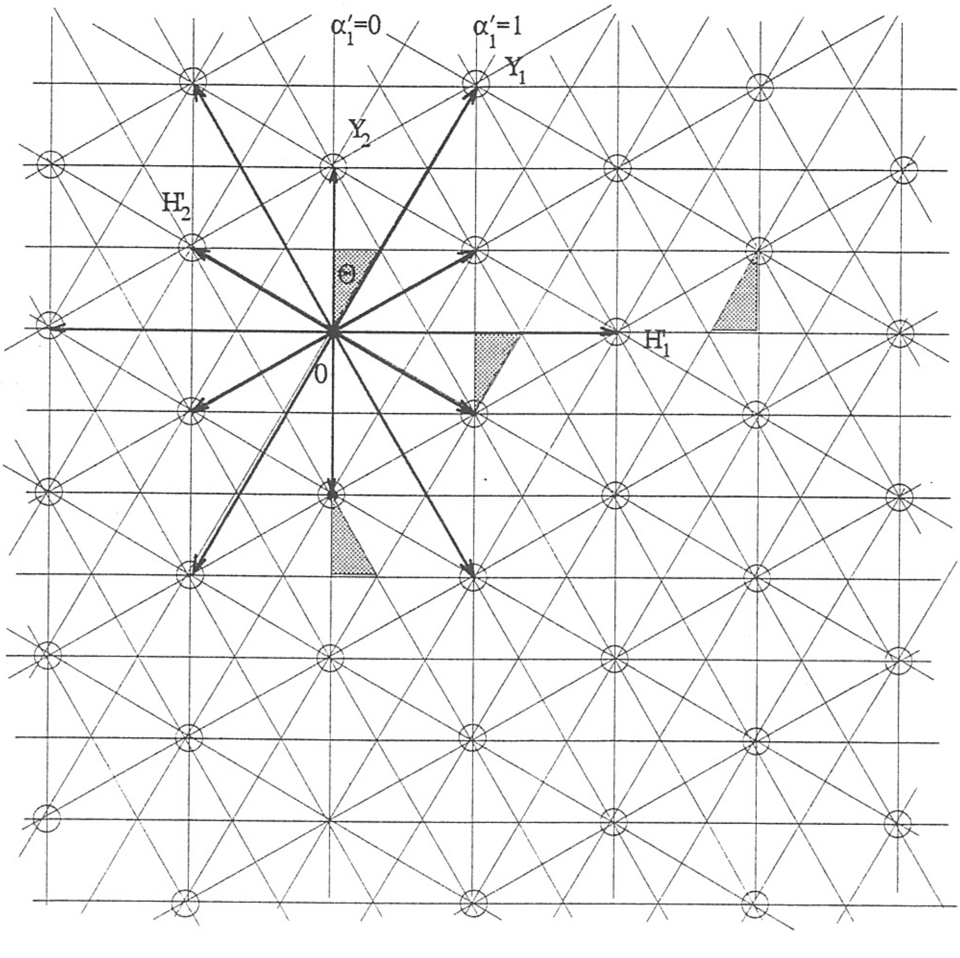
$$m < \alpha'(X) < m+1, \ m \in Z.$$

Bounded, for the same reason, in addition to the fact, that $\{\alpha' \in \Delta\}$ contains a basis of a^*.

Remark The points of Λ_Z are vertices of some cells. For some diagrams the vertices of the cells build up the whole Λ_Z. In general, however, Λ_Z is a proper subset of the set of vertices of all cells.

The following pictures display the diagrams of the different two-dimensional root systems.

G_2



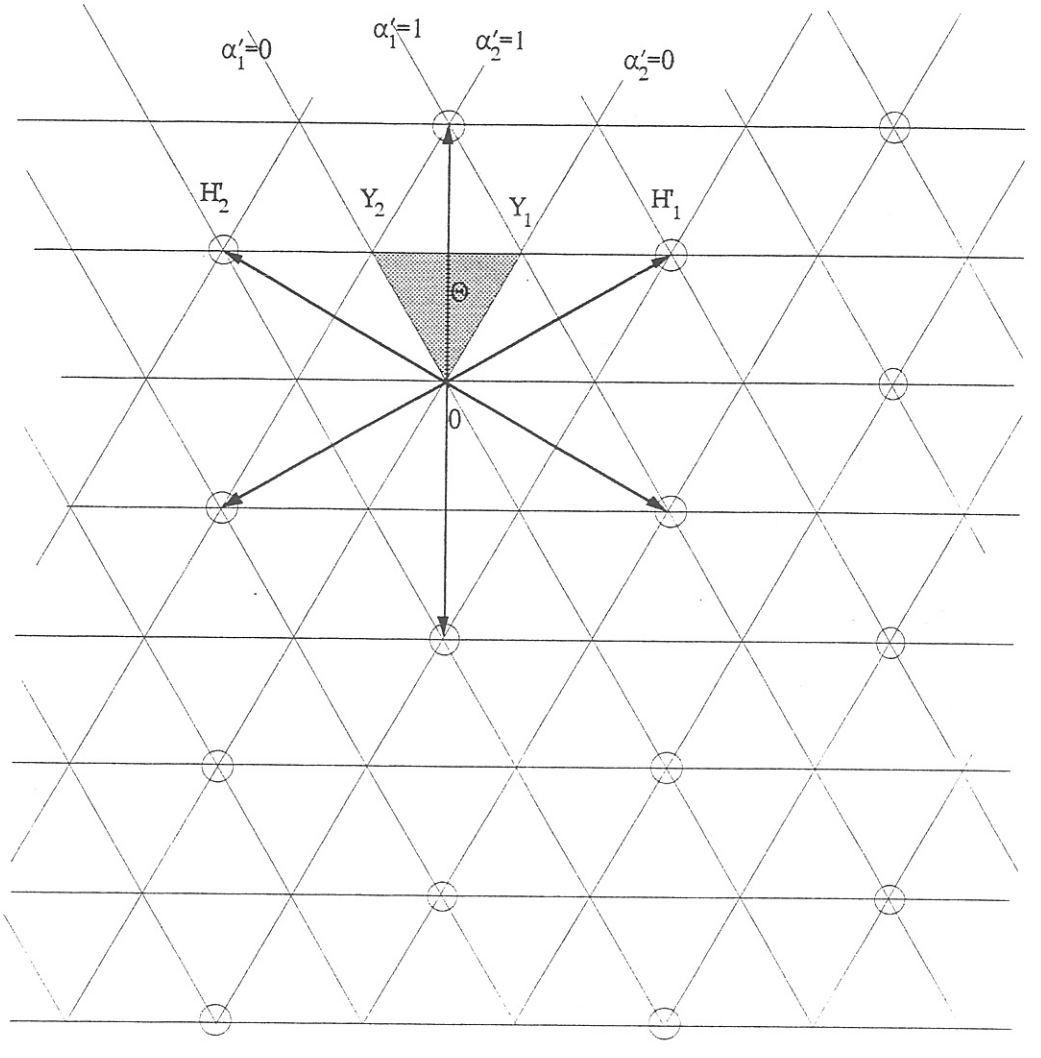
i) The vectors Y_i coincide with some coroots $H_{\alpha'}$,

ii) $\Lambda_Z = ZY_1 + ZY_2$,

iii) Θ = (fundamental) cell of the diagram,

iv) $\Lambda_Z = \Lambda_* = ZH'_1 + ZH'_2$, but here the vertices of all cells build up a latice, finer than Λ_Z.

A_2



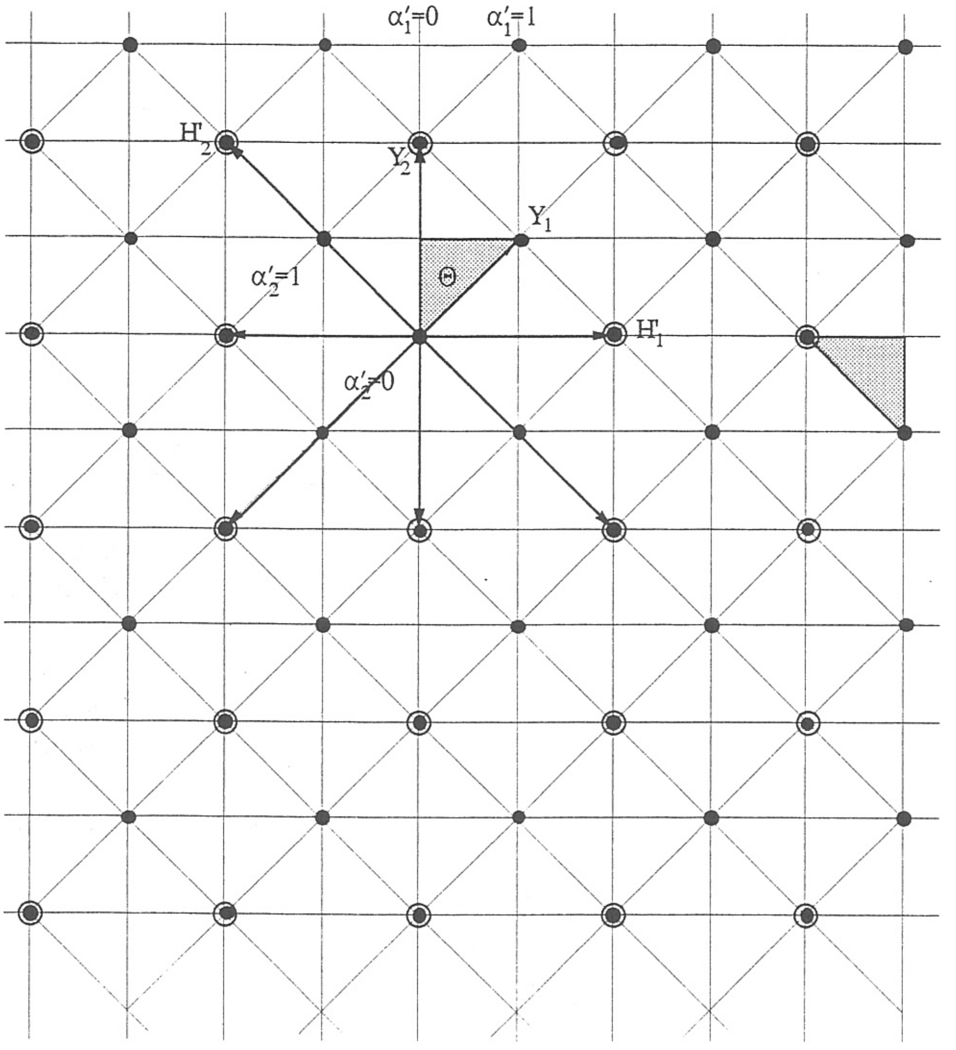
i) The vectors Y_i do not coincide with coroots $H_{\alpha'}$, $\Lambda_Z \neq \Lambda_*$,

ii) $\Lambda_Z = ZY_1 + ZY_2$, coincides with the set of vertices of all cells,

iii) Θ = (fundamental) cell of the diagram,

iv) $\Lambda_* = ZH'_1 + ZH'_2$.

B_2



i) Only Y_2 coincides with some coroot $H_{\alpha'}$, $\Lambda_* \subset \Lambda_Z$,

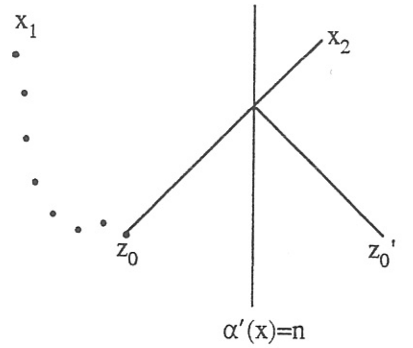
ii) $\Lambda_Z = ZY_1 + ZY_2$, do not coincides with the set of vertices of all cells,

iii) Θ = (fundamental) cell of the diagram,

iv) $\Lambda_* = ZH'_1 + ZH'_2$.

Proposition-2 *The group Γ operates simply transititvely on the set of all cells.*

Transitivity: Let x_1, x_2 belong to two different cells P_1, P_2 respectively and take z_0 in



the orbit Γx_1 of x_1, nearest to x_2. z_0 cannot be separated by some plane (α,n) of the diagram, since reflection of z_0 on that plane would produce a point closer to x_2. Thus, the point $z_0 \in \Gamma x_1$ belongs to the same cell with x_2 $\Rightarrow$ $z_0=f(x_1) \in P_2$, and this implies $f(P_1)=P_2$.

Simple transitivity: Take $X = \sum_{1\le i\le d} t_i Y_i$, with $t_i > 0$ sufficiently small, so that

$$\alpha(X) = \sum_{1\le i\le d} n_i t_i < 1,$$

for every positive root $\alpha \in \Delta^+$. Then X is contained in some cell P_0 with $cl(P_0) \ni 0$. Assume now that Γ does not act simply transitively on the set of cells. Then, there must be different $f_1, f_2 \in \Gamma$, with $f_2(P_0)=f_1(P_0)$ $\Rightarrow$ $f_2^{-1}\circ f_1(P_0)=P_0$. The map $f_2^{-1}\circ f_1$ has the form $f_2^{-1}\circ f_1 = (A,v) \in \Gamma = W \times \Lambda_*$, and $v=f_2^{-1}\circ f_1(0) \in cl(P_0)$. Hence, $v = \sum_{1\le i\le d} n_i Y_i \in cl(P_0)$ and $\alpha_i(v)=2n_i \ge 0$. If some $n_i \ge 1$, then we have a contradiction, since the cell P_0 is characterized by $\alpha_i(v) \le 1$, for all $v \in cl(P_0)$. Thus, $n_i=0$ for all i $\Rightarrow$ v=0 and consequently $f_2^{-1}\circ f_1 \in W$, with $f_2^{-1}\circ f_1(P_0)=P_0$. By the simple transitivity of W' on the Weyl chambers, we have $f_2^{-1}\circ f_1 = Id$.

q.e.d.

Proposition-3 *The group Γ is generated by the reflexions on the walls of an arbitrary cell P of the diagram.*

In fact, let Γ' be the subgroup of Γ, generated by the reflexions on the walls of some cell P_2. Take another cell P_1, $x_1 \in P_1$, $x_2 \in P_2$, and $z \in \Gamma' x_1$ such that $|z-x_2|$ is minimum. Thinking as in the preceding proposition, we find $z \in P_2$, and some $f \in \Gamma'$ with $f(P_1)=P_2$ $\Rightarrow$ $\Gamma' \supseteq \Gamma$.

q.e.d.

Fixing a root system $\Pi=\{\alpha_1, \dots, \alpha_d\}$, the cell P_0, we defined above, is characterized by the equation

$$cl(P_0) = \{X \in \mathfrak{a} \mid X=\sum_{1\le i\le d} t_i Y_i \text{ and } \sum_{1\le i\le d} n_i t_i \le 1,\ t_i \ge 0\},$$

where $\alpha = n_1\alpha_1 + \dots + n_d\alpha_d$, is the maximal root of Π.

P_0 is called **fundamental** and is contained in a fundamental Weyl chamber, with respect to Π. Recall that $n_i>0$, for all i, and every other positive root α can be written $\alpha=\sum_{1\le i\le d} m_i \alpha_i$, with $m_i \le n_i$, for all i.

Exercise-10 Show that P_0 is indeed a cell of the diagram.

Exercise-11 Show that the reflexions S, on the walls of some cell P are conjugate to the reflexions S' on the walls of the cell P', through an $f \in \Gamma$, with $fP=P'$ i.e. $S' = f\circ S\circ f^{-1}$.

Dieserwegen wird man einst (natürlich nicht, so lange ich lebe) erkennen, daß die Behandlung des selben Gegenstandes von irgend einem früheren Philosophen, gegen die meinige gehalten, flach erscheint. Daher hat die Menschheit Manches, was sie nie vergessen wird, von mir gelernt, und werden meine Schriften nicht untergehn.

Schopenhauer, Parerga ... I, p. 150

45 . Inner involutions of simple compact Lie algebras

We are interested in the classification, with respect to conjugation in Int(g), of inner involutive automorphisms of a simple compact (real) Lie algebra g. We use the diagram and the latices studied in the preceding §.

