Introduction to the Geometry of the Triangle

Paul Yiu

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Department of Mathematics
Florida Atlantic University

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Contents

Chapter 1

The Circumcircle and the Incircle

1.1 Preliminaries

1.1.1 Coordinatization of points on a line

Let B and C be two fixed points on a line $\mathcal{L}$. Every point X on $\mathcal{L}$ can be coordinatized in one of several ways:

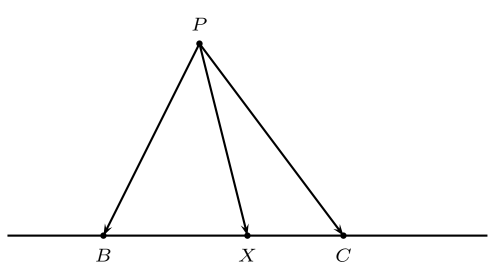
(1) the ratio of division $t = \frac{BX}{BC}$,

(2) the *absolute* barycentric coordinates: an expression of X as a *convex* combination of B and C:

$$X = (1-t)B + tC,$$

which expresses for an arbitrary point P outside the line $\mathcal{L}$, the vector $\mathbf{PX}$ as a linear combination of the vectors $\mathbf{PB}$ and $\mathbf{PC}$:

$$\mathbf{PX} = (1-t)\mathbf{PB} + t\mathbf{PC}.$$

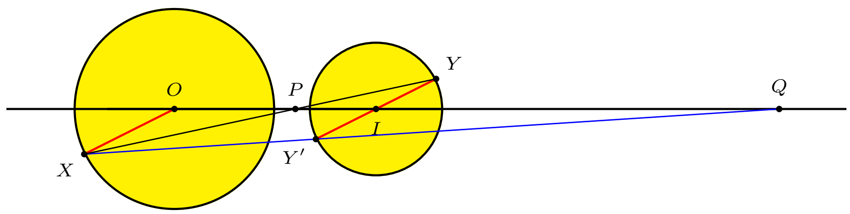


(3) the *homogeneous* barycentric coordinates: the proportion $XC : BX$, which are masses at B and C so that the resulting system (of two particles) has *balance point* at X.

1.1.2 Centers of similitude of two circles

Consider two circles $O(R)$ and $I(r)$, whose centers O and I are at a distance d apart. Animate a point X on $O(R)$ and construct a ray through I *oppositely* parallel to the ray OX to intersect the circle $I(r)$ at a point Y. You will find that the line XY always intersects the line OI at the same point P. This we call the *internal center of similitude* of the two circles. It divides the segment OI in the ratio $OP : PI = R : r$. The absolute barycentric coordinates of P with respect to OI are

$$P = \frac{R \cdot I + r \cdot O}{R + r}.$$



If, on the other hand, we construct a ray through I *directly* parallel to the ray OX to intersect the circle $I(r)$ at Y', the line XY' always intersects OI at another point Q. This is the *external center of similitude* of the two circles. It divides the segment OI in the ratio $OQ : QI = R : -r$, and has absolute barycentric coordinates

$$Q = \frac{R \cdot I - r \cdot O}{R - r}.$$

1.1.3 Harmonic division

Two points X and Y are said to divide two other points B and C *harmonically* if

$$\frac{BX}{XC} = -\frac{BY}{YC}.$$

They are *harmonic conjugates* of each other with respect to the segment BC.

Exercises

1. If X, Y divide B, C harmonically, then B, C divide X, Y harmonically.

2. Given a point X on the line BC, make use of the notion of the centers of similitude of two circles to construct the harmonic conjugate of X with respect to the segment BC. Distinguish between two cases when X divides BC internally and externally.

3. Given two fixed points B and C, the locus of the points P for which $\frac{|BP|}{|CP|} = k$ (constant) is a circle.

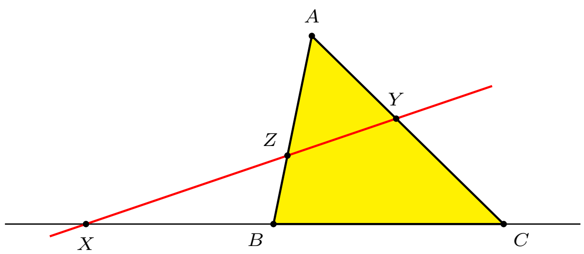
1.1.4 Menelaus and Ceva Theorems

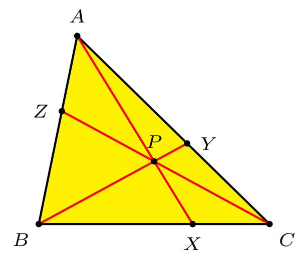
Consider a triangle ABC with points X, Y, Z on the side lines BC, CA, AB respectively.

Menelaus Theorem

The points X, Y, Z are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$





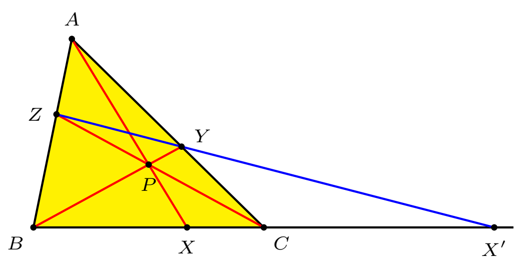
Ceva Theorem

The lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$

Ruler construction of harmonic conjugate

Let X be a point on the line BC. To construct the harmonic conjugate of X with respect to the segment BC, we proceed as follows.



(1) Take any point A outside the line BC and construct the lines AB and AC.

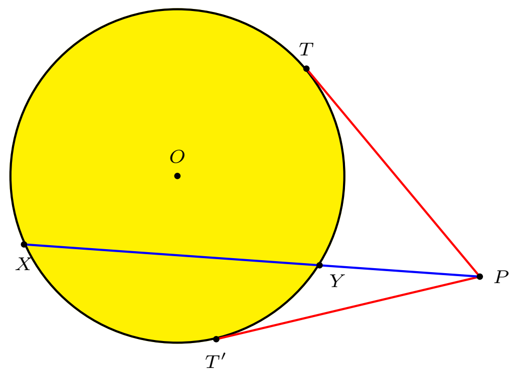
(2) Mark an arbitrary point P on the line AX and construct the lines BP and CP to intersect respectively the lines CA and AB at Y and Z.

(3) Construct the line YZ to intersect BC at X'.

Then X and X' divide B and C harmonically.

1.1.5 The power of a point with respect to a circle

The *power* of a point P with respect to a circle $\mathcal{C} = O(R)$ is the quantity $\mathcal{C}(P) := OP^2 - R^2$. This is positive, zero, or negative according as P is outside, on, or inside the circle $\mathcal{C}$. If it is positive, it is the square of the length of a tangent from P to the circle.



Theorem (Intersecting chords)

If a line $\mathcal{L}$ through P intersects a circle $\mathcal{C}$ at two points X and Y, the product $PX \cdot PY$ (of signed lengths) is equal to the power of P with respect to the circle.

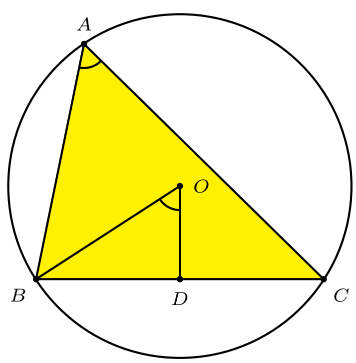
1.2 The circumcircle and the incircle of a triangle

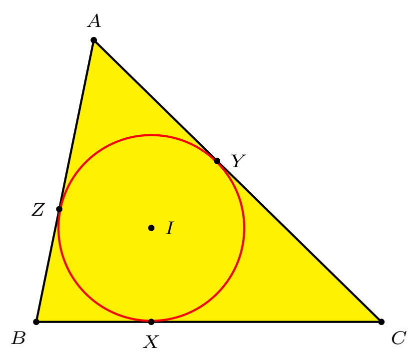
For a generic triangle ABC, we shall denote the lengths of the sides BC, CA, AB by a, b, c respectively. The symbol S denotes *twice* the area of the triangle.

1.2.1 The circumcircle

The **circumcircle** of triangle ABC is the unique circle passing through the three vertices A, B, C. Its center, the **circumcenter** O, is the intersection of the perpendicular bisectors of the three sides. The circumradius R is given by the law of sines:

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$





1.2.2 The incircle

The **incircle** is tangent to each of the three sides BC, CA, AB (without extension). Its center, the **incenter** I, is the intersection of the bisectors of the three angles. The **inradius** r is related to the area $\frac{1}{2}S$ by

$$S = (a + b + c)r.$$

If the incircle is tangent to the sides BC at X, CA at Y, and AB at Z, then

$$AY = AZ = \frac{b+c-a}{2}, \quad BZ = BX = \frac{c+a-b}{2}, \quad CX = CY = \frac{a+b-c}{2}.$$

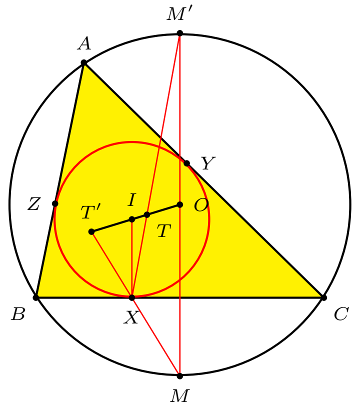
These expressions are usually simplified by introducing the *semiperimeter* $s = \frac{1}{2}(a + b + c)$:

$$AY = AZ = s - a, \qquad BZ = BX = s - b, \qquad CX = CY = s - c.$$

Also, $r = \frac{S}{2s}$.

1.2.3 The centers of similitude of (O) and (I)

Denote by T and T' respectively the internal and external centers of similitude of the circumcircle and incircle of triangle ABC.



These are points dividing the segment OI harmonically in the ratios

$$OT : TI = R : r, \qquad OT' : T'I = R : -r.$$

Exercises

1. Use the Ceva theorem to show that the lines AX, BY, CZ are concurrent. (The intersection is called the *Gergonne point* of the triangle).

2. Construct the three circles each passing through the Gergonne point and tangent to two sides of triangle ABC. The 6 points of tangency lie on a circle.

3. Given three points A, B, C not on the same line, construct three circles, with centers at A, B, C, mutually tangent to each other *externally*.

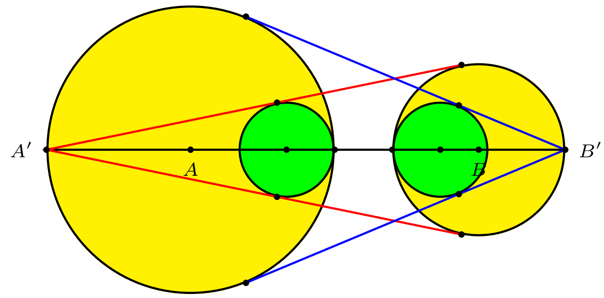
4. Two circles are orthogonal to each other if their tangents at an intersection are perpendicular to each other. Given three points A, B, C not on a line, construct three circles with these as centers and orthogonal to each other.

5. The centers A and B of two circles $A(a)$ and $B(b)$ are at a distance d apart. The line AB intersect the circles at A' and B' respectively, so that A, B are between A', B'.

   (1) Construct the tangents from A' to the circle $B(b)$, and the circle tangent to these two lines and to $A(a)$ internally.

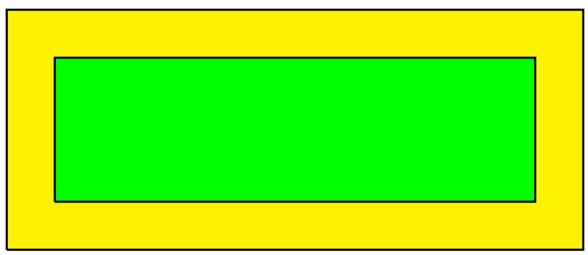
   (2) Construct the tangents from B' to the circle $A(a)$, and the circle tangent to these two lines and to $B(b)$ internally.

   (3) The two circles in (1) and (2) are congruent.



6. Given a point Z on a line segment AB, construct a right-angled triangle ABC whose incircle touches the hypotenuse AB at Z. [1]

7. (Paper Folding) The figure below shows a rectangular sheet of paper containing a border of uniform width. The paper may be any size and shape, but the border must be of such a width that the area of the inner rectangle is exactly half that of the sheet. You have no ruler or compasses, or even a pencil. You must determine the inner rectangle purely by paper folding. [2]



8. Let ABC be a triangle with incenter I.

(1a) Construct a tangent to the incircle at the point diametrically opposite to its point of contact with the side BC. Let this tangent intersect CA at Y_1 and AB at Z_1.

(1b) Same in part (a), for the side CA, and let the tangent intersect AB at Z_2 and BC at X_2.

(1c) Same in part (a), for the side AB, and let the tangent intersect BC at X_3 and CA at Y_3.

(2) Note that $AY_3 = AZ_2$. Construct the circle tangent to AC and AB at Y_3 and Z_2. How does this circle intersect the circumcircle of triangle ABC?

9. The incircle of $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D also, and touches CX and XB at Y and Z respectively.

[1] P. Yiu, G. Leversha, and T. Seimiya, Problem 2415 and solution, *Crux Math.* 25 (1999) 110; 26 (2000) 62 – 64.

[2] Problem 2519, *Journal of Recreational Mathematics*, 30 (1999-2000) 151 – 152.

(1) The four points E, F, Z, Y are concyclic. [3]

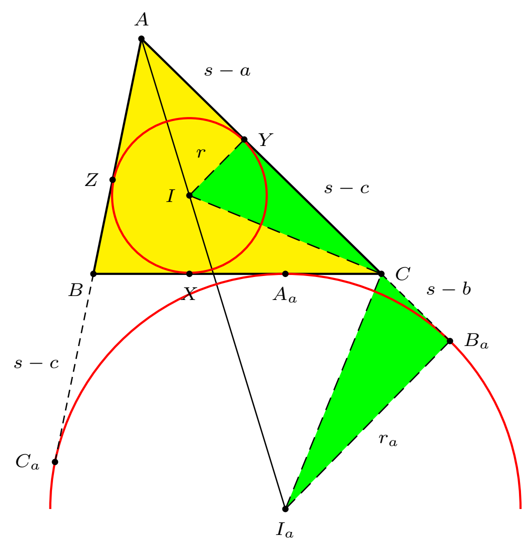
(2) What is the **locus** of the center of the circle $EFZY$? [4]

1.2.4 The Heron formula

The area of triangle ABC is given by

$$\frac{S}{2} = \sqrt{s(s-a)(s-b)(s-c)}.$$

This formula can be easily derived from a computation of the inradius r and the radius of one of the **tritangent circles** of the triangle. Consider the *excircle* $I_a(r_a)$ whose center is the intersection of the bisector of angle A and the external bisectors of angles B and C. If the incircle $I(r)$ and this excircle are tangent to the line AC at Y and Y' respectively, then



(1) from the similarity of triangles AIY and AI_aY',

$$\frac{r}{r_a} = \frac{s-a}{s};$$

(2) from the similarity of triangles CIY and I_aCY',

$$r \cdot r_a = (s-b)(s-c).$$

It follows that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

[3]International Mathematical Olympiad 1996.

[4]IMO 1996.

From this we obtain the famous Heron formula for the area of a triangle:

$$\frac{S}{2} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercises

1. $R = \frac{abc}{2S}$.

2. $r_a = \frac{S}{b+c-a}$.

3. Suppose the incircle of triangle ABC touches its sides BC, CA, AB at the points X, Y, Z respectively. Let X', Y', Z' be the antipodal points of X, Y, Z on the incircle. Construct the rays AX', BY', and CZ'.

Explain the concurrency of these rays by considering also the points of contact of the excircles of the triangle with the sides.

4. Construct the **tritangent circles** of a triangle ABC.

(1) Join each excenter to the midpoint of the corresponding side of ABC. These three lines intersect at a point P. (This is called the *Mittenpunkt* of the triangle).

(2) Join each excenter to the point of tangency of the incircle with the corresponding side. These three lines are concurrent at another point Q.

(3) The lines AP and AQ are symmetric with respect to the bisector of angle A; so are the lines BP, BQ and CP, CQ (with respect to the bisectors of angles B and C).

5. Construct the excircles of a triangle ABC.

(1) Let D, E, F be the midpoints of the sides BC, CA, AB. Construct the incenter S of triangle DEF, [5] and the **tangents from** S to each of the three excircles.

(2) The 6 points of tangency are on a circle, which is *orthogonal* to each of the excircles.

[5] This is called the Spieker point of triangle ABC.

1.3 Euler's formula and Steiner's porism

1.3.1 Euler's formula

The distance between the circumcenter and the incenter of a triangle is given by

$$OI^2 = R^2 - 2Rr.$$

Construct the **circumcircle** $O(R)$ of triangle ABC. **Bisect** angle A and **mark** the intersection M of the bisector with the circumcircle. Construct the circle $M(B)$ to intersect this bisector at a point I. This is the incenter since

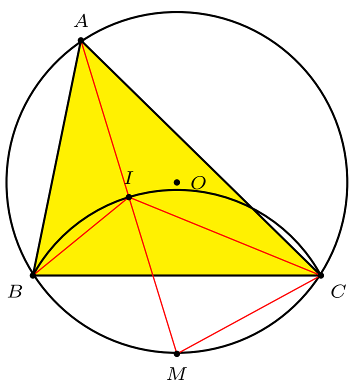
$$\angle IBC = \frac{1}{2}\angle IMC = \frac{1}{2}\angle AMC = \frac{1}{2}\angle ABC,$$

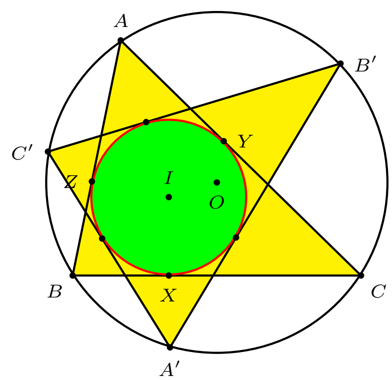
and for the same reason $\angle ICB = \frac{1}{2}\angle ACB$. Note that

(1) $IM = MB = MC = 2R\sin\frac{A}{2}$,

(2) $IA = \frac{r}{\sin\frac{A}{2}}$, and

(3) by the theorem of intersecting chords, $R^2 - OI^2 =$ the *power* of I with respect to the circumcircle = $IA \cdot IM = 2Rr$.





1.3.2 Steiner's porism

[6] Construct the circumcircle (O) and the incircle (I) of triangle ABC. **Animate** a point A' on the circumcircle, and construct the **tangents from** A' to the incircle (I). Extend these tangents to intersect the circumcircle again at B' and C'. The lines $B'C'$ is always tangent to the incircle. This is the famous theorem on Steiner porism: *if two given circles are the circumcircle and incircle of one triangle, then they are the circumcircle and incircle of a continuous family of* poristic *triangles*.

[6] Also known as Poncelet's porism.

Exercises

1. $r \leq \frac{1}{2}R$. When does equality hold?

2. Suppose $OI = d$. Show that there is a right-angled triangle whose sides are d, r and $R - r$. Which one of these is the hypotenuse?

3. Given a point I inside a circle $O(R)$, construct a circle $I(r)$ so that $O(R)$ and $I(r)$ are the circumcircle and incircle of a (family of poristic) triangle(s).

4. Given the circumcenter, incenter, and one vertex of a triangle, construct the triangle.

5. Construct an animation picture of a triangle whose circumcenter lies on the incircle. [7]

[7] Hint: $OI = r$.

1.4 Appendix: Mixtilinear incircles

A mixtilinear incircle of triangle ABC is one that is tangent to two sides of the triangle and to the circumcircle internally. Denote by A' the point of tangency of the mixtilinear incircle $K(\rho)$ in angle A with the circumcircle. The center K clearly lies on the bisector of angle A, and $AK : KI = \rho : -(\rho - r)$. In terms of barycentric coordinates,

$$K = \frac{1}{r}[-(\rho - r)A + \rho I].$$

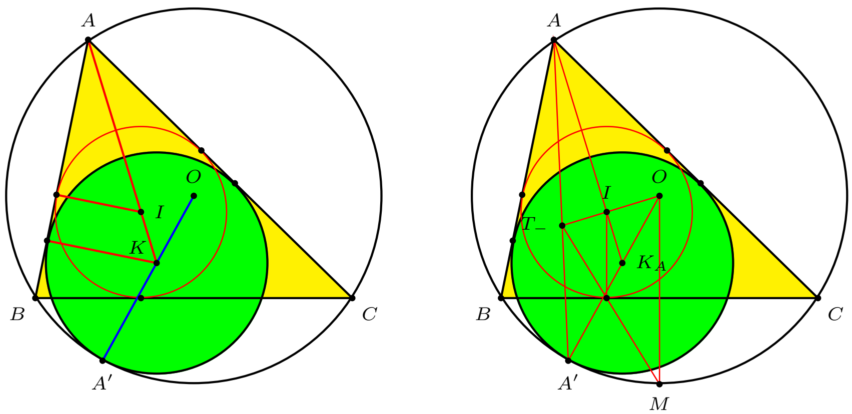
Also, since the circumcircle $O(A')$ and the mixtilinear incircle $K(A')$ touch each other at A', we have $OK : KA' = R - \rho : \rho$, where R is the circumradius. From this,

$$K = \frac{1}{R}[\rho O + (R - \rho)A'].$$

Comparing these two equations, we obtain, by rearranging terms,

$$\frac{RI - rO}{R - r} = \frac{R(\rho - r)A + r(R - \rho)A'}{\rho(R - r)}.$$

We note some interesting consequences of this formula. First of all, it gives the intersection of the lines joining AA' and OI. Note that the point on the line OI represented by the left hand side is T'.



This leads to a simple construction of the mixtilinear incircle: [8]

> Given a triangle ABC, let P be the external center of similitude of the circumcircle (O) and incircle (I). Extend AP to intersect the circumcircle at A'. The intersection of AI and $A'O$ is the center K_A of the mixtilinear incircle in angle A.

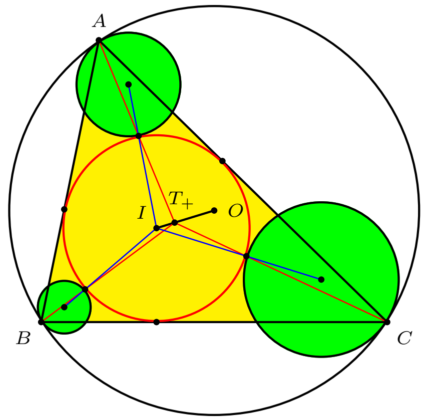
The other two mixtilinear incircles can be constructed similarly.

[8] P.Yiu, Mixtilinear incircles, *Amer. Math. Monthly* 106 (1999) 952 – 955.

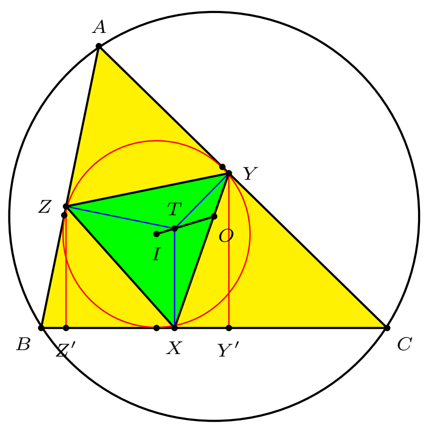
Exercises

1. Can any of the centers of similitude of (O) and (I) lie outside triangle ABC?

2. There are three circles each tangent internally to the circumcircle at a vertex, and externally to the incircle. It is known that the three lines joining the points of tangency of each circle with (O) and (I) pass through the internal center T of similitude of (O) and (I). Construct these three circles. [9]



3. Let T be the insimilicenter of (O) and (I), with pedals Y and Z on CA and AB respectively. If Y' and Z' are the pedals of Y and Z on BC, calculate the length of $Y'Z'$. [10]



[9] A.P. Hatzipolakis and P. Yiu, Triads of circles, preprint.

[10] A.P. Hatzipolakis and P. Yiu, Pedal triangles and their shadows, *Forum Geom.*, 1 (2001) 81 – 90.

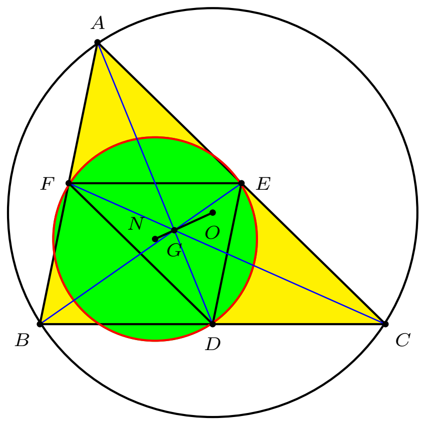
Chapter 2

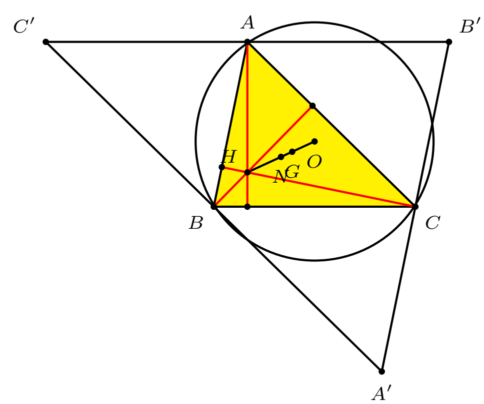
The Euler Line and the Nine-point Circle

2.1 The Euler line

2.1.1 Homothety

The similarity transformation $\mathsf{h}(T, r)$ which carries a point X to the point X' which divides $TX' : TX = r : 1$ is called the *homothety* with center T and ratio r.





2.1.2 The centroid

The three medians of a triangle intersect at the centroid, which divides each median in the ratio $2 : 1$. If D, E, F are the midpoints of the sides BC, CA, AB of triangle ABC, the centroid G divides the median AD in the ratio $AG : GD = 2 : 1$. The *medial* triangle DEF is the image of triangle ABC under the homothety $\mathsf{h}(G, -\frac{1}{2})$. The circumcircle of the medial triangle has radius $\frac{1}{2}R$. Its center is the point $N = \mathsf{h}(G, -\frac{1}{2})(O)$. This divides the segement OG in the ratio $OG : GN = 2 : 1$.

2.1.3 The orthocenter

The *dilated triangle* $A'B'C'$ is the image of ABC under the homothety $\mathsf{h}(G, -2)$. [1] Since the altitudes of triangle ABC are the perpendicular bisectors of the sides of triangle $A'B'C'$, they intersect at the homothetic image of the circumcenter O. This point is called the *orthocenter* of triangle ABC, and is usually denoted by H. Note that

$$OG : GH = 1 : 2.$$

The line containing O, G, H is called the Euler line of triangle ABC. The Euler line is undefined for the equilateral triangle, since these points coincide.

Exercises

1. A triangle is equilateral if and only if two of its circumcenter, centroid, and orthocenter coincide.

2. The circumcenter N of the medial triangle is the midpoint of OH.

3. The Euler lines of triangles HBC, HCA, HAB intersect at a point on the Euler line of triangle ABC. What is this intersection?

4. The Euler lines of triangles IBC, ICA, IAB also intersect at a point on the Euler line of triangle ABC. [2]

5. (Gossard's Theorem) Suppose the Euler line of triangle ABC intersects the side lines BC, CA, AB at X, Y, Z respectively. The Euler lines of the triangles AYZ, BZX and CXY bound a triangle homothetic to ABC with ratio -1 and with homothetic center on the Euler line of ABC.

6. What is the **locus** of the centroids of the poristic triangles with the same circumcircle and incircle of triangle ABC? How about the orthocenter?

7. Let $A'B'C'$ be a poristic triangle with the same circumcircle and incircle of triangle ABC, and let the sides of $B'C'$, $C'A'$, $A'B'$ touch the incircle at X, Y, Z.

(i) What is the **locus** of the centroid of XYZ?

(ii) What is the **locus** of the orthocenter of XYZ?

(iii) What can you say about the Euler line of the triangle XYZ?

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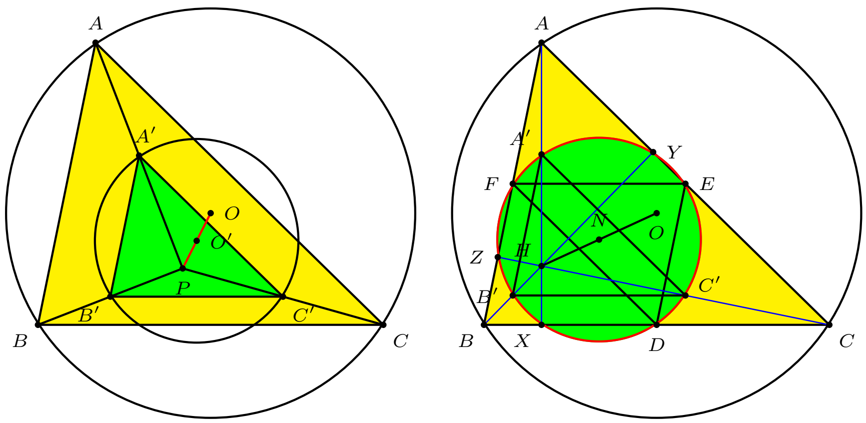
[1] It is also called the anticomplementary triangle.

[2] Problem 1018, *Crux Mathematicorum*.

2.2 The nine-point circle

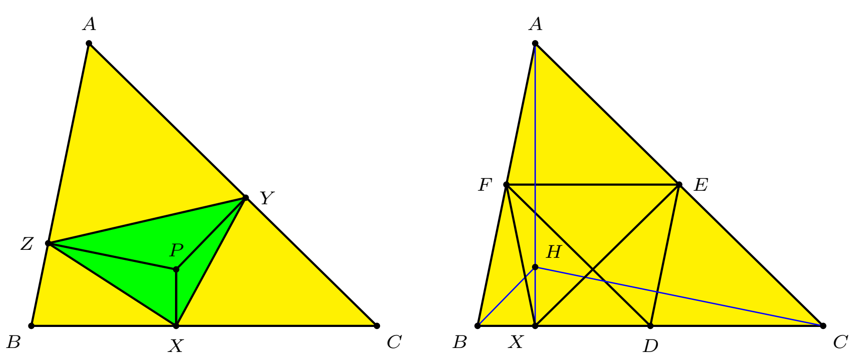
2.2.1 The Euler triangle as a midway triangle

The image of ABC under the homothety $\mathsf{h}(P, \frac{1}{2})$ is called the *midway* triangle of P. The midway triangle of the orthocenter H is called the *Euler triangle*. The circumcenter of the midway triangle of P is the midpoint of OP. In particular, the circumcenter of the Euler triangle is the midpoint of OH, which is the same as N. *The medial triangle and the Euler triangle have the same circumcircle.*



2.2.2 The orthic triangle as a pedal triangle

The *pedals* of a point are the intersections of the sidelines with the corresponding perpendiculars through P. They form the *pedal triangle* of P. The pedal triangle of the orthocenter H is called the *orthic triangle* of ABC.



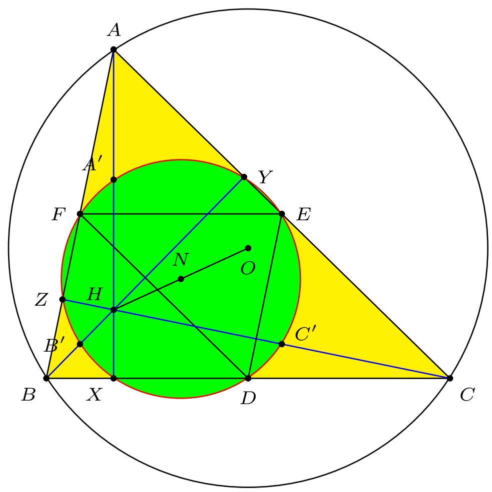
The pedal X of the orthocenter H on the side BC is also the pedal of A on the same line, and can be regarded as the *reflection* of A in the line EF. It follows that

$$\angle EXF = \angle EAF = \angle EDF,$$

since $AEDF$ is a parallelogram. From this, the point X lies on the circle DEF; similarly for the pedals Y and Z of H on the other two sides CA and AB.

2.2.3 The nine-point circle

From §2.2.1,2 above, the medial triangle, the Euler triangle, and the orthic triangle have the same circumcircle. This is called the *nine-point circle* of triangle ABC. Its center N, the midpoint of OH, is called the *nine-point center* of triangle ABC.



Exercises

1. On the Euler line,
$$OG : GN : NH = 2 : 1 : 3.$$

2. Let P be a point on the circumcircle. What is the **locus** of the midpoint of HP? Can you give a proof?

3. Let ABC be a triangle and P a point. The perpendiculars at P to PA, PB, PC intersect BC, CA, AB respectively at A', B', C'.

   (1) A', B', C' are collinear. [3]

   (2) The nine-point circles of the (right-angled) triangles PAA', PBB', PCC' are concurrent at P and another point P'. Equivalently, their centers are collinear. [4]

[3] B. Gibert, Hyacinthos 1158, 8/5/00.

[4] A.P. Hatzipolakis, Hyacinthos 3166, 6/27/01. The three midpoints of AA', BB', CC' are collinear. The three nine-point circles intersect at P and its pedal on this line.

4. If the midpoints of AP, BP, CP are all on the nine-point circle, must P be the orthocenter of triangle ABC? [5]

5. (Paper folding) Let N be the nine-point center of triangle ABC.

(1) Fold the perpendicular to AN at N to intersect CA at Y and AB at Z.

(2) Fold the reflection A' of A in the line YZ.

(3) Fold the reflections of B in $A'Z$ and C in $A'Y$.

What do you observe about these reflections?

2.2.4 Triangles with nine-point center on the circumcircle

We begin with a circle, center O and a point N on it, and construct a family of triangles with (O) as circumcircle and N as nine-point center.

(1) Construct the nine-point circle, which has center N, and passes through the midpoint M of ON.

(2) **Animate** a point D on the minor arc of the nine-point circle *inside* the circumcircle.

(3) Construct the chord BC of the circumcircle with D as midpoint. (This is simply the perpendicular to OD at D).

(4) Let X be the point on the nine-point circle antipodal to D. Complete the parallelogram $ODXA$ (by **translating** the vector **DO** to X).

The point A lies on the circumcircle and the triangle ABC has nine-point center N on the circumcircle.

Here is an curious property of triangles constructed in this way: let A', B', C' be the reflections of A, B, C in their own opposite sides. The reflection triangle $A'B'C'$ degenerates, *i.e.*, the three points A', B', C' are collinear. [6]

[5] Yes. See P. Yiu and J. Young, Problem 2437 and solution, *Crux Math.* 25 (1999) 173; 26 (2000) 192.

[6] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, Chapter 16.