Theorem-1 *Let g be a compact semi simple Lie algebra. Then every inner automorphism $f\in Int(g)$ is conjugate in Int(g) to some automorphism of the form $exp(adH_0)$ of g, with $H_0\in cl(P_0)$, where P_0 is a fundamental cell of the diagram of g.*

Every $f\in Int(g)$ can be written f=exp(adX). Let a be a maximal abelian subalgebra of g. Then, by Pro-3 §42, there is a $Y\in g$, such that X=exp(adY)H, with $H\in a$. For $A=exp(adY)\in Int(g)$ and X=AH we have, $f=exp(adAH) = exp(A(adH)A^{-1}) = Aexp(adH)A^{-1}$. By the simple transitivity of Γ on the set of cells, we conclude that $H = SH_0+v$, where $(S,v)\in W'\times_\sigma\Lambda_*$ and $H_0\in cl(P_0)$ (a fundamental cell). We have then

$$exp(adH) = exp(ad(SH_0+v)) = exp(adSH_0)exp(adv),$$

since v, $SH_0\in a$ and consequently $[SH_0, v]=0$. Since $v\in\Lambda_*$, we have adv=I, hence

$$exp(adH) = exp(ad(SH_0)) = exp(Sad(H_0)S) = S\circ exp(adH)\circ S^{-1}.$$

Thus,

$$f = (A\circ S)\circ(expadH_0)\circ(A\circ S)^{-1}. \qquad \text{q.e.d.}$$

We restrict ourselves to the simple compact Lie algebras g and consider the automor phisms of order 2 of the form f=exp(adX), with $X\in cl(P_0)$. P_0 is defined by a fundamental system of roots $\Pi=\{\alpha_1, ..., \alpha_d\}$ and the maximal root α_0

$$\alpha_0 = n_1\alpha_1 + ... + n_d\alpha_d,$$

$$cl(P_0) = \{X\in a|\ \alpha_i(X)\geq 0, \text{ for } i=1, ..., d \text{ and } \alpha_0(X)\leq 1\}. \qquad (1)$$

According to Pro-5 §42, $X\in P_0$ will define an involutive automorphism exp(adX), if and only if,

$$2X\in\Lambda_Z. \qquad (2)$$

If we write in the basis of $\{Y_1, ..., Y_d\}$

$$X=t_1Y_1+ ... +t_dY_d, \text{ with } t_i\geq 0 \text{ and}$$

$$\alpha_0(X) = t_1n_1+...+t_dn_d\leq 1,$$

we see that

$$t_i\leq 1/n_i .$$

Thus, if for some i the corresponding $n_i>2$, then

$$2t_i\leq 2/n_i<1 \quad \text{and} \quad 2X = \sum 2t_iY_i\notin\Lambda_Z.$$

We proved: The only $X\in a$, satisfying (2) are these for which

$t_i=0$, when the corresponding (in the max. root α_0) $n_i>2$. (3)

Besides $2t_i$ must be an integer ≥ 0, thus if $t_i \neq 0$, then $t_i \geq 1/2$. We conclude that there can be no more than two t_i, different from 0, since in the contrary case we'll have $\sum t_i n_i > 1$. Thus X can only have one of the following forms:

$X = Y_i/2$, and the corresponding $n_i = 1$, (4)

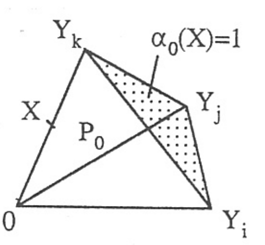
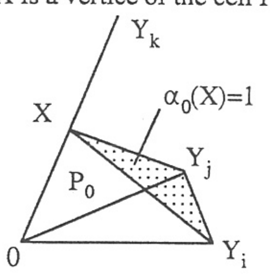
$X = Y_i/2$, and the corresponding $n_i = 2$, (5)

$X = (Y_i+Y_j)/2$, and the corresponding $n_i = n_j = 1$. (6)

Case (6) gives an involutive automorphism f=expadX which is conjugate to an automorphism f'=expadX', with X' as in (4). In fact, the translated cell $P=P_0-Y_i$ is a cell of the diagram and $0 \in cl(P)$. Hence there is some element of the Weyl group $S \in W'$, such that $S(P_0)=P_0-Y_i$. Then the isometry

$$Y \rightarrow S^{-1}(Y-Y_i) \qquad (*)$$

leaves P_0 invariant and maps Y_i to 0. At the same time Y_j is mapped onto some other $Y_k \in \Lambda_Z$ and, by linearity, $X=(Y_i+Y_j)/2$ is mapped onto $Y_k/2$. Y_k has again corresponding $n_k=1$ (coeficient in the maximal root), since it is the image of $Y_k \in \Lambda_Z$ under the isometry (*) which preserves Λ_Z and is contained in the plane $\alpha_0(X)=1$ (all vertices of P_0 different from 0 are contained in that plane). Thus, up to conjugation, the images for the different positions of X in cl(P) are the following two :

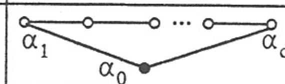
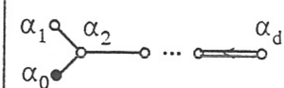
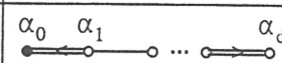
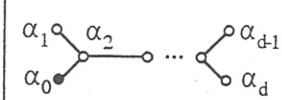
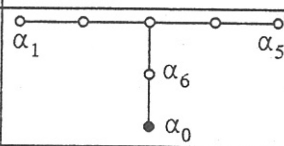
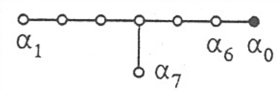
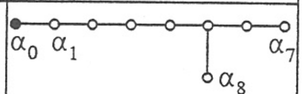
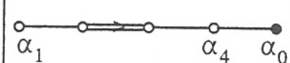
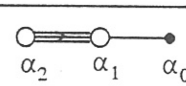
Case 1. $X = Y_k/2$, with corresponding $n_k=1$. (Y_k is a vertice of the cell P_0)	Case 2. $X = Y_k/2$, with corresponding $n_k=2$. (X is a vertice of the cell P_0)
	

The corresponding automorphisms f=expadX, for the different vertices Y_k of P_0 (falling into the two cases above) may be conjugate to each other. The investigation of this question, for simple Lie algebras, proceeds in a case by case examination, using the notion of **extended Dynkin diagram**, which results from the usual Dynkin diagram Π, by adding the maximal root α_0. We denote the corresponding extended root system by

$$\Pi_0= \{\alpha_0, \alpha_1, ..., \alpha_d\}.$$

The next table displays the extended Dynkin diagrams of the various simple Lie algebras. The calculation of the maximal root, with respect to some fundamental system, was made using the information in §§27-35.

Maximal roots and extended Dynkin diagrams

Lie Al	Positive roots	Maximal root	Extended Dynkin diagr.
A_d	$\alpha_i+\dots+\alpha_j$	$\alpha_1+\dots+\alpha_d$	
B_d	$\alpha_i+\dots+\alpha_d$ $\alpha_i+\dots+\alpha_{j-1}$ $(\alpha_i+..+\alpha_d)+(\alpha_j+..+\alpha_d)$	$\alpha_1+2\alpha_2+\dots+2\alpha_d$	
C_d	$\alpha_i+\dots+\alpha_{j-1}$ $(\alpha_i+..+\alpha_{d-1})+(\alpha_j+..+\alpha_d)$	$2\alpha_1+\dots+2\alpha_{d-1}+\alpha_d$	
D_d	$\alpha_i+\dots+\alpha_{j-1}$ $(\alpha_i+..+\alpha_{d-1})+(\alpha_j+..+\alpha_{d-2})+\alpha_d$	$\alpha_1+2\alpha_2+\dots+2\alpha_{d-2}$ $+\alpha_{d-1}+\alpha_d$	
E_6	See table on p. 33-5	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$	
E_7	See table on p. 33-5	$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5$ $+2\alpha_6+2\alpha_7$	
E_8	See table on p. 33-5	$2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+$ $6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8$	
F_4	See table on p. 35-2	$2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4$	
G_2	See picture on p. 32-3	$2\alpha_1+3\alpha_2$	

Theorem-2 *Every inner involution of a compact simple Lie algebra* $\mathfrak{g}$ *is conjugate to one of the form*

$$\theta_k = \exp(\mathrm{ad}(Y_k/2)),$$

where Y_k *satisfies one of the following conditions :*

i) $n_k=1$ *(corresponding coefficient in* α_0*) and then* Y_k *is a vertex of* P_0.

ii) $n_k=2$ *and then* $Y_k/2$ *is a vertex of* P_0.

Moreover the involutions θ_i *and* θ_j *are conjugate, if and only if either*

a) $n_i=n_j=2$ *and there is an isometry* f *of* Π, *with* $f(\alpha_i)=\alpha_j$, *or*

b) $n_i=n_j=1$ *and there is an isometry* f *of* Π_0 *with* $f(\alpha_i)=\alpha_j$.

The first two assertions are already proved. As to the last assertions we proceed as follows :

I) First we remark that every isometry f of the root system, which leaves P_0 invariant ($f(P_0) \subset P_0$), defines an isometry of Π_0 (extended root system). f is considered here as an isometry of $\mathfrak{a}$ (maximal abelian etc.) and to each root α, corresponds the root $\alpha' = \alpha \circ f$. $f(P_0) \subset P_0$ means that the walls of P_0 are mapped to similar walls. Thus, we have an isometry of Π_0. The inverse is also obvious.

II) θ_k, θ_j are conjugate in Aut(g) if and only if, $Y_k = f(Y_j)$ with f as in I). In fact, in that case we have

$$\theta_k = \exp(\mathrm{ad}(f(Y_j/2))) = f \circ \exp(\mathrm{ad}(Y_j/2)) \circ f^{-1} = f \circ \theta_j \circ f^{-1}.$$

Inversely, if $\theta_k = f \circ \theta_j \circ f^{-1}$, then

$$Y_k/2 = f(Y_j/2) + v, \quad \text{with } v \in \Lambda_Z . \qquad (*)$$

f is an automorphism, it preserves the roots and the diagram and maps the cell P_0 onto some other cell $P \ni 0$. The translate $P' = P + v$ is also a cell of the diagram, and there are two possibilities:

1) $Y_j/2$ is a vertex of P_0 ($n_j = 2$). Then $Y_k/2$ is a vertex of P_0 ($n_k = 2$) too and $Y_k/2 \in P'$. By the transitivity of Γ on the cells, there is some $\psi \in \Gamma$ with $\psi P' = P_0$ and $\psi(Y_k/2) = Y_k/2$ (Y_k belongs to both P' and P_0). In that case

$$F : X \to \psi(f(X) + v)$$

is an isometry on the roots, leaves P_0 invariant and maps $Y_j/2$ onto $Y_k/2$. Since all the non-zero vertices of P_0 lie on the plane $\alpha_0(X) = 1$, we have $\alpha_0(Y_k/2) = n_k/2 = 1$, hence $n_k = 2$.

2) Y_j is a vertex of P_0 ($n_j = 1$). Then Y_k is again a vertex of P_0 ($n_k = 1$) and $Y_k \in P'$. For the same reason as before, there is a $\psi \in \Gamma$ with $\psi P' = P$ and $\psi(Y_k) = Y_k$ and the map

$$F : X \to \psi(f(X) + v)$$

is an isometry on the roots, leaves P_0 invariant and maps Y_j onto Y_k. That $n_k = 1$ is seen by the same argument as in 1).

3) In the case 1) F preserves the plane $\alpha_0(X) = 1$. To see this note that F is not, in gen eral, linear. In fact ψ will be of the form $\psi = (\psi_0, v_0) \in W' \times_\sigma \Lambda_*$, hence

$$F(X) = \psi(f(X) + v) = \psi_0(f(X) + v) + v_0 ,$$

and since $F(Y_j/2) = Y_k/2$ we'll have (using (*))

$$F(Y_j/2) = Y_k/2 = \psi_0(f(Y_j/2) + v) + v_0 = \psi_0(Y_k/2) + v_0 \quad \Rightarrow$$

$$F(X) = \psi_0(f(X - Y_j/2)) + Y_k/2 .$$

Suppose now that $n_j = n_k = 2$ and $\alpha_0 \circ F = \alpha_i$, with $i \neq 0$, j. Then

$$\alpha_0 \circ F(X) = \alpha_i(X) = \alpha_0(\psi_0(f(X - Y_j/2))) + \alpha_0(Y_k/2),$$

which for $X = Y_j/2$ gives the contradiction $0 = 1$. q.e.d.

Lesen ist ein bloßes Surrogat des eigenen Denkens. Man läßt dabei seine Gedanken von einem Andern am Gängelbande führen. Zudem taugen viele Bücher bloß, zu zeigen, wie viel Irrwege es giebt und wie arg man sich verlaufen könnte, wenn man von ihnen sich verleiten ließe. Den aber der Genius leitet, d.h. der selbst denkt, freiwillig denkt, richtig denkt, - der hat die Boussole, den rechten Weg zu finden. - Lesen soll man also nur dann, wann die Quelle der eigenen Gedanken stockt; was auch beim besten Kopfe oft genug der Fall seyn wird. Hingegen die eigenen, urkräftigen Gedanken verscheuchen, um ein Buch zur Hand zu nehmen, ist Sünde wider den heiligen Geist.

Schopenhauer, Parerga ... II, p. 539

46. Simple real Lie algebras of inner type

These are the real Lie algebras of the type

$$g = k \oplus ip,$$

where k, p are the ±1-eigenspaces of an <u>inner</u> involution $\theta : u \to u$ of a simple compact Lie algebra u. The involutions of this type were classified in the preceding §, here we classify the characteristic subalgebras k, in dependence from u and θ.

We start with a simple compact Lie algebra u and take the complexification $g_C = u \oplus iu$. If a is a maximal abelian subalgebra and $h = a_C$ the corresponding Cartan subalgebra of g_C, then to the roots $\alpha' \in \Delta'$ of u, with respect to a, correspond the roots $\alpha = 2\pi i\alpha' \in h^*$ of g_C, with respect to h. The root vectors can be chosen so as to satisfy (see §42) :

$$u = ih_0 \oplus_{\alpha\in\Delta^+} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus_{\alpha\in\Delta^+} \mathbf{R}i(X_\alpha + X_{-\alpha}), \text{ and } ih_0 \equiv a. \qquad (0)$$

We fix a fundamental system of roots $\Pi = \{\alpha_1, \dots, \alpha_d\}$ and the corresponding extended system $\Pi = \Pi \cup \{\alpha_0\}$, where α_0 is the maximal root with respect to Π,

$$\alpha_0 = n_1\alpha_1 + \dots + n_d\alpha_d, \text{ with all } n_i > 0. \qquad (1)$$

The central latice is

$$\Lambda_Z = \mathbf{Z}Y_1 + \dots + \mathbf{Z}Y_d, \text{ where } \alpha_i(Y_j) = \delta_{ij}, \qquad (2)$$

and, by the results in §45, every inner automorphism θ of u is conjugate to one of the form

$$\theta = \mathrm{expad}(Y_k/2). \qquad (3)$$

We denote the extension of θ on g_C by the same letter and we notice the (induced) direct sum splitting $g_C = k_C + p_C$, where k_C, p_C are respectively the ±1-eigenspaces of θ. k_C is a complex subalgebra of g_C and obviously

$$h = a_C \subset k_C. \qquad (4)$$

For the different roots $\alpha = 2\pi i\alpha'$ of g_C we have:

I) $\theta(X_\alpha) = \exp(i\pi\alpha'(Y_k))X_\alpha = X_\alpha$,
when α does not contain α_k or contains it with coefficient=2. (5)

II) $\theta(X_\alpha) = \exp(i\pi\alpha'(Y_k))X_\alpha = -X_\alpha$,
when α contains α_k with coefficient=1. (6)

Remark All the Y_k appearing in (3) correspond to $n_k \le 2$ (in the maximal root). Hence all other roots containing α_k will contain it with coefficient ≤ 2 (The-5, §26). Thus (5) and (6) exhaust all possible cases.

Obviously k_C and p_C can be written as direct sums:

$$k_C = h \oplus_{\alpha \in \text{case I}} CX_\alpha ,$$
$$p_C = \oplus_{\alpha \in \text{case II}} CX_\alpha .$$

Applying The-2 of the preceding paragraph, we distinguish the two possibilities :

$n_k=1$ and consequently Y_k is a vertex of the cell P_0:

In this case

$$k_C = h \oplus_{\alpha \in \text{case I}} CX_\alpha = h \oplus_{\alpha \in \Delta_k} CX_\alpha ,$$

where Δ_k is the subset of roots, which do not contain α_k. Then the (d-1) linear equations

$$\alpha'_i(X) = 0, \; i=1, \dots , d \text{ and } i \neq k,$$

define a 1-dimensional subspace t of a, which is in the center of k. Then, taking the orthogonal complement $t^\perp$ of t in a, we see that

$$[kk] = it^\perp \oplus_{\alpha \in \Delta_k} R(X_\alpha - X_{-\alpha}) \oplus_{\alpha \in \Delta_k} Ri(X_\alpha + X_{-\alpha}),$$
$$k = t \oplus [kk], \tag{7}$$

where [kk] is a semi simple (applying the lemma of §27 to $[k_C k_C]$) compact Lie algebra. The roots of this Lie algebra may be identified with the set Δ_k and a corresponding fundamental system of [kk] with

$$\{\alpha_1, \dots , \alpha_{k-1}, \alpha_{k+1}, \dots , \alpha_d\}. \tag{8}$$

$n_k=2$ and consequently $Y_k/2$ is a vertex of the cell P_0:

In this case

$$k_C = h \oplus_{\alpha \in \text{case I}} CX_\alpha = h \oplus_{\alpha \in \Delta_k} CX_\alpha \oplus_{\alpha \in \Delta^2_k} CX_\alpha ,$$

where Δ_k is the subset of roots, which do not contain α_k and Δ^2_k is the subset of roots, which contain α_k with coefficient =2 . Δ^2_k contains the maximal root α_0 and, applying the lemma of §27, we see that k_C is semi simple compact and its root system may be identified with $\Delta_k \cup \Delta^2_k$. Since every root of g_C results from the maximal one, by substracting successively simple roots (Ex-10 §26), we see that a fundamental system of roots for k_C is given by

$$\{\alpha_0, \alpha_1, \dots , \alpha_{k-1}, \alpha_{k+1}, \dots , \alpha_d\}. \tag{9}$$

In the rest of this section we point out the different cases of compact Lie subalgebras k, arising as fixed point subalgebras of some involution θ, as in (3). This is done by a case by case examination of the 9 types of compact Lie algebras.

A_d. The maximal root (table in §44) shows that k can take all the values 1, 2, ... , d and in all these cases $n_k=1$.

By the symmetry of the Dynkin diagram, we see that only the values

$$k = 1, 2, \dots, [(d-1)/2]+1,$$

give non-conjugate involutive automorphisms ([x] denoting the integer part of x). In all these cases, a fundamental system has the form (8) and consequently the corresponding characteristic Lie algebra k has the form :

$$k = t \oplus [kk] = t \oplus A_i \oplus A_{d-i-1}\,, \text{ for } i = 1, \ldots, [(d-1)/2]. \tag{10}$$

t is the 1-dimensional center and all these k are non-isomorphic to each other.

B_d. Here the maximal root (table in §44) has the form

$$\alpha_1+2\alpha_2+ \ldots + 2\alpha_d\,.$$

There is only one case $n_1=1$, which gives the characteristic subalgebra

$$k = t \oplus [kk] = t \oplus B_{d-1}\,. \tag{11}$$

The other cases are d-1 in number and have all

$$n_k=2, \text{ for } k = 2, \ldots, d\,.$$

The type of the corresponding Lie algebra results from the fundamental system (9), which by inspecting the extended fundamental system of B_d gives :

$$k = D_k \oplus B_{d-k}\,, \text{ for } k = 2, \ldots, d. \tag{12}$$

C_d. The maximal root here has the form

$$2\alpha_1+2\alpha_2+ \ldots + 2\alpha_{d-1} + \alpha_d\,.$$

There is again one case $n_d=1$, with corresponding characteristic subalgebra

$$k = t \oplus [kk] = t \oplus A_{d-1}\,. \tag{13}$$

The other cases are d-1 in number and have all

$$n_k=2, \text{ for } k = 1, \ldots, d-1\,.$$

The type of the corresponding Lie algebra results from the fundamental system (9), which by inspecting the extended fundamental system of B_d gives :

$$k = C_k \oplus C_{d-k}\,, \text{ for } k = 1, \ldots, d-1. \tag{14}$$

By the symmetry of the extended diagram, only the first $[(d-1)/2]+1$ are non-isomorphic to each other.

D_d. The maximal root here has the form

$$\alpha_1+2\alpha_2+ \ldots + 2\alpha_{d-2} + \alpha_{d-1} + \alpha_d\,.$$

Here we have three cases with $n_k=1$, for $k = 1$, d-1 and d. For k=1 we have

$$k = t \oplus [kk] = t \oplus D_{d-1}\,. \tag{15}$$

The other two cases, for k= d-1 and k=d, give conjugate involutions, the corresponding characteristic subalgebra being

$$k = t \oplus [kk] = t \oplus A_{d-1}\,. \tag{16}$$

The other cases are d-3 in number and have all

$$n_k=2, \text{ for } k = 2, \ldots, d-2\,.$$

By the symmetry of the extended diagram, only the first $[(d-4)/2]+1=[d/2]-1$ are non-isomorphic to each other. The type of the corresponding Lie algebra results from the fundamental system (9), which by inspecting the extended fundamental system of D_d gives :

$$k = D_k \oplus D_{d-k}\,, \text{ for } k = 2, \ldots, [d/2]-1. \tag{17}$$

E_6. The maximal root has the form

$$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+ \alpha_5+ 2\alpha_6\,.$$

Here we have two cases with n_k=1, for k = 1 and 5 , which give conjugate involutions and corresponding characteristic subalgebra

$$k = t \oplus [kk] = t \oplus D_5 . \qquad (18)$$

We have also three cases with n_k=2, for k = 2, 4 and 6, which by the symmetry of the extended Dynkin diagram, give conjugate involutions. The corresponding characteristic subalgebra is given by

$$k = A_1 \oplus A_5 . \qquad (19)$$

E_7 . The maximal root has the form

$$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7 .$$

n_k=1, and corresponding characteristic subalgebra

$$k = t \oplus [kk] = t \oplus E_6 . \qquad (20)$$

n_k=2, for k = 2 and 6, which by the symmetry of the extended Dynkin diagram, give conjugate involutions. The corresponding characteristic subalgebra being given by

$$k = A_1 \oplus D_6 . \qquad (21)$$

n_7=2, and characteristic subalgebra being given by

$$k = A_7 . \qquad (22)$$

E_8 . The maximal root has the form

$$2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8 .$$

n_1=2, the characteristic subalgebra being

$$k = A_1 \oplus E_7 . \qquad (23)$$

n_7=2, and characteristic subalgebra being given by

$$k = D_8 . \qquad (24)$$

F_4 . The maximal root has the form

$$2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4 .$$

n_1=2, with characteristic subalgebra

$$k = B_4 . \qquad (25)$$

n_4=2, with characteristic subalgebra

$$k = A_1 \oplus C_3 . \qquad (26)$$

G_2 . The maximal root has the form

$$2\alpha_1+3\alpha_2 .$$

n_1=2, with characteristic subalgebra

$$k = A_1 \oplus A_1 . \qquad (26)$$

Theorem *Formulas (10) - (26) give the characteristic subalgebras of all the real simple Lie algebras of inner type. The real Lie aglebra has the Cartan decomposition*

$$g = k \oplus ip,$$

where k, p are respectively the ±1-eigenspaces of an ***inner*** *involution* $\theta : u \to u$ *of a simple compact Lie algebra u (=k⊕ p).*

If people contradict themselves, can I
 Help contradicting them, and every body,
Even my veracious self? -But that's a lie;
 I never did so, never will-how should I?
He who doubts all things, nothing can deny;
 Truth's fountains may be clear-her streams are muddy,
And cut through such canals of contradiction,
That she must often navigate o'er fiction.

Byron, Don Juan, Canto XV, 88

47 . Canonical representation of automorphisms

Since the automorphisms of the compact simple Lie algebras B_d, C_d, E_8, E_7, F_4, G_2 are all inner (§43), the real subalgebras of the previous theorem give all real forms of the corresponding complexifications. It remains to study the real forms of the remaining types of compact Lie algebras A_d, for d>1, D_4, D_d, for d>4 and E_6, which have also "outer" automorphisms. In these cases we are interested in "outer" involutive automorphisms and their characteristic subalgebras. In order to classify these automorphisms, we represent them in a "canonical form". Here we explain how this is done.

Lemma-1 *Let u be a simple compact Lie algebra and $\theta : u \to u$ be an involutive automorphism. Let $u=k\oplus p$ be the decomposition in ±1-eigenspaces of θ . Then k is a maximal Lie subalgebra of u.*

For the orthogonal decomposition $u=k\oplus p$ we know (§40) that

$$[kk] \subset k,\ [kp] \subset p,\ [pp] \subset k. \tag{1}$$

We show that

$$[pp] = k. \tag{2}$$

In fact, suppose $[pp]$ is a proper subspace of k and take the orthogonal complement $a=[pp]^{\perp}$ $\neq \{0\}$ in k. We arive at a contradiction as follows.

$$[ap] \subset p \text{ and } <[ap], p> = - <a,[pp]> = 0 \quad \Rightarrow$$
$$[ap] = 0.$$

Then $[a,[pp]] = 0$, since $[a,[pp]] = -[p,[pa]]-[p,[ap]]$.
Besides $<[aa],[pp]> = 0$, since $<[aa],[pp]> = - <a, [a,[pp]]> = 0$. Thus a is an ideal of u, which is impossible (u simple).

Suppose now there were some subalgebra k' kontaining k as a proper subalgebra, and $k'\neq k$. Then take the orthogonal complements

$$k' = k\oplus p'$$
$$u = k\oplus p' \oplus p'',\ \ p = p' \oplus p''.$$

For these subspaces we have

$$[kp']\subset p',\ [p', p']\subset k,\ [kp'']\subset p'', [p', p'']=\{0\} \text{ and } [p'', p'']\subset k.$$

It follows that $k'=k\oplus p'$ and p'' satisfy (1) and consequently also (2). Thus we should have

$$[p'', p''] = k = k\oplus p',$$

which is impossible, when $p'\neq\{0\}$. q.e.d.

Lemma-2 *Let u be a simple compact Lie algebra and $\theta : u \to u$ an "outer" automorphism (i.e. $\theta \notin Int(u)$). Let $u=k\oplus p$ be the decomposition in ±1-eigenspaces. Then k is semi simple.*

Since k is compact, we have (§42) $k=c\oplus[kk]$, where c is the center of k and $[kk]$ is semi simple. The center c is contained in every maximal abelian $m\subset k$, which extends uniquely to a maximal abelian subalgebra a of u (The-5, §41). a is not contained in k, since $\theta\notin \mathrm{Int}(u)$ (The-3, §43). Hence a and k are both contained in the centralizer of c in u:

$$z(c) = \{X\in u \mid [Xc]=0\},$$

which, consequently contains k as a proper subalgebra. By the previous lemma, we must have $z(c) = u$, and c will be abelian ideal of u, which is impossible. q.e.d.

Lemma-3 *Let u be a compact semi simple Lie algebra and $\theta{:}u\rightarrow u$ an involution, with corresponding eigenspace decomposition $u=k\oplus p$. Let $m\subset k$ be a maximal abelian subalgebra of k. Then there is an abelian subalgebra $m\subset a$ invariant under θ, $X\in m$ a regular element of u and the Weyl chamber containing X is invariant under θ.*

Given $m\subset k$, the existence and uniqueness of a is proved in §41, The-5. The invariance of a follows from $m\subset a$, which implies $\theta(m)=m\subset\theta(a)$ and by uniqueness of a, $\theta(a)=a$. As to $X\in m$, consider a generator X of m (§42). Then every maximal abelian subalgebra a with $X\in a$ contains also m (proof the same as that of Pro-2, §42). If X were not regular element of u, there would be another maximal abelian subalgebra $a'\ni X$ and $a'\neq a$. But then, by uniqueness of a, we should have $a'=a$, a contradiction.

The last statement of the lemma is a consequence of the fact, that θ leaves invariant the diagram D_0 (containing the planes through 0) hence the connected component containing X is mapped onto itself. q.e.d.

Remark From the last lemma, we see that the Weyl chamber containing X, defines a fundamental system of roots Π, mapped by θ onto itself. Thus, θ defines an isometry of Π and since it is an automorphism of u_C, for every root of the later, we'll have

$$[H,X_\alpha] = \alpha(H)X_\alpha \quad\Rightarrow$$

$$[\theta(H),\theta(X)] = \alpha(\theta^{-1}(\theta(H)))\,\theta(X).$$

Thus, as usual for automorphisms,

$$\alpha' = \alpha\circ\theta^{-1}$$

is again a root of u_C and

$$\theta(X) = c_\alpha X_{\alpha'}, \text{ where } c_\alpha c_{-\alpha}=1 \text{ and } c_\alpha c_\beta N_{\alpha'\beta'} = N_{\alpha\beta}c_{\alpha+\beta}\,. \qquad (3)$$

On the other side (see §39) an isometry θ_0 of Π is canonically extended to an automorphism of u_C, denoted by the same letter. This is done by defining the automorphism on the root vectors $\{X|\alpha_i\in\Pi\}$, which generate the Lie algbra, by taking $c_{\alpha_i}=1$, for all $\alpha_i\in\Pi$:

$$\theta_0(X_{\alpha_i}) = X_{\alpha_i'},$$
$$\theta_0(X_{-\alpha_i}) = X_{-\alpha_i'}, \text{ for all } \alpha_i\in\Pi. \qquad (4)$$

The composite automorphism

$$\Theta = \theta_0^{-1}\circ\theta$$

is of the form

$$\Theta = \exp(\mathrm{ad}(H)), \text{ with } H\in a,$$

and leaves (Lemma-2 §43) the subalgebra a pointwise fixed. Thus, we can write

$$\theta = \theta_0 \circ \exp(\mathrm{ad}(H)), \text{ with } H \in \mathfrak{a}.$$

When θ is an involution, θ_0 and Θ are involutions too. This is obvious for θ_0 and $\Theta^2=1$ follows from $\Theta^2 = \theta_0^{-1} \circ \theta \circ \theta_0^{-1} \circ \theta$ and the fact that θ, θ_0 commute, since they commute on the vectors $X_{\pm\alpha_i}$, for all $\alpha_i \in \Pi$, and coincide on $\mathfrak{a}_C$.

The following theorem shows that there is a conjugate of θ (with respect to Int(u)) which can be written in the form $\theta_0 \circ \exp(\mathrm{ad}(H))$ with $H \in \mathfrak{m}$, instead of $H \in \mathfrak{a}$.