2.3 Simson lines and reflections

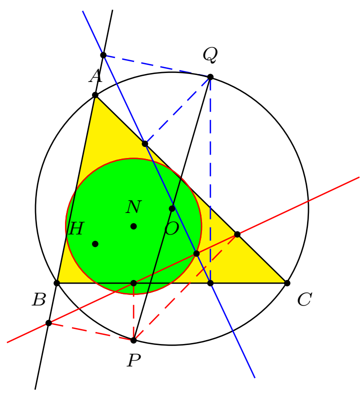
2.3.1 Simson lines

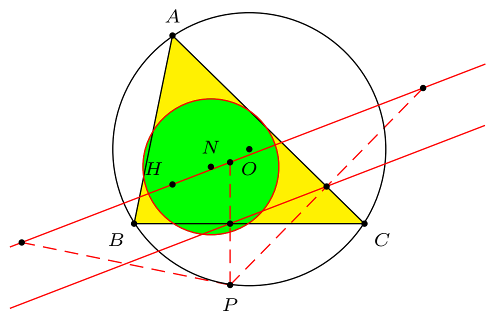
Let P on the circumcircle of triangle ABC.

(1) Construct its pedals on the side lines. These pedals are always collinear. The line containing them is called the *Simson line* $\mathsf{s}(P)$ of P.

(2) Let P' be the point on the cirucmcircle antipodal to P. Construct the Simson line (P') and **trace** the intersection point $\mathsf{s}(P) \cap (P')$. Can you identify this locus?

(3) Let the Simson line $\mathsf{s}(P)$ intersect the side lines BC, CA, AB at X, Y, Z respectively. The circumcenters of the triangles AYZ, BZX, and CXY form a triangle homothetic to ABC at P, with ratio $\frac{1}{2}$. These circumcenters therefore lie on a circle tangent to the circumcircle at P.





2.3.2 Line of reflections

Construct the **reflections** of the P in the side lines. These reflections are always collinear, and the line containing them always passes through the orthocenter H, and is parallel to the Simson line $\mathsf{s}(P)$.

2.3.3 Musselman's Theorem: Point with given line of reflections

Let $\mathcal{L}$ be a line through the orthocenter H.

(1) Choose an arbitrary point Q on the line $\mathcal{L}$ and **reflect** it in the side lines BC, CA, AB to obtain the points X, Y, Z.

(2) Construct the circumcircles of AYZ, BZX and CXY. These circles have a common point P, which happens to lie on the circumcircle.

(3) Construct the reflections of P in the side lines of triangle ABC. These are on the line $\mathcal{L}$.

2.3.4 Musselman's Theorem: Point with given line of reflections (Alternative)

Animate a point Q on the circumcircle. Let Q' be the second intersection of the line HQ with the circumcircle.

(1) The **reflections** X, Y, Z of Q on the side lines BC, CA, AB are collinear; so are those X', Y', Z' of Q'.

(2) The lines XX', YY', ZZ' intersect at a point P, which happens to be on the circumcircle.

(3) Construct the reflections of P in the side lines of triangle ABC. These are on the line HQ.

2.3.5 Blanc's Theorem

Animate a point P on the circumcircle, together with its antipodal point P'.

(1) Construct the line PP' to intersect the side lines BC, CA, AB at X, Y, Z respectively.

(2) Construct the circles with diameters AX, BY, CZ. These three circles have two common points. One of these is on the circumcircle. Label this point P^*, and the other common point Q.

(3) What is the **locus** of Q?

(4) The line P^*Q passes through the orthocenter H. As such, it is the line of reflection of a point on the circumcircle. What is this point?

(5) Construct the Simson lines of P and P'. They intersect at a point on the nine-point circle. What is this point?

Exercises

1. Let P be a given point, and $A'B'C'$ the homothetic image of ABC under $\mathsf{h}(P, -1)$ (so that P is the common midpoint of AA', BB' and CC').

(1) The circles $AB'C'$, $BC'A'$ and $CA'B'$ intersect at a point Q on the circumcircle;

(2) The circles ABC', BCA' and CAB' intersect at a point Q' such that P is the midpoint of QQ'. [7]

[7]Musselman, *Amer. Math. Monthly*, 47 (1940) 354 – 361. If $P = (u : v : w)$, the intersection of the three circles in (1) is the point

$$\left(\frac{1}{b^2(u+v-w)w - c^2(w+u-v)v} : \cdots : \cdots\right)$$

on the circumcircle. This is the isogonal conjugate of the infinite point of the line

$$\sum_{\text{cyclic}} \frac{u(v+w-u)}{a^2} x = 0.$$

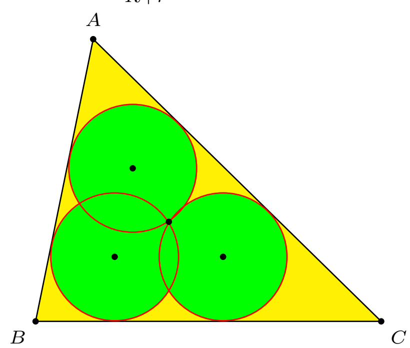
2.4 Appendix: Homothety

Two triangles are homothetic if the corresponding sides are parallel.

2.4.1 Three congruent circles with a common point and each tangent to two sides of a triangle

[8] Given a triangle ABC, to construct three congruent circles passing through a common point P, each tangent to two sides of the triangle.

Let t be the common radius of these congruent circles. The centers of these circles, I_1, I_2, I_3, lie on the bisectors IA, IB, IC respectively. Note that the lines I_2I_3 and BC are parallel; so are the pairs I_3I_1, CA, and I_1I_2, AB. It follows that $\triangle I_1I_2I_3$ and ABC are similar. Indeed, they are in *homothetic* from their common incenter I. The ratio of homothety can be determined in two ways, by considering their circumcircles and their incircles. Since the circumradii are t and R, and the inradii are $r-t$ and r, we have $\frac{r-t}{r} = \frac{r}{R}$. From this, $t = \frac{Rr}{R+r}$.



How does this help constructing the circles? Note that the line joining the circumcenters P and O passes through the center of homothety I, and indeed,

$$OI : IP = R : t = R : \frac{Rr}{R+r} = R + r : r.$$

Rewriting this as $OP : PI = R : r$, we see that P is indeed the internal center of similitude of (O) and (I).

Now the construction is easy.

2.4.2 Squares inscribed in a triangle and the Lucas circles

Given a triangle ABC, to construct the inscribed square with a side along BC we contract the square erected externally on the same side by a homothety at vertex A. The ratio of the homothety is $h_a : h_a + a$, where h_a is the altitude on BC. Since $h_a = \frac{S}{a}$, we have

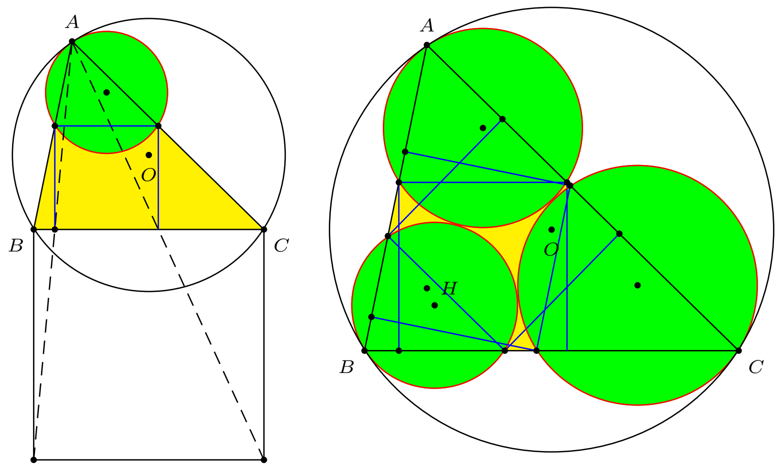
$$\frac{h_a}{h_a + a} = \frac{S}{S + a^2}.$$

[8] Problem 2137, *Crux Mathematicorum*.

The circumcircle is contracted into a circle of radius

$$R_a = R \cdot \frac{S}{S+a^2} = \frac{abc}{2S} \cdot \frac{S}{S+a^2} = \frac{abc}{2(S+a^2)},$$

and this passes through the two vertices of the inscribed on the sides AB and AC. Similarly, there are two other inscribed squares on the sides CA and AB, and two corresponding circles, tangent to the circumcircle at B and C respectively. It is remarkable that these three circles are mutually tangent to each other. These are called the Lucas circles of the triangle. [9]



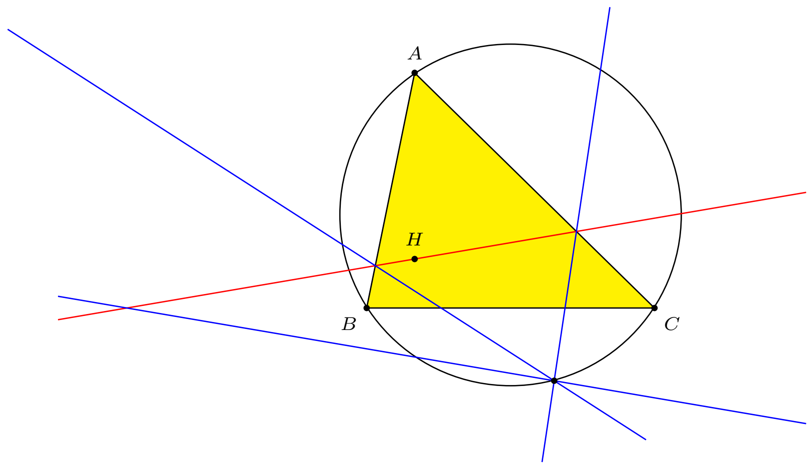
2.4.3 More on reflections

(1) The reflections of a line $\mathcal{L}$ in the side lines of triangle ABC are concurrent if and only if $\mathcal{L}$ passes through the orthocenter. In this case, the intersection is a point on the circumcircle. [10]

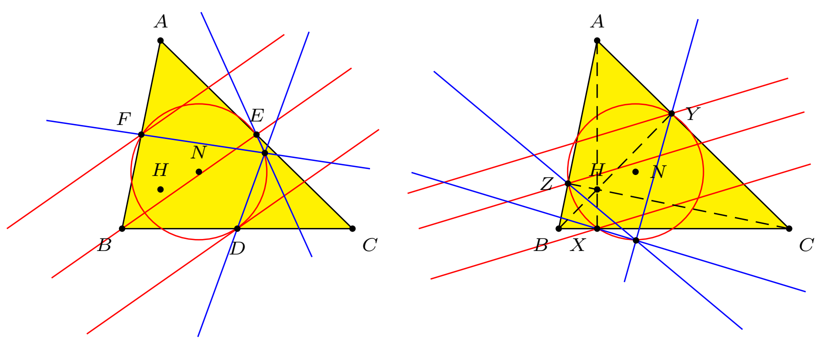
---

[9]See A.P. Hatzipolakis and P. Yiu, The Lucas circles, *Amer. Math. Monthly*, 108 (2001) 444 – 446. After the publication of this note, we recently learned that Eduoard Lucas (1842 – 1891) wrote about this triad of circles, considered by an anonymous author, as the three circles mutually tangent to each other and each tangent to the circumcircle at a vertex of ABC. The connection with the inscribed squares were found by Victor Thébault (1883 – 1960).

[10]S.N. Collings, Reflections on a triangle, part 1, *Math. Gazette*, 57 (1973) 291 – 293; M.S. Longuet-Higgins, Reflections on a triangle, part 2, ibid., 293 – 296.



(2) Construct *parallel* lines $\mathcal{L}_a$, $\mathcal{L}_b$, and $\mathcal{L}_c$ through the D, E, F be the midpoints of the sides BC, CA, AB of triangle ABC. **Reflect** the lines BC in $\mathcal{L}_a$, CA in $\mathcal{L}_b$, and AB in $\mathcal{L}_c$. These three reflection lines intersect at a point on the nine-point circle.[11]



(3) Construct *parallel* lines $\mathcal{L}_a$, $\mathcal{L}_b$, and $\mathcal{L}_c$ through the pedals of the vertices A, B, C on their opposite sides. Reflect these lines in the respective side lines of triangle ABC. The three reflection lines intersect at a point on the nine-point circle. [12]

[11] This was first discovered in May, 1999 by a high school student, Adam Bliss, in Atlanta, Georgia. A proof can be found in F.M. van Lamoen, Morley related triangles on the nine-point circle, *Amer. Math. Monthly*, 107 (2000) 941 – 945. See also, B. Shawyer, A remarkable concurrence, *Forum Geom.*, 1 (2001) 69 – 74.

[12] *Ibid.*

Chapter 3

Homogeneous Barycentric Coordinates

3.1 Barycentric coordinates with reference to a triangle

3.1.1 Homogeneous barycentric coordinates

The notion of barycentric coordinates dates back to Möbius. In a given triangle ABC, every point P is coordinatized by a triple of numbers $(u : v : w)$ in such a way that the system of masses u at A, v at B, and w at C will have its *balance point* at P. These masses can be taken in the proportions of the areas of triangle PBC, PCA and PAB. Allowing the point P to be outside the triangle, we use *signed areas* of oriented triangles. The *homogeneous barycentric coordinates* of P with reference to ABC is a triple of numbers $(x : y : z)$ such that

$$x : y : z = \triangle PBC : \triangle PCA : \triangle PAB.$$

Examples

1. The *centroid* G has homogeneous barycentric coordinates $(1 : 1 : 1)$. The areas of the triangles GBC, GCA, and GAB are equal. [1]

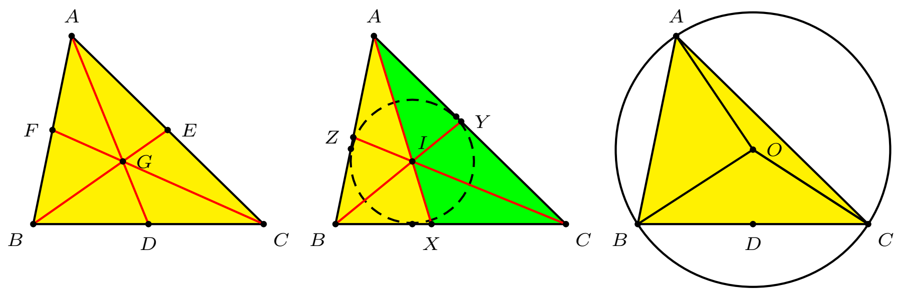
2. The *incenter* I has homogeneous barycentric coordinates $(a : b : c)$. If r denotes the inradius, the areas of triangles IBC, ICA and IAB are respectively $\frac{1}{2}ra$, $\frac{1}{2}rb$, and $\frac{1}{2}rc$. [2]

3. The *circumcenter*. If R denotes the circumradius, the coordinates of the circumcenter O are [3]

$$\triangle OBC : \triangle OCA : \triangle OAB$$

[1] In Kimberling's Encyclopedia of Triangle Centers, [ETC], the centroid appears as X_2.
[2] In ETC, the incenter appears as X_1.
[3] In ETC, the circumcenter appears as X_3.

$$
\begin{aligned}
&= \frac{1}{2}R^2\sin 2A : \frac{1}{2}R^2\sin 2B : \frac{1}{2}R^2\sin 2C\\
&= \sin A\cos A : \sin B\cos B : \sin C\cos C\\
&= a\cdot\frac{b^2+c^2-a^2}{2bc} : b\cdot\frac{c^2+a^2-b^2}{2ca} : c\cdot\frac{a^2+b^2-c^2}{2ab}\\
&= a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2).
\end{aligned}
$$



4. Points on the line BC have coordinates of the form $(0 : y : z)$. Likewise, points on CA and AB have coordinates of the forms $(x : 0 : z)$ and $(x : y : 0)$ respectively.

Exercise

1. Verify that the sum of the coordinates of the circumcenter given above is $4S^2$:

$$a^2(b^2+c^2-a^2)+b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)=4S^2,$$

where S is twice the area of triangle ABC.

2. Find the coordinates of the excenters. [4]

3.1.2 Absolute barycentric coordinates

Let P be a point with (homogeneous barycentric) coordinates $(x : y : z)$. If $x+y+z \neq 0$, we obtain the *absolute* barycentric coordinates by scaling the coefficients to have a unit sum:

$$P = \frac{x\cdot A + y\cdot B + z\cdot C}{x+y+z}.$$

If P and Q are given in absolute barycentric coordinates, the point X which divides PQ in the ratio $PX : XQ = p : q$ has absolute barycentric coordinates $\dfrac{q\cdot P + p\cdot Q}{p+q}$. It is, however, convenient to perform calculations avoiding denominators of fractions.

[4] $I_a = (-a : b : c)$, $I_b = (a : -b : c)$, $I_c = (a : b : -c)$.

We therefore adapt this formula in the following way: if $P = (u : v : w)$ and $Q = (u' : v' : w')$ are the homogeneous barycentric coordinates satisfying $u + v + w = u' + v' + w'$, the point X dividing PQ in the ratio $PX : XQ = p : q$ has homogeneous barycentric coordinates

$$(qu + pu' : qv + pv' : qw + pw').$$

Example: Internal center of similitudes of the circumcircle and the incircle

These points, T and T', divide the segment OI harmonically in the ratio of the circumradius $R = \frac{abc}{2S}$ and the inradius $r = \frac{S}{2s}$. Note that $R : r = \frac{abc}{2S} : \frac{S}{2s} = sabc : S^2$.

Since

$$O = (a^2(b^2 + c^2 - a^2) : \cdots : \cdots)$$

with coordinates sum $4S^2$ and $I = (a : b : c)$ with coordinates sum $2s$, we equalize their sums and work with

$$\begin{array}{rcl} O & = & (sa^2(b^2 + c^2 - a^2) : \cdots : \cdots), \\ I & = & (2S^2 a : 2S^2 b : 2S^2 c). \end{array}$$

The internal center of similitude T divides OI in the ratio $OT : TI = R : r$, the a-component of its homogeneous barycentric coordinates can be taken as

$$S^2 \cdot sa^2(b^2 + c^2 - a^2) + sabc \cdot 2S^2 a.$$

The simplification turns out to be easier than we would normally expect:

$$\begin{array}{rl} & S^2 \cdot sa^2(b^2 + c^2 - a^2) + sabc \cdot 2S^2 a \\ = & sS^2 a^2(b^2 + c^2 - a^2 + 2bc) \\ = & sS^2 a^2((b + c)^2 - a^2) \\ = & sS^2 a^2(b + c + a)(b + c - a) \\ = & 2s^2 S^2 \cdot a^2(b + c - a). \end{array}$$

The other two components have similar expressions obtained by *cyclically permuting* a, b, c. It is clear that $2s^2S^2$ is a factor common to the three components. Thus, the homogeneous barycentric coordinates of the internal center of similitude are [5]

$$(a^2(b + c - a) : b^2(c + a - b) : c^2(a + b - c)).$$

Exercises

1. The external center of similitude of (O) and (I) has homogeneous barycentric coordinates [6]

$$(a^2(a + b - c)(c + a - b) : b^2(b + c - a)(a + b - c) : c^2(c + a - b)(b + c - a)),$$

---

[5] In ETC, the internal center of similitude of the circumcircle and the incircle appears as the point X_{55}.

[6] In ETC, the external center of similitude of the circumcircle and the incircle appears as the point X_{56}.

which can be taken as

$$\left(\frac{a^2}{b+c-a} : \frac{b^2}{c+a-b} : \frac{c^2}{a+b-c}\right).$$

2. The orthocenter H lies on the Euler line and divides the segment OG externally in the ratio $OH : HG = 3 : -2$. [7] Show that its homogeneous barycentric coordinates can be written as

$$H = (\tan A : \tan B : \tan C),$$

or equivalently,

$$H = \left(\frac{1}{b^2+c^2-a^2} : \frac{1}{c^2+a^2-b^2} : \frac{1}{a^2+b^2-c^2}\right).$$

3. Make use of the fact that the nine-point center N divides the segment OG in the ratio $ON : NG = 3 : -1$ to show that its barycentric coordinates can be written as [8]

$$N = (a\cos(B-C) : b\cos(C-A) : c\cos(A-B)).$$

[7] In ETC, the orthocenter appears as the point X_4.
[8] In ETC, the nine-point center appears as the point X_5.

3.2 Cevians and traces

Because of the fundamental importance of the Ceva theorem in triangle geometry, we shall follow traditions and call the three lines joining a point P to the vertices of the reference triangle ABC the *cevians* of P. The intersections A_P, B_P, C_P of these cevians with the side lines are called the *traces* of P. The coordinates of the traces can be very easily written down:

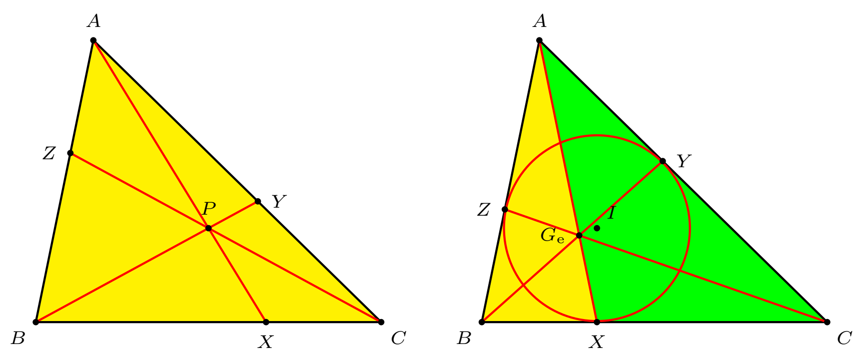
$$A_P = (0 : y : z), \qquad B_P = (x : 0 : z), \qquad C_P = (x : y : 0).$$

3.2.1 Ceva Theorem

Three points X, Y, Z on BC, CA, AB respectively are the traces of a point if and only if they have coordinates of the form

$$\begin{array}{ccccccc} X & = & 0 & : & y & : & z, \\ Y & = & x & : & 0 & : & z, \\ Z & = & x & : & y & : & 0, \end{array}$$

for some x, y, z.



3.2.2 Examples

The Gergonne point

The points of tangency of the incircle with the side lines are

$$\begin{array}{ccccccc} X & = & 0 & : & s-c & : & s-b, \\ Y & = & s-c & : & 0 & : & s-a, \\ Z & = & s-b & : & s-a & : & 0. \end{array}$$

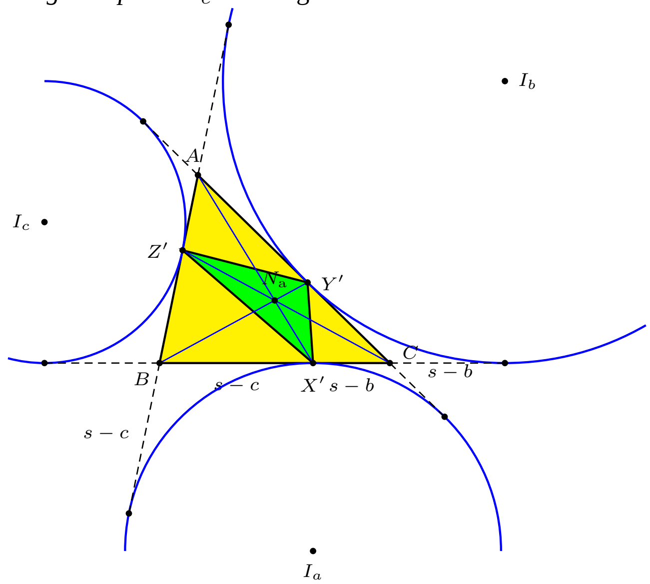
These can be reorganized as

$$\begin{array}{ccccccc} X & = & 0 & : & \frac{1}{s-b} & : & \frac{1}{s-c}, \\ Y & = & \frac{1}{s-a} & : & 0 & : & \frac{1}{s-c}, \\ Z & = & \frac{1}{s-a} & : & \frac{1}{s-b} & : & 0. \end{array}$$

It follows that AX, BY, CZ intersect at a point with coordinates

$$\left(\frac{1}{s-a}:\frac{1}{s-b}:\frac{1}{s-c}\right).$$

This is called the *Gergonne point* G_e of triangle ABC. [9]



The Nagel point

The points of tangency of the excircles with the corresponding sides have coordinates

$$\begin{aligned} X' &= (0:s-b:s-c), \\ Y' &= (s-a:0:s-c), \\ Z' &= (s-a:s-b:0). \end{aligned}$$

These are the traces of the point with coordinates

$$(s-a:s-b:s-c).$$

This is the *Nagel point* N_a of triangle ABC. [10]

Exercises

1. The Nagel point N_a lies on the line joining the incenter to the centroid; it divides IG in the ratio $IN_a : N_aG = 3 : -2$.

---

[9] The Gergonne point appears in ETC as the point X_7.
[10] The Nagel point appears in ETC as the point X_8.

3.3 Isotomic conjugates

The Gergonne and Nagel points are examples of isotomic conjugates. Two points P and Q (not on any of the side lines of the reference triangle) are said to be isotomic conjugates if their respective traces are symmetric with respect to the midpoints of the corresponding sides. Thus,

$$BA_P = A_QC, \qquad CB_P = B_QA, \qquad AC_P = C_QB.$$

We shall denote the isotomic conjugate of P by $P^\bullet$. If $P = (x : y : z)$, then

$$P^\bullet = (\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$$

.

3.3.1 Equal-parallelian point

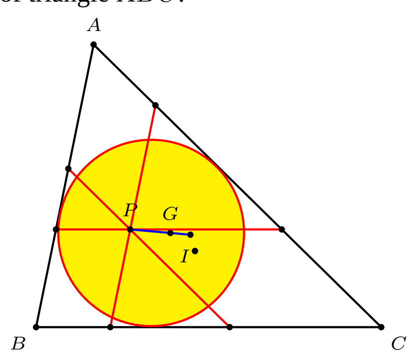
Given triangle ABC, we want to construct a point P the three lines through which parallel to the sides cut out equal intercepts. Let $P = xA + yB + zC$ in absolute barycentric coordinates. The parallel to BC cuts out an intercept of length $(1-x)a$. It follows that the three intercepts parallel to the sides are equal if and only if

$$1 - x : 1 - y : 1 - z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

The right hand side clearly gives the homogeneous barycentric coordinates of $I^\bullet$, the isotomic conjugate of the incenter I. [11] This is a point we can easily construct. Now, translating into *absolute* barycentric coordinates:

$$I^\bullet = \frac{1}{2}[(1-x)A + (1-y)B + (1-z)C] = \frac{1}{2}(3G - P).$$

we obtain $P = 3G - 2I^\bullet$, and can be easily constructed as the point dividing the segment $I^\bullet G$ externally in the ratio $I^\bullet P : PG = 3 : -2$. The point P is called the equal-parallelian point of triangle ABC. [12]



[11] The isotomic conjugate of the incenter appears in ETC as the point X_{75}.
[12] It appears in ETC as the point X_{192}.

Exercises

1. Calculate the homogeneous barycentric coordinates of the equal-parallelian point and the length of the equal parallelians. [13]

2. Let $A'B'C'$ be the midway triangle of a point P. The line $B'C'$ intersects CA at
$$\begin{array}{ll} B_a = B'C' \cap CA, & C_a = B'C' \cap AB, \\ C_b = C'A' \cap AB, & A_b = C'A' \cap BC, \\ A_c = A'B' \cap BC, & B_c = A'B' \cap CA. \end{array}$$
Determine P for which the three segments B_aC_a, C_bA_b and A_cB_c have equal lengths. [14]

3.3.2 Yff's analogue of the Brocard points

Consider a point $P = (x : y : z)$ satisfying $BA_P = CB_P = AC_P = w$. This means that
$$\frac{z}{y+z}a = \frac{x}{z+x}b = \frac{y}{x+y}c = w.$$
Elimination of x, y, z leads to
$$0 = \begin{vmatrix} 0 & -w & a-w \\ b-w & 0 & -w \\ -w & c-w & 0 \end{vmatrix} = (a-w)(b-w)(c-w) - w^3.$$
Indeed, w is the unique positive root of the cubic polynomial
$$(a-t)(b-t)(c-t) - t^3.$$
This gives the point
$$P = \left(\left(\frac{c-w}{b-w}\right)^{\frac{1}{3}} : \left(\frac{a-w}{c-w}\right)^{\frac{1}{3}} : \left(\frac{b-w}{a-w}\right)^{\frac{1}{3}}\right).$$

The isotomic conjugate
$$P^{\bullet} = \left(\left(\frac{b-w}{c-w}\right)^{\frac{1}{3}} : \left(\frac{c-w}{a-w}\right)^{\frac{1}{3}} : \left(\frac{a-w}{b-w}\right)^{\frac{1}{3}}\right)$$
satisfies
$$CA_P = AB_P = BC_P = w.$$

These points are usually called the *Yff analogues of the Brocard points*. [15] They were briefly considered by A.L. Crelle. [16]

---

[13] $(ca+ab-bc : ab+bc-ca : bc+ca-ab)$. The common length of the equal parallelians is $\frac{2abc}{ab+bc+ca}$.

[14] A.P. Hatzipolakis, Hyacinthos, message 3190, 7/13/01. $P = (3bc-ca-ab : 3ca-ab-bc : 3ab-bc-ca)$. This point is not in the current edition of ETC. It is the reflection of the equal-parallelian point in $I^{\bullet}$. In this case, the common length of the segment is $\frac{2abc}{ab+bc+ca}$, as in the equal-parallelian case.

[15] P. Yff, An analogue of the Brocard points, *Amer. Math. Monthly*, 70 (1963) 495 – 501.

[16] A.L. Crelle, 1815.

3.4 Conway's formula

3.4.1 Notation

Let S denote *twice* the area of triangle ABC. For a real number θ, denote $S \cdot \cot\theta$ by S_θ. In particular,

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

For arbitrary θ and φ, we shall simply write $S_{\theta\varphi}$ for $S_\theta \cdot S_\varphi$.

We shall mainly make use of the following relations.

Lemma

(1) $S_B + S_C = a^2$, $S_C + S_A = b^2$, $S_A + S_B = c^2$.

(2) $S_{AB} + S_{BC} + S_{CA} = S^2$.

Proof. (1) is clear. For (2), since $A + B + C = 180°$, $\cot(A + B + C)$ is infinite. Its denominator

$$\cot A \cdot \cot B + \cot B \cdot \cot C + \cot C \cdot \cot A - 1 = 0.$$

From this, $S_{AB}+S_{BC}+S_{CA} = S^2(\cot A\cdot\cot B+\cot B\cdot\cot C+\cot C\cdot\cot A) = S^2$.

Examples

(1) The orthocenter has coordinates

$$\left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) = (S_{BC} : S_{CA} : S_{AB}).$$

Note that in the last expression, the coordinate sum is $S_{BC} + S_{CA} + S_{AB} = S^2$.

(2) The circumcenter, on the other hand, is the point

$$O = (a^2S_A : b^2S_B : c^2S_C) = (S_A(S_B + S_C) : S_B(S_C + S_A) : S_C(S_A + S_B)).$$

Note that in this form, the coordinate sum is $2(S_{AB} + S_{BC} + S_{CA}) = 2S^2$.

Exercises

1. Calculate the coordinates of the nine-point center in terms of S_A, S_B, S_C. [17]

2. Calculate the coordinates of the reflection of the orthocenter in the circumcenter, *i.e.*, the point L which divides the segment HO in the ratio $HL : LO = 2 : -1$. This is called the *de Longchamps point* of triangle ABC. [18]

[17] $N = (S^2 + S_{BC} : S^2 + S_{CA} : S^2 + S_{AB})$.

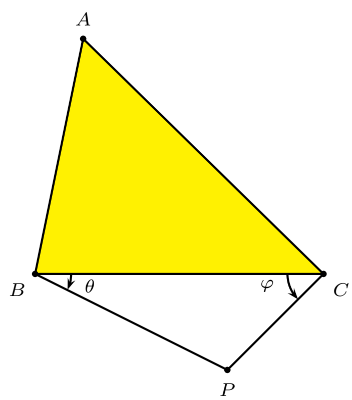
[18] $L = (S_{CA} + S_{AB} - S_{BC} : \cdots : \cdots) = (\frac{1}{S_B} + \frac{1}{S_C} - \frac{1}{S_A} : \cdots : \cdots)$. It appears in ETC as the point X_{20}.

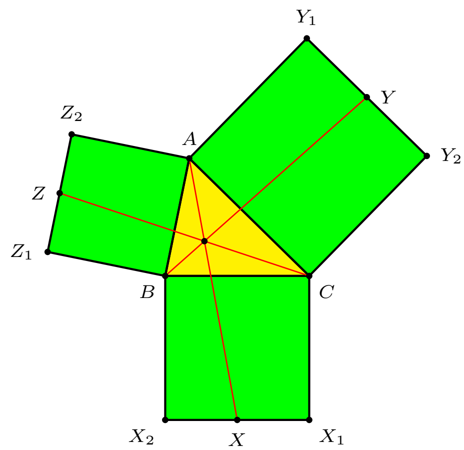
3.4.2 Conway's formula

If the swing angles of a point P on the side BC are $\angle CBP = \theta$ and $\angle BCP = \varphi$, the coordinates of P are

$$(-a^2 : S_C + S_\varphi : S_B + S_\theta).$$

The swing angles are chosen in the rangle $-\frac{\pi}{2} \leq \theta, \varphi \leq \frac{\pi}{2}$. The angle θ is positive or negative according as angles $\angle CBP$ and $\angle CBA$ have different or the same orientation.





3.4.3 Examples

Squares erected on the sides of a triangle

Consider the square BCX_1X_2 erected externally on the side BC of triangle ABC. The swing angles of X_1 with respect to the side BC are

$$\angle CBX_1 = \frac{\pi}{4}, \qquad \angle BCX_1 = \frac{\pi}{2}.$$

Since $\cot\frac{\pi}{4} = 1$ and $\cot\frac{\pi}{2} = 0$,

$$X_1 = (-a^2 : S_C : S_B + S).$$

Similarly,

$$X_2 = (-a^2 : S_C + S : S_B).$$

Exercises

1. Find the midpoint of X_1X_2.
2. Find the vertices of the inscribed squares with a side along BC. [19].

[19] Recall that this can be obtained from applying the homothety $\mathsf{h}(A, \frac{S}{S+a^2})$ to the square BCX_1X_2

3.5 The Kiepert perspectors

3.5.1 The Fermat points

Consider the equilateral triangle BCX erected externally on the side BC of triangle ABC. The swing angles are $\angle CBX = \angle BCX = \frac{\pi}{3}$. Since $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$,

$$X = \left(-a^2 : S_C + \frac{S}{\sqrt{3}} : S_B + \frac{S}{\sqrt{3}}\right),$$

which can be rearranged in the form

$$X = \left(\frac{-a^2}{(S_B + \frac{S}{\sqrt{3}})(S_C + \frac{S}{\sqrt{3}})} : \frac{1}{S_B + \frac{S}{\sqrt{3}}} : \frac{1}{S_C + \frac{S}{\sqrt{3}}}\right).$$

Similarly, we write down the coordinates of the apexes Y, Z of the equilateral triangles CAY and ABZ erected externally on the other two sides. These are

$$Y = \left(\frac{1}{S_A + \frac{S}{\sqrt{3}}} : *****: \frac{1}{S_C + \frac{S}{\sqrt{3}}}\right)$$

and

$$Z = \left(\frac{1}{S_A + \frac{S}{\sqrt{3}}} : \frac{1}{S_B + \frac{S}{\sqrt{3}}} : *****\right).$$

Here we simply write $*****$ in places where the exact values of the coordinates are not important. This is a particular case of the following general situation.

3.5.2 Perspective triangles

Suppose X, Y, Z are points whose coordinates can be written in the form

$$\begin{array}{ccccccc} X & = & ***** & : & y & : & z, \\ Y & = & x & : & ***** & : & z, \\ Z & = & x & : & y & : & *****. \end{array}$$

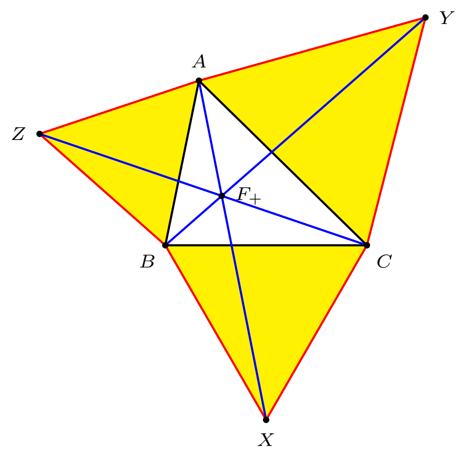
The lines AX, BY, CZ are concurrent at the point $P = (x : y : z)$.
Proof. The intersection of AX and BC is the trace of X on the side BC. It is the point $(0 : y : z)$. Similarly, the intersections $BY \cap CA$ and $CZ \cap AB$ are the points $(x : 0 : z)$ and $(x : y : 0)$. These three points are in turn the traces of $P = (x : y : z)$. Q.E.D.

We say that triangle XYZ is *perspective* with ABC, and call the point P the *perspector* of XYZ.

We conclude therefore that *the apexes of the equilateral triangles erected externally on the sides of a triangle ABC form a triangle perspective with ABC at the point*

$$F_+ = \left(\frac{1}{\sqrt{3}S_A + S} : \frac{1}{\sqrt{3}S_B + S} : \frac{1}{\sqrt{3}S_C + S}\right).$$

This is called the (positive) *Fermat point* of triangle ABC. [20]



Exercises

1. If the equilateral triangles are erected "internally" on the sides, the apexes again form a triangle with perspector

$$F_- = \left(\frac{1}{\sqrt{3}S_A - S} : \frac{1}{\sqrt{3}S_B - S} : \frac{1}{\sqrt{3}S_C - S}\right),$$

   the negative Fermat point of triangle ABC. [21]

2. Given triangle ABC, extend the sides AC to B_a and AB to C_a such that $CB_a = BC_a = a$. Similarly define C_b, A_b, A_c, and B_c.

   (a) Write down the coordinates of B_a and C_a, and the coordinates of the intersection A' of BB_a and CC_a.

   (b) Similarly define B' and C', and show that $A'B'C'$ is perspective with ABC. Calculate the coordinates of the perspector. [22]

3.5.3 Isosceles triangles erected on the sides and Kiepert perspectors

More generally, consider an isosceles triangle YCA of base angle $\angle YCA = \angle YAC = \theta$. The vertex Y has coordinates

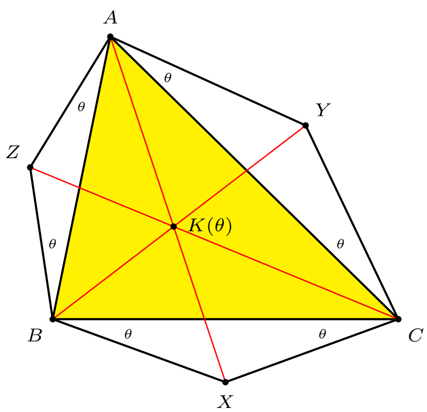
$$(S_C + S_\theta : -b^2 : S_A + S_\theta).$$

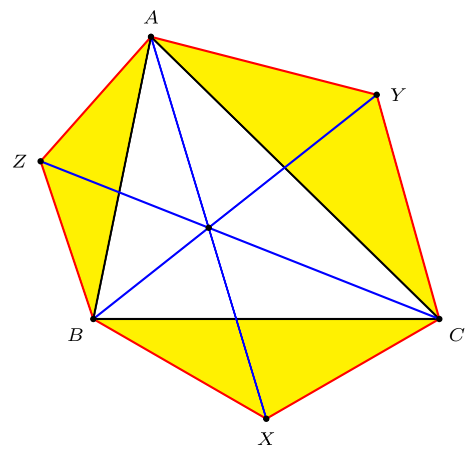
[20] The positive Fermat point is also known as the first isogonic center. It appears in ETC as the point X_{13}.
[21] The negative Fermat point is also known as the second isogonic center. It appears in ETC as the point X_{14}.
[22] The Spieker point.

If *similar* isosceles triangles XBC and ZAB are erected on the other two sides (with the same orientation), the lines AX, BY, and CZ are concurrent at the point

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

We call XYZ the Kiepert triangle and $K(\theta)$ the *Kiepert perspector* of parameter θ.





3.5.4 The Napoleon points

The famous Napoleon theorem states that *the centers of the equilateral triangles erected externally on the sides of a triangle form an equilateral triangle*. These centers are the apexes of similar isosceles triangles of base angle 30° erected externally on the sides. They give the Kiepert perspector

$$\left(\frac{1}{S_A + \sqrt{3}S} : \frac{1}{S_B + \sqrt{3}S} : \frac{1}{S_C + \sqrt{3}S}\right).$$

This is called the (positive) Napoleon point of the triangle. [23] Analogous results hold for equilateral triangles erected internally, leading to the negative Napoleon point [24]

$$\left(\frac{1}{S_A - \sqrt{3}S} : \frac{1}{S_B - \sqrt{3}S} : \frac{1}{S_C - \sqrt{3}S}\right).$$

Exercises

1. The centers of the three squares erected externally on the sides of triangle ABC form a triangle perspective with ABC. The perspector is called the (positive)

[23] The positive Napoleon point appears in ETC as the point X_{17}.
[24] The negative Napoleon point appears in ETC as the point X_{18}.

Vecten point. Why is this a Kiepert perspector? Identify its Kiepert parameter, and write down its coordinates? [25]

2. Let ABC be a given triangle. Construct a small semicircle with B as center and a diameter perpendicular to BC, intersecting the side BC. **Animate** a point T on this semicircle, and **hide** the semicircle.

(a) Construct the ray BT and let it intersect the perpendicular bisector of BC at X.

(b) **Reflect** the ray BT in the **bisector** of angle B, and construct the perpendicular bisector of AB to intersect this reflection at Z.

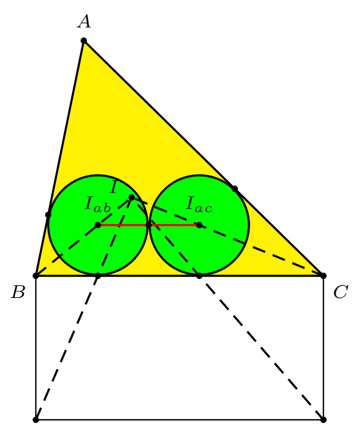
(c) **Reflect** AZ in the **bisector** of angle A, and **reflect** CX in the **bisector** of angle C. Label the intersection of these two reflections Y.

(d) Construct the perspector P of the triangle XYZ.

(e) What is the **locus** of P as T traverses the semicircle?

3. Calculate the coordinates of the midpoint of the segment F_+F_-. [26]

4. Inside triangle ABC, consider two congruent circles $I_{ab}(r_1)$ and $I_{ac}(r_1)$ tangent to each other (externally), both to the side BC, and to CA and AB respectively. Note that the centers I_{ab} and I_{ac}, together with their pedals on BC, form a rectangle of sides $2:1$. This rectangle can be constructed as the image under the homothety $\mathsf{h}(I, \frac{2r}{a})$ of a similar rectangle erected externally on the side BC.



(a) Make use of these to construct the two circles.

(b) Calculate the homogeneous barycentric coordinates of the point of tangency of the two circles. [27]

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[25] This is $K(\frac{\pi}{4})$, the positive Vecten point. It appears in ETC as X_{485}.

[26] $((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$. This points appears in ETC as X_{115}. It lies on the nine-point circle.

[27] This divides ID (D = midpoint of BC) in the ratio $2r : a$ and has coordinates $(a^2 : ab+S : ac+S)$.

(c) Similarly, there are two other pairs of congruent circles on the sides CA and AB. The points of tangency of the three pairs have a perspector [28]

$$\left(\frac{1}{bc+S} : \frac{1}{ca+S} : \frac{1}{ab+S}\right).$$

(d) Show that the pedals of the points of tangency on the respective side lines of ABC are the traces of [29]

$$\left(\frac{1}{bc+S+S_A} : \frac{1}{ca+S+S_B} : \frac{1}{ab+S+S_C}\right).$$

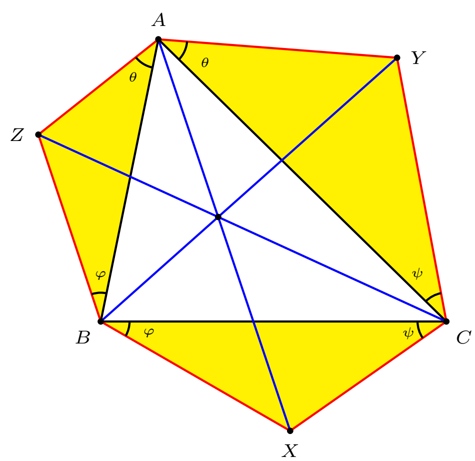
3.5.5 Nagel's Theorem

Suppose X, Y, Z are such that

$$\begin{aligned}\angle CAY = \angle BAZ = \theta,\\ \angle ABZ = \angle CBX = \varphi,\\ \angle BCX = \angle ACY = \psi.\end{aligned}$$

The lines AX, BY, CZ are concurrent at the point

$$\left(\frac{1}{S_A+S_\theta} : \frac{1}{S_B+S_\varphi} : \frac{1}{S_C+S_\psi}\right).$$



Exercises

1. Let X', Y', Z' be respectively the pedals of X on BC, Y on CA, and Z on AB. Show that $X'Y'Z'$ is a cevian triangle. [30]

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[28] This point is not in the current edition of ETC.
[29] This point is not in the current edition of ETC.
[30] Floor van Lamoen.

2. For $i = 1, 2$, let $X_iY_iZ_i$ be the triangle formed with given angles θ_i, φ_i and ψ_i. Show that the intersections

$$X = X_1X_2 \cap BC, \quad Y = Y_1Y_2 \cap CA, \qquad Z = Z_1Z_2 \cap AB$$

form a cevian triangle. [31]

[31] Floor van Lamoen. $X = (0 : S_{\psi_1} - S_{\psi_2} : S_{\varphi_1} - S_{\varphi_2})$.

Chapter 4

Straight Lines

4.1 The equation of a line

4.1.1 Two-point form

The equation of the line joining two points with coordinates $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix} = 0,$$

or

$$(y_1z_2 - y_2z_1)x + (z_1x_2 - z_2x_1)y + (x_1y_2 - x_2y_1)z = 0.$$

4.1.2 Examples

1. The equations of the side lines BC, CA, AB are respectively $x = 0$, $y = 0$, $z = 0$.

2. The perpendicular bisector of BC is the line joining the circumcenter $O = (a^2S_A : b^2S_B : c^2S_C)$ to the midpoint of BC, which has coordinates $(0 : 1 : 1)$. By the two point form, it has equation

$$(b^2S_B - c^2S_C)x - a^2S_Ay + a^2S_Az = 0,$$

Since $b^2S_B - c^2S_C = \cdots = S_A(S_B - S_C) = -S_A(b^2 - c^2)$, this equation can be rewritten as

$$(b^2 - c^2)x + a^2(y - z) = 0.$$

3. The equation of the Euler line, as the line joining the centroid $(1 : 1 : 1)$ to the orthocenter $(S_{BC} : S_{CA} : S_{AB})$ is

$$(S_{AB} - S_{CA})x + (S_{BC} - S_{AB})y + (S_{CA} - S_{BC})z = 0,$$

or
$$\sum_{\text{cyclic}} S_A(S_B - S_C)x = 0.$$

4. The equation of the OI-line joining the circumcenter $(a^2S_A : b^2S_B : c^2S_C)$ to and the incenter $(a : b : c)$ is
$$0 = \sum_{\text{cyclic}} (b^2S_Bc - c^2S_Cb)x = \sum_{\text{cyclic}} bc(bS_B - cS_C)x.$$
Since $bS_B - cS_C = \cdots = -2(b-c)s(s-a)$ (exercise), this equation can be rewritten as
$$\sum_{\text{cyclic}} bc(b-c)s(s-a)x = 0.$$
or
$$\sum_{\text{cyclic}} \frac{(b-c)(s-a)}{a}x = 0.$$

5. The line joining the two Fermat points
$$\begin{aligned} F_{\pm} &= \left(\frac{1}{\sqrt{3}S_A \pm S} : \frac{1}{\sqrt{3}S_B \pm S} : \frac{1}{\sqrt{3}S_C \pm S}\right) \\ &= ((\sqrt{3}S_B \pm S)(\sqrt{3}S_C \pm S) : \cdots : \cdots) \end{aligned}$$
has equation
$$\begin{aligned} 0 &= \sum_{\text{cyclic}} \left(\frac{1}{(\sqrt{3}S_B + S)(\sqrt{3}S_C - S)} - \frac{1}{(\sqrt{3}S_B - S)(\sqrt{3}S_C + S)}\right)x \\ &= \sum_{\text{cyclic}} \left(\frac{(\sqrt{3}S_B - S)(\sqrt{3}S_C + S) - (\sqrt{3}S_B + S)(\sqrt{3}S_C - S)}{(3S_{BB} - S^2)(3S_{CC} - S^2)}\right)x \\ &= \sum_{\text{cyclic}} \left(\frac{2\sqrt{3}(S_B - S_C)S}{(3S_{BB} - S^2)(3S_{CC} - S^2)}\right)x. \end{aligned}$$
Clearing denominators, we obtain
$$\sum_{\text{cyclic}} (S_B - S_C)(3S_{AA} - S^2)x = 0.$$

4.1.3 Intercept form: tripole and tripolar

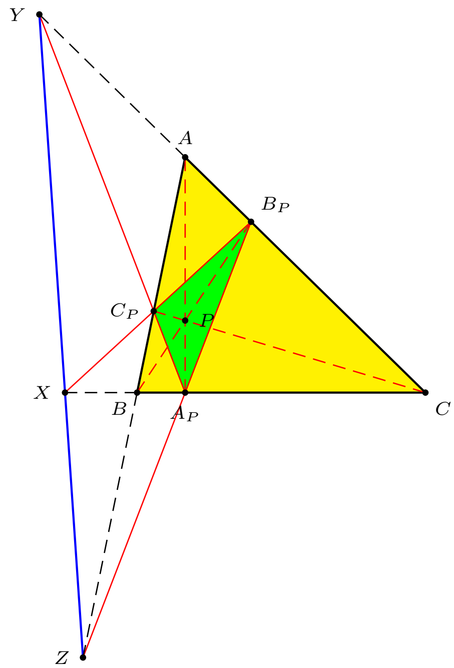
If the intersections of a line $\mathcal{L}$ with the side lines are
$$X = (0 : v : -w), \qquad Y = (-u : 0 : w), \qquad Z = (u : -v : 0),$$
the equation of the line $\mathcal{L}$ is
$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$
We shall call the point $P = (u : v : w)$ the *tripole* of $\mathcal{L}$, and the line $\mathcal{L}$ the *tripolar* of P.

Construction of tripole

Given a line $\mathcal{L}$ intersecting BC, CA, AB at X, Y, Z respectively, let

$$A' = BY \cap CZ, \qquad B' = CZ \cap AX, \qquad C' = AX \cap BY.$$

The lines AA', BB' and CC' intersect at the tripole P of $\mathcal{L}$.



Construction of tripolar

Given P with traces A_P, B_P, and C_P on the side lines, let

$$X = B_PC_P \cap BC, \qquad Y = C_PA_P \cap CA, \qquad Z = A_PB_P \cap AB.$$

These points X, Y, Z lie on the tripolar of P.

Exercises

1. Find the equation of the line joining the centroid to a given point $P = (u : v : w)$. [1]

2. Find the equations of the cevians of a point $P = (u : v : w)$.

3. Find the equations of the angle bisectors.

---

[1] Equation: $(v - w)x + (w - u)y + (u - v)z = 0$.

4.2 Infinite points and parallel lines

4.2.1 The infinite point of a line

The infinite point of a line $\mathcal{L}$ has homogeneous coordinates given by the difference of the *absolute* barycentric coordinates of two distinct points on the line. As such, the coordinate sum of an infinite point is zero. We think of all infinite points constituting the line at infinity, $\mathcal{L}_\infty$, which has equation $x + y + z = 0$.

Examples

1. The infinite points of the side lines BC, CA, AB are $(0 : -1 : 1)$, $(1 : 0 : -1)$, $(-1 : 1 : 0)$ respectively.

2. The infinite point of the $A-$altitude has homogeneous coordinates

$$(0 : S_C : S_B) - a^2(1 : 0 : 0) = (-a^2 : S_C : S_B).$$

3. More generally, the infinite point of the line $px + qy + rz = 0$ is

$$(q - r : r - p : p - q).$$

4. The infinite point of the Euler line is the point

$$3(S_{BC} : S_{CA} : S_{AB}) - SS(1 : 1 : 1) \sim (3S_{BC} - SS : 3S_{CA} - SS : 3S_{AB} - SS).$$

5. The infinite point of the OI-line is

$$\begin{array}{rl} & (ca(c-a)(s-b) - ab(a-b)(s-c) : \cdots : \cdots) \\ \sim & (a(a^2(b+c) - 2abc - (b+c)(b-c)^2) : \cdots : \cdots). \end{array}$$

4.2.2 Parallel lines

Parallel lines have the same infinite point. The line through $P = (u : v : w)$ parallel to $\mathcal{L} : px + qy + rz = 0$ has equation

$$\begin{vmatrix} q-r & r-p & p-q \\ u & v & w \\ x & y & z \end{vmatrix} = 0.$$

Exercises

1. Find the equations of the lines through $P = (u : v : w)$ parallel to the side lines.

2. Let DEF be the medial triangle of ABC, and P a point with cevian triangle XYZ (with respect to ABC. Find P such that the lines DX, EY, FZ are parallel to the internal bisectors of angles A, B, C respectively. [2]

[2]The Nagel point $P = (b + c - a : c + a - b : a + b - c)$. N. Dergiades, Hyacinthos, message 3677, 8/31/01.

4.3 Intersection of two lines

The intersection of the two lines

$$p_1x + q_1y + r_1z = 0,$$
$$p_2x + q_2y + r_2z = 0$$

is the point

$$(q_1r_2 - q_2r_1 : r_1p_2 - r_2p_1 : p_1q_2 - p_2q_1).$$

The infinite point of a line $\mathcal{L}$ can be regarded as the intersection of $\mathcal{L}$ with the line at infinity $\mathcal{L}_\infty : x + y + z = 0$.

Theorem

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$, are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

4.3.1 Intersection of the Euler and Fermat lines

Recall that these lines have equations

$$\sum_{\text{cyclic}} S_A(S_B - S_C)x = 0,$$

and

$$\sum_{\text{cyclic}} (S_B - S_C)(3S_{AA} - S^2)x = 0.$$

The A-coordinate of their intersection

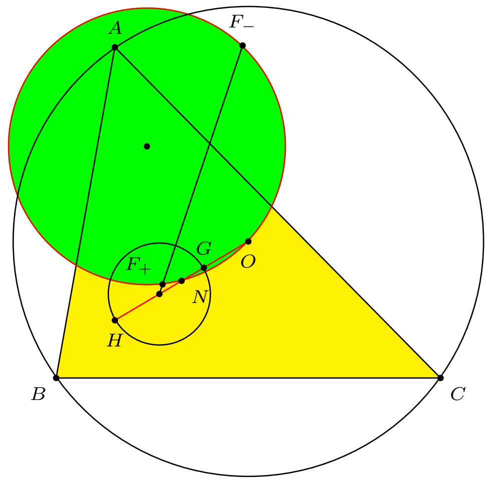
$$\begin{aligned}
&= S_B(S_C - S_A)(S_A - S_B)(3S_{CC} - S^2) \\
&\quad -S_C(S_A - S_B)(S_C - S_A)(3S_{BB} - S^2) \\
&= (S_C - S_A)(S_A - S_B)(S_B(3S_{CC} - S^2) - S_C(3S_{BB} - S^2)) \\
&= (S_C - S_A)(S_A - S_B)(3S_{BC}(S_C - S_B) - S^2(S_B - S_C)) \\
&= -(S_B - S_C)(S_C - S_A)(S_A - S_B)(3S_{BC} + S^2).
\end{aligned}$$

This intersection is the point

$$(3S_{BC} + S^2 : 3S_{CA} + S^2 : 3S_{AB} + S^2).$$

Since $(3S_{BC} : 3S_{CA} : 3S_{AB})$ and $(S^2 : S^2 : S^2)$ represent H and G, with equal coordinate sums, this point is the midpoint of GH. [3]

[3] This point appears in ETC as X_{381}.



Remark

Lester has discovered that there is a circle passing the two Fermat points, the circumcenter, and the nine-point center. [4] The circle with GH as diameter, whose center is the intersection of the Fermat and Euler line as we have shown above, is orthogonal to the Lester circle. [5] It is also interesting to note that the midpoint between the Fermat points is a point on the nine-point circle. It has coordinates $((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$. [6]

4.3.2 Triangle bounded by the outer side lines of the squares erected externally

Consider the square BCX_cX_b erected externally on BC. Since $X_c = (-a^2 : S_C : S_B + S)$, and the line X_bX_c, being parallel to BC, has infinite point $(0 : -1 : 1)$, this line has equation

$$(S_C + S_B + S)x + a^2y + a^2z = 0.$$

Since $S_B + S_C = a^2$, this can be rewritten as

$$a^2(x+y+z) + Sx = 0.$$

Similarly, if CAY_aY_c and ABZ_bZ_a are squares erected externally on the other two sides, the lines Y_cY_a and Z_aZ_b have equations

$$b^2(x+y+z) + Sy = 0$$

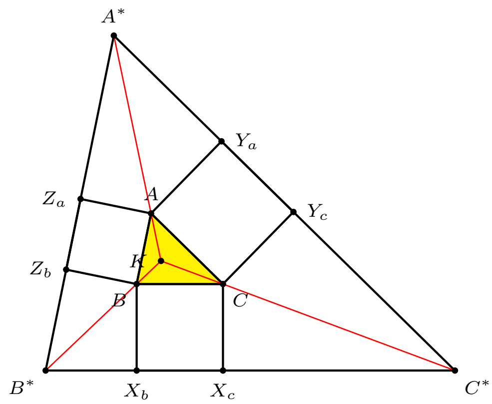
and

$$c^2(x+y+z) + Sz = 0$$

[4] J.A. Lester, Triangles, III: complex centre functions and Ceva's theorem, *Aequationes Math.*, 53 (1997) 4–35.

[5] P. Yiu, Hyacinthos, message 1258, August 21, 2000.

[6] This point appears as X_{115} in ETC.



respectively. These two latter lines intersect at the point

$$A^* = (-(b^2 + c^2 + S) : b^2 : c^2).$$

Similarly, the lines Z_aZ_b and X_bX_c intersect at

$$B^* = (a^2 : -(c^2 + a^2 + S) : c^2),$$

and the lines X_bX_c and Y_cY_a intersect at

$$C^* = (a^2 : b^2 : -(a^2 + b^2 + S)).$$

The triangle $A^*B^*C^*$ is perspective with ABC at the point

$$K = (a^2 : b^2 : c^2).$$

This is called the *symmedian point* of triangle ABC. [7]

Exercises

1. The symmedian point lies on the line joining the Fermat points.

2. The line joining the two Kiepert perspectors $K(\pm\theta)$ has equation

$$\sum_{\text{cyclic}} (S_B - S_C)(S_{AA} - S^2 \cot^2 \theta)x = 0.$$

Show that this line passes through a fixed point. [8]

3. Show that triangle $A^\theta B^\theta C^\theta$ has the same centroid as triangle ABC.

4. Construct the parallels to the side lines through the symmedian point. The 6 intersections on the side lines lie on a circle. The symmedian point is the unique point with this property. [9]

---

[7] It is also known as the Grebe point, and appears in ETC as the point X_6.
[8] The symmedian point.
[9] This was first discovered by Lemoine in 1883.

5. Let DEF be the medial triangle of ABC. Find the equation of the line joining D to the excenter $I_a = (-a : b : c)$. Similarly write down the equation of the lines joining to E to I_b and F to I_c. Show that these three lines are concurrent by working out the coordinates of their common point. [10]

6. The perpendiculars from the excenters to the corresponding sides are concurrent. Find the coordinates of the intersection by noting how it is related to the circumcenter and the incenter. [11]

7. Let D, E, F be the midpoints of the sides BC, CA, AB of triangle ABC. For a point P with traces A_P, B_P, C_P, let X, Y, Z be the midpoints of B_PC_P, C_PA_P, A_PB_P respectively. Find the equations of the lines DX, EY, FZ, and show that they are concurrent. What are the coordinates of their intersection? [12]

8. Let D, E, F be the midpoints of the sides of BC, CA, AB of triangle ABC, and X, Y, Z the midpoints of the altitudes from A, B, C respeectively. Find the equations of the lines DX, EY, FZ, and show that they are concurrent. What are the coordinates of their intersection? [13]

9. Given triangle ABC, extend the sides AC to B_a and AB to C_a such that $CB_a = BC_a = a$. Similarly define C_b, A_b, A_c, and B_c. The lines B_aC_a, C_bA_b, and A_cB_c bound a triangle perspective with ABC. Calculate the coordinate of the perspector. [14]

[10] This is the Mittenpunkt $(a(s-a) : \cdots : \cdots)$; it appears in ETC as X_9.

[11] This is the reflection of I in O. As such, it is the point $2O - I$, and has coordinates

$$(a(a^3 + a^2(b+c) - a(b+c)^2 - (b+c)(b-c)^2) : \cdots : \cdots).$$

This point appears as X_{40} in ETC.

[12] The intersection is the point dividing the segment PG in the ratio $3:1$.

[13] This intersection is the symmedian point $K = (a^2 : b^2 : c^2)$.

[14] $(\frac{a(b+c)}{b+c-a} : \cdots : \cdots)$. This appears in ETC as X_{65}.

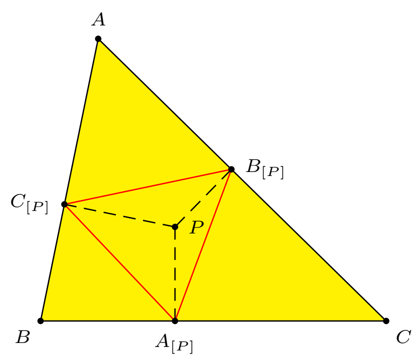
4.4 Pedal triangle

The pedals of a point $P = (u : v : w)$ are the intersections of the side lines with the corresponding perpendiculars through P. The $A-$altitude has infinite point $A_H - A = (0 : S_C : S_B) - (S_B + S_C : 0 : 0) = (-a^2 : S_C : S_B)$. The perpendicular through P to BC is the line

$$\begin{vmatrix} -a^2 & S_C & S_B \\ u & v & w \\ x & y & z \end{vmatrix} = 0,$$

or

$$-(S_B v - S_C w)x + (S_B u + a^2 w)y - (S_C u + a^2 v)z = 0.$$



This intersects BC at the point

$$A_{[P]} = (0 : S_C u + a^2 v : S_B u + a^2 w).$$

Similarly the coordinates of the pedals on CA and AB can be written down. The triangle $A_{[P]}B_{[P]}C_{[P]}$ is called the *pedal triangle* of triangle ABC:

$$\begin{pmatrix} A_{[P]} \\ B_{[P]} \\ C_{[P]} \end{pmatrix} = \begin{pmatrix} 0 & S_C u + a^2 v & S_B u + a^2 w \\ S_C v + b^2 u & 0 & S_A v + b^2 w \\ S_B w + c^2 u & S_A w + c^2 v & 0 \end{pmatrix}$$

4.4.1 Examples

1. The pedal triangle of the circumcenter is clearly the medial triangle.

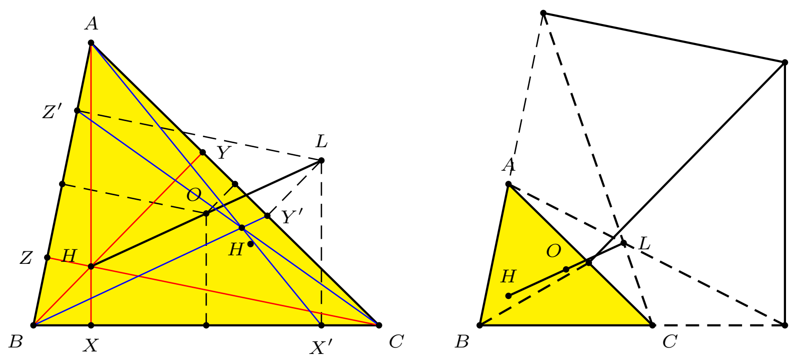
2. The pedal triangle of the orthocenter is called the *orthic* triangle. Its vertices are clearly the traces of H, namely, the points $(0 : S_C : S_B)$, $(S_C : 0 : S_A)$, and $(S_B : S_A : 0)$.

3. Let L be the reflection of the orthocenter H in the circumcenter O. This is called the de Longchamps point. [15] Show that the pedal triangle of L is the cevian triangle of some point P. What are the coordinates of P? [16]

---

[15] The de Longchamps point appears as X_{20} in ETC.

[16] $P = (S_A : S_B : S_C)$ is the isotomic conjugate of the orthocenter. It appears in ETC as the point X_{69}.



4. Let L be the de Longchamps point again, with homogeneous barycentric coordinates

$$(S_{CA}+S_{AB}-S_{BC}:S_{AB}+S_{BC}-S_{CA}:S_{BC}+S_{CA}-S_{AB}).$$

Find the equations of the perpendiculars to the side lines at the corresponding traces of L. Show that these are concurrent, and find the coordinates of the intersection.

The perpendicular to BC at $A_L=(0:S_{AB}+S_{BC}-S_{CA}:S_{BC}+S_{CA}-S_{AB})$ is the line

$$\begin{vmatrix} -(S_B+S_C) & S_C & S_B \\ 0 & S_{AB}+S_{BC}-S_{CA} & S_{BC}+S_{CA}-S_{AB} \\ x & y & z \end{vmatrix}=0.$$

This is

$$S^2(S_B-S_C)x-a^2(S_{BC}+S_{CA}-S_{AB})y+a^2(S_{BC}-S_{CA}+S_{AB})z=0.$$

Similarly, we write down the equations of the perpendiculars at the other two traces. The three perpendiculars intersect at the point [17]

$$(a^2(S_C^2S_A^2+S_A^2S_B^2-S_B^2S_C^2):\cdots:\cdots).$$

Exercises

1. Let D, E, F be the midpoints of the sides BC, CA, AB, and A', B', C' the pedals of A, B, C on their opposite sides. Show that $X=EC'\cap FB'$, $Y=FA'\cap DC'$, and $Z=DB'\cap EC'$ are collinear. [18]

2. Let X be the pedal of A on the side BC of triangle ABC. Complete the squares AXX_bA_b and AXX_cA_c with X_b and X_c on the line BC. [19]

[17] This point appears in ETC as X_{1078}. Conway calls this point the *logarithm of the de Longchamps point*.

[18] These are all on the Euler line. See G. Leversha, Problem 2358 and solution, *Crux Mathematicorum*, 24 (1998) 303; 25 (1999) 371 –372.

[19] A.P. Hatzipolakis, Hyacinthos, message 3370, 8/7/01.

(a) Calculate the coordinates of A_b and A_c. [20]

(b) Calculate the coordinates of $A' = BA_c \cap CA_b$. [21]

(c) Similarly define B' and C'. Triangle $A'B'C'$ is perspective with ABC. What is the perspector? [22]

(d) Let A'' be the pedal of A' on the side BC. Similarly define B'' and C''. Show that $A''B''C''$ is perspective with ABC by calculating the coordinates of the perspector. [23]

---

[20] $A_b = (a^2 : -S : S)$ and $A_c = (a^2 : S : -S)$.

[21] $A' = (a^2 : S : S)$.

[22] The centroid.

[23] $(\frac{1}{S_A+S} : \frac{1}{S_B+S} : \frac{1}{S_C+S})$. This is called the first Vecten point; it appears as X_{485} in ETC.

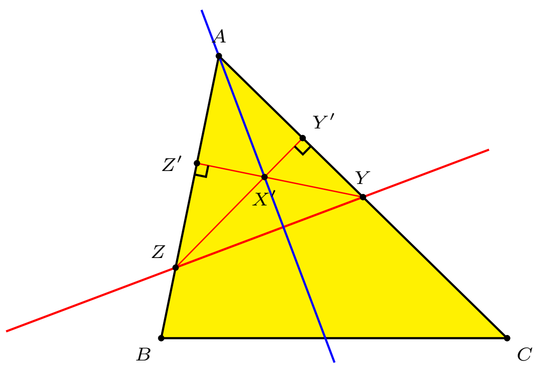
4.5 Perpendicular lines

Given a line $\mathcal{L}: px+qy+rz=0$, we determine the infinite point of lines perpendicular to it. [24] The line $\mathcal{L}$ intersects the side lines CA and AB at the points $Y=(-r:0:p)$ and $Z=(q:-p:0)$. To find the perpendicular from A to $\mathcal{L}$, we first find the equations of the perpendiculars from Y to AB and from Z to CA. These are

$$\begin{vmatrix} S_B & S_A & -c^2 \\ -r & 0 & p \\ x & y & z \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} S_C & -b^2 & S_A \\ q & -p & 0 \\ x & y & z \end{vmatrix} = 0$$

These are

$$\begin{aligned} S_Apx + (c^2r - S_Bp)y + S_Arz &= 0, \\ S_Apx + S_Aqy + (b^2q - S_Cp)z &= 0. \end{aligned}$$



These two perpendiculars intersect at the orthocenter of triangle AYZ, which is the point

$$\begin{aligned} X' &= (*****: S_Ap(S_Ar - b^2q + S_Cp) : S_Ap(S_Aq + S_Bp - c^2r) \\ &\sim (*****: S_C(p-q) - S_A(q-r) : S_A(q-r) - S_B(r-p)). \end{aligned}$$

The perpendicular from A to $\mathcal{L}$ is the line AX', which has equation

$$\begin{vmatrix} 1 & 0 & 0 \\ *** & S_C(p-q) - S_A(q-r) & -S_A(q-r) + S_B(r-p) \\ x & y & z \end{vmatrix} = 0,$$

or

$$-(S_A(q-r) - S_B(r-p))y + (S_C(p-q) - S_A(q-r))z = 0.$$

This has infinite point

$$(S_B(r-p) - S_C(p-q) : S_C(p-q) - S_A(q-r) : S_A(q-r) - S_B(r-p)).$$

Note that the infinite point of $\mathcal{L}$ is $(q-r : r-p : p-q)$. We summarize this in the following theorem.