Theorem-1 *Let $\mathfrak{u}$ be a simple compact Lie algebra and $\theta:\mathfrak{u} \to \mathfrak{u}$ an involutive automorphism. Then there is a maximal abelian subalgebra $\mathfrak{a}$ which is invariant by θ, a fundamental system Π of roots, with respect to $\mathfrak{a}$ in which θ induces an isometry θ_0 which extends canonicaly to an automorphism of $\mathfrak{u}_C$ (denoted by the same letter). Then, there is a $H \in \mathfrak{a}$ fixed by θ_0, such that $\theta_0 \circ$ expadH is conjugate to θ.*

Let $\mathfrak{k}$ be the +1-eigenspace of θ, $\mathfrak{m} \subset \mathfrak{k}$ a maximal abelian subalgebra, $\mathfrak{a}$ and θ_0 as in the preceding remark. Π is the disjoint union of the two following subsets :

$$\Pi' = \{\alpha_1, \dots, \alpha_s\}, \text{ with the property } \alpha_i' = \alpha_i,$$

$$\Pi'' = \{\xi_1, \xi_1', \dots, \xi_r, \xi_r'\}, \text{ where } \xi_r \neq \xi_r',$$

and 2r+s = d is the rank of $\mathfrak{u}$. For the corresponding coefficients c_α in (3) we'll have (Cor §43) :

$$c_\alpha = \exp(2i\pi\rho_\alpha).$$

In particular taking $\alpha = \xi_i \in \Pi''$, the linear system of r equations

$$(\xi_i - \xi_i')(H) = \rho_\alpha,$$

has solutions $H \in \mathfrak{a}$, and for such an H we can define the inner automorphism of $\mathfrak{u}$

$$V = \exp(\mathrm{ad}H).$$

$\theta' = V \circ \theta \circ V^{-1}$ has the following properties :

i) θ' is involutive and leaves $\mathfrak{a}$ invariant,

ii) θ' induces in Π the same isometry as θ ,

iii) Let

$$\theta'(X_\alpha) = \mu_\alpha X_{\alpha'}.$$

For the constants μ_α we have the following relations :

$$|\mu_\alpha| = 1, \quad \overline{\mu}_\alpha = \mu_{-\alpha},$$

and since θ' is involutive

$$\mu_\alpha \mu_{\alpha'} = 1 \quad \mu_{\alpha'} = \mu_{-\alpha}.$$

Thus,

$$\mu_\alpha^2 = 1, \text{ for } \alpha' = \alpha \in \Pi' \quad \Rightarrow$$

$$\theta'(X_{\alpha_i}) = \pm X_{\alpha_i'}, \text{ for } \alpha_i \in \Pi'.$$

For the roots $\alpha = \xi_i \in \Pi''$ we have

$$\theta'(X_{\xi_i}) = V \circ \theta \circ \exp(\mathrm{ad}(-H))(X_{\xi_i}) = V \circ \theta\, (\exp(-2i\pi\xi_i(H))X_{\xi_i}) = V c_\alpha (\exp(-2i\pi\xi_i(H)))X_{\xi_i'}$$

$$= c_\alpha(\exp(2i\pi(\xi_i'(H) - \xi_i(H))))X_{\xi_i} \quad \Rightarrow$$

$$\theta'(X_{\xi_i}) = X_{\xi_i}$$

and since $c_{\xi_1} c_{\xi_1'} = 1$ (involution)

$$\theta'(X_{\xi_1'}) = X_{\xi_1'} \ .$$

These calculations show that θ' and θ are commuting automorphisms, since they commute in $\mathfrak{a}$ and in $X_{\pm\alpha}$, $\alpha \in \Pi$, which generate $\mathfrak{u}_C$. Thus, we have

$$\theta(\mathfrak{k}') \subset \mathfrak{k}', \text{ where } \mathfrak{k}' = V\mathfrak{k} \text{ is the } +1\text{-eigenspace of } \theta'.$$

Then

$$\Theta = \theta_0^{-1} \circ \theta'$$

defines an automorphism of $\mathfrak{k}'$, which leaves both $\mathfrak{m}' = V\mathfrak{m}$ (maximal abelian subalgebra of $\mathfrak{k}'$) and $\mathfrak{a}' \supset \mathfrak{m}'$ pointwise fixed. For this automorphism of $\mathfrak{u}_C$ we have

$$\Theta(X_{\alpha_1}) = c_{\alpha_1} X_{\alpha_1} ,$$

$$\Theta(X_{\xi_1}) = X_{\xi_1} ,$$

$$\Theta(X_{\xi_1'}) = X_{\xi_1'} .$$

We define then $H \in \mathfrak{a}'$ so as to satisfy the equations

$$\xi_i'(H) = \xi_i(H) = 0,$$

$$\exp(\alpha_i(H)) = c_{\alpha_i} .$$

Then Θ and exp(ad$H$) coincide on X_α, $\alpha \in \Pi$, consequently they coincide everywhere. In addition $\xi_i'(H) = \xi_i(H)$ implies (see following Ex.) $H \in \mathfrak{m}'$.

q.e.d.

Exercise-1 Show that $\mathfrak{m} \subset \mathfrak{a}$ above, is characterized by the equations $\xi_i'(H) = \xi_i(H)$, for $i = 1,.., r$.

Remark In the case of "outer" involutions, $\theta : \mathfrak{u} \to \mathfrak{u}$, $\mathfrak{k}$ is semi simple and the same is true for $\mathfrak{k}' = V\mathfrak{k}$. In addition $\Theta = \theta_0^{-1} \circ \theta'$ leaves $\mathfrak{m}'$ pointwise fixed, thus by applying lemma-2 of §43, we get $\Theta = \exp(\text{ad}H)$ for some $H \in \mathfrak{m}'$.

Das Beschneiden der Bäume zu nützen, Taxus usw. Buchsbaum, so werden Gelehrte am Hofe und im Staate behandelt.

Lichtenberg, Sudelbücher p. 513

48 . Real Lie algebras of outer type

We study here real forms resulting from outer automorphisms $\theta: u \to u$ of a simple compact Lie algebra u . Such automorphisms exist only in the cases of compact simple Lie algebras of type A_d, D_d and E_6 . From the preceding paragraph, we know that each involution θ is conjugate in Int(u) to some involution of the form $\theta_0 \circ \exp(\mathrm{ad}H)$, where θ_0 is the canonical extension (to an involutive automorphism) of an involutive isometry of the Dynkin diagram, and H is an element of a maximal abelian subalgebra a of u, which is invariant under θ_0 and in addition $\theta_0(H)=H$.

The classification below, proceeds in two steps. First we classify the Lie algebras resulting from involutions, whose canonical representation has H=0. In this case we proceed by describing the corresponding characteristic Lie algebra

$$k_0 = \{X \in u | \theta_0(X)=X\}.$$

In the second case, exp(adH) is non trivial and can be considered as an element of Int(k_0). By the methods of §46, we bring it to a canonical form. There is here a subtle point, in the identification of $\exp(\mathrm{ad}H) \in \mathrm{Int}(u)$ with an element of Int(k_0) :

k_0 is a subalgebra of u, and u decomposes in a direct sum

$$u = k_0 \oplus p_0,$$
$$p_0 = \{X \in u | \theta_0(X)=-X\},$$
$$k_0 = \{X \in u | \theta_0(X)=X\},$$
$$[k_0 k_0] \subset k_0, \ [k_0 p_0] \subset p_0, \ [p_0 p_0] \subset k_0 . \qquad (1)$$

The elements of Aut(u) of the form $\{\exp(\mathrm{ad}X) | X \in k_0\}$ generate a subgroup of Aut(u), which consists of automorphisms of u, which leave k_0 and p_0 invariant. The connected component K of this subgroup is canonicaly isomorphic with Int(k_0). The isomorphism is defined using the decomposition of such an element exp(adX) :

$$\exp(\mathrm{ad}X) = \exp(\mathrm{ad}X)|k_0 \oplus \exp(\mathrm{ad}X)|p_0 , \qquad (2)$$

the isomorphism being given by

$$j(\exp(\mathrm{ad}X)) = \exp(\mathrm{ad}X)|k_0 .$$

This is "proved" by the following remarks :

i.) K is a closed subgroup of Int(u) and corresponds to the subalgebra k_0. K coincides with the connected component of the unity of the subgroup of automorphisms of u, which commute with θ_0 . Besides k_0 is the Lie algebra of Int(k_0).

ii.) $j : K \to \mathrm{Int}(k_0)$ is a covering of compact groups and a homomorphism whose kernel is a discrete, normal subgroup of K, hence contained in the center of K (Chevalley p. 50).

iii.) The center of K is $\{1\}$, thus j is an isomorphism. To see this, consider the center $c \subset K$

and $z(c) \subset \mathrm{Int}(u)$ the centralizer of c :

$$z(c) = \{g \in \mathrm{Int}(u) | g \circ c \circ g^{-1} = 1\}.$$

$K \subset z(c)$ and $T = \exp(a) \subset z(c)$, since the torus $S = \exp(m)$ (notation of the preceding §) contains c and $T \supset S$. Besides T is not contained in K, since $\theta_0 \notin \mathrm{Int}(u)$. Thus, K is a proper subgroup of z(c), hence $z(c) = \mathrm{Int}(u)$ and c is contained in the center of Int(u). But this is $\{1\}$. q.e.d.

Notice that the reasoning here is that of lemmas-1, -2 of §47 transfered from the Lie algebra to the Lie group level.

Lemma-1 *Let $\theta: u \to u$ be an involutive automorphism of a simple compact Lie algebra u, $u = k \oplus p$, the corresponding eigenspace decomposition, $m \subset k$ a maximal abelian subalgebra of k and $a \supset m$ a maximal abelian subalgebra of u, invariant with respect to θ. Let Δ the set of roots of u and for each root $\alpha \in \Delta$, denote by $\alpha' = \alpha \circ \theta^{-1} \in \Delta$ and by X_α the root vectors Define also k_α :*

$$\theta(X) = k_\alpha X_{\alpha'} .$$

Then $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where

$$\Delta_1 = \{\alpha \in \Delta | \alpha' = \alpha \text{ and } k_\alpha = 1\},$$
$$\Delta_2 = \{\alpha \in \Delta | \alpha' = \alpha \text{ and } k_\alpha = -1\},$$
$$\Delta_3 = \{\alpha \in \Delta | \alpha' \neq \alpha\}.$$

That $\alpha' = \alpha$ implies $k_\alpha = \pm 1$, was proved in the previous §. The rest is obvious.

The same equations defining Δ_i's define also subsets of a fundamental system Π :

$$\Pi_1 = \{\alpha_1, \alpha_2, \dots, \alpha_p\},$$
$$\Pi_2 = \{\beta_1, \beta_2, \dots, \beta_q\},$$
$$\Pi_3 = \{\xi_1, \xi_1', \xi_2, \xi_2', \dots, \xi_r, \xi_r'\}.$$

From the general theory on compact real forms, we know that the corresponding complexifications give a decomposition :

$$u_C = a_C \oplus_{\alpha \in \Delta} CX_\alpha = k_C \oplus p_C , \tag{3}$$

where k_C and p_C are the ± 1-eigenspaces of the extension of θ on u_C. From (3) and the lemma above, we get immediately

$$k_C = m_C \oplus_{\alpha \in \Delta_1} CX_\alpha \oplus_{\xi \in \Delta_3} C(X_\xi + k_\xi X_{\xi'}) ,$$
$$p_C = n_C \oplus_{\beta \in \Delta_2} CX_\beta \oplus_{\xi \in \Delta_3} C(X_\xi - k_\xi X_{\xi'}) . \tag{4}$$

Here $a = m \oplus n$ is the decomposition of a, with respect to $k \oplus p$.

$$m = \{X \in a | \xi(X) = \xi'(X) \text{ for } \xi \in \Delta_3\}. \tag{5}$$

The following is obvious :

Lemma-2 *Let $\underline{\alpha} = \alpha | m$ denote the restriction of a root on m. Then*

$$\{\underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_p, \underline{\beta}_1, \underline{\beta}_2, \dots, \underline{\beta}_q, \underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_r\}$$

are independent linear forms on $m \subset k$ and $p+q+r$ is the dimension of m ($\dim a = p+q+2r$).

Lemma-3 *The subset of roots of k*

$$\{\underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_p, \underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_r\}$$

consists of simple roots.

In fact, suppose $\underline{\omega}$ belongs to this subset and it can be written

$$\underline{\omega} = \underline{\gamma}+\underline{\delta},$$

where $\underline{\gamma}$, $\underline{\delta}$ are positive roots, with respect to an ordering of $\mathfrak{m}^*$ which is compatible with an ordering of $\mathfrak{a}^*$. Such orderings are defined by extending a basis of $\mathfrak{m}$ to a basis of $\mathfrak{a}$. A consequence of such an ordering is that $\underline{\alpha}>0 \Rightarrow \alpha>0$. For the corresponding root vectors of $\mathfrak{k}_C$ we'll have

$$[X_\gamma+\theta X_\gamma , X_\delta+\theta X_\delta] \text{ and } X_\omega+\theta X_\omega \text{ are linear dependend.}$$

By the independence of $X_{\gamma+\delta}$, $X_{\gamma+\delta'}$, ... (when they are non-zero) follows that ω will be equal to some of the $\gamma+\delta$, $\gamma+\delta'$, $\gamma'+\delta'$, ... etc. but this is impossible. q.e.d.

In the special case $\theta=\theta_0$ (in the canonical decomposition of θ) we have $\Delta_2= \emptyset$ (from the definition of the canonical extension) and for $H \in \mathfrak{m}_C$:

$$[HX_\alpha] = \underline{\alpha}(H)X_\alpha ,$$

$$[H, X_\xi+k_\xi X_{\xi'}] = [H, X_\xi+\theta_0 X_\xi] = \xi(H)X_\xi+\theta_0[\theta_0^{-1}H, X]$$

$$= \underline{\xi}(H) +\theta_0(\underline{\xi}'(H)X_\xi)$$

$$= \underline{\xi}(H)(X+k_\xi X_{\xi'}) . \quad (\xi(H)=\xi'(H))$$

Here (apply the lemma of §27),

$$\underline{\Delta} = \{\underline{\alpha} \mid \alpha \in \Delta_1 \cup \Delta_3\}$$

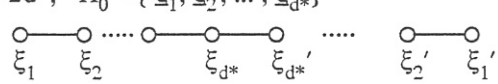
is the set of roots of (the complexifications of) $\mathfrak{k}_0$ with respect to $\mathfrak{m}$.

Lemma-4 *In the case* $\theta=\theta_0$ *(q=0) the set of roots* $\Pi_0 = \{\underline{\alpha}_1, \underline{\alpha}_2, \dots , \underline{\alpha}_p, \underline{\xi}_1, \underline{\xi}_2 , \dots , \underline{\xi}_r\}$ *is a fundamental system of roots for* $\mathfrak{k}_0$.

This is a consequence of lemma-3 and the subsequent remarks.

We turn now in a case by case examination of outer involutions of the form $\theta=\theta_0$.

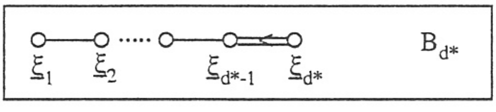
A_d d even, $d = 2d^*$, $\Pi_0 = \{ \underline{\xi}_1, \underline{\xi}_2, \dots , \underline{\xi}_{d^*}\}$



If $t < d^*-1$, $\underline{\xi}_t+n\, \underline{\xi}_{t+1}$ is a root of $(\mathfrak{k}_0)_C$, only for n=1.

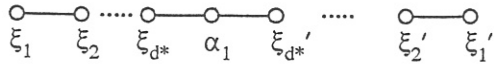
$\underline{\xi}_{d^*-1}+n\, \underline{\xi}_{d^*}$ is a root of $(\mathfrak{k}_0)_C$, only for n=1, n=2.