[24] I learned of this method from Floor van Lamoen.

Theorem

If a line $\mathcal{L}$ has infinite point $(f:g:h)$, the lines perpendicular to $\mathcal{L}$ have infinite points

$$(f':g':h') = (S_Bg - S_Ch : S_Ch - S_Af : S_Af - S_Bg).$$

Equivalently, two lines with infinite points $(f:g:h)$ and $(f':g':h')$ are perpendicular to each other if and only if

$$S_Aff' + S_Bgg' + S_Chh' = 0.$$

4.5.1 The tangential triangle

Consider the tangents to the circumcircle at the vertices. The radius OA has infinite point

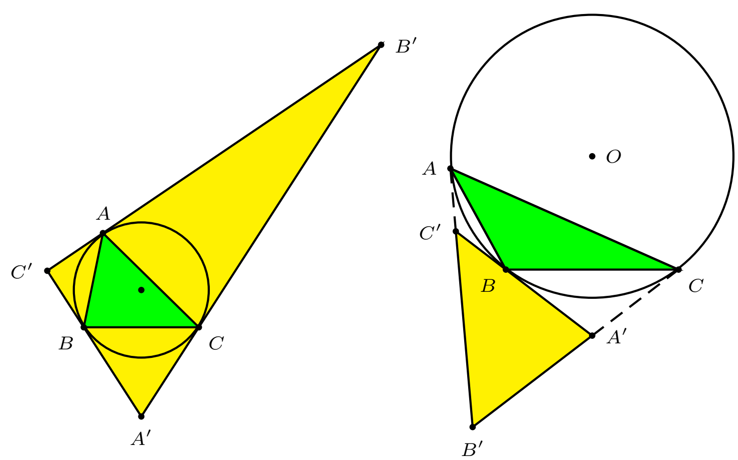
$$(a^2S_A : b^2S_B : c^2S_C) - (2S^2 : 0 : 0) = (-(b^2S_B + c^2S_C) : b^2S_B : c^2S_C).$$

The infinite point of the tangent at A is

$$(b^2S_{BB} - c^2S_{CC} : c^2S_{CC} + S_A(b^2S_B + c^2S_C) : -S_A(b^2S_B + c^2S_C) - b^2S_{BB}).$$

Consider the B-coordinate:

$$c^2S_{CC} + S_A(b^2S_B + c^2S_C) = c^2S_C(S_C + S_A) + b^2S_{AB} = b^2(c^2S_C + S_{AB}) = b^2S^2.$$



Similarly, the C-coordinate = $-c^2S^2$. It follows that this infinite point is $(-(b^2 - c^2) : b^2 : -c^2)$, and the tangent at A is the line

$$\begin{vmatrix} 1 & 0 & 0 \\ -(b^2-c^2) & b^2 & -c^2 \\ x & y & z \end{vmatrix} = 0,$$

or simply $c^2y+b^2z=0$. The other two tangents are $c^2x+a^2z=0$, and $b^2x+a^2y=0$. These three tangents bound a triangle with vertices

$$A'=(-a^2:b^2:c^2),\quad B'=(a^2:-b^2:c^2),\quad C'=(a^2:b^2:-c^2).$$

This is called the *tangential triangle* of ABC. It is perspective with ABC at the point $(a^2:b^2:c^2)$, the symmedian point.

4.5.2 Line of ortho-intercepts

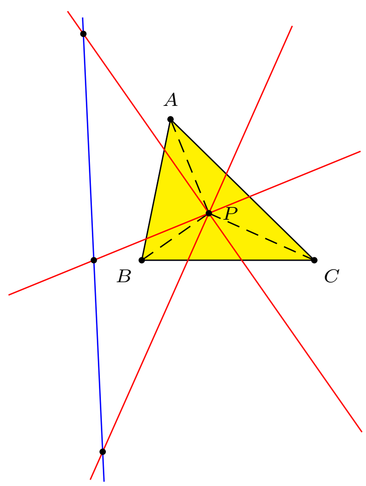
[25]

Let $P=(u:v:w)$. We consider the line perpendicular to AP at P. Since the line AP has equation $wy-vz=0$ and infinite point $(-(v+w):v:w)$, the perpendicular has infinite point $(S_Bv-S_Cw : S_Cw+S_A(v+w) : -S_A(v+w)-S_Bv) \sim (S_Bv-S_Cw : S_Av+b^2w : -S_Aw-c^2v)$. It is the line

$$\begin{vmatrix} u & v & w \\ S_Bv-S_Cw & S_Av+b^2w & -S_Aw-c^2v \\ x & y & z \end{vmatrix}=0.$$

This perpendicular line intersects the side line BC at the point

$$\begin{array}{rl} & (0:u(S_Av+b^2w)-v(S_Bv-S_Cw):-u(S_Aw+c^2v)-w(S_Bv-S_Cw)) \\ \sim & (0:(S_Au-S_Bv+S_Cw)v+b^2wu:-((S_Au+S_Bv-S_Cw)w+c^2uv)). \end{array}$$



Similarly, the perpendicular to BP at P intersects CA at

$$(-((-S_Au+S_Bv+S_Cw)u+a^2vw):0:(S_Au+S_Bv-S_Cw)w+c^2uv)$$

[25]B. Gibert, Hyacinthos, message 1158, August 5, 2000.

and the perpendicular to CP at P intersects AB at

$$((-S_Au+S_Bv+S_Cw)u+a^2vw:-((S_Au-S_Bv+S_Cw)v+b^2wu):0).$$

These three points are collinear. The line containing them has equation

$$\sum_{\text{cyclic}}\frac{x}{(-S_Au+S_Bv+S_Cw)u+a^2vw}=0.$$

Exercises

1. If triangle ABC is acute-angled, the symmedian point is the Gergonne point of the tangential triangle.

2. Given a line $\mathcal{L}$, construct the two points each having $\mathcal{L}$ as its line of ortho-intercepts. [26]

3. The tripole of the line of ortho-intercepts of the incenter is the point $(\frac{a}{s-a}:\frac{b}{s-b}:\frac{c}{s-c})$. [27]

4. Calculate the coordinates of the tripole of the line of ortho-intercepts of the nine-point center. [28]

5. Consider a line $\mathcal{L}: px+qy+rz=0$.

(1) Calculate the coordinates of the pedals of A, B, C on the line $\mathcal{L}$. Label these points X, Y, Z.

(2) Find the equations of the perpendiculars from X, Y, Z to the corresponding side lines.

(3) Show that these three perpendiculars are concurrent, and determine the coordinates of the common point.

This is called the *orthopole* of $\mathcal{L}$.

6. **Animate** a point P on the circumcircle. Contruct the orthopole of the diameter OP. This orthopole lies on the nine-point circle.

7. Consider triangle ABC with its incircle $I(r)$.

(a) Construct a circle $X_b(\rho_b)$ tangent to BC at B, and also externally to the incircle.

(b) Show that the radius of the circle (X_b) is $\rho_b=\frac{(-sb)^2}{4r}$.

(c) Let $X_c(\rho_c)$ be the circle tangent to BC at C, and also externally to the incircle. Calculate the coordinates of the pedal A' of the intersection $BX_c\cap CX_b$ on the line BC. [29]

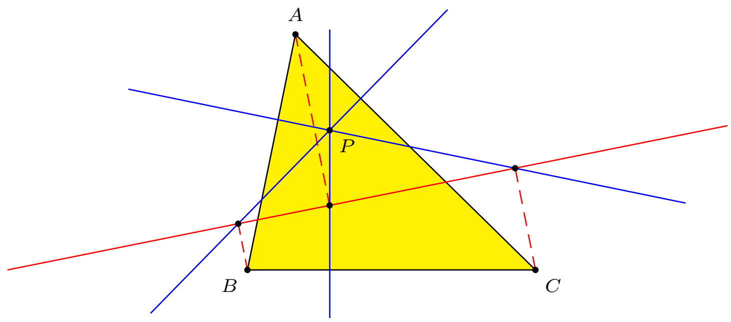
---

[26] One of these points lies on the circumcircle, and the other on the nine-point circle.

[27] This is a point on the OI-line of triangle ABC. It appears in ETC as X_{57}. This point divides OI in the ratio $OX_{57}:OI=2R+r:2R-r$.

[28] $(a^2(3S^2-S_{AA}):\cdots:\cdots)$. This point is not in the current edition of ETC.

[29] $(0:(s-c)^2:(s-b)^2)$.



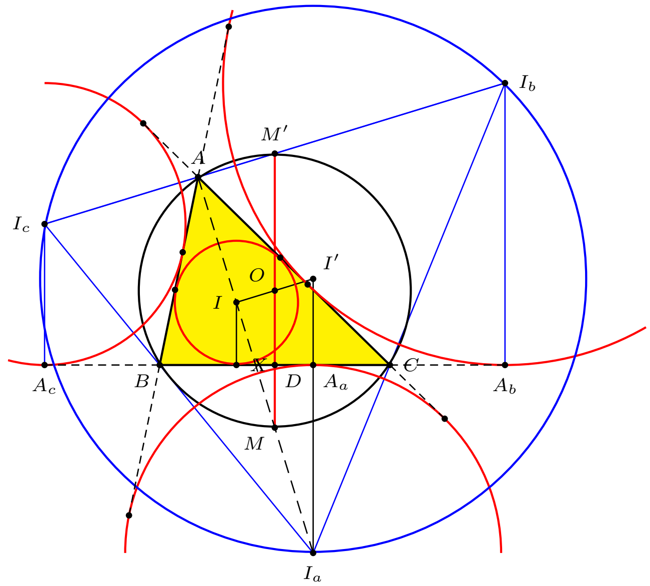
(d) Define B' and C'. Show that $A'B'C'$ is perspective with ABC and find the perspector. [30]

[30] $\left(\frac{1}{(s-a)^2} : \frac{1}{(s-b)^2} : \frac{1}{(s-c)^2}\right)$. This point appears in ETC as X_{279}. See P. Yiu, Hyacinthos, message 3359, 8/6/01.

4.6 Appendices

4.6.1 The excentral triangle

The vertices of the excentral triangle of ABC are the excenters I_a, I_b, I_c.



(1) Identify the following elements of the excentral triangle in terms of the elements of triangle ABC.

Excentral triangle $I_aI_bI_c$	Triangle ABC
Orthocenter	I
Orthic triangle	Triangle ABC
Nine-point circle	Circumcircle
Euler line	OI-line
Circumradius	$2R$
Circumcenter	I' = Reflection of I in O
Centroid	divides OI in the ratio $-1:4$.

The centroid of the excentral triangle is also the centroid of LIN_a.[31]

(2) Let Y be the intersection of the circumcircle (O) with the line I_cI_a (other than B). Note that Y is the midpoint of I_cI_a. The line YO intersects CA at its midpoint E and the circumcircle again at its antipode Y'. Since E is the common midpoint of the segments Q_cQ_a and QQ_b,

(i) $YE = \frac{1}{2}(r_c + r_a)$;

(ii) $EY' = \frac{1}{2}(r_a - r)$.

Since $YY' = 2R$, we obtain the relation

$$r_a + r_b + r_c = 4R + r.$$

[31] Problem 10763 and solution, *Amer. Math. Monthly* 108 (2001) 671.

4.6.2 Centroid of pedal triangle

We determine the centroid of the pedal triangle of P by first equalizing the coordinate sums of the pedals:

$$\begin{aligned}
A_{[P]} &= (0 : S_C u + a^2 v : S_B u + a^2 w) \sim (0 : b^2c^2(S_C u + a^2 v) : b^2c^2(S_B u + a^2 w))\\
B_{[P]} &= (S_C v + b^2 u : 0 : S_A v + b^2 w) \sim (c^2a^2(S_C v + b^2 u) : 0 : c^2a^2(S_A v + b^2 w))\\
C_{[P]} &= (S_B w + c^2 u : S_A w + c^2 v : 0) \sim (a^2b^2(S_B w + c^2 u) : a^2b^2(S_A w + c^2 v) : 0).
\end{aligned}$$

The centroid is the point

$$(2a^2b^2c^2u + a^2c^2S_Cv + a^2b^2S_Bw : b^2c^2S_Cu + 2a^2b^2c^2v + a^2b^2S_Aw : b^2c^2S_Bu + c^a a^2S_Av + 2a^2b^2c^2w).$$

This is the same point as P if and only if

$$\begin{array}{ccccccc}
2a^2b^2c^2u & + & a^2c^2S_Cv & + & a^2b^2S_Bw & = & ku,\\
b^2c^2S_Cu & + & 2a^2b^2c^2v & + & a^2b^2S_Aw & = & kv,\\
b^2c^2S_Bu & + & c^2a^2S_Av & + & 2a^2b^2c^2w & = & kw
\end{array}$$

for some k. Adding these equations, we obtain

$$3a^2b^2c^2(u+v+w) = k(u+v+w).$$

If $P = (u : v : w)$ is a finite point, we must have $k = 3a^2b^2c^2$. The system of equations becomes

$$\begin{array}{ccccccc}
-a^2b^2c^2u & + & a^2c^2S_Cv & + & a^2b^2S_Bw & = & 0,\\
b^2c^2S_Cu & - & a^2b^2c^2v & + & a^2b^2S_Aw & = & 0,\\
b^2c^2S_Bu & + & c^2a^2S_Av & - & a^2b^2c^2w & = & 0.
\end{array}$$

Now it it easy to see that

$$\begin{aligned}
b^2c^2u : c^2a^2v : a^2b^2w &= \begin{vmatrix} -b^2 & S_A\\ S_A & -c^2 \end{vmatrix} : -\begin{vmatrix} S_C & S_A\\ S_B & -c^2 \end{vmatrix} : \begin{vmatrix} S_C & -b^2\\ S_B & S_A \end{vmatrix}\\
&= b^2c^2 - S_{AA} : c^2S_C + S_{AB} : S_{CA} + b^2S_B\\
&= S^2 : S^2 : S^2\\
&= 1 : 1 : 1.
\end{aligned}$$

It follows that $u : v : w = a^2 : b^2 : c^2$, and P is the symmedian point.

Theorem (Lemoine)

The symmedian point is the only point which is the centroid of its own pedal triangle.

4.6.3 Perspectors associated with inscribed squares

Consider the square $A_bA_cA'_cA'_b$ inscribed in triangle ABC, with A_b, A_c on BC. These have coordinates

$$\begin{array}{ll}
A_b = (0 : S_C + S : S_B), & A_c = (0 : S_C : S_B + S),\\
A'_b = (a^2 : S : 0), & A'_c = (a^2 : 0 : S).
\end{array}$$

Similarly, there are inscribed squares $B_cB_aB'_aB'_c$ and $C_aC_bC'_bC'_a$ on the other two sides.

Here is a number of perspective triangles associated with these squares. [32] In each case, we give the definition of A_n only.

n	A_n	Perspector of $A_nB_nC_n$
1	$BB_c \cap CC_b$	orthocenter
2	$BA'_c \cap CA'_b$	circumcenter
3	$BC'_a \cap CB'_a$	symmedian point
4	$B''_cB''_a \cap C''_aC''_b$	symmedian point
5	$B'_cB'_a \cap C'_aC'_b$	$X_{493} = (\frac{a^2}{S+b^2} : \cdots : \cdots)$
6	$C_bA_b \cap A_cB_c$	Kiepert perspector $K(\arctan 2)$
7	$C_aA_c \cap A_bB_a$	Kiepert perspector $K(\arctan 2)$
8	$C_aA'_c \cap B_aA'_b$	$(\frac{S_A+S}{S_A} : \cdots : \cdots)$
9	$C'_aA'_b \cap B'_aA'_c$	$X_{394} = (a^2S_{AA} : b^2S_{BB} : c^2S_{CC})$

For A_4, $BCA''_cA''_b$, $CAB''_aB''_c$ and $ABC''_bC''_a$ are the squares constructed externally on the sides of triangle ABC.

[32] K.R. Dean, Hyacinthos, message 3247, July 18, 2001.

Chapter 5

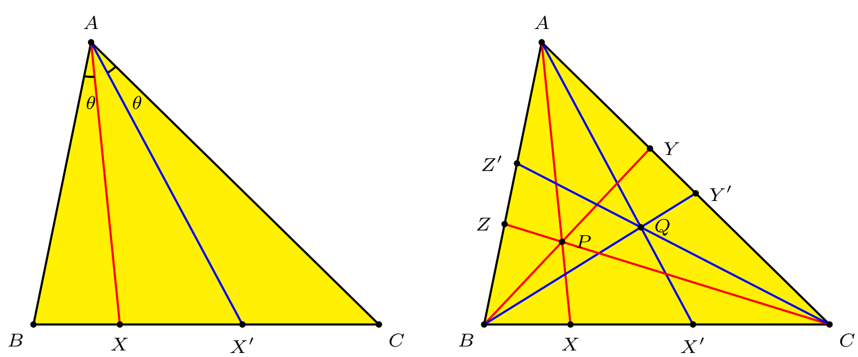
Circles I

5.1 Isogonal conjugates

Let P be a point with homogeneous barycentric coordinates $(x : y : z)$.

(1) The reflection of the cevian AP in the bisector of angle A intersects the line BC at the point $X' = (0 : \frac{b^2}{y} : \frac{c^2}{z})$.

Proof. Let X be the A-trace of P, with $\angle BAP = \theta$. This is the point $X = (0 : y : z) = (0 : S_A - S_\theta : -c^2)$ in Conway's notation. It follows that $S_A - S_\theta : -c^2 = y : z$. If the reflection of AX (in the bisector of angle A) intersects BC at X', we have $X' = (0 : -b^2 : S_A - S_\theta) = (0 : -b^2c^2 : c^2(S_A - S_\theta)) = (0 : b^2z : c^2y) = (0 : \frac{b^2}{y} : \frac{c^2}{z})$.



(2) Similarly, the reflections of the cevians BP and CP in the respective angle bisectors intersect CA at $Y' = (\frac{a^2}{x} : 0 : \frac{c^2}{z})$ and AB at $Z' = (\frac{a^2}{x} : \frac{b^2}{y} : 0)$.

(3) These points X', Y', Z' are the traces of

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right) = (a^2yz : b^2zx : c^2xy).$$

The point P^* is called the *isogonal conjugate* of P. Clearly, P is the isogonal conjugate of P^*.

5.1.1 Examples

1. The isogonal conjugate of the centroid G is the symmedian point $K = (a^2 : b^2 : c^2)$.

2. The incenter is its own isogonal conjugate; so are the excenters.

3. The isogonal conjugate of the orthocenter $H = (\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C})$ is $(a^2 S_A : b^2 S_B : c^2 S_C)$, the circumcenter.

4. The isogonal conjugate of the Gergonne point $G_e = (\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c})$ is the point $(a^2(s-a) : b^2(s-b) : c^2(s-c))$, the internal center of similitude of the circumcircle and the incircle.

5. The isogonal conjugate of the Nagel point is the external center of similitude of (O) and (I).

Exercises

1. Let A', B', C' be the circumcenters of the triangles OBC, OCA, OAB. The triangle $A'B'C'$ has perspector the isogonal conjugate of the nine-point center. [1]

2. Let P be a given point. Construct the circumcircles of the pedal triangles of P and of P^*. What can you say about these circles and their centers?

3. The *isodynamic points* are the isogonal conjugates of the Fermat points. [2]

(a) Construct the positive isodynamic point F_+^*. This is a point on the line joining O and K. How does this point divide the segment OK?

(b) Construct the pedal triangle of F_+^*. What can you say about this triangle?

4. Show that the isogonal conjugate of the Kiepert perspector $K(\theta) = (\frac{1}{S_A+S_\theta} : \frac{1}{S_B+S_\theta} : \frac{1}{S_C+S_\theta})$ is always on the line OK. How does this point divide the segment OK?

5. The perpendiculars from the vertices of ABC to the corresponding sides of the pedal triangle of a point P concur at the isogonal conjugate of P.

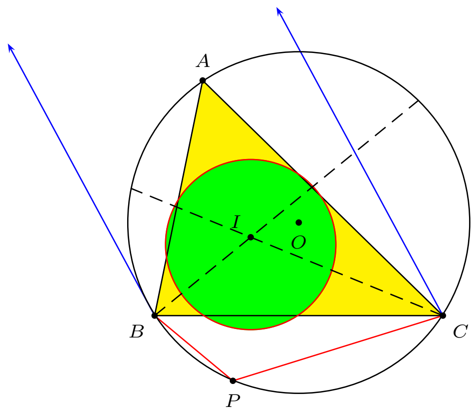
[1] This is also known as the Kosnita point, and appears in ETC as the point X_{54}.

[2] These appear in ETC as the points X_{15} and X_{16}.

5.2 The circumcircle as the isogonal conjugate of the line at infinity

Let P be a point on the circumcircle.

(1) If AX and AP are symmetric with respect to the bisector of angle A, and BY, BP symmetric with respect to the bisector of angle B, then AX and BY are parallel.



Proof. Suppose $\angle PAB = \theta$ and $\angle PBA = \varphi$. Note that $\theta + \varphi = C$. Since $\angle XAB = A + \theta$ and $\angle YBA = B + \varphi$, we have $\angle XAB + \angle YBA = 180°$ and AX, BY are parallel.

(2) Similarly, if CZ and CP are symmetric with respect to the bisector of angle C, then CZ is parallel to AX and BY.

It follows that the isogonal conjugate of a point on the circumcircle is an infinite point, and conversely. We may therefore regard the circumcircle as the isogonal conjugate of the line at infinity. As such, the circumcircle has equation

$$a^2yz + b^2zx + c^2xy = 0.$$

Exercises

1. **Animate** a point P on the circumcircle.

   (1) Construct the **locus** of **isogonal conjugates** of points on the line OP.

   (2) Construct the isogonal conjugate Q of the infinite point of the line OP.

   The point lies on the locus in (1).

2. **Animate** a point P on the circumcircle. Find the **locus** of the **isotomic conjugate** $P^\bullet$. [3]

[3] The line $a^2x + b^2y + c^2z = 0$.

3. Let P and Q be antipodal points on the circumcircle. The lines $PQ^\bullet$ and $QP^\bullet$ joining each of these points to the **isotomic conjugate** of the other intersect orthogonally on the circumcircle.

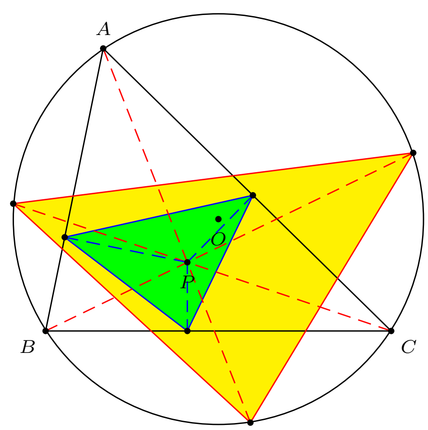
4. Let P and Q be antipodal points on the circumcircle. What is the locus of the intersection of $PP^\bullet$ and $QQ^\bullet$?

5. Let $P = (u : v : w)$. The lines AP, BP, CP intersect the circumcircle again at the points

$$\begin{aligned} A^{(P)} &= \left(\frac{-a^2vw}{c^2v+b^2w} : v : w\right), \\ B^{(P)} &= \left(u : \frac{-b^2wu}{a^2w+c^2u} : w\right), \\ C^{(P)} &= \left(u : v : \frac{-c^2uv}{b^2u+a^2v}\right). \end{aligned}$$

These form the vertices of the *Circumcevian triangle* of P.

(a) The circumcevian triangle of P is always similar to the pedal triangle.



(b) The circumcevian triangle of the incenter is perspective with the intouch triangle. What is the perspector? [4]

(c) The circumcevian triangle of P is always perspective with the tangential triangle. What is the perspector? [5]

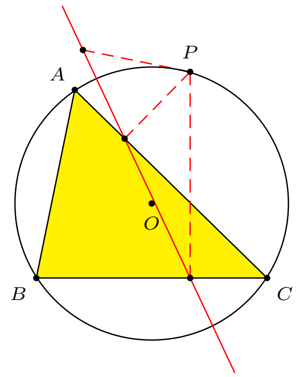
[4]The external center of similitude of the circumcircle and the incircle.

[5]$(a^2(-\frac{a^4}{u^2}+\frac{b^4}{v^2}+\frac{c^4}{w^2}) : \cdots : \cdots)$.

5.3 Simson lines

Consider the pedals of a point $P = (u : v : w)$:

$$\begin{aligned} A_{[P]} &= (0 : S_C u + a^2 v : S_B u + a^2 w), \\ B_{[P]} &= (S_C v + b^2 u : 0 : S_A v + b^2 w), \\ C_{[P]} &= (S_B w + c^2 u : S_A w + c^2 v : 0). \end{aligned}$$



These pedals of P are collinear if and only if P lies on the circumcircle, since

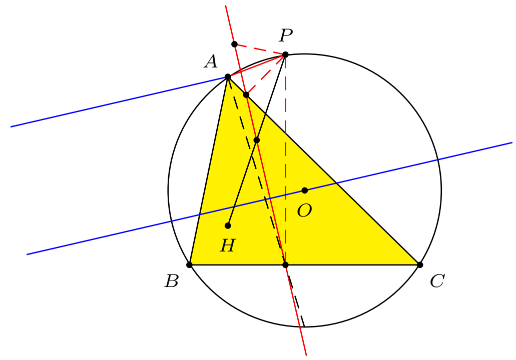
$$\begin{aligned} & \begin{vmatrix} 0 & S_C u + a^2 v & S_B u + a^2 w \\ S_C v + b^2 u & 0 & S_A v + b^2 w \\ S_B w + c^2 u & S_A w + c^2 v & 0 \end{vmatrix} \\ = \; & (u+v+w) \begin{vmatrix} a^2 & S_C u + a^2 v & S_B u + a^2 w \\ b^2 & 0 & S_A v + b^2 w \\ c^2 & S_A w + c^2 v & 0 \end{vmatrix} \\ \vdots \; & \cdots \\ = \; & (u+v+w)(S_{AB} + S_{BC} + S_{CA})(a^2 vw + b^2 wu + c^2 uv). \end{aligned}$$

If P lies on the circumcircle, the line containing the pedals is called the *Simson line* $\mathsf{s}(P)$ of P. If we write the coordinates of P in the form $(\frac{a^2}{f} : \frac{b^2}{g} : \frac{c^2}{h}) = (a^2gh : b^2hf : c^2fg)$ for an infinite point $(f : g : h)$, then

$$\begin{aligned} A_{[P]} &= (0 : a^2 S_C gh + a^2 b^2 hf : a^2 S_B gh + a^2 c^2 fg) \\ &\sim (0 : h(-S_C(h+f) + (S_C + S_A)f) : g(-S_B(f+g) + (S_A + S_B)f)) \\ &\sim (0 : -h(S_C h - S_A f) : g(S_A f - S_B g)). \end{aligned}$$

This becomes $A_{[P]} = (0 : -hg' : gh')$ if we write $(f' : g' : h') = (S_B g - S_C h : S_C h - S_A f : S_A f - S_B g)$ for the infinite point of lines in the direction perpendicular to $(f : g : h)$. Similarly, $B_{[P]} = (hf' : 0 : -fh')$ and $C_{[P]} = (-gf' : fg' : 0)$. The equation of the Simson line is

$$\frac{f}{f'}x + \frac{g}{g'}y + \frac{h}{h'}z = 0.$$



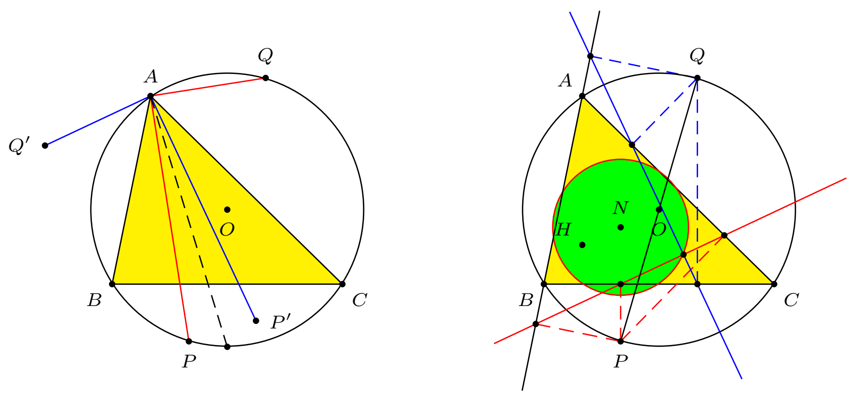
It is easy to determine the infinite point of the Simson line:

$$\begin{aligned}
B_{P]} - C_{[P]} &= c^2(S_C v + b^2 u : 0 : S_A v + b^2 w) - b^2(S_B w + c^2 u : S_A w + c^2 v : 0) \\
&= (*** : -b^2(S_A w + c^2 v) : c^2(S_A v + b^2 w)) \\
&\vdots \quad \cdots \\
&= (*** : S_C h - S_A f : S_A f - S_B g) \\
&= (f' : g' : h').
\end{aligned}$$

The Simson line $\mathsf{s}(P)$ is therefore perpendicular to the line defining P. It passes through, as we have noted, the midpoint between H and P, which lies on the nine-point circle.

5.3.1 Simson lines of antipodal points

Let P and Q be antipodal points on the circumcircle. They are isogonal conjugates of the infinite points of perpendicular lines.



Therefore, the Simson lines $\mathsf{s}(P)$ and $\mathsf{s}(Q)$ are perpendicular to each other. Since the midpoints of HP and HQ are antipodal on the nine-point circle, the two Simson lines intersect on the nine-point circle.

Exercises

1. **Animate** a point P on the circumcircle of triangle ABC and **trace** its Simson line.

2. Let H be the orthocenter of triangle ABC, and P a point on the circumcircle. Show that the midpoint of HP lies on the Simson line $\mathsf{s}(P)$ and on the nine-point circle of triangle ABC.

3. Let $\mathcal{L}$ be the line $\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=0$, intersecting the side lines BC, CA, AB of triangle ABC at U, V, W respectively.

(a) Find the equation of the perpendiculars to BC, CA, AB at U, V, W respectively. [6]

(b) Find the coordinates of the vertices of the triangle bounded by these three perpendiculars. [7]

(c) Show that this triangle is perspective with ABC at a point P on the circumcircle. [8]

(d) Show that the Simson line of the point P is parallel to $\mathcal{L}$.

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[6] $(S_Bv+S_Cw)x+a^2wy+a^2vz=0$, etc.

[7] $(-S^2u^2+S_{AB}uv+S_{BC}vw+S_{CA}wu : b^2(c^2uv-S_Auw-S_Bvw) : c^2(b^2uw-S_Auv-S_Cvw)$, etc.

[8] $P=\left(\frac{a^2}{-a^2vw+S_Buv+S_Cuw}:\cdots:\cdots\right)$.

5.4 Equation of the nine-point circle

To find the equation of the nine-point circle, we make use of the fact that it is obtained from the circumcircle by applying the homothety $\mathsf{h}(G, -\frac{1}{2})$. If $P = (x : y : z)$ is a point on the nine-point circle, then the point

$$Q = 3G - 2P = (x+y+z)(1:1:1) - 2(x:y:z) = (y+z-x : z+x-y : x+y-z)$$

is on the circumcircle. From the equation of the circumcircle, we obtain

$$a^2(z+x-y)(x+y-z) + b^2(x+y-z)(y+z-x) + c^2(y+z-x)(z+x-y) = 0.$$

Simplifying this equation, we have

$$0 = \sum_{\text{cyclic}} a^2(x^2 - y^2 + 2yz - z^2) = \sum_{\text{cyclic}} (a^2 - c^2 - b^2)x^2 + 2a^2yz,$$

or

$$\sum_{\text{cyclic}} S_A x^2 - a^2 yz = 0.$$

Exercises

1. Verify that the midpoint between the Fermat points, namely, the point with coordinates

$$((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2),$$

lies on the nine-point circle.

5.5 Equation of a general circle

Every circle $\mathcal{C}$ is homothetic to the circumcircle by a homothety, say $\mathsf{h}(T,k)$, where $T = uA + vB + wC$ (in absolute barycentric coordinate) is a center of similitude of $\mathcal{C}$ and the circumcircle. This means that if $P(x : y : z)$ is a point on the circle $\mathcal{C}$, then

$$\mathsf{h}(T,k)(P) = kP+(1-k)T \sim (x+tu(x+y+z) : y+tv(x+y+z) : z+tw(x+y+z)),$$

where $t = \frac{1-k}{k}$, lies on the circumcircle. In other words,

$$\begin{aligned}
0 &= \sum_{\text{cyclic}} a^2(ty + v(x+y+z))(tz + w(x+y+z)) \\
&= \sum_{\text{cyclic}} a^2(yz + t(wy + vz)(x+y+z) + t^2vw(x+y+z)^2) \\
&= (a^2yz + b^2zx + c^2xy) + t(\sum_{\text{cyclic}} a^2(wy+vz))(x+y+z) \\
&\quad +t^2(a^2vw + b^2wu + c^2uv)(x+y+z)^2
\end{aligned}$$

Note that the last two terms factor as the product of $x + y + z$ and another *linear form*. It follows that every circle can be represented by an equation of the form

$$a^2yz + b^2zx + c^2xy + (x+y+z)(px+qy+rz) = 0.$$

The line $px + qy + rz = 0$ is the *radical axis* of $\mathcal{C}$ and the circumcircle.

Exercises

1. The radical axis of the circumcircle and the nine-point circle is the line

$$S_Ax + S_By + S_Cz = 0.$$

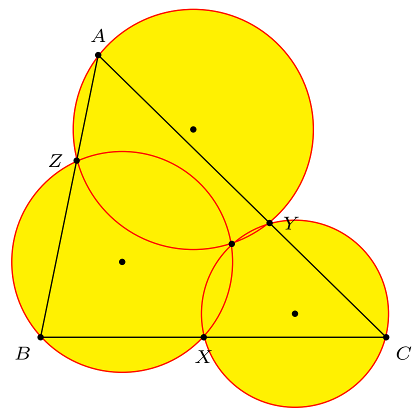
2. The circle through the excenters has center at the reflection of I in O, and radius $2R$. Find its equation. [9]

[9] $a^2yz + b^2zx + c^2xy + (x+y+z)(bcx + cay + abz) = 0$.

5.6 Appendix: Miquel Theory

5.6.1 Miquel Theorem

Let X, Y, Z be points on the lines BC, CA, and AB respectively. The three circles AYZ, BZX, and CXY pass through a common point.



5.6.2 Miquel associate

Suppose X, Y, Z are the traces of $P = (u : v : w)$. We determine the equation of the circle AYZ. [10] Writing it in the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(px + qy + rz) = 0$$

we note that $p = 0$ since it passes through $A = (1 : 0 : 0)$. Also, with $(x : y : z) = (u : 0 : w)$, we obtain $r = -\frac{b^2u}{w+u}$. Similarly, with $(x : y : z) = (u : v : 0)$, we obtain $q = -\frac{c^2u}{u+v}$. The equation of the circle

$$\mathcal{C}_{AYZ}: \qquad a^2yz + b^2zx + c^2xy - (x + y + z)\left(\frac{c^2u}{u+v}y + \frac{b^2u}{w+u}z\right) = 0.$$

Likewise, the equations of the other two circles are

$$\mathcal{C}_{BZX}: \qquad a^2yz + b^2zx + c^2xy - (x + y + z)(\frac{c^2v}{u+v}x + \frac{a^2v}{v+w}z) = 0,$$

and the one through C, X, and Y has equation

$$\mathcal{C}_{CXY}: \qquad a^2yz + b^2zx + c^2xy - (x + y + z)(\frac{b^2w}{w+u}x + \frac{a^2w}{v+w}y) = 0.$$

By Miquel's Theorem, the three circles intersect at a point P', which we call the *Miquel associate* of P. The coordinates of P' satisfy the equations

$$\frac{c^2u}{u+v}y + \frac{b^2u}{w+u}z = \frac{c^2v}{u+v}x + \frac{a^2v}{v+w}z = \frac{b^2w}{w+u}x + \frac{a^2w}{v+w}y.$$

[10] For the case when X, Y, Z are the intercepts of a line, see J.P. Ehrmann, *Steiner's theorems on the complete quadrilateral*, Forum Geometricorum, forthcoming.

Solving these equations, we have

$$\begin{aligned} P' &= \left(\frac{a^2}{v+w}\left(-\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}\right),\right.\\ &\quad :\frac{b^2}{w+u}\left(\frac{a^2vw}{v+w}-\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}\right),\\ &\quad \left.:\frac{c^2}{u+v}\left(\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}-\frac{c^2uv}{u+v}\right)\right). \end{aligned}$$

Examples

P	Miquel associate P'
centroid	circumcenter
orthocenter	orthocenter
Gergonne point	incenter
incenter	$(\frac{a^2(a^3+a^2(b+c)-a(b^2+bc+c^2)-(b+c)(b^2+c^2))}{b+c}:\cdots:\cdots)$
Nagel Point	$(a(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2):\cdots:\cdots)$

5.6.3 Cevian circumcircle

The cevian circumcircle of P is the circle through its traces. This has equation

$$(a^2yz+b^2zx+c^2xy)-(x+y+z)(px+qy+rz)=0,$$

where

$$vq+wr=\frac{a^2vw}{v+w},\qquad up+wr=\frac{b^2wu}{w+u},\qquad up+vq=\frac{c^2uv}{u+v}.$$

Solving these equations, we have

$$\begin{aligned} p &= \frac{1}{2u}\left(-\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}\right),\\ q &= \frac{1}{2v}\left(\frac{a^2vw}{v+w}-\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}\right),\\ r &= \frac{1}{2w}\left(\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}-\frac{c^2uv}{u+v}\right). \end{aligned}$$

5.6.4 Cyclocevian conjugate

The cevian circumcircle intersects the line BC at the points given by

$$a^2yz-(y+z)(qy+rz)=0.$$

This can be rearranged as

$$qy^2+(q+r-a^2)yz+rz^2=0.$$

The product of the two roots of $y : z$ is $\frac{r}{q}$. Since one of the roots $y : z = v : w$, the other root is $\frac{rw}{qv}$. The second intersection is therefore the point

$$X' = 0 : rw : qv = 0 : \frac{1}{qv} : \frac{1}{rw}.$$

Similarly, the "second" intersections of the circle XYZ with the other two sides can be found. The cevians AX', BY', and CZ' intersect at the point $(\frac{1}{pu} : \frac{1}{qv} : \frac{1}{rw})$. We denote this by $\mathsf{c}(P)$ and call it the *cyclocevian conjugate* of P. Explicitly,

$$\mathsf{c}(P) = \left(\frac{1}{-\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}} : \frac{1}{\frac{a^2vw}{v+w}-\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}} : \frac{1}{\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}-\frac{c^2uv}{u+v}}\right).$$

Examples

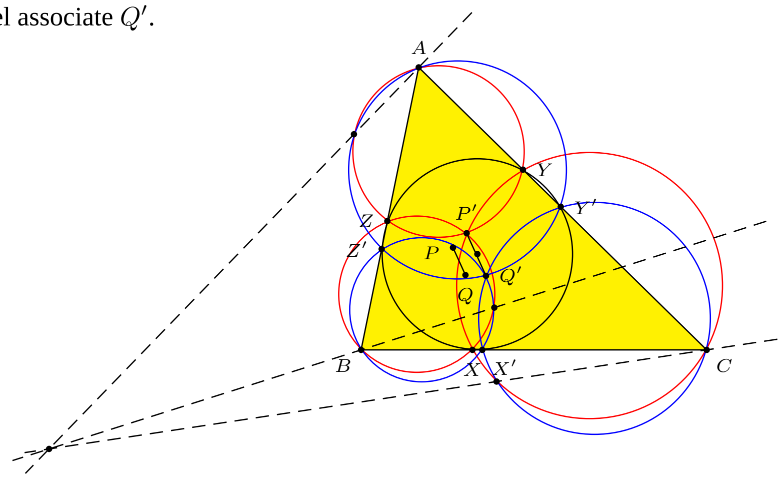
1. The centroid and the orthocenter are cyclocevian conjugates, their common cevian circumcircle being the nine-point circle.

2. The cyclocevian conjugate of the incenter is the point

$$\left(\frac{1}{a^3+a^2(b+c)-a(b^2+bc+c^2)-(b+c)(b^2+c^2)} : \cdots : \cdots\right).$$

Theorem

Given a point P, let P' be its Miquel associate and Q its cyclocevian conjugate, with Miquel associate Q'.



(a) P' and Q' are isogonal conjugates.
(b) The lines PQ and $P'Q'$ are parallel.
(c) The "second intersections" of the pairs of circles AYZ, $AY'Z'$; BZX, $BZ'X'$; and CXY, $CX'Y'$ form a triangle $A'B'C'$ perspective with ABC.
(d) The "Miquel perspector" in (c) is the intersection of the trilinear polars of P and Q with respect to triangle ABC.

Exercises

1. For a real number t, we consider the triad of points

$$X_t = (0 : 1 - t : t), \qquad Y_t = (t : 0 : 1 - t), \qquad Z_t = (1 - t : t : 0)$$

on the sides of the reference triangle.

(a) The circles AY_tZ_t, BZ_tX_t and CX_tY_t intersect at the point

$$\begin{aligned} M_t \quad = \quad & (a^2(-a^2t(1-t) + b^2t^2 + c^2(1-t)^2) \\ & : b^2(a^2(1-t)^2 - b^2t(1-t) + c^2t^2) \\ & : c^2(a^2t^2 + b^2(1-t)^2 - c^2t(1-t))). \end{aligned}$$

(b) Writing $M_t = (x : y : z)$, eliminate t to obtain the following equation in x, y, z:

$$b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2 - c^4xy - b^4zx - a^4yz = 0.$$

(c) Show that the locus of M_t is a circle.

(d) Verify that this circle contains the circumcenter, the symmedian point, and the two *Brocard points*

$$\Omega_{\leftarrow} = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right),$$

and

$$\Omega_{\rightarrow} = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right).$$

Chapter 6

Circles II

6.1 Equation of the incircle

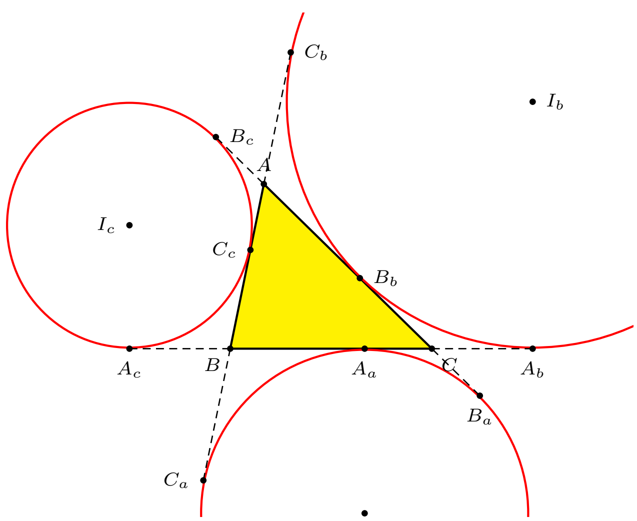
Write the equation of the incircle in the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) = 0$$

for some undetermined coefficients p, q, r. Since the incircle touches the side BC at the point $(0 : s - c : s - b)$, $y : z = s - c : s - b$ is the only root of the quadratic equation $a^2yz + (y + z)(qy + rz) = 0$. This means that

$$qy^2 + (q + r - a^2)yz + rz^2 = k((s - b)y - (s - c)z)^2$$

for some scalar k.



Comparison of coefficients gives $k = 1$ and $q = (s-b)^2$, $r = (s-c)^2$. Similarly, by considering the tangency with the line CA, we obtain $p = (s - a)^2$ and (consistently)

$r = (s-c)^2$. It follows that the equation of the incircle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)((s-a)^2x + (s-b)^2y + (s-c)^2z) = 0.$$

The radical axis with the circumcircle is the line

$$(s-a)^2x + (s-b)^2y + (s-c)^2z = 0.$$

6.1.1 The excircles

The same method gives the equations of the excircles:

$$\begin{aligned}
a^2yz + b^2zx + c^2xy - (x+y+z)(s^2x + (s-c)^2y + (s-b)^2z) &= 0,\\
a^2yz + b^2zx + c^2xy - (x+y+z)((s-c)^2x + s^2y + (s-a)^2z) &= 0,\\
a^2yz + b^2zx + c^2xy - (x+y+z)((s-b)^2x + (s-a)^2y + s^2z) &= 0.
\end{aligned}$$

Exercises

1. Show that the Nagel point of triangle ABC lies *on* its incircle if and only if one of its sides is equal to $\frac{s}{2}$. Make use of this to design an animation picture showing a triangle with its Nagel point on the incircle.

2. (a) Show that the centroid of triangle ABC lies on the incircle if and only if $5(a^2+b^2+c^2) = 6(ab+bc+ca)$.

(b) Let ABC be an *equilateral* triangle with center O, and $\mathcal{C}$ the circle, center O, radius half that of the incirle of ABC. Show that the distances from an arbitrary point P on $\mathcal{C}$ to the sidelines of ABC are the lengths of the sides of a triangle whose centroid is on the incircle.

6.2 Intersection of the incircle and the nine-point circle

We consider how the incircle and the nine-point circle intersect. The intersections of the two circles can be found by solving their equations simultaneously:

$$\begin{aligned}
&a^2yz + b^2zx + c^2xy - (x+y+z)((s-a)^2x + (s-b)^2y + (s-c)^2z) = 0,\\
&a^2yz + b^2zx + c^2xy - \frac{1}{2}(x+y+z)(S_Ax + S_By + S_Cz) = 0.
\end{aligned}$$

6.2.1 Radical axis of (I) and (N)

Note that

$$(s-a)^2 - \frac{1}{2}S_A = \frac{1}{4}((b+c-a)^2 - (b^2+c^2-a^2)) = \frac{1}{2}(a^2 - a(b+c) + bc) = \frac{1}{2}(a-b)(a-c).$$

Subtracting the two equations we obtain the equation of the radical axis of the two circles:

$$\mathcal{L}: \qquad (a-b)(a-c)x + (b-a)(b-c)y + (c-a)(c-b)z = 0.$$

We rewrite this as

$$\frac{x}{b-c} + \frac{y}{c-a} + \frac{z}{a-b} = 0.$$

There are two points with simple coordinates on this line:

$$P = ((b-c)^2 : (c-a)^2 : (a-b)^2),$$

and

$$Q = (a(b-c)^2 : b(c-a)^2 : c(a-b)^2).$$

Making use of these points we obtain a very simple parametrization of points on the radical axis $\mathcal{L}$, except P:

$$(x:y:z) = ((a+t)(b-c)^2 : (b+t)(c-a)^2 : (c+t)(a-b)^2)$$

for some t.

6.2.2 The line joining the incenter and the nine-point center

We find the intersection of the radical axis $\mathcal{L}$ and the line joining the centers I and N. It is convenient to write the coordinates of the nine-point center in terms of a, b, c. Thus,

$$N = (a^2(b^2+c^2) - (b^2-c^2)^2 : b^2(c^2+a^2) - (c^2-a^2)^2 : c^2(a^2+b^2) - (a^2-b^2)^2)$$

with coordinate sum $8S^2$.[1]

[1] Start with $N = (S^2 + S_{BC} : \cdots : \cdots)$ (with coordinate sum $4S^2$) and rewrite $S^2 + S_{BC} = \cdots = \frac{1}{2}(a^2(b^2+c^2) - (b^2-c^2)^2)$.

We seek a real number k for which the point

$$\begin{array}{ll} & (a^2(b^2+c^2)-(b^2-c^2)^2+ka \\ : & b^2(c^2+a^2)-(c^2-a^2)^2+kb \\ : & c^2(a^2+b^2)-(a^2-b^2)^2+kc) \end{array}$$

on the line IN also lies on the radical axis $\mathcal{L}$. With $k=-2abc$, we have

$$\begin{array}{ll} & a^2(b^2+c^2)-(b^2-c^2)^2-2a^2bc \\ = & a^2(b-c)^2-(b^2-c^2)^2 \\ = & (b-c)^2(a^2-(b+c)^2) \\ = & 4s(a-s)(b-c)^2, \end{array}$$

and two analogous expressions by cyclic permutations of a, b, c. These give the coordinates of a point on $\mathcal{L}$ with $t=-s$, and we conclude that the two lines intersect at the *Feuerbach point*

$$F=((s-a)(b-c)^2:(s-b)(c-a)^2:(s-c)(a-b)^2).$$

We proceed to determine the ratio of division $IF:FN$. From the choice of k, we have

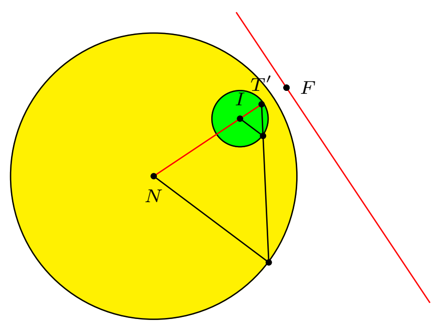
$$F\sim 8S^2\cdot N-2abc\cdot 2s\cdot I=8S^2\cdot N-4sabc\cdot I.$$

This means that

$$NF:FI=-4sabc:8S^2=-8sRS:8S^2=-sR:S=R:-2r=\frac{R}{2}:-r.$$

The point F is the external center of similitude of the nine-point circle and the incircle.

However, if a center of similitude of two circles lies on their radical axis, the circles must be tangent to each other (at that center). [2]



[2]Proof: Consider two circles of radii p and q, centers at a distance d apart. Suppose the intersection of the radical axis and the center line is at a distance x from the center of the circle of radius p, then $x^2-p^2=(d-x)^2-q^2$. From this, $x=\frac{d^2+p^2-q^2}{2d}$, and $d-x=\frac{d^2-p^2+q^2}{2d}$. The division ratio is $x:d-x=d^2+p^2-q^2:d^2-p^2+q^2$. If this is equal to $p:-q$, then $p(d^2-p^2+q^2)+q(d^2+p^2-q^2)=0$, $(p+q)(d^2-(p-q)^2)=0$. From this $d=|p-q|$, and the circles are tangent internally.

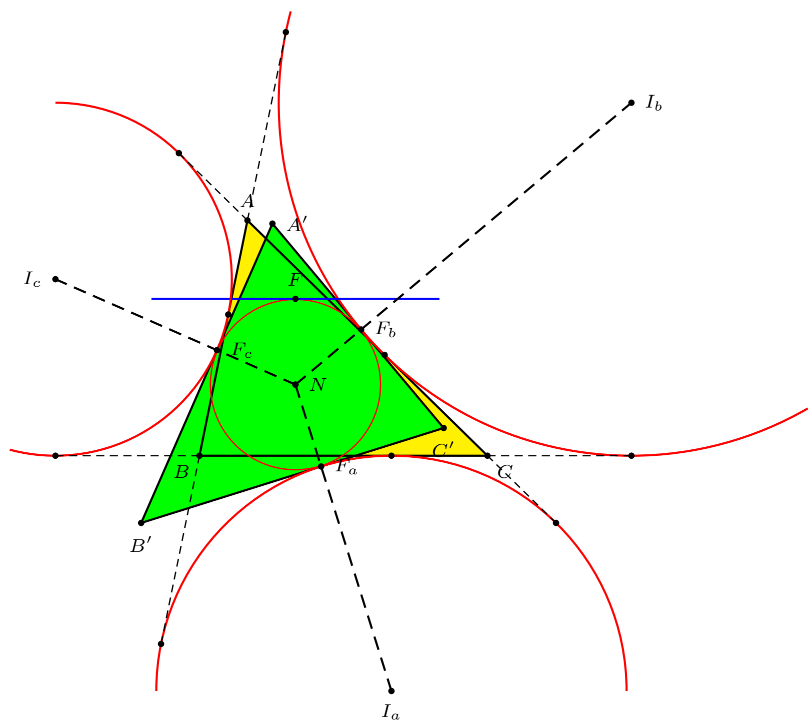
Feuerbach's Theorem

The nine-point circle and the incircle are tangent internally to each other at the point F, the common tangent being the line

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0.$$

The nine-point circle is tangent to each of the excircles externally. The points of tangency form a triangle perspective with ABC at the point

$$F'=\left(\frac{(b+c)^2}{s-a}:\frac{(c+a)^2}{s-b}:\frac{(a+b)^2}{s-c}\right).$$



Exercises

1. Show that F and F' divide I and N harmonically.

2. Find the equations of the common tangent of the nine-point circle and the excircles. [3]

3. Apart from their common tangent at F_a, the nine-point circle and the A-excircle have another pair of common tangents, intersecting at their external center of

[3]Tangent to the A-excircle: $\frac{x}{b-c}+\frac{y}{c+a}-\frac{z}{a+b}=0$.

similitude A'. Similarly define B' and C'. The triangle $A'B'C'$ is perspective with ABC. What is the perspector? [4]

4. Let ℓ be a diameter of the circumcircle of triangle ABC. Animate a point P on ℓ and construct its *pedal circle*, the circle through the pedals of P on the side lines. The pedal circle always passes through a fixed point on the nine-point circle.

What is this fixed point if the diameter passes through the incenter?

[4]The Feuerbach point.

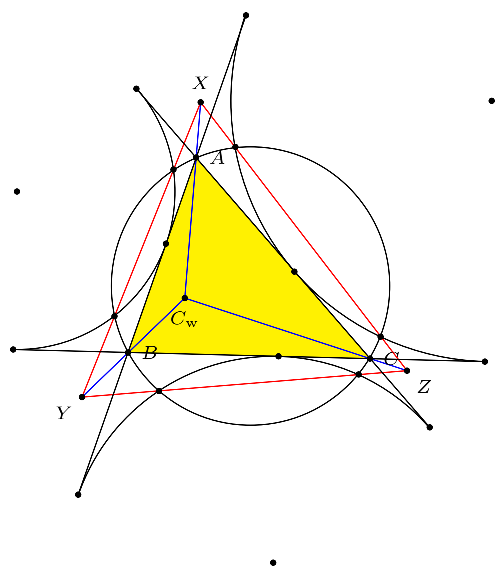
6.3 The excircles

Consider the radical axes of the excircles with the circumcircle. These are the lines

$$\begin{aligned}
s^2x + (s-c)^2y + (s-b)^2z &= 0,\\
(s-c)^2x + s^2y + (s-a)^2z &= 0,\\
(s-b)^2x + (s-a)^2y + s^2z &= 0.
\end{aligned}$$

These three lines bound a triangle with vertices

$$\begin{aligned}
A' &= (-(b+c)(a^2+(b+c)^2) : b(a^2+b^2-c^2) : c(c^2+a^2-b^2)),\\
B' &= (a(a^2+b^2-c^2) : -(c+a)(b^2+(c+a)^2) : c(b^2+c^2-a^2)),\\
C' &= (a(c^2+a^2-b^2) : b(b^2+c^2-a^2) : -(a+b)(c^2+(a+b)^2)).
\end{aligned}$$



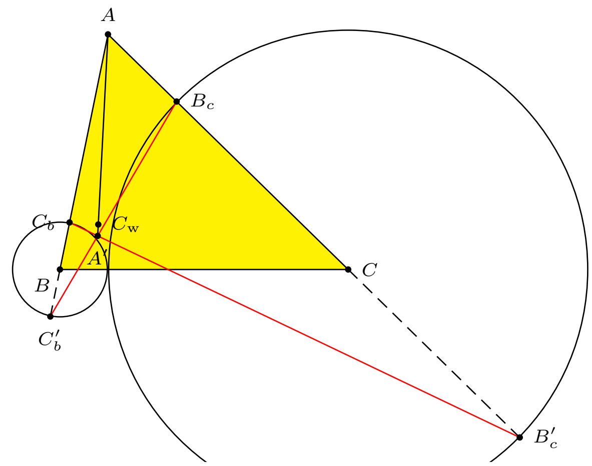
The triangle $A'B'C'$ is perspective with ABC at the Clawson point [5]

$$\left(\frac{a}{S_A} : \frac{b}{S_B} : \frac{c}{S_C}\right).$$

Exercises

1. Let A_H be the pedal of A on the opposite side BC of triangle ABC. Construct circle $B(A_H)$ to intersect AB at C_b and C_b' (so that C_b' in on the extension of AB), and circle $C(A_H)$ to intersect AC at and B_c and B_c' (so that B_c' in on the extension of AC).

[5]This point appears in ETC as the point X_{19}.



(a) Let A_1 be the intersection of the lines $B_cC'_b$ and $C_bB'_c$. Similarly define B_1 and C_1. Show that $A_1B_1C_1$ is perspective with ABC at the Clawson point. [6]

(b) Let $A_2 = BB_c \cap CC_b$, $B_2 = CC_a \cap AA_c$, and $C_2 = AA_b \cap BB_a$. Show that $A_2B_2C_2$ is perspective with ABC. Calculate the coordinates of the perspector. [7]

(c) Let $A_3 = BB'_c \cap CC'_b$, $B_3 = CC'_a \cap AA'_c$, and $C_3 = AA'_b \cap BB'_a$. Show that $A_3B_3C_3$ is perspective with ABC. Calculate the coordinates of the perspector. [8]

2. Consider the B- and C-excircles of triangle ABC. Three of their common tangents are the side lines of triangle ABC. The fourth common tangent is the reflection of the line BC in the line joining the excenters I_b and I_c.

(a) Find the equation of this fourth common tangent, and write down the equations of the fourth common tangents of the other two pairs of excircles.

(b) Show that the triangle bounded by these 3 fourth common tangents is homothetic to the orthic triangle, and determine the homothetic center. [9]

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[6]A.P.Hatzipolakis, Hyacinthos, message 1663, October 25, 2000.

[7]$X_{278} = (\frac{1}{(s-a)S_A} : \cdots : \cdots)$

[8]$X_{281} = (\frac{s-a}{S_A} : \cdots : \cdots)$

[9]The Clawson point. See R. Lyness and G.R. Veldkamp, Problem 682 and solution, *Crux Math.* 9 (1983) 23 – 24.

6.4 The Brocard points

Consider the circle through the vertices A and B and tangent to the side AC at the vertex A. Since the circle passes through A and B, its equation is of the form

$$a^2yz + b^2zx + c^2xy - rz(x+y+z) = 0$$

for some constant r. Since it is tangent to AC at A, when we set $y = 0$, the equation should reduce to $z^2 = 0$. This means that $r = b^2$ and the circle is

$$\mathcal{C}_{AAB}: \qquad a^2yz + b^2zx + c^2xy - b^2z(x+y+z) = 0.$$

Similarly, we consider the analogous circles

$$\mathcal{C}_{BBC}: \qquad a^2yz + b^2zx + c^2xy - c^2x(x+y+z) = 0.$$

and

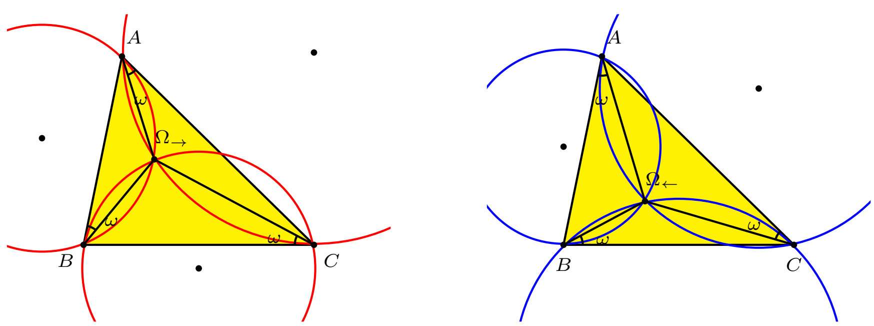
$$\mathcal{C}_{CCA}: \qquad a^2yz + b^2zx + c^2xy - a^2y(x+y+z) = 0.$$

These three circles intersect at the *forward Brocard point*

$$\Omega_{\rightarrow} = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right).$$

This point has the property that

$$\angle AB\Omega_{\rightarrow} = \angle BC\Omega_{\rightarrow} = \angle CA\Omega_{\rightarrow}.$$



In reverse orientations there are three circles $\mathcal{C}_{ABB}$, $\mathcal{C}_{BCC}$, and $\mathcal{C}_{CAA}$ intersecting at the *backward Brocard point*

$$\Omega_{\leftarrow} = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right).$$

satisfying

$$\angle BA\Omega_{\leftarrow} = \angle CB\Omega_{\leftarrow} = \angle AC\Omega_{\leftarrow}.$$

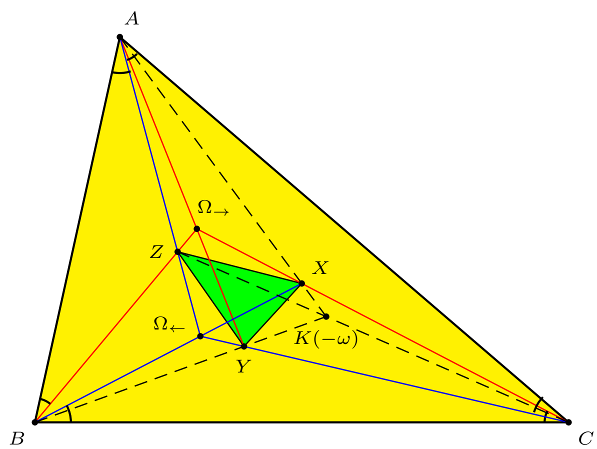
Note from their coordinates that the two Brocard points are isogonal conjugates. This means that the 6 angles listed above are all equal. We denote the common value by ω and call this the *Brocard angle* of triangle ABC. By writing the coordinates of $\Omega_{\rightarrow}$ in Conway's notation, it is easy to see that

$$\cot\omega = \frac{1}{2}(S_A + S_B + S_C).$$

The lines $B\Omega_{\leftarrow}$ and $C\Omega_{\rightarrow}$ intersect at $A_{-\omega}$. Similarly, we have $B_{-\omega} = C\Omega_{\leftarrow} \cap A\Omega_{\rightarrow}$, and $C_{-\omega} = A\Omega_{\leftarrow} \cap B\Omega_{\rightarrow}$. Clearly the triangle $A_{-\omega}B_{-\omega}C_{-\omega}$ is perspective to ABC at the point

$$K(-\omega) = \left(\frac{1}{S_A - S_\omega} : \cdots : \cdots\right) \sim \cdots \sim \left(\frac{1}{a^2} : \cdots : \cdots\right),$$

which is the isotomic conjugate of the symmedian point. [10]



Exercises

1. The midpoint of the segment $\Omega_{\rightarrow}\Omega_{\leftarrow}$ is the *Brocard midpoint* [11]

$$(a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2)).$$

Show that this is a point on the line OK.

2. The Brocard circle is the circle through the three points $A_{-\omega}$, $B_{-\omega}$, and $C_{-\omega}$. It has equation

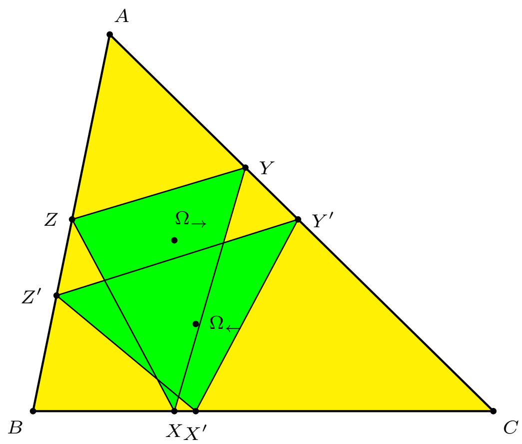
$$a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{a^2+b^2+c^2}(x+y+z)\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.$$

Show that this circle also contains the two Brocard point $\Omega_{\rightarrow}$ and $\Omega_{\leftarrow}$, the circumcenter, and the symmedian point.

[10] This is also known as the *third Brocard point*. It appears as the point X_{76} in ETC.

[11] The Brocard midpoint appears in ETC as the point X_{39}.

3. Let XYZ be the pedal triangle of $\Omega_{\rightarrow}$ and $X'Y'Z'$ be that of $\Omega_{\leftarrow}$.



(a) Find the coordinates of these pedals.

(b) Show that $Y'Z$ is parallel to BC.

(c) The triangle bounded by the three lines $Y'Z$, $Z'X$ and $X'Y$ is homothetic to triangle ABC. What is the homothetic center? [12]

(d) The triangles XYZ and $Y'Z'X'$ are congruent.

[12] The symmedian point.

6.5 Appendix: The circle triad $(A(a), B(b), C(c))$

Consider the circle $A(a)$. This circle intersects the line AB at the two points $(c+a : -a : 0)$, $(c-a : a : 0)$, and AC at $(a+b : 0 : -a)$ and $(b-a : 0 : a)$. It has equation

$$\mathcal{C}_a : \quad a^2yz + b^2zx + c^2xy + (x+y+z)(a^2x + (a^2-c^2)y + (a^2-b^2)z) = 0.$$

Similarly, the circles $B(b)$ and $C(c)$ have equations

$$\mathcal{C}_b : \quad a^2yz + b^2zx + c^2xy + (x+y+z)((b^2-c^2)x + b^2y + (b^2-a^2)z) = 0,$$

and

$$\mathcal{C}_c : \quad a^2yz + b^2zx + c^2xy + (x+y+z)((c^2-b^2)x + (c^2-a^2)y + c^2z) = 0.$$

These are called the de Longchamps circles of triangle ABC. The radical center L of the circles is the point $(x : y : z)$ given by

$$a^2x+(a^2-c^2)y+(a^2-b^2)z = (b^2-c^2)x+b^2y+(b^2-a^2)z = (c^2-b^2)x+(c^2-a^2)y+c^2z.$$

Forming the pairwise sums of these expressions we obtain

$$S_A(y+z) = S_B(z+x) = S_C(x+y).$$

From these,

$$y+z : z+x : x+y = \frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C} = S_{BC} : S_{CA} : S_{AB},$$

and

$$x : y : z = S_{CA}+S_{AB}-S_{BC} : S_{AB}+S_{BC}-S_{CA} : S_{BC}+S_{CA}-S_{AB}.$$

This is called the *de Longchamps point* of the triangle. [13] It is the reflection of the orthocenter in the circumcenter, *i.e.*, $L = 2 \cdot O - H$.

Exercises

1. Show that the intersections of $\mathcal{C}_b$ and $\mathcal{C}_c$ are the reflections of A

(i) in the midpoint of BC, and

(ii) in the perpendicular bisector of BC.

What are the coordinates of these points? [14]

2. The circle $\mathcal{C}_a$ intersects the circumcircle at B' and C'.

3. The de Longchamps point L is the orthocenter of the anticomplementary triangle, and triangle $A'B'C'$ is the orthic triangle.

[13] The de Longchamps point appears as the point X_{20} in ETC.
[14] $(-1 : 1 : 1)$ and $A' = (-a^2 : b^2-c^2 : c^2-b^2)$.

6.5.1 The Steiner point

The radical axis of the circumcircle and the circle $\mathcal{C}_a$ is the line

$$a^2x + (a^2 - c^2)y + (a^2 - b^2)z = 0.$$

This line intersects the side line BC at point

$$A' = \left(0 : \frac{1}{c^2 - a^2} : \frac{1}{a^2 - b^2}\right).$$

Similarly, the radical axis of $\mathcal{C}_b$ has b-intercept

$$B' = \left(\frac{1}{b^2 - c^2} : 0 : \frac{1}{a^2 - b^2}\right),$$

and that of $\mathcal{C}_c$ has c-intercept

$$C' = \left(\frac{1}{b^2 - c^2} : \frac{1}{c^2 - a^2} : 0\right).$$

These three points A', B', C' are the traces of the point with coordinates

$$\left(\frac{1}{b^2 - c^2} : \frac{1}{c^2 - a^2} : \frac{1}{a^2 - b^2}\right).$$

This is a point on the circumcircle, called the *Steiner point*. [15]

Exercises

1. The antipode of the Steiner point on the circumcircle is called the *Tarry point*. Calculate its coordinates. [16]

2. Reflect the vertices A, B, C in the centroid G to form the points A', B', C' respectively. Use the **five-point conic** command to construct the conic through A, B, C, A', B',C". This is the *Steiner circum-ellipse*. Apart from the vertices, it intersects the circumcircle at the Steiner point.