Thus, the Dynkin diagram for $(\mathfrak{k}_0)_C$ is

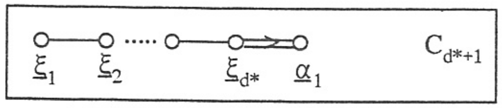


(6)

A_d d odd, $d = 2d^*+1$, $\Pi_0 = \{ \underline{\xi}_1, \underline{\xi}_2, \dots , \underline{\xi}_{d^*} , \underline{\alpha}_1\}$



If $t < d^*$, $\underline{\xi}_t + n\,\underline{\xi}_{t+1}$ is a root of k_C, only for n=1.
$n\,\underline{\xi}_{d*} + \underline{\alpha}_1$ is a root of $(k_0)_C$, only for n=1, n=2.
Thus, the Dynkin diagram for $(k_0)_C$ is

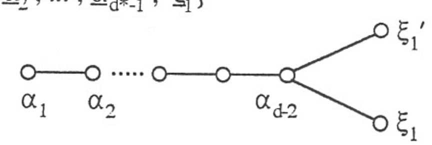


(7)

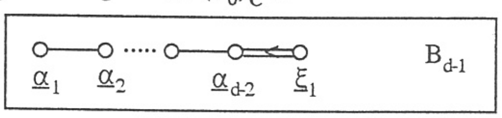
with maximal root

$$2\,\underline{\xi}_1 + 2\,\underline{\xi}_2 + \ldots + 2\,\underline{\xi}_{d*} + \underline{\alpha}_1 \ .$$

D_d $\Pi_0 = \{\, \underline{\alpha}_1, \underline{\alpha}_2, \ldots, \underline{\alpha}_{d*-1}, \underline{\xi}_1 \}$



If $t < d-2$, $\underline{\alpha}_t + n\,\underline{\alpha}_{t+1}$ is a root of k_C, only for n=1.
$\underline{\alpha}_{d-2} + n\,\underline{\xi}_1$ is a root of $(k_0)_C$, only for n=1, n=2. $2\,\underline{\alpha}_{d-2} + \underline{\xi}_1$ is not a root.
Thus, the Dynkin diagram for $(k_0)_C$ is

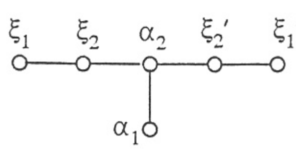


(8)

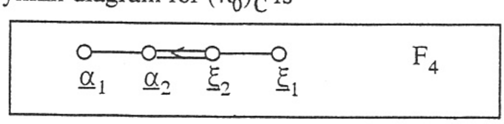
with maximal root

$$\underline{\alpha}_1 + 2\,\underline{\alpha}_2 + \ldots + 2\,\underline{\alpha}_{d-2} + \underline{\xi}_1 \ .$$

E_6 $\Pi_0 = \{\, \underline{\alpha}_1, \underline{\alpha}_2, \underline{\xi}_1, \underline{\xi}_2 \}$



$\underline{\alpha}_1 + \underline{\alpha}_2$, $\underline{\alpha}_2 + \underline{\xi}_2$ and $\underline{\xi}_1 + \underline{\xi}_2$ are roots of $(k_0)_C$.
Thus, the Dynkin diagram for $(k_0)_C$ is



(9)

with maximal root

$$2\,\underline{\xi}_1 + 4\,\underline{\xi}_2 + 3\,\underline{\alpha}_2 + 2\,\underline{\alpha}_1 \ .$$

The computation of Dynkin diagrams has been done using strings of roots. In all these cases $(k_0)_C$ is proved to be simple. This has a technical advantage: The restriction of the Killing form of u on k_0 is adk_0-invariant, hence a multiple of the Killing form of k_0. Thus, we can compute angles and compare lengths into k_0 using the Killing form of u.

We come now to the second step, of classification of involutive automorphisms of the form

$$\theta = \theta_0 \circ \exp(\mathrm{ad}H), \text{ with } \theta_0(H) = H. \qquad (10)$$

From (10) we have $\theta_0 \circ \theta = \theta \circ \theta_0$ and $(\exp \mathrm{ad}H)^2 = 1$. We have also $\exp\mathrm{ad}H(k_0) \subset k_0$, hence $\exp\mathrm{ad}H | k_0 = \exp\mathrm{ad}_{k_0} H$ (ad_{k_0} being the adjoint representation of k_0) is an inner involutive automorphism of k_0. Thus, we can apply to k_0 the results of §§45, 46. In particular, expadH is conjugate in Int(k_0) and consequently (introductory remarks of this §) also in Int(u), to an inner involution ϕ of u, which leaves k_0 invariant and induces on this an involution $\underline{\phi}$, determined by the coefficients of the maximal root of k_0

$$\underline{\alpha} = n'_1 \underline{\alpha}_1 + \dots + n'_p \underline{\alpha}_p + n''_1 \underline{\xi}_1 + \dots + n''_r \underline{\xi}_r .$$

Using the notation of §§45, 46, the involution $\underline{\phi}$ is defined by some Y_i, where the corresponding n'_i or n''_i in the maximal root $\underline{\alpha}$ is 1 or 2. Thus, Y_i is defined by the equations

$$\underline{\alpha}_a(Y'_i) = \delta_{ai}/2 \text{ and } \underline{\xi}_b(Y'_i) = 0, \qquad (11)$$

or by the equations

$$\underline{\alpha}_a(Y''_i) = 0 \text{ and } \underline{\xi}_b(Y''_i) = \delta_{bi}/2,$$
$$\text{for } a = 1, \dots, p \text{ and } b = 1, \dots, r. \qquad (12)$$

Lemma-5 *For a Y''_i satisfying (12), the corresponding involution $\theta_0 \circ \exp(\mathrm{ad}Y''_i)$ is conjugate in Int(u) to θ_0.*

In fact, the equations

$$\alpha_a(Y_i) = 0, \quad \xi_b(Y_i) = \delta_{bi}/2, \quad \xi_b'(Y_i) = 0,$$

define an element of a = max. abelian subalgebra of u (invariant under θ_0 etc.). We have

$$Y''_i = Y_i + \theta_0(Y_i).$$

This because both sides belong to m and

$$\alpha_a(Y''_i) = \alpha_a(Y_i + \theta_0(Y_i)) = 0,$$
$$\xi_b(Y''_i) = \xi_b(Y_i) = \delta_{bi}/2. \qquad (\text{since } \{\underline{\alpha}_a, \underline{\xi}_b\} \text{ is a basis of } m^*)$$

By its definition, since $2Y_i \in \Lambda_Z$ (of u) $\exp\mathrm{ad}Y_i$ is an involutive automorphism of u. Thus we have

$$\exp\mathrm{ad}(-Y_i) = \exp\mathrm{ad}(Y_i),$$
$$\exp\mathrm{ad}(Y_i)\circ\theta_0\circ \exp\mathrm{ad}(Y_i) = \theta_0\circ(\theta_0\circ \exp\mathrm{ad}(Y_i)\circ\theta_0)\circ\exp\mathrm{ad}(Y_i)$$
$$=\theta_0\circ \exp\mathrm{ad}(\theta_0 Y_i)\circ\exp\mathrm{ad}(Y_i) = \theta_0\circ \exp\mathrm{ad}(Y_i+\theta_0 Y_i)$$
$$= \theta_0\circ \exp\mathrm{ad}(Y''_i). \qquad \text{q.e.d.}$$

The lemma shows that in the case of A_d, with $d=2d^*$, where k_0 is given by (6), only the case (12) is possible, hence there are no other outer involutions non-conjugate to the canonical one θ_0.

In all other cases there are roots of k_0 of the form $\underline{\alpha}_i$, which are restrictions $\underline{\alpha}_i=\alpha_i|m$ of roots of u, with the property $\alpha_i'=\alpha_i\circ\theta_0=\alpha_i$. Thus, we assume now that

$$\theta=\theta_0\circ\mathrm{expad}(Y'_i),$$

and without loss of generality, we may also assume that i=1: In fact, in the cases (7) and (9) there is no other root but $\underline{\alpha}_1$ with coefficient (in the corresponding maximal root of k_0) 1 or 2. In the case (8) our reasoning depends only from the part of the diagram, which starts from $\underline{\alpha}_i$ and ends at $\underline{\xi}_1$ and all these parts of the diagram have an obvious similarity. For convenience, we suppress the prime from Y'_i and write instead

$$\theta_1=\theta_0\circ\mathrm{expad}Y_1,$$

where $Y_1=Y'_1$ satisfies (11) for i=1. We consider the corresponding subsets (of roots of u) $\Delta_1\cup\Delta_2\cup\Delta_3$ of lemma-1, with respect to $\theta=\theta_1$. By lemma-1, a fundamental system of roots of u consists of the roots

$$\Pi=\{\beta,\alpha_2,\dots,\alpha_p,\xi_1,\xi_1',\dots,\xi_r,\xi_r'\}. \qquad (13)$$

This results from the defining relations (11) and the formula

$$(\mathrm{expad}Y_1)(X_{\alpha_j})=(\exp 2i\pi a_j(Y_1))X_{\alpha_j}\text{ , for } j=1,\dots,p.$$

By lemma-2, $\{\underline{\beta},\underline{\alpha}_2,\dots,\underline{\alpha}_p,\underline{\xi}_1,\dots,\underline{\xi}_r\}$ consists of independent linear forms and by lemma 3, there is a fundamental system of roots of the Lie algebra

$$k=\{X\in u\,|\,\theta_1(X)=X\},$$

of the form

$$\Pi_k=\{\underline{\eta},\underline{\alpha}_2,\dots,\underline{\alpha}_p,\underline{\xi}_1,\dots,\underline{\xi}_r\}, \qquad (14)$$

where $\underline{\eta}$ is a simple root of k which is also the restriction on m of a root $\eta\in\Delta_1\cup\Delta_3$ of u (see lemma-1 and lemma-3). To determine η we use the following :

Lemma-6 *Let $\Pi=\{\alpha_1,\dots,\alpha_d\}$ be a fundamental system of roots of a semi simple complex Lie algebra g. Let also α be a root which can be written as a sum*

$$\alpha=\alpha_{i_1}+\dots+\alpha_{i_n},$$

of n different simple roots of Π. Let also α_j be different from all the summands of α_{i_r}. Then $\alpha+\alpha_j$ is again a root, if and only if there is some root $\alpha_{i_k}\in\{\alpha_{i_1},\dots,\alpha_{i_n}\}$ with $<\alpha_{i_k},\alpha_j><0$.

This because for simple roots $<\alpha_i,\alpha_j><0$ and the Cartan integer $2<\alpha_i,\alpha_j>/<\alpha_j,\alpha_j>=q-p<0\Leftrightarrow p>q$. $\alpha-\alpha_j$ is not a root, hence q=0, and consequently $\alpha+\alpha_j$ is a root $\Leftrightarrow p>0\Leftrightarrow$ $<\alpha_{i_k},\alpha_j><0$, for some i_k. q.e.d.

We turn now to the determination of η :

Since u is simple, there is a sequence of simple roots, which connects β with the "nearest" to β (in the Dynkin diagram) ξ_k (connectedness of the Dynkin diagram):

$$\beta,\alpha_{i_1},\dots,\alpha_{i_t},\xi_k. \qquad (15)$$

1) This sequence is uniquely determined by ξ_k, since there are no "loops" in the Dynkin

diagrams. Two different sequences from β to ξ_k would produce such a loop.

2) The sequence from β to ξ_k is a connected part of the Dynkin diagram of $\mathfrak{u}$. If we delete some α_{i_n} (n=1, ... , t) the two remaining summs are orthogonal to each other and at the same time they are roots of $\mathfrak{u}$, by lemma-6.

3) $\eta = \beta + \alpha_{i_1} + ... + \alpha_{i_t} + \xi_k$ is a root of $\mathfrak{u}$, by lemma-6.

4) $\eta \in \Delta_3$, hence $\underline{\eta} \in \Delta_k$ = root system of $\mathfrak{k}$.

5) $\underline{\eta}$ is a simple positive root of $\mathfrak{k}_C$, hence it is the one appearing in (14). To see this, we need a litle bit of work :

If $\underline{\eta}$ which is a positive root of $\mathfrak{k}_C$, were not simple, then there would exist positive roots γ, δ of $\mathfrak{u}$ with the properties

$$\gamma, \delta \in \Delta_1 \cup \Delta_3 \text{ and } \underline{\eta} = \underline{\gamma} + \underline{\delta}. \qquad (16)$$

Repeating the reasoning of lemma-3, we see that then we should have

$$\eta=\gamma+\delta \text{ or } =\gamma'+\delta \text{ or } =\gamma+\delta' \text{ or } =\gamma'+\delta'.$$

Since

$$\eta = \beta + \alpha_{i_1} + ... + \alpha_{i_t} + \xi_k , \qquad (17)$$

we may assume that γ or γ' are sums of certain of the summands of η and δ or δ' are sums of the remaining summands. We may assume that ξ_k is contained in the summands of γ. If β were not contained in γ too, then δ (or δ' etc.) would contain β and some α_{i_s} . Consequently we should have $\delta \in \Delta_2$ ($X_\delta=\lambda[X_\beta[X_{\alpha_\cdot}[...]]]$ and all the partial products are $\neq 0$, since δ is a sum of successive roots of the diagram (a connected part of this)). This contra-dicts to the assumptions (16). Thus, we should have

$$\gamma = \beta + \alpha_{i_1} + ... + \alpha_{i_t} + \xi_k ,$$

and no summand α_{i_s} of η would be missing (lemma-6) i.e. $\gamma=\eta$, which is impossible. This shows that η is simple.

Using the analysis above, we can computethe Dynkin diagrams of the characteristic algebras $\mathfrak{k}$, which are fixed by the involutions

$$\theta_1 = \theta_0 \circ \exp \operatorname{ad} Y_1.$$

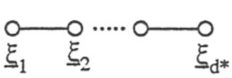
This is done using the fundamental system of $\mathfrak{k}$:

$$\Pi_k = \{ \underline{\eta}, \underline{\alpha}_2, ... , \underline{\alpha}_p, \underline{\xi}_1, ... , \underline{\xi}_r \}.$$

Since θ_0 and θ_1 induce the same isometry in the fundamental system Π of $\mathfrak{u}$, the subalgebra $\mathfrak{m}$ and the restrictions $\underline{\alpha}_2, ... , \underline{\alpha}_p, \underline{\xi}_1, ... , \underline{\xi}_r$ are the same for both θ_0 and θ_1 . The Dynkin diagram of Π_k results from that of Π_0, through substitution of $\underline{\alpha}_1$ by $\underline{\eta}$, constructed from the (connected) part of the diagram which connects α_1 and ξ_k. We have the three cases :

A_d $d=2d^*+1$, $\eta = \alpha_1 + \xi_{d^*}$, $\Pi_k = \{ \underline{\eta}, \underline{\xi}_{d^*}, ... , \underline{\xi}_1 \}$.

The part of the Dynkin diagram of k corresponding to $\{\underline{\xi}_{d^*}, ... , \underline{\xi}_1\}$ will be the same with that of $\mathfrak{k}_0$ (given by (7)) :



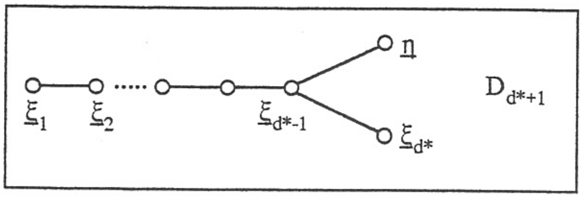
For the rest of the diagram, we compute the strings of roots :

$$\underline{\eta}+\underline{\xi}_{d*} = (\underline{\alpha}_1+\underline{\xi}_{d*}+\underline{\xi}_{d*}') \text{ and } (\alpha_1+\xi_{d*}+\xi_{d*}')\in\Delta_2 ,$$

thus $\underline{\eta}+\underline{\xi}_{d*}$ is not a root of k . In fact,

$$\theta_1(X_{\alpha_1+\xi_{d*}+\xi_{d*'}}) = \theta_1(\lambda[X_{\xi_{d*}}[X_{\alpha_1}X_{\xi_{d*'}}]]) = -\lambda[X_{\xi_{d*'}}[X_{\alpha_1}X_{\xi_{d*}}]]$$
$$=\lambda([X_{\alpha_1}[X_{\xi_{d*}}X_{\xi_{d*'}}]]+[X_{\xi_{d*}}[X_{\xi_{d*'}}X_{\alpha_1}]]) = -X_{\alpha_1+\xi_{d*}+\xi_{d*'}} \cdot$$

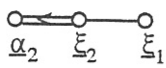
$[X_{\xi_{d*}}X_{\xi_{d*'}}]=0$, because $\xi_{d*}+\xi_{d*}'$ is not a root of A_d . Analogously, we see that $\underline{\eta}+\underline{\xi}_{d*-1}$ is a root and finally that the diagram of k is



(18)

E_6 $\eta = \alpha_1+\alpha_2+\xi_2$, $\Pi_k = \{ \underline{\eta}, \underline{\alpha}_2, \underline{\xi}_2 , \underline{\xi}_1 \}$.