3. Use the **five-point conic** command to construct the conic through the vertices of triangle ABC, its centroid, and orthocenter. This is a rectangular hyperbola called the *Kiepert hyperbola* which intersect the circumcircle, apart from the vertices, at the Tarry point.

[15] This point appears as X_{99} in ETC.

[16] $\left(\frac{1}{a^2(b^2+c^2)-(b^4+c^4)} : \cdots : \cdots\right)$. The Tarry point appears the point X_{98} in ETC.

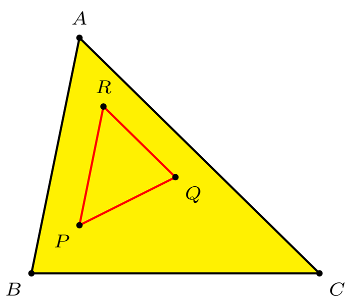
Chapter 7

Circles III

7.1 The distance formula

Let $P = uA + vB + wC$ and $Q = u'A + v'B + w'C$ be given in absolute barycentric coordinates. The distance between them is given by

$$PQ^2 = S_A(u - u')^2 + S_B(v - v')^2 + S_C(w - w')^2$$



Proof. Through P and Q draw lines parallel to AB and AC respectively, intersecting at R. The barycentric coordinates of R can be determined in two ways. $R = P + h(B - C) = Q + k(A - C)$ for some h and k. It follows that $R = uA + (v+h)B + (w-h)C = (u' + k)A + v'B + (w' - k)C$, from which $h = -(v - v')$ and $k = u - u'$. Applying the law of cosines to triangle PQR, we have

$$\begin{aligned}
PQ^2 &= (ha)^2 + (kb)^2 - 2(ha)(kb)\cos C \\
&= h^2a^2 + k^2b^2 - 2hkS_C \\
&= (S_B + S_C)(v - v')^2 + (S_C + S_A)(u - u')^2 + 2(u - u')(v - v')S_C \\
&= S_A(u - u')^2 + S_B(v - v')^2 \\
&\quad + S_C[(u - u')^2 + 2(u - u')(v - v') + (v - v')^2].
\end{aligned}$$

The result follows since

$$(u - u') + (v - v') = (u + v) - (u' + v') = (1 - w) - (1 - w') = -(w - w').$$

The distance formula in homogeneous coordinates

If $P = (x : y : z)$ and $Q = (u : v : w)$, the distance between P and Q is given by

$$|PQ|^2 = \frac{1}{(u+v+w)^2(x+y+z)^2} \sum_{\text{cyclic}} S_A((v+w)x - u(y+z))^2.$$

Exercises

1. The distance from $P = (x : y : z)$ to the vertices of triangle ABC are given by

$$\begin{aligned} AP^2 &= \frac{c^2y^2 + 2S_Ayz + b^2z^2}{(x+y+z)^2}, \\ BP^2 &= \frac{a^2z^2 + 2S_Bzx + c^2x^2}{(x+y+z)^2}, \\ CP^2 &= \frac{b^2x^2 + 2S_Cxy + a^2y^2}{(x+y+z)^2}. \end{aligned}$$

2. The distance between $P = (x : y : z)$ and $Q = (u : v : w)$ can be written as

$$|PQ|^2 = \frac{1}{x+y+z} \cdot \left(\sum_{\text{cyclic}} \frac{c^2v^2 + 2S_Avw + b^2w^2}{(u+v+w)^2} x \right) - \frac{a^2yz + b^2zx + c^2xy}{(x+y+z)^2}.$$

3. Compute the distance between the incenter and the nine-point center $N = (S^2 + S_A : S^2 + S_B : S^2 + S_C)$. Deduce Feuerbach's theorem by showing that this is $\frac{R}{2} - r$. Find the coordinates of the Feuerbach point F as the point dividing NI externally in the ratio $R : -2r$.

7.2 Circle equations

7.2.1 Equation of circle with center $(u : v : w)$ and radius ρ:

$$a^2yz + b^2zx + c^2xy - (x+y+z)\sum_{\text{cyclic}}\left(\frac{c^2v^2 + 2S_Avw + b^2w^2}{(u+v+w)^2} - \rho^2\right)x = 0.$$

7.2.2 The power of a point with respect to a circle

Consider a circle $\mathcal{C} := O(\rho)$ and a point P. By the theorem on intersecting chords, for any line through P intersecting $\mathcal{C}$ at two points X and Y, the product $|PX||PY|$ of *signed* lengths is constant. We call this product the *power* of P with respect to $\mathcal{C}$. By considering the diameter through P, we obtain $|OP|^2 - \rho^2$ for the power of a point P with respect to $O(\rho)$.

7.2.3 Proposition

Let p, q, r be the powers of A, B, C with respect to a circle $\mathcal{C}$.

(1) The equation of the circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0.$$

(2) The center of the circle is the point

$$(a^2S_A + S_B(r-p) - S_C(p-q) : b^2S_B + S_C(p-q) - S_A(r-p) : c^2S_C + S_A(q-r) - S_B(r-p).$$

(3) The radius ρ of the circle is given by

$$\rho^2 = \frac{a^2b^2c^2 - 2(a^2S_Ap + b^2S_Bq + c^2S_Cr) + S_A(q-r)^2 + S_B(r-p)^2 + S_C(p-q)^2}{4S^2}.$$

Exercises

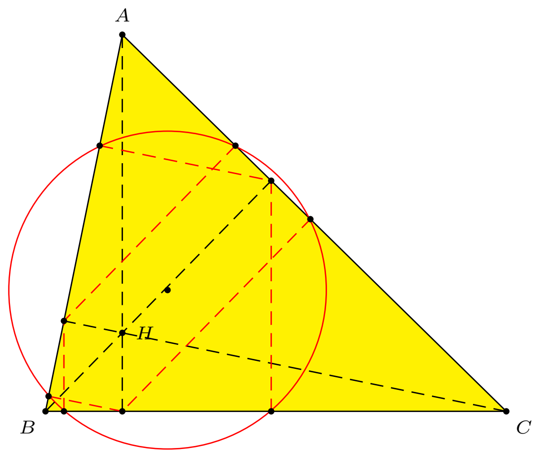
1. Let X, Y, Z be the pedals of A, B, C on their opposite sides. The pedals of X on CA and AB, Y on AB, BC, and Z on CA, BC are on a circle. Show that the equation of the circle is [1]

$$a^2yz + b^2zx + c^2xy - \frac{1}{4R^2}(x+y+z)(S_{AA}x + S_{BB}y + S_{CC}z) = 0.$$

2. Let $P = (u : v : w)$ with cevian triangle XYZ.

   (a) Find the equations of the circles ABY and ACZ, and the coordinates of their second intersection A'.

[1] This is called the *Taylor circle* of triangle ABC. Its center is the point X_{389} in ETC. This point is also the intersection of the three lines through the midpoint of each side of the *orthic triangle* perpendicular to the corresponding side of ABC.



(b) Similarly define B' and C'. Show that triangle $A'B'C'$ is perspective with ABC. Identify the perspector. [2]

[2] $\left(\frac{a^2}{v+w} : \cdots : \cdots\right)$. See Tatiana Emelyanov, Hyacinthos, message 3309, 7/27/01.

7.3 Radical circle of a triad of circles

Consider three circles with equations

$$a^2yz + b^2zx + c^2xy - (x+y+z)(p_ix + q_iy + r_iz) = 0, \qquad i = 1, 2, 3.$$

7.3.1 Radical center

The *radical center* P is the point with equal powers with respect to the three circles. Its coordinates are given by the solutions of the system of equations.

$$p_1x + q_1y + r_1z = p_2x + q_2y + r_2z = p_3x + q_3y + r_3z.$$

Explicitly, if we write

$$M = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix},$$

then, $P = (u : v : w)$ with [3]

$$u = \begin{vmatrix} 1 & q_1 & r_1 \\ 1 & q_2 & r_2 \\ 1 & q_3 & r_3 \end{vmatrix}, \quad v = \begin{vmatrix} p_1 & 1 & r_1 \\ p_2 & 1 & r_2 \\ p_3 & 1 & r_3 \end{vmatrix}, \quad w = \begin{vmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{vmatrix}.$$

7.3.2 Radical circle

There is a circle orthogonal to each of the circles $\mathcal{C}_i$, $i = 1, 2, 3$. The center is the radical center P above, and its square radius is the *negative* of the common power of P with respect to the circles, ***i.e.***,

$$\frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2} - \frac{\det M}{u+v+w}.$$

This circle, which we call the *radical circle* of the given triad, has equation

$$\sum_{\text{cyclic}} (c^2v + b^2w)x^2 + 2S_Auyz - \det(M)(x+y+z)^2 = 0.$$

In standard form, it is

$$a^2yz + b^2zx + c^2xy - \frac{1}{u+v+w} \cdot (x+y+z)(\sum_{\text{cyclic}} (c^2v + b^2w - \det(M))x) = 0.$$

The radical circle is real if and only if

$$(u+v+w)(p_iu + q_iv + r_iw) - (a^2vw + b^2wu + c^2uv) \geq 0$$

for *any* $i = 1, 2, 3$.

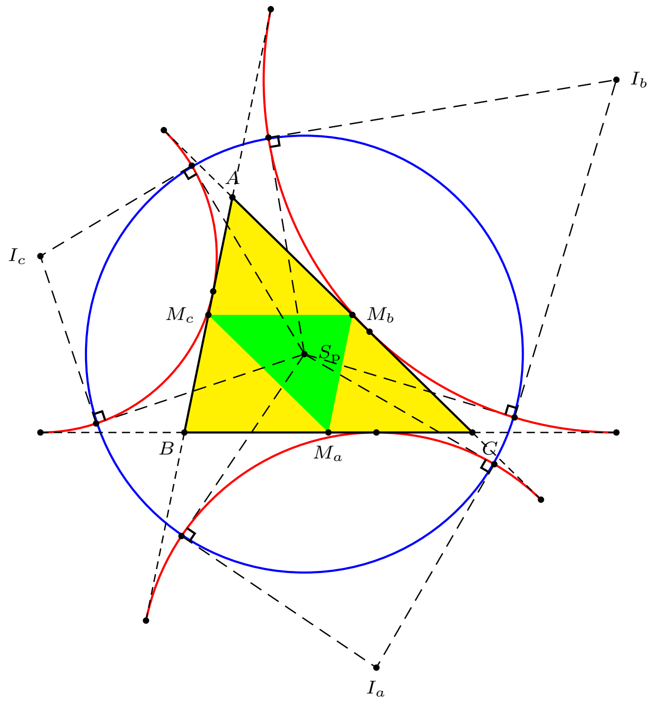
[3]Proof: $p_1u + q_1v + r_1w = p_2u + q_2v + r_2w = p_3u + q_3v + r_3w = \det M$.

7.3.3 The excircles

The radical center of the excircles is the point $P = (u : v : w)$ given by

$$\begin{aligned} u &= \begin{pmatrix} 1 & (s-c)^2 & (s-b)^2 \\ 1 & s^2 & (s-a)^2 \\ 1 & (s-a)^2 & s^2 \end{pmatrix} = \begin{pmatrix} 1 & (s-c)^2 & (s-a)^2 \\ 0 & c(a+b) & -c(a-b) \\ 0 & b(c-a) & b(c+a) \end{pmatrix} \\ &= bc(a+b)(c+a) + bc(a-b)(c-a) = 2abc(b+c), \end{aligned}$$

and, likewise, $v = 2abc(c+a)$ and $w = 2abc(a+b)$. This is the point $(b+c : c+a : a+b)$, called the *Spieker center*. It is the incenter of the medial triangle.



Since, with $(u, v, w) = (b+c, c+a, a+b)$,

$$\begin{aligned} & (u+v+w)(s^2u + (s-c)^2v + (s-b)^2w) - (a^2vw + b^2wu + c^2uv) \\ = \; & (a+b+c)(2abc + \sum a^3 + \sum a^2(b+c)) - (a+b+c)(abc + \sum a^3) \\ = \; & (a+b+c)(abc + \sum a^2(b+c)), \end{aligned}$$

the square radius of the orthogonal circle is

$$\frac{abc + \sum a^2(b+c)}{a+b+c} = \cdots = \frac{1}{4}(r^2 + s^2).$$

The equation of the radical circle can be written as

$$\sum_{\text{cyclic}} (s-b)(s-c)x^2 + asyz = 0.$$

7.3.4 The de Longchamps circle

The radical center L of the circle triad $(A(a), B(b), C(c))$ is the point $(x : y : z)$ given by

$$a^2x+(a^2-c^2)y+(a^2-b^2)z = (b^2-c^2)x+b^2y+(b^2-a^2)z = (c^2-b^2)x+(c^2-a^2)y+c^2z.$$

Forming the pairwise sums of these expressions we obtain

$$S_A(y+z) = S_B(z+x) = S_C(x+y).$$

From these,

$$y+z : z+x : x+y = \frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C} = S_{BC} : S_{CA} : S_{AB},$$

and

$$x : y : z = S_{CA}+S_{AB}-S_{BC} : S_{AB}+S_{BC}-S_{CA} : S_{BC}+S_{CA}-S_{AB}.$$

This is called the *de Longchamps point* of the triangle. [4] It is the reflection of the orthocenter in the circumcenter, *i.e.*, $L = 2\cdot O - H$. The de Longchamps circle is the radical circle of the triad $A(a)$, $B(b)$ and $C(c)$. It has equation

$$a^2yz + b^2zx + c^2xy - (x+y+z)(a^2x+b^2y+c^2z) = 0.$$

This circle is real if and only if triangle ABC is obtuse - angled.

It is also orthogonal to the triad of circles $(D(A), E(B), F(C))$. [5]

Exercises

1. The radical center of the triad of circles $A(R_a)$, $B(R_b)$, and $C(R_c)$ is the point

$$2S^2\cdot O - a^2R_a^2(A-A_H) - b^2R_b^2(B-B_H) - c^2R_c^2(C-C_H).$$

[4] The de Longchamps point appears as the point X_{20} in ETC.

[5] G. de Longchamps, Sur un nouveau cercle remarquable du plan d'un triangle, *Journal de Math. Spéciales*, 1886, pp. 57 – 60, 85 – 87, 100 – 104, 126 – 134.

7.4 The Lucas circles

[6]

Consider the square $A_bA_cA'_cA'_b$ inscribed in triangle ABC, with A_b, A_c on BC. Since this square can be obtained from the square erected externally on BC via the homothety $\mathsf{h}(A, \frac{S}{a^2+S})$, the equation of the circle $\mathcal{C}_A$ through A, A'_b and A'_c can be easily written down:

$$\mathcal{C}_A: \qquad a^2yz + b^2zx + c^2xy - \frac{a^2}{a^2+S}\cdot(x+y+z)(c^2y+b^2z) = 0.$$

Likewise if we construct inscribed squares $B_cB_aB'_aB'_c$ and $C_aC_bC'_bC'_a$ on the other two sides, the corresponding Lucas circles are

$$\mathcal{C}_B: \qquad a^2yz + b^2zx + c^2xy - \frac{b^2}{b^2+S}\cdot(x+y+z)(c^2x+a^2z) = 0,$$

and

$$\mathcal{C}_C: \qquad a^2yz + b^2zx + c^2xy - \frac{c^2}{c^2+S}\cdot(x+y+z)(b^2x+a^2y) = 0.$$

The coordinates of the radical center satisfy the equations

$$\frac{a^2(c^2y+b^2z)}{a^2+S} = \frac{b^2(a^2z+c^2x)}{b^2+S} = \frac{c^2(b^2x+a^2y)}{c^2+S}.$$

Since this can be rewritten as

$$\frac{y}{b^2}+\frac{z}{c^2} : \frac{z}{c^2}+\frac{x}{a^2} : \frac{x}{a^2}+\frac{y}{b^2} = a^2+S : b^2+S : c^2+S,$$

it follows that

$$\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2} = b^2+c^2-a^2+S : c^2+a^2-b^2+S : a^2+b^2-c^2+S,$$

and the radical center is the point

$$(a^2(2S_A+S) : b^2(2S_B+S) : c^2(2S_C+S)).$$

The three Lucas circles are mutually tangent to each other, the points of tangency being

$$\begin{aligned} A' &= (a^2S_A : b^2(S_B+S) : c^2(S_C+S)), \\ B' &= (a^2(S_A+S) : b^2S_B : c^2(S_C+S)), \\ C' &= (a^2(S_A+S) : b^2(S_B+S) : c^2S_C). \end{aligned}$$

Exercises

1. These point of tangency form a triangle perspective with ABC. Calculate the coordinates of the perspector. [7]

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[6] A.P. Hatzipolakis and P. Yiu, The Lucas circles, *Amer. Math. Monthly*, 108 (2001) 444 – 446.

[7] $(a^2(S_A+S) : b^2(S_B+S) : c^2(S_C+S))$. This point appears in ETC as X_{371}, and is called the Kenmotu point. It is the isogonal conjugate of the Vecten point $(\frac{1}{S_A+S} : \frac{1}{S_B+S} : \frac{1}{S_C+S})$.

7.5 Appendix: More triads of circles

1. (a) Construct the circle tangent to the circumcircle *internally* at A and also to the side BC.

(b) Find the coordinates of the point of tangency with the side BC.

(c) Find the equation of the circle. [8]

(d) Similarly, construct the two other circles, each tangent internally to the circumcircle at a vertex and also to the opposite side.

(e) Find the coordinates of the radical center of the three circles. [9]

2. Construct the three circles each tangent to the circumcircle *externally* at a vertex and also to the opposite side. Identify the radical center, which is a point on the circumcircle. [10]

3. Let X, Y, Z be the traces of a point P on the side lines BC, CA, AB of triangle ABC.

(a) Construct the three circles, each passing through a vertex of ABC and tangent to opposite side at the trace of P.

(b) Find the equations of these three circles.

(c) The radical center of these three circles is a point independent of P. What is this point?

4. Find the equations of the three circles each through a vertex and the traces of the incenter and the Gergonne point on the opposite side. What is the radical center of the triad of circles? [11]

5. Let $P = (u : v : w)$. Find the equations of the three circles with the cevian segments AA_P, BB_P, CC_P as diameters. What is the radical center of the triad ? [12]

6. Given a point P. The perpendicular from P to BC intersects CA at Y_a and AB at Z_a. Similarly define Z_b, X_b, and X_c, Y_c. Show that the circles AY_aZ_A, BZ_bX_b and CX_cY_c intersect at a point on the circumcircle of ABC. [13]

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[8] $a^2yz + b^2zx + c^2xy - \frac{a^2}{(b+c)^2}(x+y+z)(c^2y + b^2z) = 0$.

[9] $(a^2(a^2 + a(b+c) - bc) : \cdots : \cdots)$. This point appears as X_{595} in ETC.

[10] $\left(\frac{a^2}{b-c} : \frac{b^2}{c-a} : \frac{c^2}{a-b}\right)$. This point appears as X_{110} in ETC.

[11] The external center of similitude of the circumcircle and incircle.

[12] Floor van Lamoen, Hyacinthos, message 214, 1/24/00.

[13] If $P = (u : v : w)$, this intersection is $(\frac{a^2}{vS_B - wS_C} : \frac{b^2}{wS_C - uS_A} : \frac{c^2}{uS_A - vS_B})$; it is the infinite point of the line perpendicular to HP. A.P. Hatzipolakis and P. Yiu, Hyacinthos, messages 1213, 1214, 1215, 8/17/00.

Exercises

Consider triangle ABC with three circles $A(R_a)$, $B(R_b)$, and $C(R_c)$. The circle $B(R_b)$ intersects AB at $Z_{a+} = (R_b : c - R_b : 0)$ and $Z_{a-} = (-R_b : c + R_b : 0)$. Similarly, $C(R_c)$ intersects AC at $Y_{a+} = (R_c : 0 : b - R_c)$ and $Y_{a-} = (-R_c : 0 : b + R_c)$. [14]

1. Show that the centers of the circles $AY_{a+}Z_{a+}$ and $AY_{a-}Z_{a-}$ are symmetric with respect to the circumcenter O.

2. Find the equations of the circles $AY_{a+}Z_{a+}$ and $AY_{a-}Z_{a-}$. [15]

3. Show that these two circles intersect at

$$Q = \left(\frac{-a^2}{bR_b - cR_c} : \frac{b}{R_b} : \frac{-c}{R_c}\right)$$

on the circumcircle.

4. Find the equations of the circles $AY_{a+}Z_{a-}$ and $AY_{a-}Z_{a+}$ and show that they intersect at

$$Q' = \left(\frac{-a^2}{bR_b + cR_c} : \frac{b}{R_b} : \frac{c}{R_c}\right)$$

on the circumcircle. [16]

5. Show that the line QQ' passes through the points $(-a^2 : b^2 : c^2)$ and [17]

$$P = (a^2(-a^2R_a^2 + b^2R_b^2 + c^2R_c^2) : \cdots : \cdots).$$

6. If W is the radical center of the three circles $A(R_a)$, $B(R_b)$, and $C(R_c)$, then $P = (1-t)O + t \cdot W$ for

$$t = \frac{2a^2b^2c^2}{R_a^2a^2S_A + R_b^2b^2S_B + R_c^2c^2S_C}.$$

7. Find P if $R_a = a$, $R_b = b$, and $R_c = c$. [18]

8. Find P if $R_a = s-a$, $R_b = s-b$, and $R_c = s-c$. [19]

9. If the three circles $A(R_a)$, $B(R_b)$, and $C(R_c)$ intersect at $W = (u : v : w)$, then

$$P = (a^2(b^2c^2u^2 - a^2S_Avw + b^2S_Bwu + c^2S_Cuv) : \cdots : \cdots).$$

10. Find P if W is the incenter. [20]

11. If $W = (u : v : w)$ is on the circumcircle, then $P = Q = Q' = W$.

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[14] A.P. Hatzipolakis, Hyacinthos, message 3408, 8/10/01.

[15] $a^2yz + b^2zx + c^2xy - \epsilon(x+y+z)(c \cdot R_b y + b \cdot R_c z) = 0$ for $\epsilon = \pm 1$.

[16] $a^2yz + b^2zx + c^2xy - \epsilon(x+y+z)(c \cdot R_b y - b \cdot R_c z) = 0$ for $\epsilon = \pm 1$.

[17] QQ' : $(b^2R_b^2 - c^2R_c^2)x + a^2(R_b^2y - R_c^2z) = 0$.

[18] $(a^2(b^4 + c^4 - a^4) : b^2(c^4 + a^4 - b^4) : c^2(a^4 + b^4 - c^4))$. This point appears as X_{22} in ETC.

[19] $(\frac{a^2(a^2 - 2a(b+c) + (b^2+c^2))}{s-a} : \cdots : \cdots)$. This point does not appear in the current edition of ETC.

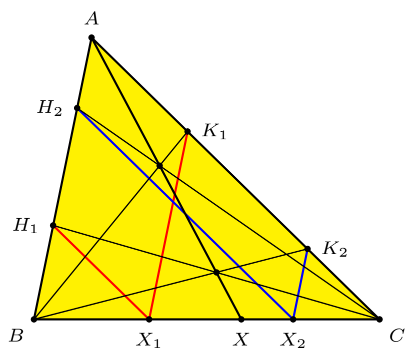
[20] $(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c})$.

Chapter 8

Some Basic Constructions

8.1 Barycentric product

Let X_1, X_2 be two points on the line BC, distinct from the vertices B, C, with homogeneous coordinates $(0 : y_1 : z_1)$ and $(0 : y_2 : z_2)$. For $i = 1, 2$, complete parallelograms $AK_iX_iH_i$ with K_i on AB and H_i on AC. The coordinates of the points H_i, K_i are



$$H_1 = (y_1 : 0 : z_1), \qquad K_1 = (z_1 : y_1 : 0);$$
$$H_2 = (y_2 : 0 : z_2), \qquad K_2 = (z_2 : y_2 : 0).$$

From these,

$$\begin{aligned} BH_1 \cap CK_2 &= (y_1 z_2 : y_1 y_2 : z_1 z_2), \\ BH_2 \cap CK_1 &= (y_2 z_1 : y_1 y_2 : z_1 z_2). \end{aligned}$$

Both of these points have A-trace $(0 : y_1y_2 : z_1z_2)$. This means that the line joining these intersections passes through A.

Given two points $P = (x : y : z)$ and $Q = (u : v : w)$, the above construction (applied to the traces on each side line) gives the traces of the point with coordinates $(xu : yv : zw)$. We shall call this point the *barycentric product* of P and Q, and denote it by $P \cdot Q$.

In particular, the *barycentric square* of a point $P = (u : v : w)$, with coordinates $(u^2 : v^2 : w^2)$ can be constructed as follows:

(1) Complete a parallelogram $AB_aA_PC_a$ with B_a on CA and C_a on AB.

(2) Construct $BB_a \cap CC_a$, and join it to A to intersect BC at X.

(3) Repeat the same constructions using the traces on CA and AB respectively to obtain Y on CA and Z on AB.

Then, X, Y, Z are the traces of the barycentric square of P.

8.1.1 Examples

(1) The Clawson point $(\frac{a}{S_A} : \frac{b}{S_B} : \frac{c}{S_C})$ can be constructed as the barycentric product of the incenter and the orthocenter.

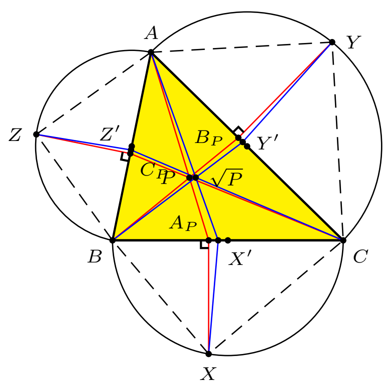
(2) The symmedian point can be constructed as the barycentric square of the incenter.

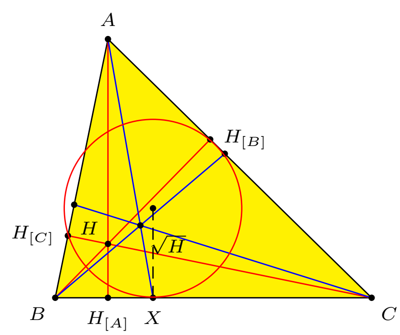
(3) If $P = (u + v + w)$ is an infinite point, its barycentric square can also be constructed as the barycentric product of P and its inferior $(v + w : w + u : u + v)$:

$$
\begin{aligned}
P^2 &= (u^2 : v^2 : w^2) \\
&= (-u(v+w) : -v(w+u) : -w(u+v)) \\
&= (u : v : w) \cdot (v + w : w + u : u + v).
\end{aligned}
$$

8.1.2 Barycentric square root

Let $P = (u : v : w)$ be a point in the interior of triangle ABC, the barycentric square root $\sqrt{P}$ is the point Q in the interior such that $Q^2 = P$. This can be constructed as follows.





(1) Construct the circle with BC as diameter.

(2) Construct the perpendicular to BC at the trace A_P to intersect the circle at X. [1] Bisect angle BXC to intersect BC at X'.

(3) Similarly obtain Y' on CA and Z' on AB.

The points X', Y', Z' are the traces of the barycentric square root of P.

[1] It does not matter which of the two intersections is chosen.

The square root of the orthocenter

Let ABC be an acute angled triangle so that the orthocenter H is an interior point. Let X be the A-trace of $\sqrt{H}$. The circle through the pedals $B_{[H]}$, $C_{[H]}$ and X is tangent to the side BC.

8.1.3 Exercises

1. Construct a point whose distances from the side lines are proportional to the radii of the excircles. [2]

2. Find the equation of the circle through B and C, tangent (internally) to incircle. Show that the point of tangency has coordinates

$$\left(\frac{a^2}{s-a} : \frac{(s-c)^2}{s-b} : \frac{(s-b)^2}{s-c}\right).$$

Construct this circle by making use of the barycentric "third power" of the Gergonne point.

3. Construct the square of an infinite point.

4. A circle is tangent to the side BC of triangle ABC at the $A-$trace of a point $P = (u : v : w)$ and internally to the circumcircle at A'. Show that the line AA' passes through the point $(au : bv : vw)$.

Make use of this to construct the three circles each tangent internally to the circumcircle and to the side lines at the traces of P.

5. Two circles each passing through the incenter I are tangent to BC at B and C respectively. A circle (J_a) is tangent externally to each of these, and to BC at X. Similarly define Y and Z. Show that XYZ is perspective with ABC, and find the perspector. [3]

6. Let $P_1 = (f_1 : g_1 : h_1)$ and $P_2 = (f_2 : g_2 : h_2)$ be two given points. Denote by X_i, Y_i, Z_i the traces of these points on the sides of the reference triangle ABC.

(a) Find the coordinates of the intersections $X_+ = BY_1 \cap CZ_2$ and $X_- = BY_2 \cap CZ_1$. [4]

(b) Find the equation of the line X_+X_-. [5]

(c) Similarly define points Y_+, Y_-, Z_+ and Z_-. Show that the three lines X_+X_-, Y_+Y_-, and Z_+Z_- intersect at the point

$$(f_1f_2(g_1h_2 + h_1g_2) : g_1g_2(h_1f_2 + f_1h_2) : h_1h_2(f_1g_2 + g_1f_2)).$$

---

[2] This has coordindates $(\frac{a}{s-a} : \cdots : \cdots)$ and can be constructed as the barycentric product of the incenter and the Gergonne point.

[3] The barycentric square root of $(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c})$. See Hyacinthos, message 3394, 8/9/01.

[4] $X_+ = f_1f_2 : f_1g_2 : h_1f_2$; $X_- = f_1f_2 : g_1f_2 : f_1h_2$.

[5] $(f_1^2g_2h_2 - f_2^2g_1h_1)x - f_1f_2(f_1h_2 - h_1f_2)y + f_1f_2(g_1f_2 - f_1g_2)z = 0$..

8.2 Harmonic associates

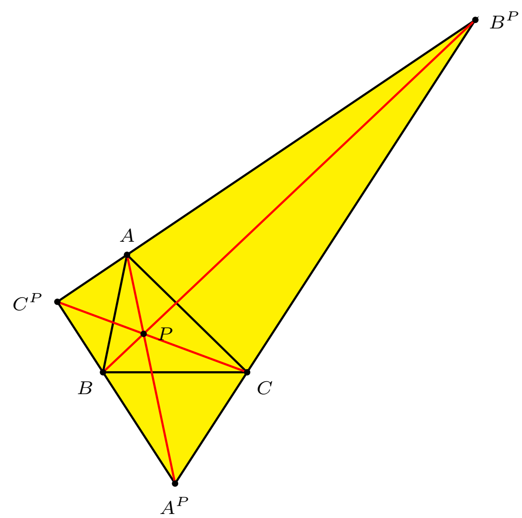
The *harmonic associates* of a point $P = (u : v : w)$ are the points

$$A^P = (-u : v : w), \qquad B^P = (u : -v : w), \qquad C^P = (u : v : -w).$$

The point A^P is the harmonic conjugate of P with respect to the cevian segment AA_P, *i.e.*,

$$AP : PA_P = -AA^P : A^P A_P;$$

similarly for B^P and C^P. The triangle $A^P C^P C^P$ is called the *precevian triangle* of P. This terminology is justified by the fact that ABC is the cevian triangle P in $A^P B^P C^P$. It is also convenient to regard P, A^P, B^P, C^P as a harmonic quadruple in the sense that any three of the points constitute the harmonic associates of the remaining point.



Examples

(1) The harmonic associates of the centroid, can be constructed as the intersection of the parallels to the side lines through their opposite vertices. They form the *superior triangle* of ABC.

(2) The harmonic associates of the incenter are the excenters.

(3) If P is an interior point with square root Q. The harmonic associates of Q can also be regarded as square roots of the same point.

8.2.1 Superior and inferior triangles

The precevian triangle of the centroid is called the *superior* triangle of ABC. If $P = (u : v : w)$, the point $(-u + v + w : u - v + w : u + v - w)$, which divides PG in the ratio $3 : -2$, has coordinates $(u : v : w)$ relative to the superior triangle, and is called the *superior* of P.

Along with the superior triangle, we also consider the cevian triangle of G as the *inferior* triangle. The point $(v + w : w + u : u + v)$, which divides PG in the ratio $3 : -1$, has coordinates $(u : v : w)$ relative to the inferior triangle, and is called the *inferior* of P.

Exercises

1. If P is the centroid of its precevian triangle, show that P is the centroid of triangle ABC.

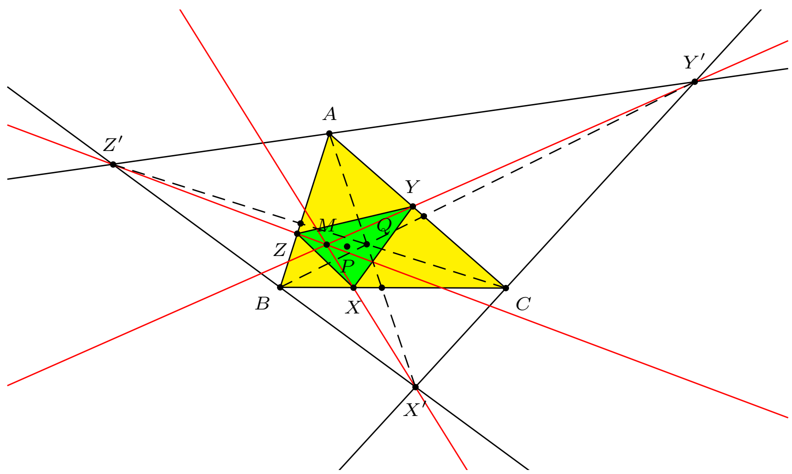
2. The incenter and the excenters form the only harmonic quadruple which is also orthocentric, *i.e.*, each one of them is the orthocenter of the triangle formed by the remaining three points.

8.3 Cevian quotient

Theorem

For any two points P and Q not on the side lines of ABC, the cevian triangle of P and precevian triangle Q are perspective. If $P = (u : v : w)$ and $Q = (x : y : z)$, the perspector is the point

$$P/Q = \left(x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)\right).$$



Proposition

$P/(P/Q) = Q$.

Proof. Direct verification.

This means that if $P/Q = Q'$, then $P/Q' = Q$.

Exercises

1. Show that $P/(P \cdot P) = P \cdot (G/P)$.

2. Identify the following cevian quotients.

P	Q	P/Q
incenter	centroid	
incenter	symmedian point	
incenter	Feuerbach point	
centroid	circumcenter	
centroid	symmedian point	
centroid	Feuerbach point	
orthocenter	symmedian point	
orthocenter	$(a(b-c) : \cdots : \cdots)$	
Gergonne point	incenter	

3. Let $P = (u : v : w)$ and $Q = (u' : v' : w')$ be two given points. If

$$X = B_P C_P \cap AA_Q, \qquad Y = C_P A_P \cap BB_Q, \qquad Z = A_P B_P \cap CC_Q,$$

show that $A_P X$, $B_P Y$ and $C_P Z$ are concurrent. Calculate the coordinates of the intersection. [6]

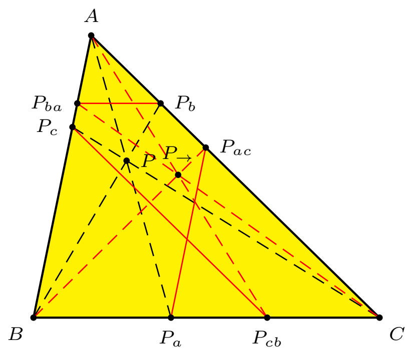
[6] $(uu'(vw' + wv') : \cdots : \cdots)$; see J. H. Tummers, Points remarquables, associés à un triangle, *Nieuw Archief voor Wiskunde* IV 4 (1956) 132 – 139. O. Bottema, Une construction par rapport à un triangle, ibid., IV 5 (1957) 68 – 70, has subsequently shown that this is the pole of the line PQ with respect to the circumconic through P and Q.

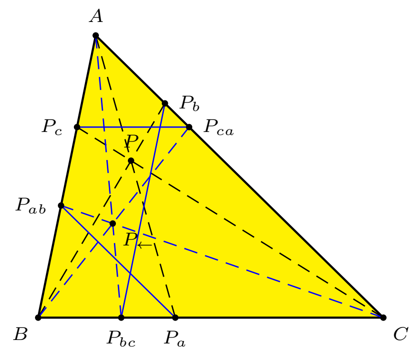
8.4 The Brocardians

The Brocardians of a point $P = (u : v : w)$ are the points

$$P_{\rightarrow} = \left(\frac{1}{w} : \frac{1}{u} : \frac{1}{v}\right) \quad \text{and} \quad P_{\leftarrow} = \left(\frac{1}{v} : \frac{1}{w} : \frac{1}{u}\right).$$

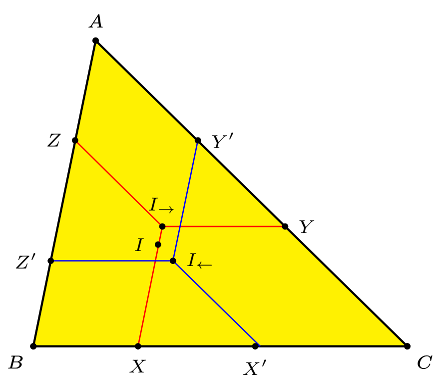
Construction of Brocardian points

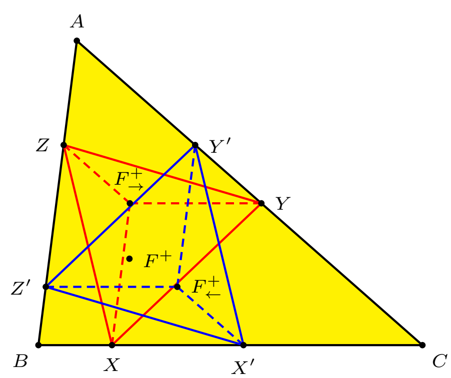




Examples

(1) The Brocard points $\Omega_{\rightarrow}$ and $\Omega_{\leftarrow}$ are the Brocardians of the symmedian point K.





(2) The Brocardians of the incenter are called the *Jerabek points*:

$$I_{\rightarrow} = \left(\frac{1}{c} : \frac{1}{a} : \frac{1}{b}\right) \quad \text{and} \quad I_{\leftarrow} = \left(\frac{1}{b} : \frac{1}{c} : \frac{1}{a}\right).$$

The *oriented* parallels through $I_{\rightarrow}$ to BC, CA, AB intersect the sides CA, AB, BC at Y, Z, X such that $I_{\rightarrow}Y = I_{\rightarrow}Z = I_{\rightarrow}X$. Likewise, the parallels through $I_{\leftarrow}$ to BC,

CA, AB intersect the sides AB, BC, CA at Z, X, Y such that $I_{\leftarrow}Z = I_{\leftarrow}X = I_{\leftarrow}Y$. These 6 segments have length ℓ satisfying $\frac{1}{\ell} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, one half of the length of the equal parallelians drawn through $(-\frac{1}{a} + \frac{1}{b} + \frac{1}{c} : \cdots : \cdots)$.

(3) If oriented parallels are drawn through the forward Brocardian point of the (positive) Fermat point F_+, and intersect the sides CA, AB, BC at X, Y, Z respectively, then the triangle XYZ is equilateral. [7]

[7] S. Bier, Equilateral triangles formed by oriented parallelians, *Forum Geometricorum*, 1 (2001) 25 – 32.

Chapter 9

Circumconics

9.1 Circumconics as isogonal transforms of lines

A circumconic is one that passes through the vertices of the reference triangle. As such it is represented by an equation of the form

$$\mathcal{C}: \qquad pyz + qzx + rxy = 0,$$

and can be regarded as the isogonal transform of the line

$$\mathcal{L}: \qquad \frac{p}{a^2}x + \frac{q}{b^2}y + \frac{r}{c^2}z = 0.$$

The circumcircle is the isogonal transform of the line at infinity. Therefore, a circumconic is an ellipse, a parabola, or a hyperbola according as its isogonal transform intersects the circumcircle at 0, 1, or 2 real points.

Apart from the three vertices, the circumconic intersects the circumcircle at the isogonal conjugate of the infinite point of the line $\mathcal{L}$:

$$\left(\frac{1}{b^2r - c^2q} : \frac{1}{c^2p - a^2r} : \frac{1}{a^2q - b^2p}\right).$$

We call this the fourth intersection of the circumconic with the circumcircle.

Examples

(1) The Lemoine axis is the tripolar of the Lemoine (symmedian) point, the line with equation

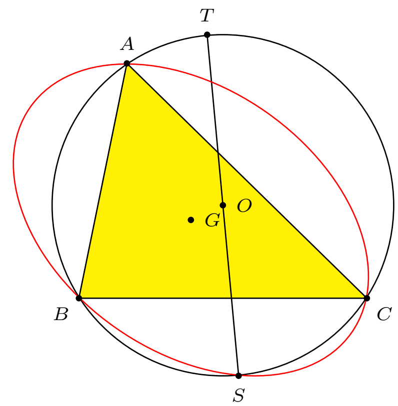
$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0.$$

Its isogonal transform is the Steiner circum-ellipse

$$yz + zx + xy = 0.$$

The fourth intersection with the circumcircle at the Steiner point [1]

$$\left(\frac{1}{b^2-c^2}:\frac{1}{c^2-a^2}:\frac{1}{a^2-b^2}\right).$$

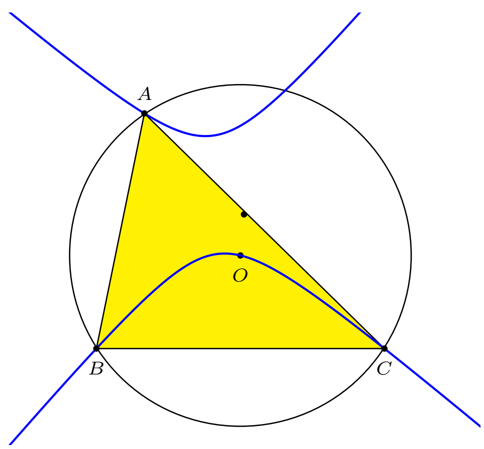


(2) The Euler line $\sum_{\text{cyclic}}(b^2-c^2)S_Ax=0$ transforms into the *Jerabek hyperbola*

$$\sum_{\text{cyclic}} a^2(b^2-c^2)S_Ayz=0.$$

Since the Euler infinity point $=(SS-3S_{BC}:SS-3S_{CA}:SS-3S_{AB})=(S_{CA}+S_{AB}-2S_{BC}:\cdots:\cdots)$, the fourth intersection with the circumcircle is the point [2]

$$\left(\frac{a^2}{S_{CA}+S_{AB}-2S_{BC}}:\cdots:\cdots\right).$$



[1] The Steiner point appears as X_{99} in ETC.

[2] This is the point X_{74} in ETC.

(3) The Brocard axis OK has equation

$$b^2c^2(b^2-c^2)x+c^2a^2(c^2-a^2)y+a^2b^2(a^2-b^2)z=0.$$

Its isogonal transform is the *Kiepert hyperbola*

$$(b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy=0.$$

The fourth intersection with the circumcircle is the *Tarry point* [3]

$$\left(\frac{1}{S_{BC}-S_{AA}}:\frac{1}{S_{CA}-S_{BB}}:\frac{1}{S_{AB}-S_{CC}}\right).$$

This is antipodal to the Steiner point, since the Euler line and the Lemoine axis are perpendicular to each other. [4]

(4) Recall that the tangent to the nine-point circle at the Feuerbach point $F=((b-c)^2(b+c-a):(c-a)^2(c+a-b):(a-b)^2(a+b-c))$ is the line

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0.$$

Applying the homothety $\mathsf{h}(G,-2)$, we obtain the line

$$(b-c)^2x+(c-a)^2y+(a-b)^2z=0$$

tangent to the point $(\frac{a}{b-c}:\frac{b}{c-a}:\frac{c}{a-b})$ at the circumcircle. [5]

The isogonal transform of this line is the parabola

$$a^2(b-c)^2yz+b^2(c-a)^2zx+c^2(a-b)^2xy=0.$$

Exercises

1. Let P be a point. The first trisection point of the cevian AP is the point A' dividing AA_P in the ratio $1:2$, *i.e.*, $AA':A'A_P=1:2$. Find the locus of P for which the first trisection points of the three cevians are collinear. For each such P, the line containing the first trisection points always passes through the centroid.

2. Show that the Tarry point as a Kiepert perspector is $K(-(\frac{\pi}{2}-\omega))$.

3. Show that the circumconic $pyz+qzx+rxy=0$ is a parabola if and only if

$$p^2+q^2+r^2-2qr-2rp-2pq=0.$$

[3] The Tarry point appears as the point X_{98} in ETC.

[4] The Lemoine axis is the radical axis of the circumcircle and the nine-point; it is perpendicular to the Euler line joining the centers of the two circles.

[5] This point appears as X_{100} in ETC.

4. **Animate** a point P on the circumcircle of triangle ABC and draw the line OP.

(a) Construct the point Q on the circumcircle which is the isogonal conjugate of the infinite point of OP.

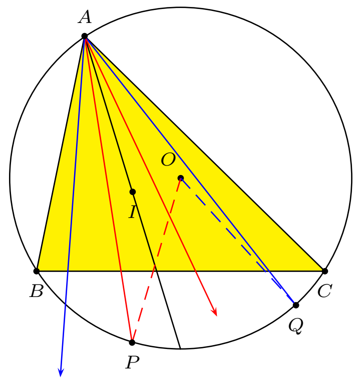
(b) Construct the tangent at Q.

(c) **Choose** a point X on the tangent line at Q, and construct the **isogonal conjugate** X^* of X.

(d) Find the **locus** of X^*.

9.2 The infinite points of a circum-hyperbola

Consider a line $\mathcal{L}$ intersecting the circumcircle at two points P and Q. The isogonal transform of $\mathcal{L}$ is a circum-hyperbola $\mathcal{C}$. The directions of the asymptotes of the hyperbola are given by its two infinite points, which are the isogonal conjugates of P and Q. The angle between them is one half of that of the arc PQ.



These asymptotes are perpendicular to each other if and only if P and Q are antipodal. In other words, the circum-hyperbola is rectangular, if and only if its isogonal transform is a diameter of the circumcircle. This is also equivalent to saying that the circum-hyperbola is rectangular if and only if it contains the orthocenter of triangle ABC.

Theorem

Let P and Q be antipodal points on the circumcircle. The asymptotes of the rectangular circum-hyperbola which is the isogonal transform of PQ are the Simson lines of P and Q.

It follows that the center of the circum-hyperbola is the intersection of these Simson lines, and is a point on the nine-point circle.

Exercises

1. Let $P = (u : v : w)$ be a point other than the orthocenter and the vertices of triangle ABC. The rectangular circum-hyperbola through P has equation

$$\sum_{\text{cyclic}} u(S_B v - S_C w)yz = 0.$$

9.3 The perspector and center of a circumconic

The tangents at the vertices of the circumconic

$$pyz + qzx + rxy = 0$$

are the lines

$$ry + qz = 0, \qquad rx + pz = 0, \qquad qx + py = 0.$$

These bound the triangle with vertices

$$(-p : q : r), \qquad (p : -q : r), \qquad (p : q : -r).$$

This is perspective with ABC at the point $P = (p : q : r)$, which we shall call the perspector of the circumconic.

We shall show in a later section that the center of the circumconic is the cevian quotient

$$Q = G/P = (u(v + w - u) : v(w + u - v) : w(u + v - w)).$$

Here we consider some interesting examples based on the fact that $P = G/Q$ if $Q = G/P$. This means that the circumconics with centers P and Q have perspectors at the other point. The two circumconics intersect at

$$\left(\frac{u}{v-w} : \frac{v}{w-u} : \frac{w}{u-v}\right).$$

9.3.1 Examples

Circumconic with center K

Since the circumcircle (with center O) has perspector at the symmedian point K, the circumconic with center K has O as perspector. This intersects the circumcircle at the point [6]

$$\left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right).$$

This point can be constructed as the antipode of the isogonal conjugate of the Euler infinity point.

Circumconic with incenter as perspector

The circumconic with incenter as perspector has equation

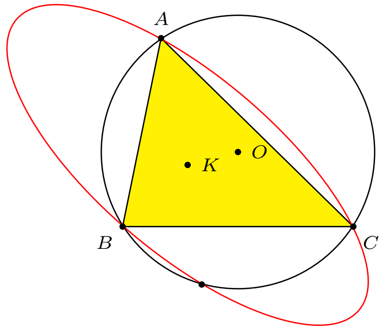
$$ayz + bzx + cxy = 0.$$

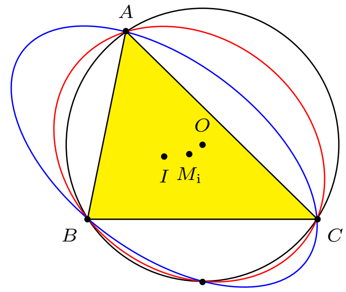
This has center $G/I = (a(b + c - a) : b(c + a - b) : c(a + b - c))$, the Mittenpunkt. The circumconic with the incenter as center has equation

$$a(s - a)yz + b(s - b)zx + c(s - c)xy = 0.$$

The two intersect at the point $\left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$ which is a point on the circumcircle. [7]

[6] This point appears as X_{110} in ETC.
[7] This point appears as X_{100} in ETC.





Exercises

1. Let P be the Spieker center, with coordinates $(b+c : c+a : a+b)$.

(a) Show that the circumconic with perspector P is an ellipse.

(b) Find the center Q of the conic. [8]

(c) Show that the circumconic with center P (and perspector Q) is also an ellipse.

(d) Find the intersection of the two conics. [9]

2. If P is the midpoint of the Brocard points $\Omega_{\rightarrow}$ and $\Omega_{\leftarrow}$, what is the point $Q = G/P$? What is the common point of the two circumconics with centers and perspectors at P and Q? [10]

3. Let P and Q be the center and perspector of the Kiepert hyperbola. Why is the circumconic with center Q and perspector P a parabola? What is the intersection of the two conics? [11]

4. Animate a point P on the circumcircle and construct the **circumconic** with P as center. What can you say about the type of the conic as P varies on the circumcircle?

5. Animate a point P on the circumcircle and construct the **circumconic** with P as perspector. What can you say about the type of the conic as P varies on the circumcircle?

---

[8] $Q = (a(b+c) : b(c+a) : c(a+b))$. This point appears in ETC as X_{37}.

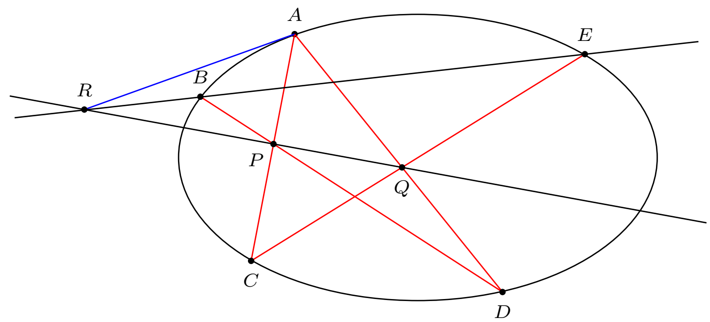
[9] $(\frac{b-c}{b+c} : \frac{c-a}{c+a} : \frac{a-b}{a+b})$. This point does not appear in the current edition of ETC.

[10] Q = symmedian point of medial triangle; common point = $(\frac{b^2-c^2}{b^2+c^2} : \cdots : \cdots)$. This point does not appear in the current edition of ETC.

[11] $(\frac{b^2-c^2}{b^2+c^2-2a^2} : \cdots : \cdots)$. This point does not appear in the current edition of ETC.

9.4 Appendix: Ruler construction of tangent at A

(1) $P = AC \cap BD$;
(2) $Q = AD \cap CE$;
(3) $R = PQ \cap BE$.
Then AR is the tangent at A.



Chapter 10

General Conics

10.1 Equation of conics

10.1.1 Carnot's Theorem

Suppose a conic $\mathcal{C}$ intersect the side lines BC at X, X', CA at Y, Y', and AB at Z, Z', then

$$\frac{BX}{XC} \cdot \frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} \cdot \frac{AZ'}{Z'B} = 1.$$

Proof. Write the equation of the conic as

$$fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2rxy = 0.$$

The intersections with the line BC are the two points $(0 : y_1 : z_1)$ and $(0 : y_2 : z_2)$ satisfying

$$gy^2 + hz^2 + 2pyz = 0.$$

From this,

$$\frac{BX}{XC} \cdot \frac{BX'}{X'C} = \frac{z_1 z_2}{y_1 y_2} = \frac{g}{h}.$$

Similarly, for the other two pairs of intersections, we have

$$\frac{CY}{YA} \cdot \frac{CY'}{Y'A} = \frac{h}{f}, \qquad \frac{AZ}{ZB} \cdot \frac{AZ'}{Z'B} = \frac{f}{g}.$$

The product of these division ratios is clearly 1.

The converse of Carnot's theorem is also true: if X, X', Y, Y', Z, Z' are points on the side lines such that

$$\frac{BX}{XC} \cdot \frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} \cdot \frac{AZ'}{Z'B} = 1,$$

then the 6 points are on a conic.

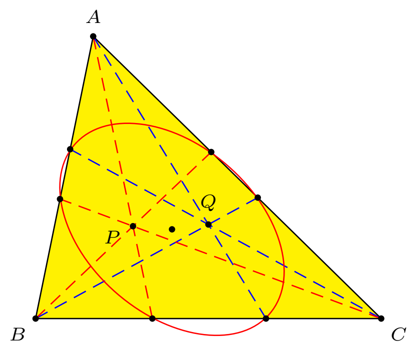
Corollary

If X, Y, Z are the traces of a point P, then X', Y', Z' are the traces of another point Q.

10.1.2 Conic through the traces of P and Q

Let $P = (u : v : w)$ and $Q = (u' : v' : w')$. By Carnot's theorem, there is a conic through the 6 points. The equation of the conic is

$$\sum_{\text{cyclic}} \frac{x^2}{uu'} - \left(\frac{1}{vw'} + \frac{1}{v'w}\right) yz = 0.$$



Exercises

1. Show that the points of tangency of the A-excircle with AB, AC, the B-excircle with BC, AB, and the C-excircle with CA, CB lie on a conic. Find the equation of the conic. [1]

2. Let $P = (u : v : w)$ be a point not on the side lines of triangle ABC.

   (a) Find the equation of the conic through the traces of P and the midpoints of the three sides. [2]

   (b) Show that this conic passes through the midpoints of AP, BP and CP.

   (c) For which points is the conic an ellipse, a hyperbola?

3. Given a point $P = (u : v : w)$ and a line $\mathcal{L} : \frac{x}{u'} + \frac{y}{v'} + \frac{z}{w'} = 0$, find the locus of the pole of $\mathcal{L}$ with respect to the circumconics through P. [3]

---

[1] $\sum_{\text{cyclic}} x^2 + \frac{s^2+(s-a)^2}{s(s-a)} yz = 0.$

[2] $\sum_{\text{cyclic}} -vwx^2 + u(v+w)yz = 0.$

[3] The conic through the traces of P and $Q = (u' : v' : w')$; Jean-Pierre Ehrmann, Hyacinthos, message 1326, 9/1/00.

10.2 Inscribed conics

An *inscribed* conic is one tangent to the three side lines of triangle ABC. By Carnot's theorem, the points of tangency must either be the traces of a point P (Ceva Theorem) or the intercepts of a line (Menelaus Theorem). Indeed, if the conic is non-degenerate, the former is always the case. If the conic is tangent to BC at $(0:q:r)$ and to CA at $(p:0:r)$, then its equation must be

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}-\frac{2yz}{qr}-\frac{2zx}{rp}-\epsilon\frac{2xy}{pq}=0$$

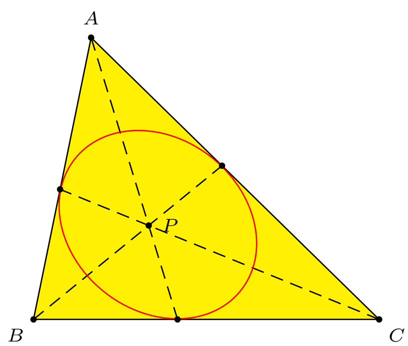
for $\epsilon=\pm1$. If $\epsilon=-1$, then the equation becomes

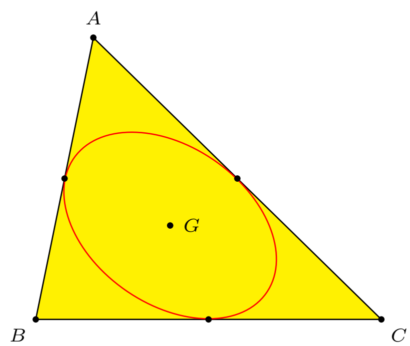
$$\left(-\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)^2=0,$$

and the conic is degenerate. The inscribed conic therefore has equation

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}-\frac{2yz}{qr}-\frac{2zx}{rp}-\frac{2xy}{pq}=0$$

and touches BC at $(0:q:r)$. The points of tangency form a triangle perspective with ABC at $(p:q:r)$, which we call the perspector of the inscribed conic.





10.2.1 The Steiner in-ellipse

The Steiner in-ellipse is the inscribed conic with perspector G. It has equation

$$x^2+y^2+z^2-2yz-2zx-2xy=0.$$

Exercises

1. The locus of the squares of infinite points is the Steiner in-ellipse

$$x^2+y^2+z^2-2yz-2zx-2xy=0.$$

2. Let $\mathcal{C}$ be the inscribed conic

$$\sum_{\text{cyclic}} \frac{x^2}{p^2} - \frac{2yz}{qr} = 0,$$

tangent to the side lines at $X = (0 : q : r)$, $Y = (p : 0 : r)$, and $Z = (p : q : 0)$ respectively. Consider an arbitrary point $Q = (u : v : w)$.

(a) Find the coordinates of the second intersection A' of $\mathcal{C}$ with XQ. [4]

(b) Similarly define B' and C'. Show that triangle $A'B'C'$ is perspective with ABC, and find the perspector. [5]

[4] $(\frac{4u^2}{p} : q(\frac{u}{p} + \frac{v}{q} - \frac{w}{r})^2 : r(\frac{u}{p} - \frac{v}{q} + \frac{w}{r})^2)$.

[5] $(\frac{p}{(-\frac{u}{p} + \frac{v}{q} + \frac{w}{r})^2} : \cdots : \cdots)$.

10.3 The adjoint of a matrix

The *adjoint* of a matrix (not necessarily symmetric)

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is the *transpose* of the matrix formed by the cofactors of M:

$$M^{\#} = \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{22}a_{13} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{21}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & -a_{11}a_{32} + a_{31}a_{12} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

Proposition

(1) $MM^{\#} = M^{\#}M = \det(M)I$.
(2) $M^{\#\#} = (\det M)M$.

Proposition

Let (i, j, k) be a permutation of the indices 1, 2, 3.
(1) If the rows of a matrix M are the coordinates of three points, the line joining P_i and P_j has coordinates given by the k-th column of $M^{\#}$.
(2) If the columns of a matrix M are the coordinates of three lines, the intersection of L_i and L_j is given by the k-row of $M^{\#}$.

10.4 Conics parametrized by quadratic functions

Suppose

$$x : y : z = a_0 + a_1t + a_2t^2 : b_0 + b_1t + b_2t^2 : c_0 + c_1t + c_2t^2$$

Elimination of t gives

$$(p_1x + q_1y + r_1z)^2 - (p_0x + q_0y + r_0z)(p_2x + q_2y + r_2z) = 0,$$

where the coefficients are given by the entries of the adjoint of the matrix

$$M = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix},$$

namely,

$$M^{\#} = \begin{pmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix}.$$

This conic is nondegenerate provided $\det(M) \neq 0$.

10.4.1 Locus of Kiepert perspectors

Recall that the apexes of similar isosceles triangles of base angles θ constructed on the sides of triangle ABC form a triangle $A^\theta B^\theta C^\theta$ with perspector

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

Writing $t = S_\theta$, and clearing denominators, we may take

$$(x : y : z) = (S_{BC} + a^2t + t^2 : S_{CA} + b^2t + t^2 : S_{AB} + c^2t + t^2).$$

With

$$M = \begin{pmatrix} S_{BC} & a^2 & 1 \\ S_{CA} & b^2 & 1 \\ S_{AB} & c^2 & 1 \end{pmatrix},$$

we have

$$M^{\#} = \begin{pmatrix} b^2 - c^2 & c^2 - a^2 & a^2 - b^2 \\ -S_A(b^2 - c^2) & -S_B(c^2 - a^2) & -S_C(a^2 - b^2) \\ S_{AA}(b^2 - c^2) & S_{BB}(c^2 - a^2) & S_{CC}(a^2 - b^2) \end{pmatrix}$$

Writing $u = (b^2 - c^2)x$, $v = (c^2 - a^2)y$, and $w = (a^2 - b^2)z$, we have

$$(S_Au + S_Bv + S_Cw)^2 - (u + v + w)(S_{AA}u + S_{BB}v + S_{CC}w) = 0,$$

which simplifies into

$$0 = \sum_{\text{cyclic}} (2S_{BC} - S_{BB} - S_{CC})vw = -\sum_{\text{cyclic}} (b^2 - c^2)^2 vw.$$

In terms of x, y, z, we have, after deleting a common factor $-(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$,

$$\sum_{\text{cyclic}} (b^2 - c^2)yz = 0.$$

This is the circum-hyperbola which is the isogonal transform of the line

$$\sum_{\text{cyclic}} b^2c^2(b^2 - c^2)x = 0.$$

10.5 The matrix of a conic

10.5.1 Line coordinates

In working with conics, we shall find it convenient to use matrix notations. We shall identify the homogeneous coordinates of a point $P = (x : y : z)$ with the *row matrix* $\begin{pmatrix} x & y & z \end{pmatrix}$, and denote it by the same P. A line $\mathcal{L}$ with equation $px + qy + rz = 0$ is represented by the *column matrix*

$$L = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

(so that $PL = 0$). We shall call L the *line coordinates* of $\mathcal{L}$.

10.5.2 The matrix of a conic

A conic given by a quadratic equation

$$fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2rxy = 0$$

can be represented by in matrix form $PMP^t = 0$, with

$$M = \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix}.$$

We shall denote the conic by $\mathcal{C}(M)$.

10.5.3 Tangent at a point

Let P be a point on the conic $\mathcal{C}$. The *tangent* at P is the line MP^t.

10.6 The dual conic

10.6.1 Pole and polar

The *polar* of a point P (with respect to the conic $\mathcal{C}(M)$) is the line MP^t, and the *pole* of a line L is the point $L^tM^\#$. Conversely, if L intersects a conic $\mathcal{C}$ at two points P and Q, the *pole* of L with respect to $\mathcal{C}$ is the intersection of the tangents at P and Q.

Exercises

1. A conic is self-polar if each vertex is the pole of its opposite side. Show that the matrix of a self-polar conic is a diagonal matrix.

2. If P lies on the polar of Q, then Q lies on the polar of P.

10.6.2 Condition for a line to be tangent to a conic

A line $L : px + qy + rz = 0$ is tangent to the conic $\mathcal{C}(M)$ if and only if $L^tM^\#L = 0$. If this condition is satisfied, the point of tangency is $L^tM^\#$.

10.6.3 The dual conic

Let M be the symmetric matrix

$$\begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix}.$$

The *dual conic* of $\mathcal{C} = \mathcal{C}(M)$ is the conic represented by the adjoint matrix

$$M^\# = \begin{pmatrix} gh - p^2 & pq - rh & rp - gq \\ pq - hr & hf - q^2 & qr - fp \\ rp - gq & qr - fp & fg - r^2 \end{pmatrix}.$$

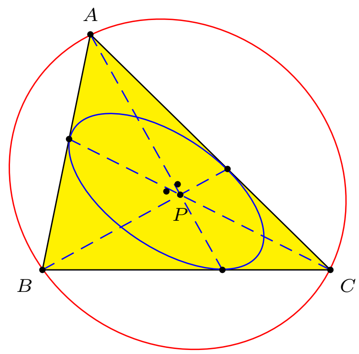
Therefore, a line $L : px + qy + rz = 0$ is tangent to $\mathcal{C}(M)$ if and only if the point $L^t = (p : q : r)$ is on the dual conic $\mathcal{C}(M^\#)$.

10.6.4 The dual conic of a circumconic

The dual conic of the circumconic $pyz + qzx + rxy = 0$ (with perspector $P = (p : q : r)$) is the inscribed conic

$$\sum_{\text{cyclic}} -p^2x^2 + 2qryz = 0$$

with perspector $P^\bullet = (\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$. The center is the point $(q + r : r + p : p + q)$.



Exercises

1. The polar of $(u : v : w)$ with respect to the circumconic $pyz + qzx + rxy = 0$ is the line
$$p(wy + vz) + q(uz + wx) + r(vx + uy) = 0.$$

2. Find the equation of the dual conic of the incircle. Deduce Feuerbach's theorem by showing that the radical axis of the nine-point circle and the incircle, namely, the line
$$\frac{x}{b-c} + \frac{y}{c-a} + \frac{z}{a-b} = 0$$
is tangent to the incircle. [6]

3. Show that the common tangent to the incircle and the nine-point circle is also tangent to the Steiner in-ellipse. Find the coordinates of the point of tangency. [7]

4. Let $P = (u : v : w)$ and $Q = (u' : v' : w')$ be two given points. If
$$X = B_PC_P \cap AA_Q, \qquad Y = C_PA_P \cap BB_Q, \qquad Z = A_PB_P \cap CC_Q,$$
show that A_PX, B_PY and C_PZ are concurrent at the pole of PQ with respect to the circumconic through P and Q. [8]

5. The tangents at the vertices to the circumcircle of triangle ABC intersect the side lines BC, CA, AB at A', B', C' respectively. The *second* tagents from A', B', C' to the circumcircle have points of tangency X, Y, Z respectively. Show that XYZ is perspective with ABC and find the perspector. [9]

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[6] $\sum_{\text{cyclic}}(s-a)yz = 0$.

[7] $((b-c)^2 : (c-a)^2 : (a-b)^2)$. This point appears as X_{1086} in ETC.

[8] O. Bottema, Une construction par rapport à un triangle, *Nieuw Archief voor Wiskunde*, IV 5 (1957) 68–70.

[9] $(a^2(b^4+c^4-a^4) : \cdots : \cdots)$. This is a point on the Euler line. It appears as X_{22} in ETC. See D.J. Smeenk and C.J. Bradley, Problem 2096 and solution, *Crux Mathematicorum*, 21 (1995) 344; 22(1996) 374 – 375.

10.7 The type, center and perspector of a conic

10.7.1 The type of a conic

The conic $\mathcal{C}(M)$ is an ellipse, a parabola, or a hyperbola according as the *characteristic* $GM^{\#}G$ is positive, zero, or negative.

Proof. Setting $z = -(x+y)$, we reduce the equation of the conic into

$$(h+f-2q)x^2 + 2(h-p-q+r)xy + (g+h-2p)y^2 = 0.$$

This has discriminant

$$\begin{aligned} & (h-p-q+r)^2 - (g+h-2p)(h+f-2q) \\ = \;& h^2 - (g+h)(h+f) - 2h(p+q-r) \\ & +2(h+f)p + 2(g+h)q + (p+q-r)^2 + 4pq \\ = \;& -(fg+gh+hf) + 2(fp+gq+hr) + (p^2+q^2+r^2-2pq-2qr-2rp) \end{aligned}$$

which is the negative of the sum of the entries of $M^{\#}$. From this the result follows.

10.7.2 The center of a conic

The center of a conic is the pole of the line at infinity. As such, the center of $\mathcal{C}(M)$ has coordinates $GM^{\#}$, formed by the column sums of $M^{\#}$:

$$(p(q+r-p)-(qg+rh)+gh : q(r+p-q)-(rh+pf)+hf : r(p+q-r)-(pf+qg)+fg).$$

10.7.3 The perspector of a conic

Theorem (Conway)

Let $\mathcal{C} = \mathcal{C}(M)$ be a nondegenerate, non-self-polar conic. The triangle formed by the poles of the sidelines is perspective with ABC, and has perspector

$$\left(\frac{1}{qr-pf} : \frac{1}{rp-qg} : \frac{1}{pq-rh}\right).$$

Proof. The coordinates of these poles are given by the columns of the adjoint matrix $M^{\#}$. These are the points

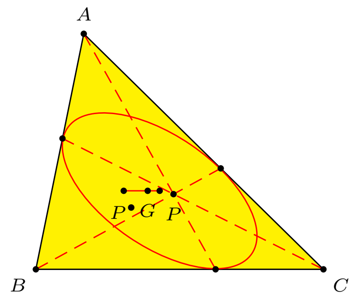
$$\begin{aligned} A' &= \left(*****: \frac{1}{rp-qg} : \frac{1}{pq-rh}\right), \\ B' &= \left(\frac{1}{qr-pf} : ***** : \frac{1}{pq-rh}\right), \\ C' &= \left(\frac{1}{qr-pf} : \frac{1}{rp-qg} : *****\right). \end{aligned}$$

From these it is clear that $A'B'C'$ is perspective with ABC at the point given above.

The point $\left(\frac{1}{qr-pf} : \frac{1}{rp-qg} : \frac{1}{pq-rh}\right)$ is called the *perspector* of the conic $\mathcal{C}(M)$.