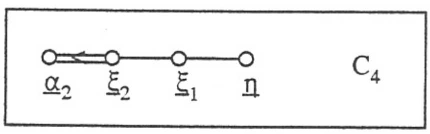
The part of the Dynkin diagram of k corresponding to $\{ \underline{\alpha}_2, \underline{\xi}_2 , \underline{\xi}_1\}$ will be the same with that of k_0 (given by (9)) :



For the rest of the diagram, we compute the strings of roots :

$$\underline{\eta}+\underline{\xi}_1 = (\underline{\alpha}_1+\underline{\alpha}_2+\underline{\xi}_2+\underline{\xi}_1) \text{ with } (\alpha_1+\alpha_2+\xi_2+\xi_1)\in\Delta_1 .$$

Thus $\underline{\eta}+\underline{\xi}_1$ is a root of k and its diagram is



(19)

D_d In this case we have several choices for β :

$$\beta = \alpha_i , \alpha_{i+1} , \dots , \alpha_{d-2} ,$$

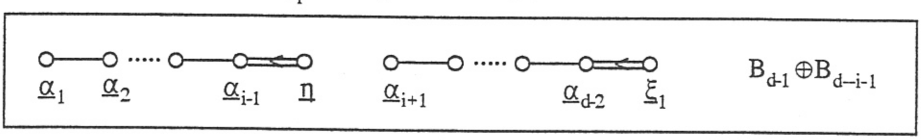
for which the corresponding η is

$$\eta = \alpha_i + \alpha_{i+1} + \dots + \alpha_{d-2} + \xi_1$$

and the corresponding root system is

$$\Pi_{k_i} = \{\underline{\eta}, \underline{\alpha}_1, \underline{\alpha}_2 , \dots , \underline{\alpha}_{i-1} , \underline{\alpha}_{i+1} , \dots , \underline{\alpha}_{d-2} , \underline{\xi}_1\}.$$

The Dynkin diagram of Π_{k_i} is computed from (8):

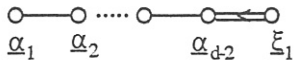


(20)

Lemma-7 *The involutions* $\theta_i=\theta_0{}^\circ\, expad(Y_i)$ and $\theta_{d-i-1}=\theta_0{}^\circ expad(Y_{d-i-1})$ *(the* Y_i *as in the (11)) are conjugate in Int(u), hence they give isomorphic decompositions of* u, u=k⊕p *and* *isomorphic real forms* $k \oplus ip$ *of* u_C.

The isomorphism between k_i and k_{d-i-1} is obvious from (20). The lemma guarantees the isomorphism of the corresponding real forms $k_j \oplus ip_j$. In this case we have

$$k_0 = B_{d-1}$$



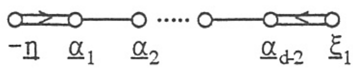
and, by lemma-6

$$\underline{\eta} = \underline{\alpha}_1 + \underline{\alpha}_2 + \dots + \underline{\alpha}_{d-2} + \underline{\xi}_1 .$$

is a root of $k_0 = B_{d-1}$. Looking at the roots of B_{d-1}, we see that the mutual inner products in the set

$$\{-\underline{\eta}\, ,\ \underline{\alpha}_1\, ,\ \underline{\alpha}_2\, , \dots ,\ \underline{\alpha}_{d-2}\, ,\ \underline{\xi}_1\}$$

are given by the diagram



This means that $\{-\underline{\eta}\, ,\ \underline{\alpha}_1\, ,\ \underline{\alpha}_2\, , \dots ,\ \underline{\alpha}_{d-2}\}$ is also a fundamental system of B_{d-1}. Thus, there is a uniquely defined element S of the corresponding Weyl group, with the property

$$S(\underline{\xi}_1) = -\underline{\eta},$$
$$S(\underline{\alpha}_{d-2}) = \underline{\alpha}_1 ,$$
$$\dots\dots\dots$$
$$S(\underline{\alpha}_i) = \underline{\alpha}_{d-i-1} .$$
$$\dots\dots\dots$$

We have also the following relation :

$$Y_{d-i-1} = S^{-1}(Y_i) + \theta_0(Y_d) + Y_d, \qquad (*)$$

where $Y_d \in a$ is defined by the equations :

$$\alpha_a(Y_d) = 0, \text{ for } a=1,\dots,d-2 \text{ and } \xi_1(Y_d) = 1/2,\ \xi_1'(Y_d) = 0.$$

(*) is verified by applying on its sides $\alpha_1, \alpha_2, \dots, \alpha_{d-2}, \xi_1, \xi_1'$, which build a basis of a:

$$\alpha_a(Y_{d-i-1}) = \alpha_a(S^{-1}(Y_i)) + \alpha_a(\theta_0(Y_d)) + \alpha_a(Y_d),$$

which is correct, since

$$\alpha_a(\theta_0(Y_d)) = \alpha_a(Y_d) = 0 \text{ and}$$
$$\delta_{a,d-i-1}/2 = \alpha_a(Y_{d-i-1})$$
$$= \alpha_a(S^{-1}(Y_i)) = \underline{\alpha}_a(S^{-1}(Y_i)) = \underline{\alpha}_{d-a-1}(Y_i) = \delta_{i,d-a-1}/2 .$$
$$0 = \xi_1(Y_{d-i-1}) = \xi_1(S^{-1}(Y_i)) + \xi_1(\theta_0(Y_d)) + \xi_1(Y_d)$$
$$= -\eta(Y_i)) + \xi_1(Y_d) = -1/2 + 1/2 = 0. \text{ etc.}$$

By the leading remarks of this paragraph, S extends to an inner automorphism (denoted by the same letter) of u, which commutes with θ_0 and we have :

$$(\mathrm{expad}(Y_d) \circ S^{-1}) \circ (\theta_0 \circ \mathrm{expad}(Y_i)) \circ (S \circ \mathrm{expad}(Y_d))$$

$$= (\mathrm{expad}(Y_d)\circ\theta_0)\circ(S^{-1}\circ\mathrm{expad}(Y_i)\circ S)\circ(\mathrm{expad}(Y_d))$$
$$= \theta_0\circ\mathrm{expad}(\theta_0(Y_d))\circ\mathrm{expad}(S^{-1}(Y_i))\circ\mathrm{expad}(Y_d)$$
$$= \theta_0\circ\mathrm{expad}(S^{-1}(Y_i) + \theta_0(Y_d) + Y_d).$$

This completes the proof of lemma, since by the definition of Y_d, we have $\mathrm{expad}(2Y_d) = 1$ $\Rightarrow$ $\mathrm{expad}(-Y_d) = \mathrm{expad}(Y_d)$. q.e.d.

The last lemma completes the classification of the real simple Lie algebras of "outer" type. The general classification of all simple real Lie algebras is given by the results of §46 as well as of this paragraph. Except the simple real Lie algebras which are real forms of other simple complex Lie algebras, there is also another category of simple real Lie algebras. It consists of the realifications of complex simple Lie algebras. The two categories are disjoint (except the cases $so(3,1) \approx sl(2;\mathbb{C})$) : a Lie algebra of one category cannot be isomorphic to any Lie algebra of the other. Inspecting the list of real forms and their characteristic Lie algebras, we see that the pair $(\mathfrak{u},\mathfrak{k})$ determines uniquely the real form i.e. inside a fixed $\mathfrak{u}$, the real forms are uniquely characterized by their characteristic subalgebras.

But I must crowd all into one grand mess
  Or mass; for should I strech into detail,
My Muse would run much more into excess,
  Than when some squeamish people deem her frail.
But though a "bonne vivante," 'I must confess
  Her stomach's not her peccant part: this tale
However doth require some slight refection,
Just to relieve her spirits from dejection.

Byron, Don Juan, Canto XV, 64

49 . Real forms of A_d

The standard model of this type of complex Lie algebra is sl(d+1;C), studied in §12 and §27. Here we give concrete models of all real forms of this Lie algebra:

sl(d+1;**R**) The so-called **normal** real form of sl(d+1;C).

This form remains invariant under the involutive automorphism of sl(d+1;C)

$$\theta(X) = -X^t. \qquad (1)$$

This induces an involution in sl(d+1;**R**) and the corresponding (characteristic) subalgebra of fixed points is:

$$o(d+1) = \{X \in sl(d+1;\mathbf{R}) \mid X = -X^t\}. \qquad (2)$$

$$sl(d+1;\mathbf{R}) = o(d+1) \oplus Sym(d+1;\mathbf{R}), \qquad (3)$$

$$X = (1/2)(X-X^t)+(1/2)(X+X^t)$$

$$= (1/2)(X+\theta(X))+(1/2)(X-\theta(X)),$$

Sym(d+1;**R**)⊂sl(d+1;**R**) being the (vector) subspace of symmetric matrices in sl(d+1;**R**). (3) is a Cartan decomposition $k \oplus p$ of sl(d+1;**R**) and the corresponding compact real form is

$$u = k \oplus ip = o(d+1) \oplus iSym(d+1;\mathbf{R}). \qquad (4)$$

Considered as a subset of sl(d+1;C), this is identical with the matrices X satisfying

$$X^* + X = 0, \qquad (5)$$

where

$$X^* = \overline{X^t} \text{ is the adjoint of X. } (6)$$

u is called the **special unitary** Lie algebra and is denoted by su(d+1).

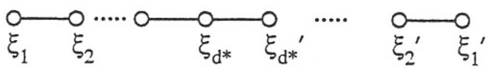
$$su(d+1) = \{X \in sl(d+1;C) \mid X^* + X = 0\}. \qquad (7)$$

Lemma-1 *su(d+1) is a compact real form of sl(d+1;C), and remains invariant under θ, the later being an involutive automorphism of exterior type.*

To see this consider the characteristic subalgebra k of θ, which is o(d+1) and whose complexification is given by:

$$k_C = o(d+1;C) = B_{d^*} \text{ or } D_{d^*} \quad \text{according to } d=2d^* \text{ or } d=2d\text{-}1. \qquad (8)$$

This means that for d=2d*, θ is conjugate of the canonical extension of the symmetry of the Dynkin diagram of sl(2d*+1;C):



(9)

sl(2d*+1;**R**) is the real form corresponding to the characteristic algebra (6) §48.

For d= 2d*-1, sl(d+1;**R**) is the real form of sl(d+1;C) which corresponds to the exterior involution and characteristic subalgebra (18) §48. q.e.d.

Looking at the restriction of θ onto the Cartan subalgebra h, of diagonal matrices, we see that

$$\theta|\mathfrak{h} = -\,\mathrm{Id},$$

which, by the analysis we made in §27, does not belong to the Weyl group of sl(d+1;C). This gives another proof of the fact that θ is not an inner involution (combine with The-1, §39).

Let us discuss for a while, what this θ is, in terms of our theory, developed in §48. Ford=2d*, we know that θ is conjugate, in Int(u), to the canonical extension of the obvious symmetry of the Dynkin diagram suggested by (9). For d=2d*-1, the form of the chararacteristic subalgebra in (8), shows that θ defines an outer involution of u=su(d+1), which is not conjugate to the canonical extension of the symmetry of the Dynkin diagram. In order to find the canonical extension of the symmetry of the Dynkin diagram, in this case, we use the so-called **opposition element** of the Weyl group, defined by the permutation of coordinates of diagonal matrices (see §27):

$$\phi(H_1, \dots, H_{d+1}) = (H_{d+1}, \dots, H_1). \tag{10}$$

This maps the fundamental roots

$$\alpha_i(H) = H_i - H_{i+1},$$

to the roots

$$\alpha'_i = \alpha_i \circ \phi,\ \alpha'_i(H) = \alpha_i(H_{d+1}, \dots, H_1) = H_{d+2-i} - H_{d+1-i} \quad \Rightarrow$$

$$\alpha'_i = -\,\alpha_{d+1-i}\,.$$

This is related to the complex structure of R^{2d^*}! In fact, using the arguments of lemma-4, §39, we can compute an inner automorphism of sl(2d*;C) which induces φ on h. This lemma suggests to use expadA, where A is the matrix of sl(2d*;C) given by

$$A = \frac{\pi}{2} B, \quad \text{where } B = \left(\begin{array}{c|c} 0 \dots 0 & 0 \dots 1 \\ \dots & \dots \\ 0 \dots 0 & 1 \dots 0 \\ \hline 0 \dots -1 & 0 \dots 0 \\ \dots & \dots \\ -1 \dots 0 & 0 \dots 0 \end{array}\right).$$

A belongs to su(2d*) and has expA = B. We have also, $\exp ad(A)C = Ad_{\exp A}C = Ad_B C = BCB^{-1}$ (see §9), and since $B^2 = -1$, $B^{-1} = -B$, φ can be extended (easy to see) to the <u>inner</u> involution of su(d+1) defined by (use of the same letter)

$$\phi(C) = -BCB = BCB^{-1}\,. \tag{11}$$

In order to bring everything in standard form, we conjugate su(d+1) by the matrix

$$D = \left(\begin{array}{c|c} 0 \dots 1 & 0 \dots 0 \\ \dots & \dots \\ 1 \dots 0 & 0 \dots 0 \\ \hline 0 \dots 0 & 0 \dots 1 \\ \dots & \dots \\ 0 \dots 0 & 1 \dots 0 \end{array}\right)$$

which leaves su(d+1) invariant. J = B·D is the matrix defining the standard complex strucwhich leaves su(d+1) invariant. J = B·D is the matrix defining the standard complex structure in R^{d+1} :

$$J = B\cdot D = \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & 1 & \dots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & 0 & 0 & \dots & 1 \\ \hline -1 & \dots & 0 & 0 & \dots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & -1 & 0 & \dots & 0 \end{array}\right)$$

$$\theta' = -JA^tJ^{-1} . \tag{12}$$

is an outer automorphism of su(d+1). The characteristic subalgebra (of fixed points) of su(d+1) is given by

$$k = \{A \in su(d+1) \mid -JA^tJ^{-1} = A\} = \{A \in su(d+1) \mid JA^t + AJ = 0\},$$

whose complexification is

$$k_C = \{A \in sl(d+1;C) \mid JA^t + AJ = 0\} = sp(d^*;C) = C_{d^*} \quad (d+1=2d^*).$$

Thus, we have the case (7) §48. The corresponding real form is described by the next two exercises.

Exercise-1 Find the Cartan decomposition of the real form, whose characteristic subalgebra is given by the preceding k. Show that the real form $g = k \oplus ip$ of $sl(2d^*;C)$ is described by the set of matrices :

$$\left(\begin{array}{c|c} A+iA_1 & B+iB_1 \\ \hline -\overline{B}+i\overline{B}_1 & \overline{A}-i\overline{A}_1 \end{array}\right)$$

where A, A_1, B, B_1 are $d^* \times d^*$ complex matrices, satisfying the relations

$$\overline{A}+A^t = 0, \ \overline{A}_1+A_1^t = 0,$$
$$B^t = B \ , \ B_1^t = -B_1 \ \text{ and}$$
$$tr(A_1) = 0.$$

Exercise-2 Show that the set of matrices, described above, coincides with the real Lie subalgebra of $sl(2d^*;C)$:

$$su^*(2d^*) = \left\{ \begin{pmatrix} X & Y \\ -\overline{Y} & \overline{X} \end{pmatrix} \,\middle|\, \begin{array}{l} X, Y: d^* \times d^* \text{ complex matrices} \\ \text{with } trX+tr\overline{X} = 0. \end{array} \right\} \tag{13}$$

Our arguments show that (13) is the real form corresponding to the canonical extension of the symmetry of the Dynkin diagram, in the case $d=2d^*-1$. The characteristic subalgebra of this form (the previous k) is denoted by $sp(d^*)$:

$$sp(d^*) = su(2d^*) \cap sp(d^*;C). \tag{14}$$

We turn now to the remaining real forms of A_d, which come all from <u>inner</u> involutive automorphisms of the compact form su(d+1). Thus, these involutions are of the form:

$$\theta(A) = J\cdot A\cdot J^{-1} \ , \ \text{where } J = \exp B, \text{ and } B \in su(d+1).$$

Since θ is an involution, we must have

$$J^2\cdot A\cdot J^{-2} = A, \ \text{ for every } A \in su(d+1),$$

and by extending to the complexification, we'll have

$$J^2 \cdot A \cdot J^{-2} = A, \text{ for every } A \in sl(d+1;C) \quad \Rightarrow \quad J^2 = cI.$$

Dividing J by a complex number, does not alter θ and we may assume that

$$J^2 = I.$$

From elementary linear algebra, we know that C^{d+1} decomposes into the direct sum of the two eigenspaces V_1, V_{-1}, corresponding to the ±1 eigenvalues

$$C^{d+1} = V_1 \oplus V_{-1} .$$

Choosing a basis of eigenvectors of J, we get the matrix representation:

$$J = \begin{pmatrix} 1 & 0 \ldots\ldots\ldots & 0 \\ \ldots\ldots\ldots & & \\ 0 \ldots 0 & 1 & 0 \ldots 0 \\ 0 \ldots 0 & -1 & \ldots 0 \\ \ldots\ldots\ldots & & \\ 0 & \ldots\ldots\ldots & -1 \end{pmatrix} \begin{matrix} \}p \\ \\ \}q \end{matrix} \tag{15}$$

We conclude that the initial involution is conjugate to $\theta(A) = J \cdot A \cdot J^{-1}$, where J ($J^{-1}=J$) is given by (15). The corresponding characteristic subalgebra k is found by the relations:

$$A \in su(d+1) \text{ and } JAJ = A \quad \Leftrightarrow$$

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \text{ where } \begin{cases} B^*+B = 0 & (B: p\times p \text{ dimensions}), \\ C^*+C = 0 & (C: q\times q \text{ dimensions}), \text{ and} \end{cases}$$
$$trB+trC = 0.$$

The matrices A can be decomposed into the sum:

$$A = \begin{pmatrix} B-(trB/p)I_p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C-(trC/q)I_q \end{pmatrix} + \begin{pmatrix} (trB/p)I_p & 0 \\ 0 & (trC/q)I_q \end{pmatrix}$$

which, by trB+trC=0, gives the characteristic subalgebra:

$$k = su(p) \oplus su(q) \oplus t \quad (p+q=d+1). \tag{16}$$

t is the (real) 1-dimensional abelian ideal of k, consisting of matrices of the form:

$$t = \left\{ \lambda \begin{pmatrix} (1/p)I_p & 0 \\ 0 & (-1/q)I_q \end{pmatrix} \,\middle|\, \lambda \in \mathbf{R} \right\}.$$

The corresponding -1-eigenspace p of the involution is:

$$p = \{A \in su(d+1) \mid JAJ = -A\} = \left\{ \begin{pmatrix} 0 & Y \\ -\overline{Y}^t & 0 \end{pmatrix} \,\middle|\, Y: p\times q \text{ complex matrix} \right\}$$

and the corresponding real form denoted by su(p,q) is given by:

$$su(p,q) = k \oplus ip = \left\{ \begin{pmatrix} A & Y \\ \overline{Y}^t & B \end{pmatrix} \,\middle|\, \begin{matrix} Y: p\times q \text{ complex matrix}, A^*+A=0, \\ B^*+B=0 \text{ and } trA+trB=0. \end{matrix} \right\}. \tag{17}$$

Exercise-3 Show that the matrices in su(p,q) coincide with the matrices of sl(d+1;C), which leave, infinitesimaly (i.e. $h_{pq}(AZ,W)+h_{pq}(Z,AW)=0$), invariant the hermitian form of C^{d+1}:

$$h_{pq}(Z,W) = z_1\overline{w}_1 + \ldots + z_p\overline{w}_p - z_{p+1}\overline{w}_{p+1} - \ldots - z_{p+q}\overline{w}_{p+q} .$$

Denn der Charakter ist schlechthin inkorrigibel; weil alle Handlungen des Menschen aus einem innern Princip fließen, vermöge dessen er, unter gleichen Umständen, stets das Gleiche thun muß und nicht anders kann. Man lese meine Preisschrift über die sogenannte Freiheit des Willens und befreie sich vom Wahn.

Schopenhauer, Aphorismen ... p. 494

50. Real forms of B_d

A model of this type of complex Lie algebra is

$$\{A \in sl(2d+1;\mathbf{C}) \mid JA^t + AJ = 0\}, \qquad (1)$$

where J is the matrix (see §§21, 30):

$$J = \left(\begin{array}{c|cc} 1 & & \\ \hline & \begin{array}{c} 0 \dots 0 \\ \dots \\ 0 \dots 0 \end{array} & \begin{array}{c} 1 \dots 0 \\ \dots \\ 0 \dots 1 \end{array} \\ \cline{2-3} & \begin{array}{c} 1 \dots 0 \\ \dots \\ 0 \dots 1 \end{array} & \begin{array}{c} 0 \dots 0 \\ \dots \\ 0 \dots 0 \end{array} \end{array}\right)$$

As we saw in §21, this model is conjugate to the standard one, of skew-symmetric matrices:

$$o(2d+1;\mathbf{C}) = \{A \in sl(2d+1;\mathbf{C}) \mid A^t + A = 0\}. \qquad (2)$$

Restricting to the real matrices we get the compact real form of this Lie algebra:

$$o(2d+1;\mathbf{R}).$$

Since all automorphisms of $o(2d+1;\mathbf{R})$ are inner, making analogous computations with those of the previous §, we see that every involution of $o(2d+1;\mathbf{R})$ is conjugate to one of the form:

$$\theta_{pq}(A) = J_{pq} A J_{pq}, \qquad (3)$$

where J_{pq} ($J_{pq}^2 = 1$) is the matrix:

$$J_{pq} = \begin{pmatrix} 1 & 0 \dots\dots\dots\dots 0 \\ \dots\dots\dots\dots\dots\dots \\ 0 \dots 0 \; 1 & 0 \dots 0 \\ 0 \dots 0 & -1 \dots 0 \\ \dots\dots\dots\dots\dots\dots \\ 0 \dots\dots\dots\dots\dots & -1 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} q \end{array}$$

The corresponding characteristic subalgebras are given by:

$$k = \{A \in o(2d+1;\mathbf{R}) \mid J_{pq} A J_{pq} = A\} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \,\middle|\, B^t + B = C^t + C = 0 \right\}$$

$$= o(p;\mathbf{R}) \oplus o(q;\mathbf{R}).$$

Since $p+q = 2d+1$ is odd, one of $\{p, q\}$ must be odd, the other being then even. Thus, we get the cases (10) and (11) of §46. More precisely, (10) for which

$$k = t \oplus B_{d-1},$$

corresponds to $p=2$ and $q=2d-1=2(d-1)+1$. For all other cases we have

$$k = D_r \oplus B_{d-r},$$

for appropriate r, depending on p, q.

The corresponding -1-eigenspace of θ_{pq} is given by:

$$\mathfrak{p} = \{A \in o(2d+1;\mathbf{R}) \mid J_{pq} A J_{pq} = -A\} = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \middle| X = \text{real } p\times q \text{ matrix} \right\} \quad (4)$$

the corresponding real form being

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p} = \left\{ \begin{pmatrix} B & iX \\ -iX^t & C \end{pmatrix} \middle| \begin{array}{l} X = \text{real } p\times q \text{ matrix}, \\ B^t+B=C^t+C=0. \end{array} \right\} \quad (5)$$

This is conjugate to the so-called **Lorentz** Lie algebra o(p,q), defined by:

$$o(p,q) = \left\{ \begin{pmatrix} B & X \\ X^t & C \end{pmatrix} \middle| \begin{array}{l} X = \text{real } p\times q \text{ matrix}, \\ B^t+B=C^t+C=0. \end{array} \right\} \quad (6)$$

The isomorphism between (5) and (6) is given by:

$$A \to \begin{pmatrix} -iI_p & 0 \\ 0 & I_q \end{pmatrix} A \begin{pmatrix} iI_p & 0 \\ 0 & I_q \end{pmatrix},$$

$$\begin{pmatrix} B & iX \\ -iX^t & C \end{pmatrix} \to \begin{pmatrix} B & X \\ X^t & C \end{pmatrix}.$$

Remark In order to have $J_{pq} \in SO(2d+1)$, q must be even. When q is odd, then p is even (since p+q=2d+1) and $-J_{pq} \in SO(2d+1)$ and the involution defined by $-J_{pq}$ is identical with that in (3).

Nihil est ab omni parte beatum.

Horace, Ode II, 16

51. Real forms of D_d

A model of this type of complex Lie algebra is

$$\{A \in sl(2d;C) \mid JA^t + AJ = 0\}, \qquad (1)$$

where J is the matrix (see §§21, 29):

$$J = \left(\begin{array}{c|c} 0 \dots 0 & 1 \dots 0 \\ \dots & \dots \\ 0 \dots 0 & 0 \dots 1 \\ \hline 1 \dots 0 & 0 \dots 0 \\ \dots & \dots \\ 0 \dots 1 & 0 \dots 0 \end{array}\right)$$

As we saw in §21, this model is conjugate (in gl(2d;C)) to the standard one, of skew-symmetric matrices:

$$o(2d;C) = \{A \in sl(2d+1;C) \mid A^t + A = 0\}. \qquad (2)$$

Restricting to the real matrices, we get the compact real form of this Lie algebra:

$$o(2d) = o(2d;\mathbf{R}) = \{A \in sl(2d+1;\mathbf{R}) \mid A^t + A = 0\}.$$

The inner involutive automorphisms of this compact Lie algebra may be computed as in the preceding §. They are conjugate to one of the following:

$$\theta_{pq}(A) = J_{pq} A J_{pq}, \qquad (3)$$

where J_{pq} ($J_{pq}^2 = 1$)is the matrix:

$$J_{pq} = \begin{pmatrix} 1 & 0 \dots\dots\dots\dots & 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ 0 \dots 0 & 1 \quad 0 \dots & 0 \\ 0 \dots\dots 0 & -1 \dots & 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ 0 & \dots\dots\dots\dots\dots & -1 \end{pmatrix} \begin{matrix} \}p \\ \\ \}q \end{matrix}$$

and the integers p, q are <u>both</u> even. In fact $J_{pq} \in SO(2d)$, only when p, q are both even. The corresponding characteristic subalgebras are given by:

$$k = \{A \in o(2d;\mathbf{R}) \mid J_{pq} A J_{pq} = A\} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \middle| \; B^t + B = C^t + C = 0 \right\}$$
$$= o(p;\mathbf{R}) \oplus o(q;\mathbf{R}).$$

Since p, q are both even, we get the cases (15) and (17) of §46, for which

$$k_C = D_r \oplus D_{d-r}, \quad r = 1, 2, 3, \dots, [d/2].$$

The corresponding -1-eigenspace of θ_{pq} is given by:

$$p = \{A \in o(2d;\mathbf{R}) \mid J_{pq} A J_{pq} = -A\} = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \middle| \; X = \text{real } p \times q \text{ matrix} \right\} \qquad (4)$$

the corresponding real form being

$$g = k \oplus ip = \left\{ \begin{pmatrix} B & iX \\ -iX^t & C \end{pmatrix} \middle| \; \begin{matrix} X = \text{real } p \times q \text{ matrix}, \\ B^t + B = C^t + C = 0. \end{matrix} \right\} \qquad (5)$$

This is conjugate to the Lorentz Lie algebra o(p,q), defined by:

$$o(p,q) = \left\{ \begin{pmatrix} B & X \\ X^t & C \end{pmatrix} \middle| \begin{array}{l} X = \text{real } p\times q \text{ matrix}, \\ B^t+B=C^t+C=0. \end{array} \right\} \qquad (6)$$

where p, q are both even. For r = 1 we have the particular case (15) §46, which has

$$k = o(p;\mathbf{R})\oplus o(q;\mathbf{R}),$$

$$k_C = t\oplus D_{d-1}.$$

There is still another inner involution of o(2d) given by

$$\theta(A) = -JAJ,$$

with J as in ((1), §52). The corresponding characteristic subalgebra is given by:

$$k = \{A\in o(2d;\mathbf{R}) |\ AJ = JA\} = \left\{ \begin{pmatrix} B & C \\ -C & B \end{pmatrix} \middle|\ B^t+B=0,\ C^t=C \right\}.$$

The map

$$\begin{pmatrix} B & C \\ -C & B \end{pmatrix} \rightarrow B+iC$$

defines an **R**-linear isomorphism of k with the **unitary** Lie algebra u(d)

$$u(d) = \mathbf{R}\oplus su(d),$$

whose complexification is $t\oplus A_{d-1}$, which corresponds to the case (16) §46.
The corresponding -1-eigenspace of θ is given by:

$$p = \{A\in o(2d;\mathbf{R}) |\ AJ = -JA\} = \left\{ \begin{pmatrix} B & C \\ C & -B \end{pmatrix} \middle|\ B^t+B=C^t+C=0 \right\} \qquad (4)$$

the corresponding real form being

$$g = k\oplus ip = \left\{ \begin{pmatrix} B+iC & D+iE \\ -D+iE & B-iC \end{pmatrix} \middle|\ B^t+B=C^t+C=E^t+E=0,\ D^t=D \right\}$$

$$= \left\{ \begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \middle|\ X\in o(d;\mathbf{C}),\ Y^*=Y \right\}.$$

The last real Lie algebra is denoted by

$$so^*(2d).$$

Exercise-1 Show that so*(2d) is the Lie algebra of endomorphisms of $\mathbf{C}^{2d}$, which leave (infinitesimally) invariant the skew-hermitian form of $\mathbf{C}^{2d}$ (J as in (1), §52):

$$h(Z,W) = Z^t J\,\overline{W}.$$

As proved in §46, these are all real forms of inner type of D_d. By the results of the same §, we know that the only exterior real forms are these which correspond to (3) for p, q both odd (≠1) numbers. These have characteristic subalgebras (in accordance with (20) §48):

$$k = o(p;\mathbf{R})\oplus o(q;\mathbf{R}),\ \ k_C = B_{[p/2]}\oplus B_{[q/2]},$$

the corresponding real form being the Lorentz Lie algebra o(p,q).
For q=1, p=2d-1, we have the characteristic subalgebra

$$k = o(2d-1),\ \text{with}\ \ k_C = B_{d-1}$$

corresponding to the case (8), § 48, whose real form is the Lorentz Lie algebra o(2d-1,1).

The mind is lost in mighty contemplation
  Of intellect expended on two courses;
And indigestion's grand multiplication
  Requires arithmetic beyond my forces.
Who would suppose, from Adam's simple ration,
  That cookery could have call'd forth such resources,
As form a science and a nomenclature
From out the commonest demands of nature?

Byron, Don Juan, Canto XV, 69

52. Real forms of C_d

A model of this type of complex Lie algebra is

$$sp(d;C) = \{A \in sl(2d;C) \mid JA^t + AJ = 0\},$$

where J is the matrix (see §§21, 28):

$$J = \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & 1 & \dots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & 0 & 0 & \dots & 1 \\ \hline -1 & \dots & 0 & 0 & \dots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & -1 & 0 & \dots & 0 \end{array}\right) \qquad (1)$$

Restricting to the real matrices we get the real form of this Lie algebra:

$$sp(d;\mathbf{R}).$$

An obvious involution which leaves this real form invariant is defined by

$$\theta(A) = JAJ^{-1} = -A^t. \qquad (2)$$

The corresponding Cartan decomposition is given by $g = k \oplus p$, where

$$k = \{A \in sp(d;\mathbf{R}) \mid -A^t = A\} = \left\{ \begin{pmatrix} B & C \\ -C & B \end{pmatrix} \middle| \ B^t + B = 0,\ C^t = C \right\},$$

which by the map

$$\begin{pmatrix} B & C \\ -C & B \end{pmatrix} \rightarrow B + iC \in u(d),$$

is proved to be isomorphic to the unitary Lie algebra u(d), whose complexification is, in accordance with (13) §46:

$$k_C = t \oplus A_{d-1}.$$

The corresponding -1-eigenspace being

$$p = \{A \in sp(d;\mathbf{R}) \mid A^t = A\} = \left\{ \begin{pmatrix} B & C \\ C & -B \end{pmatrix} \middle| \ B^t = B,\ C^t = C \right\},$$

which gives the real compact form

$$u = k \oplus ip = \left\{ \begin{pmatrix} B+iC & D+iE \\ -D+iE & B-iC \end{pmatrix} \middle| \ B^t + B = 0,\ C^t = C,\ D^t = D,\ E^t = E \right\}.$$

We see easily that this Lie algebra is equal to

$$su(2d) \cap sp(d;C) = sp(d). \qquad (3)$$

We met already this as the characteristic subalgebra of the real form su*(2d) of sl(2d;C).

The inner involutive automorphisms of this compact Lie algebra may be computed as in the preceding §. They are conjugate to one of the following:

$$\theta_{pq}(A) = J_{pq} A J_{pq}, \qquad (4)$$

where J_{pq} ($J_{pq}^2 = 1$)is the matrix:

$$J_{pq} = \begin{pmatrix} I_p & & & \\ & -I_q & & \\ & & I_p & \\ & & & -I_q \end{pmatrix}$$

where p+q=d. The corresponding characteristic subalgebras are given by:

$$k = \{A \in sp(d) \mid J_{pq} A J_{pq} = A\} = sp(p) \oplus sp(q),$$

which correspond to the cases (14) of §46, for which

$$k_C = C_p \oplus C_{d-p}, \; p = 1, 2, 3, \dots, d-1.$$

The corresponding real forms are denoted by

$$sp(p,q).$$

Remark The Lie algebra sp(d) may be identified with the Lie algebra of the group of the isometries of quaternionic d-dimensional space H^d, with respect to the "hermitian" form:

$$h(Z,W) = z_1 \overline{w}_1 + \dots + z_d \overline{w}_d .$$

Here the twelfth Canto of our introduction
 Ends. When the body of the book's begun,
You'll find it of a different construction
 From what some people say 'twill be when done:
The plan at present's simply in concoction.
 I can't oblige you, reader! to read on;
That's your affair, not mine: a real spirit
Should neither court neglect nor dread to bear it.
Byron, Don Juan, Canto XII, 87

53 . Signature and normal real form

The **signature** of a real form $\mathfrak{g}$ of a complex semi simple Lie algebra $\mathfrak{g}_C$ (= complexification of $\mathfrak{g}$) is defined to be the signature of the Killing form of $\mathfrak{g}$

$$\delta(\mathfrak{g}) = \dim V^+ - \dim V^- , \qquad (1)$$

where V^+ (resp. V^-) is the maximal dimensional subspace, on which the Killing form is positive (resp. negative) definite. If $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition of the real form, the signature of $\mathfrak{g}$ is given by

$$\delta(\mathfrak{g}) = \dim\mathfrak{p} - \dim\mathfrak{k}. \qquad (2)$$

We see that the signature of the compact real form $\mathfrak{u}$, $\delta(\mathfrak{u}) = -\dim(\mathfrak{u})$, is the minimum value that the signature can attain. We'll see that the maximum value which $\delta(\mathfrak{g})$ can attain is rank($\mathfrak{g}$) and this happens for certain special real forms called **normal.**

Examples of such normal real forms (of semi simple complex Lie algebras) are constructed using Weyl-Chevalley bases (§37) of the complex Lie algebras:

$$\mathfrak{g}_C = \mathfrak{h} \oplus_{\alpha\in\Delta} CX_\alpha ,$$

and defining the real form to be

$$\mathfrak{g} = \mathfrak{h}_0 \oplus_{\alpha\in\Delta^+} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus_{\alpha\in\Delta^+} \mathbf{R}(X_\alpha + X_{-\alpha}). \qquad (3)$$

Using the computations of The-1, §40, we see that the signature of $\mathfrak{g}$ is indeed d=dim($\mathfrak{h}$).

Proposition *The signature of a semi simple real form $\mathfrak{g}$ satisfies the inequality:*

$$-\dim(\mathfrak{g}) \le \delta(\mathfrak{g}) \le \mathrm{rank}(\mathfrak{g}).$$

The left inequality is obvious. To prove the right one, consider a Cartan decomposition of the $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ of $\mathfrak{g}$, take a maximal abelian subalgebra $\mathfrak{e}$ of $\mathfrak{p}$ and an element $Y\in\mathfrak{e}$. Y defines two linear maps:

$$A = \mathrm{ad}Y: \mathfrak{p} \to \mathfrak{k}, \quad \text{and} \quad B = -\mathrm{ad}Y: \mathfrak{k} \to \mathfrak{p},$$

for which we have

$$\langle AZ,W\rangle = \langle Z,BW\rangle, \quad \text{for every } Z\in\mathfrak{p}, W\in\mathfrak{k} . \qquad (4)$$

A and B are conjugate with respect to the (definite) metrics in $\mathfrak{p}$ and $\mathfrak{k}$. Thus, using orthonormal bases, they are represented by transposed matrices and consequently they have the same rank. This gives the relations:

$$\begin{aligned}
&\mathrm{rank}A = \mathrm{rank}B && \Rightarrow \\
&\dim\mathfrak{p} - \dim(\ker A) = \dim\mathfrak{k} - \dim(\ker B) && \Rightarrow \\
&\dim\mathfrak{p} - \dim\mathfrak{k} = \dim(\ker A) - \dim(\ker B) && \\
&\qquad\qquad \le \dim\mathfrak{e} - \dim(\ker B) &&
\end{aligned}$$

$$\leq \dim \mathfrak{e} \leq \dim \mathfrak{a} = \mathrm{rank}\mathfrak{g}.$$

Where $\mathfrak{a} \supset \mathfrak{e}$ is a maximal abelian subalgebra of $\mathfrak{g}$, extending $\mathfrak{e}$. q.e.d.

We define a real form g to be **normal**, when the space $\mathfrak{p}$ of the Cartan decomposition contains a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. One can easily verify that the examples (3) fit into this definition.

Theorem *Two normal real forms of the same complex semi simple Lie algebra $\mathfrak{g}_C$, are isomorphic to each other.*

In fact, the existence of one normal form in $\mathfrak{g}$ is proved by (3). We denote by $\mathfrak{u}_0$ this special normal form and by $\mathfrak{u}_0=\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ the corresponding compact form. If $\mathfrak{g}_1=\mathfrak{k}_1 \oplus i\mathfrak{p}_1$ is another normal real form, and $\mathfrak{h}_1 \subset \mathfrak{p}_1$ is a maximal abelian subalgebra, with $\dim\mathfrak{h}_1=\mathrm{rank}\mathfrak{g}_1$, then $i\mathfrak{h}_1$ is a maximal abelian subalgebra of the corresponding compact real form $\mathfrak{u}_1=\mathfrak{k}_1 \oplus i\mathfrak{p}_1$. By §42, $\mathfrak{u}_0$ and $\mathfrak{u}_1$ are isomorphic through an automorphism $f: \mathfrak{u}_1 \to \mathfrak{u}_0$. By composing with an automorphism of $\mathfrak{u}_0$, if necessary, we may assume that $f(i\mathfrak{h}_1)=\mathfrak{h}_0$. Let

$$\theta_i : X+Y \to X-Y, \quad \text{for } X+Y \in \mathfrak{u}_i=\mathfrak{k}_i \oplus i\mathfrak{p}_i ,$$

be the conjugations with respect to the real forms $\mathfrak{g}_i$. Then

$$\theta_f = f \circ \theta_1 \circ f^{-1} : \mathfrak{u}_0 \to \mathfrak{u}_0$$

is an involution and for every $X \in \mathfrak{h}_0$ we'll have

$$\theta_f X = \theta_0 X = -X.$$

Applying the lemma below, we conclude that there is $\sigma \in \mathrm{Int}(\mathfrak{u}_0)$ with the property

$$\theta_f = \sigma \circ \theta_0 \circ \sigma^{-1},$$

which means that θ_f and θ_0 define isomorphic real forms. But the real forms defined by θ_f and θ_1 are also isomorphic. q.e.d.

Lemma *Let θ_1, θ_2 be involutive automorphisms of the compact semi simple real form $\mathfrak{u}$ of $\mathfrak{g}_C$. Let also $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{u}$, on which the automorphisms coincide and have*

$$\theta_1(X) = \theta_2(X) = -X.$$

Then there is $\sigma \in Int(\mathfrak{u})$ such that $\theta_2 = \sigma \circ \theta_1 \circ \sigma^{-1}$.

We work with the Weyl-Chevalley basis of $\mathfrak{g}_C$, with respect to $\mathfrak{a}_C$. For every $\alpha \in \Delta$

$$\theta_1 X = c_\alpha X_{-\alpha}, \text{ where the constants satisfy}$$

$$c_\alpha c_{-\alpha}=1, |c_\alpha| = 1, c_\alpha c_\beta = -c_{\alpha+\beta} . \tag{5}$$

The two first of (5), because of the corollary in §43 and the last being a consequence of the following computation

$$[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta} \quad \Rightarrow$$

$$\theta_1 [X_\alpha, X_\beta] = N_{\alpha\beta} c_{\alpha+\beta} X_{-\alpha-\beta}$$

$= c_\alpha c_\beta [X_{-\alpha}, X_{-\beta}] = c_\alpha c_\beta N_{-\alpha,-\beta} X_{-\alpha-\beta}$ and $N_{-\alpha,-\beta} = -N_{\alpha\beta}$ ((16) §37).

We can define (as we did in lemma-1 §39) the elements

$H_1, H_2 \in h$, such that

$c_\alpha = -e^{\alpha(H_1)}$ and analogously for $\theta_2 X_\alpha = c'_\alpha X_\alpha$,

$c'_\alpha = -e^{\alpha(H_2)}$, for all $\alpha \in \Delta$.

Then $\sigma = \exp(\mathrm{ad}(H\text{-}H)/2)$ does the work. q.e.d.

Remark Notice that the (inner) involutions $A \to J_{pq} A J_{pq}$ in §49, leave the Cartan subalgebra of diagonal matrices of the compact form su(d+1) of sl(d+1;C) pointwise fixed. However they are not conjugate to each other since they define different real forms.

Exercise Show that the signature of the real form $g = k \oplus p$ is equal to $\dim_C g_C - 2\dim_C k_C$.

But for the present, gentle reader! and
 Still gentler purchaser! the bard-that's I-
Must, with permission, shake you by the hand,
 And so your humble servant, and good bye!
We meet again, if we should understand
 Each other; and if not, I shall not try
Your patience further than by this short sample-
'Twere well if others follow'd my example.

Byron, Don Juan, Canto I, 221

"What I have written, I have written: let
The rest be on his head or mine!" So spoke
Old "Nominis Umbra;" and while speaking yet,
Away he melted in celestial smoke.
Byron, The vision of Judgment, 84

54. Catalog of simple real Lie algebras

Real forms of sl(d+1;C) (A_d)

su(d+1) = compact real form. Signature $= -\dim_{\mathbf{R}} su(d+1) = -((d+1)^2-1)$.

sl(d+1;**R**) = normal real form. Signature = d. Outer type.
o(d+1) = Characteristic subalgebra.
$sl(d+1;\mathbf{R}) = o(d+1) \oplus Sym(d+1;\mathbf{R})$ = Cartan decomposition.
Helgason's type = AI

su*(2d*) In the case d+1=2d*. Signature = -4d*-1 = -2d-3. Outer type.
sp(d*) = Characteristic subalgebra.
Helgason's type = AII

su(p,q) With p+q=d+1. Signature $= 1-(p-q)^2$ Inner type.
$su(p) \oplus su(q) \oplus t$ = Characteristic subalgebra.
Helgason's type = AIII

Real forms of o(2d+1;C) (B_d)

o(2d+1) = compact real form. Signature = -d(2d+1).

o(p,q) With p+q=2d+1. Signature $= (p+q-(p-q)^2)/2$. Inner type.
$o(p) \oplus o(q)$ = Characteristic subalgebra.
Helgason's type = BDI

Real forms of sp(d;C) (C_d)

sp(d) = compact real form. Signature $= -(2d^2+d)$.

sp(d;**R**) = normal real form. Signature = d. Inner type.
u(d) = Characteristic subalgebra.
Helgason's type = CI

sp(p,q) With p+q=d. Signature $= -(p+q+2(p-q)^2)$. Inner type.
$sp(p) \oplus sp(q)$ = Characteristic subalgebra.
Helgason's type = CII

Real forms of $o(2d;C)$ (D_d)

$o(2d)$ = compact real form. Signature $= -d(2d-1)$.

$o(p,q)$ With $p+q=2d$, p, q even. Signature $= (p+q-(p-q)^2)/2$. Inner type.
$o(p) \oplus o(q)$ = Characteristic subalgebra.
Helgason's type = BDI

$so^*(2d)$ Signature $= -d$. Inner type.
$u(d)$ = Characteristic subalgebra.
Helgason's type = DIII

$o(p,q)$ With $p+q=2d$, p, q odd. Signature $= (p+q-(p-q)^2)/2$. Outer type.
$o(p) \oplus o(q)$ = Characteristic subalgebra.
Helgason's type = BDI

Real forms of E_8

E_8 Compact form $\dim_R E_8 = 248$. Signature $= -248$.

$E_8(VIII)$ = normal real form. Signature $= 8$. Inner type.
D_8 = Characteristic subalgebra.
Helgason's type = EVIII

$E_8(IX)$ Signature $= -24$. Inner type.
$A_1 \oplus E_7$ = Characteristic subalgebra.
Helgason's type = EIX

Real forms of E_7

E_7 Compact form $\dim_R E_7 = 133$. Signature $= -133$.

$E_7(V)$ = normal real form. Signature $= 7$. Inner type.
A_7 = Characteristic subalgebra.
Helgason's type = EV

$E_7(VI)$ Signature $= -5$. Inner type.
$A_1 \oplus D_6$ = Characteristic subalgebra.
Helgason's type = EVI

$E_7(VII)$ Signature $= -25$. Inner type.
$t \oplus E_6$ = Characteristic subalgebra.
Helgason's type = EVII

Real forms of E_6

E_6 Compact form $\dim_{\mathbf{R}} E_6=78$. Signature $= -78$.

$E_6(I)$ = normal real form Signature $= 6$. Outer type.
C_4 = Characteristic subalgebra.
Helgason's type = EI

$E_6(II)$ Signature $= 2$. Inner type.
$A_1 \oplus A_5$ = Characteristic subalgebra.
Helgason's type = EII

$E_6(III)$ Signature $= -14$. Inner type.
$t \oplus D_5$ = Characteristic subalgebra.
Helgason's type = EIII

$E_6(IV)$ Signature $= -26$. Outer type.
F_4 = Characteristic subalgebra.
Helgason's type = EIV

Real forms of F_4

F_4 Compact form $\dim_{\mathbf{R}} F_4=52$. Signature $= -52$.

$F_4(I)$ = normal real form. Signature $= 4$. Inner type.
$A_1 \oplus C_3$ = Characteristic subalgebra.
Helgason's type = FI

$F_4(II)$ Signature $= -20$. Inner type.
B_4 = Characteristic subalgebra.
Helgason's type = FII

Real forms of G_2

G_2 Compact form $\dim_{\mathbf{R}} G_2=14$. Signature $= -14$.

$G_2(I)$ = normal real form. Signature $= 2$. Inner type.
$A_1 \oplus A_1$ = Characteristic subalgebra.
Helgason's type = G

The Helgason's type has to do with an other story, to be narrated elsewhere.

The other class of simple real Lie algebras consists of the realifications of simple complex Lie algebras.

Warning In using the above catalog, one must be cautious in low dimensions. In fact, in the case of real forms of low dimensional classical Lie algebras we have coincidences, for some values of d, p and q, pointed out already by Cartan. Here is the list, taken from Helgason. The geometric meaning (having to do with quaternions, octonions and all that) of these coincidences may be found in the book of Porteous "Topological geometry", Van Nostrand.

i) $su(2) = so(3) = sp(1)$,
$sl(2;\mathbf{R}) = su(1,1) = so(2,1) = sp(1;\mathbf{R})$.

ii) $so(5) = sp(2)$,
$so(3,2) = sp(2;\mathbf{R})$

iii) $so(4) = sp(1)\times sp(1)$,
$so(4,1) = sp(1,1)$.

iv) $su(4) = so(6)$,
$sl(4;\mathbf{R}) = so(3,3)$.

v) $su^*(4) = so(5,1)$.

vi) $su(2,2) = so(4,2)$.

vii) $su(3,1) = so^*(6)$.

viii) $so^*(8) = so(6,2)$.

ix) $so(3,1) = sl(2;\mathbf{C})$.

x) $so(2,2) = sl(2;\mathbf{R})\times sl(2;\mathbf{R})$

xi) $so^*(4) = su(2)\times sl(2;\mathbf{R})$.

END

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