Proposition

The center of the inscribed conic with perspector P is the inferior of $P^\bullet$.



Proof. The inscribed conic with perspector P has equation

$$\sum_{\text{cyclic}} \frac{x^2}{p^2} - \frac{2yz}{qr} = 0.$$

Exercises

1. Let $(f : g : h)$ be an infinite point. What type of conic does the equation
$$\frac{a^2x^2}{f} + \frac{b^2y^2}{g} + \frac{c^2z^2}{h} = 0$$
represent? [10]

2. Find the perspector of the conic through the traces of P and Q.

3. Find the perspector of the conic through the 6 points of tangency of the excircles with the side lines. [11]

4. A circumconic is an ellipse, a parabola or a hyperbola according as the perspector is inside, on, or outside the Steiner in-ellipse.

5. Let $\mathcal{C}$ be a conic tangent to the side lines AB and AC at B and C respectively.
   (a) Show that the equation of $\mathcal{C}$ is of the form $x^2 - kyz = 0$ for some k.
   (b) Show that the center of the conic lies on the A-median.
   (c) Construct the parabola in this family as a five-point conic. [12]
   (d) Design an animation of the conic as its center traverses the A-median. [13]

6. Prove that the locus of the centers of circumconics through P is the conic through the traces of P and the midpoints of the sides. [14]

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[10] Parabola.
[11] $(\frac{a^2+(b+c)^2}{b+c-a} : \cdots : \cdots)$. This points appears in ETC as X_{388}.
[12] The parabola has equation $x^2 - 4yz = 0$.
[13] If the center is $(t : 1 : 1)$, then the conic contains $(t : -2 : t)$.
[14] Floor van Lamoen and Paul Yiu, Conics loci associated with conics, *Forum Geometricorum*, forthcoming.

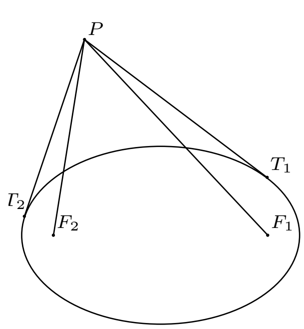
Chapter 11

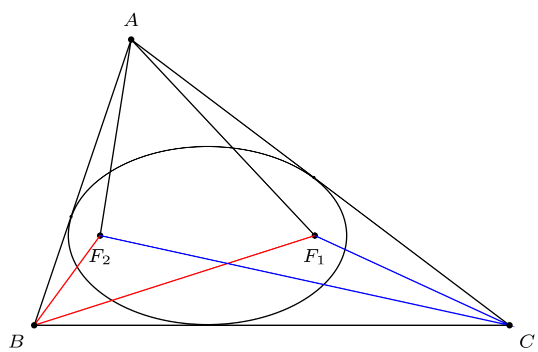
Some Special Conics

11.1 Inscribed conic with prescribed foci

11.1.1 Theorem

The foci of an inscribed central conic are isogonal conjugates.
Proof. Let F_1 and F_2 be the foci of a conic, and T_1, T_2 the points of tangency from a point P. Then $\angle F_1PT_1 = \angle F_2PT_2$. Indeed, if Q_1, Q_2 are the pedals of F_1, F_2 on the tangents, the product of the distances F_1Q_1 and F_2Q_2 to the tangents is constant, being the square of the semi-minor axis.





Given a pair of isogonal conjugates, there is an inscribed conic with foci at the two points. The center of the conic is the midpoint of the segment.

11.1.2 The Brocard ellipse

$$\sum_{\text{cyclic}} b^4c^4x^2 - 2a^4b^2c^2yz = 0$$

The Brocard ellipse is the inscribed ellipse with foci at the Brocard points

$$\begin{aligned}\Omega_{\rightarrow} &= (a^2b^2 : b^2c^2 : c^2a^2),\\ \Omega_{\leftarrow} &= (c^2a^2 : a^2b^2 : b^2c^2).\end{aligned}$$

Its center is the Brocard midpoint

$$(a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2)),$$

which is the inferior of $(b^2c^2 : c^2a^2 : a^2b^2)$, the isotomic conjugate of the symmedian point. It follows that the perspector is the symmedian point.

Exercises

1. Show that the equation of the Brocard ellipse is as given above.
2. The minor auxiliary circle is tangent to the nine-point circle. [1] What is the point of tangency? [2]

11.1.3 The de Longchamps ellipse

[3]

$$\sum_{\text{cyclic}} b^2c^2(b+c-a)x^2 - 2a^3bcyz = 0,$$

The de Longchamps ellipse is the conic through the traces of the incenter I, and has center at I.

Exercises

1. Given that the equation of the conic is show that it is always an ellipse.
2. By Carnot's theorem, the "second" intersections of the ellipse with the side lines are the traces of a point P. What is this point? [4]
3. The minor axis is the ellipse is along the line OI. What are the lengths of the semi-major and semi-minor axes of the ellipse? [5]

11.1.4 The Lemoine ellipse

Construct the inscribed conic with foci G and K.

Find the coordinates of the center and the perspector.

The points of tangency with the side lines are the traces of the G-symmedians of triangles GBC, GCA, and GAB.

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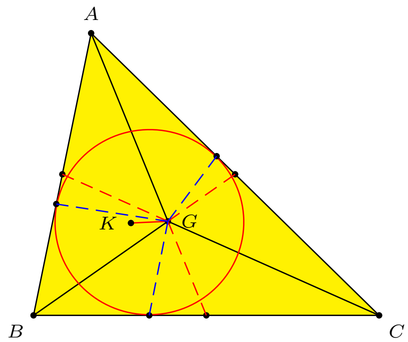
[1] V. Thébault, Problem 3857, *American Mathematical Monthly*, APH,205.

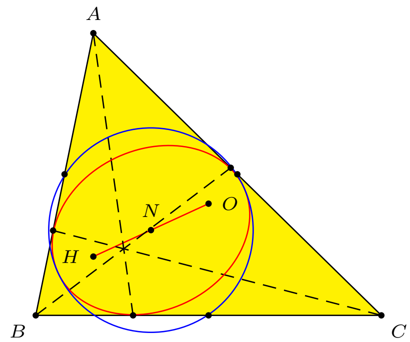
[2] Jean-Pierre Ehrmann, Hyacinthos, message 209, 1/22/00.

[3] E. Catalan, Note sur l'ellipse de Longchamps, *Journal Math. Spéciales*, IV 2 (1893) 28–30.

[4] $(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c})$.

[5] $\frac{R}{2}$ and r





11.1.5 The inscribed conic with center N

This has foci O and H. The perspector is the isotomic conjugate of the circumcenter. It is the envelope of the perpendicular bisectors of the segments joining H to a point on the circumcircle. The major auxiliary circle is the nine-point circle.

Exercises

1. Show that the equation of the Lemoine ellipse is

$$\sum_{\text{cyclic}} m_a^4 x^2 - 2m_b^2 m_c^2 yz = 0$$

where m_a, m_b, m_c are the lengths of the medians of triangle ABC.

11.2 Inscribed parabola

Consider the inscribed parabola tangent to a given line, which we regard as the tripolar of a point $P = (u : v : w)$. Thus, $\ell : \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$. The dual conic is the circumconic passes through the centroid $(1 : 1 : 1)$ and $P^\bullet = (\frac{1}{u} : \frac{1}{v} : \frac{1}{w})$. It is the circumconic

$$\mathcal{C}^\# \qquad \frac{v-w}{x} + \frac{w-u}{y} + \frac{u-v}{z} = 0.$$

The inscribed parabola, being the dual of $\mathcal{C}^\#$, is

$$\sum_{\text{cyclic}} -(v-w)^2x^2 + 2(w-u)(u-v)yz = 0.$$

The perspector is the isotomic conjugate of that of its dual. This is the point

$$\left(\frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v}\right)$$

on the Steiner circum-ellipse.

The center of the parabola is the infinite point $(v - w : w - u : u - v)$. This gives the direction of the axis of the parabola. It can also be regarded the infinite focus of the parabola. The other focus is the isogonal conjugate

$$\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}$$

on the circumcircle.

The axis is the line through this point parallel to $ux+vy+wz = 0$. The intersection of the axis with the parabola is the vertex

$$\left(\frac{(S_B(w-u) - S_C(u-v))^2}{v-w} : \cdots : \cdots\right).$$

The directrix, being the polar of the focus, is the line

$$S_A(v-w)x + S_B(w-u)y + S_C(u-v)z = 0.$$

This passes through the orthocenter, and is perpendicular to the line

$$ux + vy + wz = 0.$$

It is in fact the line of reflections of the focus. The tangent at the vertex is the Simson line of the focus.

Where does the parabola touch the given line?

$$(u^2(v-w) : v^2(w-u) : w^2(u-v)),$$

the barycentric product of P and the infinite point of its tripolar, the given tangent, or equivalently the barycentric product of the infinite point of the tangent and its tripole.

Exercises

1. Animate a point P on the Steiner circum-ellipse and construct the inscribed parabola with perspector P.

11.3 Some special conics

11.3.1 The Steiner circum-ellipse $xy + yz + zx = 0$

Construct the Steiner circum-ellipse which has center at the centroid G.

The fourth intersection with the circumcircle is the Steiner point, which has coordinates

$$\left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right).$$

Construct this point as the isotomic conjugate of an infinite point.

The axes of the ellipse are the bisectors of the angle KGS. [6] Construct these axes, and the vertices of the ellipse.

Construct the foci of the ellipse. [7]

These foci are called the Bickart points. Each of them has the property that three cevian segments are equal in length. [8]

11.3.2 The Steiner in-ellipse $\sum_{\text{cyclic}} x^2 - 2yz = 0$

Exercises

1. Let $\mathcal{C}$ be a circumconic through the centroid G. The tangents at A, B, C intersect the sidelines BC, CA, AB at A', B', C' respectively. Show that the line $A'B'C'$ is tangent to the Steiner in-ellipse at the center of $\mathcal{C}$. [9]

11.3.3 The Kiepert hyperbola $\sum_{\text{cyclic}}(b^2 - c^2)yz = 0$

The asymptotes are the Simson lines of the intersections of the Brocard axis OK with the circumcircle. [10] These intersect at the center which is on the nine-point circle. An easy way to construct the center as the intersection of the nine-point circle with the pedal circle of the centroid, *nearer to the orthocenter*. [11]

Exercises

1. Find the fourth intersection of the Kiepert hyperbola with the circumcircle, and show that it is antipodal to the Steiner point. [12]

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[6] J.H. Conway, Hyacinthos, message 1237, 8/18/00.

[7] The principal axis of the Steiner circum-ellipse containing the foci is the *least square line* for the three vertices of the triangle. See F. Gremmen, Hyacinthos, message 260, 2/1/00.

[8] O. Bottema, On some remarkable points of a triangle, *Nieuw Archief voor Wiskunde*, 19 (1971) 46 – 57; J.R. Pounder, Equal cevians, *Crux Mathematicorum*, 6 (1980) 98 – 104; postscript, *ibid.* 239 – 240.

[9] J.H. Tummers, Problem 32, *Wiskundige Opgaven met de Oplossingen*, 20-1 (1955) 31–32.

[10] These asymptotes are also parallel to the axes of the Steiner ellipses. See, J.H. Conway, Hyacinthos, message 1237, 8/18/00.

[11] The other intersection is the center of the Jerabek hyperbola. This is based on the following theorem: Let P be a point on a rectangular circum-hyperbola $\mathcal{C}$. The pedal circle of P intersects the nine-point circle at the centers of $\mathcal{C}$ and of (the rectangular circum- hyperbola which is) the isogonal conjugate of the line OP. See A.P. Hatzipolakis and P. Yiu, Hyacinthos, messages 1243 and 1249, 8/19/00.

[12] The Tarry point.

2. Show that the Kiepert hyperbola is the locus of points whose tripolars are perpendicular to the Euler line. [13]

3. Let $A'B'C'$ be the orthic triangle. The Brocard axes (the line joining the circumcenter and the symmedian point) of the triangles $AB'C'$, $A'BC'$, and $A'B'C$ intersect at the Kiepert center. [14]

11.3.4 The superior Kiepert hyperbola $\sum_{\text{cyclic}}(b^2-c^2)x^2 = 0$

Consider the locus of points P for which the three points P, $P^\bullet$ (isotomic conjugate) and P^* (isogonal conjugate) are collinear. If $P = (x : y : z)$, then we require

$$\begin{aligned} 0 &= \begin{vmatrix} x & y & z \\ yz & zx & xy \\ a^2yz & b^2zx & c^2xy \end{vmatrix} \\ &= a^2xyz(y^2-z^2) + b^2zxy(z^2-x^2) + c^2xyz(x^2-y^2) \\ &= -xyz((b^2-c^2)x^2 + (c^2-a^2)y^2 + (a^2-b^2)z^2). \end{aligned}$$

Excluding points on the side lines, the locus of P is the conic

$$(b^2-c^2)x^2 + (c^2-a^2)y^2 + (a^2-b^2)z^2 = 0.$$

We note some interesting properties of this conic:

- It passes through the centroid and the vertices of the superior triangle, namely, the four points $(\pm 1 : \pm 1 : \pm 1)$.
- It passes through the four incenters, namely, the four points $(\pm a : \pm b : \pm c)$. Since these four points form an orthocentric quadruple, the conic is a rectangular hyperbola.
- Since the matrix representing the conic is diagonal, the center of the conic has coordinates $(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2})$, which is the Steiner point.

Exercises

1. All conics passing through the four incenters are tangent to four fixed straight lines. What are these lines? [15]

2. Let P be a given point other than the incenters. Show that the center of the conic through P and the four incenters is the fourth intersection of the circumcircle and the circumconic with perspector $P \cdot P$ (barycentric square of P). [16]

[13] O. Bottema and M.C. van Hoorn, Problem 664, *Nieuw Archief voor Wiskunde*, IV 1 (1983) 79. See also R.H. Eddy and R. Fritsch, On a problem of Bottema and van Hoorn, ibid., IV 13 (1995) 165 – 172.

[14] Floor van Lamoen, Hyacinthos, message 1251, 8/19/00.

[15] The conic $\mathcal{C}$ is self-polar. Its dual conic passes through the four incenters. This means that the conic $\mathcal{C}$ are tangent to the 4 lines $\pm ax + \pm by + \pm cz = 0$.

[16] Floor van Lamoen, Hyacinthos, message 1401, 9/11/00.

3. Let X be the pedal of A on the side BC of triangle ABC. For a real number t, let A_t be the point on the altitude through A such that $XA_t = t \cdot XA$. Complete the squares $A_tXX_bA_b$ and $A_tXX_cA_c$ with X_b and X_c on the line BC. [17] Let $A'_t = BA_c \cap CA_b$, and A''_t be the pedal of A'_t on the side BC. Similarly define B''_t andC''_t. Show that as t varies, triangle $A''_tB''_tC''_t$ is perspective with ABC, and the perspector traverses the Kiepert hyperbola. [18]

11.3.5 The Feuerbach hyperbola

$$\sum_{\text{cyclic}} a(b-c)(s-a)yz = 0$$

This is the isogonal transform of the OI-line. The rectangular hyperbola through the incenter. Its center is the Feuerbach point.

11.3.6 The Jerabek hyperbola

The Jerabek hyperbola

$$\sum_{\text{cyclic}} \frac{a^2(b^2-c^2)S_A}{x} = 0$$

is the isogonal transform of the Euler line. Its center is the point

$$((b^2-c^2)^2S_A : (c^2-a^2)^2S_B : (a^2-b^2)^2S_C)$$

on the nine-point circle. [19]

Exercises

1. Find the coordinates of the fourth intersection of the Feuerbach hyperbola with the circumcircle. [20]

2. Animate a point P on the Feuerbach hyperbola, and construct its pedal circle. This pedal circle always passes through the Feuerbach point.

3. Three particles are moving at equal speeds along the perpendiculars from I to the side lines. They form a triangle perspective with ABC. The locus of the perspector is the Feuerbach hyperbola.

4. The Feuerbach hyperbola is the locus of point P for which the cevian quotient I/P lies on the OI-line. [21]

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[17] A.P. Hatzipolakis, Hyacinthos, message 3370, 8/7/01.
[18] A.P. Hatzipolakis, Hyacinthos, message 3370, 8/7/01.
[19] The Jerabek center appears as X_{125} in ETC.
[20] $(\frac{a}{a^2(b+c)-2abc-(b+c)(b-c)^2} : \cdots : \cdots)$. This point appears as X_{104} in ETC.
[21] P. Yiu, Hyacinthos, message 1013, 6/13/00.

5. Find the fourth intersection of the Jerabek hyperbola with the circumcircle. [22]

6. Let ℓ be a line through O. The tangent at H to the rectangular hyperbola which is the isogonal conjugate of ℓ intersects ℓ at a point on the Jerabek hyperbola. [23]

[22] $(\frac{a^2}{2a^4-a^2(b^2+c^2)-(b^2-c^2)^2}:\cdots:\cdots)$. This point appears as X_{74} in ETC.

[23] B. Gibert, Hyacinthos, message 4247, 10/30/01.

11.4 Envelopes

The envelope of the parametrized family of lines

$$(a_0 + a_1t + a_2t^2)x + (b_0 + b_1t + b_2t^2)y + (c_0 + c_1t + c_2t^2)z = 0$$

is the conic[24]

$$(a_1x + b_1y + c_1z)^2 - 4(a_0x + b_0y + c_0z)(a_2x + b_2y + c_2z) = 0,$$

provided that the determinant

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \neq 0.$$

Proof. This is the dual conic of the conic parametrized by

$$x : y : z = a_0 + a_1t + a_2t^2 : b_0 + b_1t + b_2t^2 : c_0 + c_1t + c_2t^2.$$

11.4.1 The Artzt parabolas

Consider similar isosceles triangles $A^\theta BC$, $AB^\theta C$ and ABC^θ constructed on the sides of triangle ABC. The equation of the line $B^\theta C^\theta$ is

$$(S^2 - 2S_At - t^2)x + (S^2 + 2(S_A + S_B)t + t^2)y + (S^2 + 2(S_C + S_A)t + t^2)z = 0,$$

where $t = S_\theta = S \cdot \cot\theta$. As θ varies, this envelopes the conic

$$(-S_Ax + c^2y + b^2z)^2 - S^2(x + y + z)(-x + y + z) = 0$$

11.4.2 Envelope of area-bisecting lines

Let Y be a point on the line AC. There is a unique point Z on AB such that the signed area of AZY is half of triangle ABC. We call YZ an area-bisecting line. If $Y = (1 - t : 0 : t)$, then $Z = (1 - \frac{1}{2t} : \frac{1}{2t} : 0) = (2t - 1 : 1 : 0$. The line YZ has equation

$$0 = \begin{vmatrix} 1-t & 0 & t \\ 2t-1 & 1 & 0 \\ x & y & z \end{vmatrix} = -tx + (-t + 2t^2)y + (1 - t)z.$$

This envelopes the conic

$$(x + y + z)^2 - 8yz = 0.$$

This conic has representing matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix}$$

[24] This can be rewritten as $\sum(4a_0a_2 - a_1^2)x^2 + 2(2(b_0c_2 + b_2c_0) - b_1c_1)yz = 0$.

with adjoint matrix

$$M^{\#} = -4\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

This is a *hyperbola* with center at the vertex A.

To construct this as a 5-point conic, we need only find 3 points on the hyperbola. Here are three obvious points: the centroid G, $(1:-1:0)$ and $(1:0:-1)$. Unfortunately the latter two are infinite point: they give the lines AB and AC as asymptotes of the hyperbola. This means that the axes of the hyperbola are the bisectors of angle A. Thus images of G in these axes give three more points on the hyperbola. To find a fifth point, we set $x=0$ and obtain $(y+z)^2-8yz=0, \dots, y-3z:z=\pm 2\sqrt{2}:1$,

$$y:z = 3\pm 2\sqrt{2}:1 = (\sqrt{2}\pm 1)^2:1 = \sqrt{2}\pm 1:\sqrt{2}\mp 1.$$

11.4.3 Envelope of perimeter-bisecting lines

Let Y be a point on the line AC. There is a unique point Z on AB such that the (signed) lengths of the segments AY and AZ add up to the semiperimeter of triangle ABC. We call YZ a perimeter-bisecting line. If $AY=t$, then $AZ=s-t$. The coordinates of the points are $Y=(b-t:0:t)$ and $Z=(c-s+t:s-t:0)$. The line YZ has equation

$$(t^2-st)x+(t^2-(s-c)t)y+(t^2-(s+b)t+bs)z=0.$$

These lines envelopes the conic

$$(sx+(s-c)y+(s+b)z)^2-4bsz(x+y+z)=0$$

with representing matrix

$$\begin{pmatrix} s^2 & s(s-c) & s(s-b) \\ s(s-c) & (s-c)^2 & (s-b)(s-c) \\ s(s-b) & (s-b)(s-c) & (s-b)^2 \end{pmatrix}$$

with adjoint matrix

$$M^{\#} = -8bcs\begin{pmatrix} 2(s-a) & s-b & s-c \\ s-b & 0 & -s \\ s-c & -s & 0 \end{pmatrix}.$$

This conic is a parabola tangent to the lines CA and AB at the points $(-(s-b):0:s)$ and $(-(s-c):s:0)$. [25]

[25] These are the points of tangency of the A-excircle with the side lines.

11.4.4 The tripolars of points on the Euler line

A typical point on the Euler line

$$\sum_{\text{cyclic}} S_A(S_B - S_C)x = 0$$

has coordinates $(S_{BC} + t : S_{CA} + t : S_{AB} + t)$, with tripolar

$$\sum_{\text{cyclic}} \frac{1}{S_{BC} + t} x = 0,$$

or

$$0 = \sum_{\text{cyclic}} (v + t)(w + t)x = \sum_{\text{cyclic}} (S_{BC} + a^2 S_A t + t^2)x.$$

The envelope is the conic

$$(a^2 S_A x + b^2 S_B y + c^2 S_C z)^2 - 4S_{ABC}(x + y + z)(S_A x + S_B y + S_C z) = 0.$$

This can be rewritten as

$$\sum_{\text{cyclic}} S_{AA}(S_B - S_C)^2 x^2 - 2S_{BC}(S_C - S_A)(S_A - S_B)yz = 0.$$

This can be rewritten as

$$\sum_{\text{cyclic}} S_{AA}(S_B - S_C)^2 x^2 - 2S_{BC}(S_C - S_A)(S_A - S_B)yz = 0.$$

It is represented by the matrix

$$M = \begin{pmatrix} S_{AA}(S_B - S_C)^2 & -S_{AB}(S_B - S_C)(S_C - S_A) & -S_{CA}(S_A - S_B)(S_B - S_C) \\ -S_{AB}(S_B - S_C)(S_C - S_A) & S_{BB}(S_C - S_A) & -S_{BC}(S_C - S_A)(S_A - S_B) \\ S_{CA}(S_A - S_B)(S_B - S_C) & -S_{BC}(S_C - S_A)(S_A - S_B) & S_{CC}(S_A - S_B) \end{pmatrix}.$$

This is clearly an inscribed conic, tangent to the side lines at the points $(0 : S_C(S_A - S_B) : S_B(S_C - S_A))$, $(S_C(S_A - S_B) : 0 : S_A(S_B - S_C))$, and $(S_B(S_C - S_A) : S_A(S_B - S_C) : 0)$. The perspector is the point [26]

$$\left(\frac{1}{S_A(S_B - S_C)} : \frac{1}{S_B(S_C - S_A)} : \frac{1}{S_C(S_A - S_B)}\right).$$

The isotomic conjugate of this perspector being an infinite point, the conic is a parabola. [27]

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[26]This point appears as X_{648} in ETC.

[27]The focus is the point X_{112} in ETC:

$$\left(\frac{a^2}{S_A(S_B - S_C)} : \frac{b^2}{S_B(S_C - S_A)} : \frac{c^2}{S_C(S_A - S_B)}\right).$$

Its directrix is the line of reflection of the focus, *i.e.*,

$$\sum_{\text{cyclic}} S_{AA}(S_B - S_C)x = 0.$$

Exercises

1. **Animate** a point P on the circumcircle, and construct a circle $\mathcal{C}(P)$, center P, and radius half of the inradius. Find the envelope of the radical axis of $\mathcal{C}(P)$ and the incircle.

2. **Animate** a point P on the circumcircle. Construct the isotomic conjugate of its isogonal conjugate, *i.e.*, the point $Q = (P^*)^\bullet$. What is the envelope of the line joining PQ? [28]

[28]The Steiner point.

Chapter 12

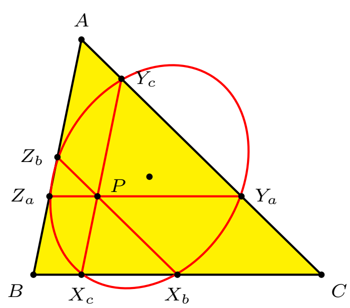
Some More Conics

12.1 Conics associated with parallel intercepts

12.1.1 Lemoine's thorem

Let $P = (u : v : w)$ be a given point. Construct parallels through P to the side lines, intersecting the side lines at the points

$$\begin{array}{ll} Y_a = (u : 0 : v + w), & Z_a = (u : v + w : 0); \\ Z_b = (w + u : v : 0), & X_b = (0 : v : w + u); \\ X_c = (0 : u + v : w), & Y_c = (u + v : 0 : w). \end{array}$$



These 6 points lie on a conic $\mathcal{C}_P$, with equation

$$\sum_{\text{cyclic}} vw(v + w)x^2 - u(vw + (w + u)(u + v))yz = 0.$$

This equation can be rewritten as

$$\begin{array}{ll} - & (u + v + w)^2(uyz + vzx + wxy) \\ + & (x + y + z)(vw(v + w)x + wu(w + u)y + uv(u + v)z) = 0. \end{array}$$

From this we obtain

Theorem (Lemoine)

The conic through the 6 parallel intercepts of P is a circle if and only if P is the symmedian point.

Exercises

1. Show that the conic $\mathcal{C}_P$ through the 6 parallel intercepts through P is an ellipse, a parabola, or a hyperbola according as P is inside, on, or outside the Steiner in-ellipse, and that its center is the midpoint of the P and the cevian quotient G/P. [1]

2. Show that the Lemoine circle is concentric with the Brocard circle. [2]

12.1.2 A conic inscribed in the hexagon $W(P)$

While $\mathcal{C}_P$ is a conic circumscribing the hexagon $W(P) = Y_aY_cZ_bZ_aX_cX_b$, there is another conic inscribed in the same hexagon. The sides of the hexagon have equations

$$\begin{array}{rlrl} Y_aY_c: & y=0; & Y_cZ_b: & -vwx+w(w+u)y+v(u+v)z=0;\\ Z_bZ_a: & z=0; & Z_aX_c: & w(v+w)x-wuy+u(u+v)z=0;\\ X_cX_b: & x=0; & X_bY_a: & v(v+w)x+u(w+u)y-uvz=0. \end{array}$$

These correspond to the following points on the dual conic: the vertices and

$$\left(-1:\frac{w+u}{v}:\frac{u+v}{w}\right),\quad \left(\frac{v+w}{u}:-1:\frac{u+v}{w}\right),\quad \left(\frac{v+w}{u}:\frac{w+u}{v}:-1\right).$$

It is easy to note that these six points lie on the circumconic

$$\frac{v+w}{x}+\frac{w+u}{y}+\frac{u+v}{z}=0.$$

It follows that the 6 lines are tangent to the incribed conic

$$\sum_{\text{cyclic}}(v+w)^2x^2-2(w+u)(u+v)yz=0,$$

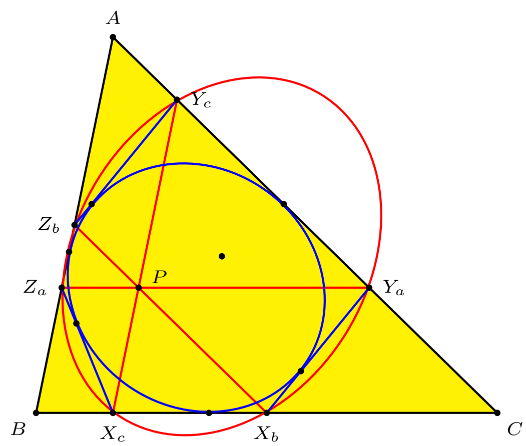
with center $(2u+v+w:u+2v+w:u+v+2w)$ and perspector

$$\left(\frac{1}{v+w}:\frac{1}{w+u}:\frac{1}{u+v}\right).$$

---

[1] The center has coordinates $(u(2vw+u(v+w-u)):v(2wu+v(w+u-v)):w(2uv+w(u+v-w))$.

[2] The center of the Lemoine circle is the midpoint between K and $G/K=O$.



Exercises

1. Find the coordinates of the points of tangency of this inscribed conic with the Y_cZ_b, Z_aX_c and X_bY_a, and show that they form a triangle perspective with ABC at [3]

$$\left(\frac{u^2}{v+w}:\frac{v^2}{w+u}:\frac{w^2}{u+v}\right).$$

12.1.3 Centers of inscribed rectangles

Let $P=(x:y:z)$ be a given point. Construct the inscribed rectangle whose top edge is the parallel to BC through P. The vertices of the rectangle on the sides AC and AB are the points $(x:y+z:0)$ and $(x:0:y+z)$.

The center of the rectangle is the point

$$A' = (a^2x : a^2(x+y+z)-S_Bx : a^2(x+y+z)-S_Cx).$$

Similarly, consider the two other rectangles with top edges through P parallel to CA and AB respectively, with centers B' and C'. The triangle $A'B'C'$ is perspective with ABC if and only if

$$\begin{array}{rl} & (a^2(x+y+z)-S_Bx)(b^2(x+y+z)-S_Cy)(c^2(x+y+z)-S_Az) \\ = & (a^2(x+y+z)-S_Cx)(b^2(x+y+z)-S_Ay)(c^2(x+y+z)-S_Bz). \end{array}$$

The first terms of these expressions cancel one another, so do the last terms. Further cancelling a common factor $x+y+z$, we obtain the quadratic equation

$$\sum a^2S_A(S_B-S_C)yz + (x+y+z)\sum_{\text{cyclic}} b^2c^2(S_B-S_C)x = 0.$$

[3] $(v+w:\frac{v^2}{w+u}:\frac{w^2}{u+v})$, $(\frac{u^2}{v+w}:w+u:\frac{w^2}{u+v})$, and $(\frac{u^2}{v+w}:\frac{v^2}{w+u}:u+v)$.

This means that the locus of P for which the centers of the inscribed rectangles form a perspective triangle is a hyperbola in the pencil generated by the Jerabek hyperbola

$$\sum a^2 S_A(S_B - S_C)yz = 0$$

and the Brocard axis OK

$$\sum_{\text{cyclic}} b^2c^2(S_B - S_C)x = 0.$$

Since the Jerabek hyperbola is the isogonal transform of the Euler line, it contains the point $H^* = O$ and $G^* = K$. The conic therefore passes through O and K. It also contains the de Longchamps point $L = (-S_{BC} + S_{CA} + S_{AB} : \cdots : \cdots)$ and the point $(S_B + S_C - S_A : S_C + S_A - S_B : S_A + S_B - S_C)$. [4]

P	Perspector
circumcenter	$(\frac{1}{2S^2 - S_{BC}} : \frac{1}{2S^2 - S_{CA}} : \frac{1}{2S^2 - S_{AB}})$
symmedian point	$(3a^2 + b^2 + c^2 : a^2 + 3b^2 + c^2 : a^2 + b^2 + 3c^2)$
de Longchamps point	$(S_{BC}(S^2 + 2S_{AA}) : \cdots : \cdots)$
$(3a^2 - b^2 - c^2 : \cdots : \cdots)$	$(\frac{1}{S^2 + S_{AA} + S_{BC}} : \cdots : \cdots)$

Exercises

1. Show that the three inscribed rectangles are similar if and only if P is the point

$$\left(\frac{a^2}{t + a^2} : \frac{b^2}{t + b^2} : \frac{c^2}{t + c^2}\right),$$

where t is the unique positive root of the cubic equation [5]

$$t^3 - (a^2b^2 + b^2c^2 + c^2a^2)t^2 - 2a^2b^2c^2 = 0.$$

[4] None of these perspectors appears in the current edition of ETC.

[5] Corrected by Peter Moses, 11/10/04.

12.2 Lines simultaneously bisecting perimeter and area

Recall from §11.4.2 that the A-area-bisecting lines envelope the conic whose dual is represented by the matrix

$$M_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

On the other hand, the A-perimeter-bisecting lines envelope another conic whose dual is represented by

$$M_2 = \begin{pmatrix} 2(s-a) & s-b & s-c \\ s-b & 0 & -s \\ s-c & -s & 0 \end{pmatrix}.$$

To find a line simultaneously bisecting the area and perimeter, we seek an intersection of of the two dual conics represented by M_1 and M_2. In the pencil of conics generated by these two, namely, the conics represented by matrices of the form $tM_1 + M_2$, there is at least one member which degenerates into a union of two lines. The intersections of the conics are the same as those of these lines with any one of them. Now, for any real parameter t,

$$\begin{aligned} \det(tM_1 + M_2) &= \begin{vmatrix} 2(t+s-a) & t+s-b & t+s-c \\ t+s-b & 0 & -(t+s) \\ t+s-c & -(t+s) & 0 \end{vmatrix} \\ &= -2(t+s)(t+s-b)(t+s-c) - 2(t+s)^2(t+s-a) \\ &= -2(t+s)[(t+s-b)(t+s-c) + (t+s)(t+s-a)] \\ &= -2(t+s)[2(t+s)^2 - 2s(t+s) + bc] \end{aligned}$$

By choosing $t = -s$, we obtain

$$-sM_1 + M_2 = \begin{pmatrix} -2a & -b & -c \\ -b & 0 & 0 \\ -c & 0 & 0 \end{pmatrix}$$

which represents the degenerate conic

$$2ax^2 + 2bxy + 2cxy = 2x(ax + by + cz) = 0.$$

In other words, the intersections of the two dual conics are the same as those

$$x^2 + xy + xz - yz = 0$$

(represented by M_1) and the lines $x = 0$ and $ax + by + cz = 0$.

With $x = 0$ we obtain $yz = 0$, and hence the points $(0 : 0 : 1)$ and $(0 : 1 : 0)$ on the dual conic. These correspond to the lines CA and AB. These clearly are not area bisecting lines. This means that such a line must pass through the incenter I, and with corresponding t satisfying

$$2bt^2 - (a+b+c)t + c = 0.$$

From this,

$$t = \frac{(a+b+c) \pm \sqrt{(a+b+c)^2 - 8bc}}{4b} = \frac{s \pm \sqrt{s^2 - 2bc}}{2b}.$$

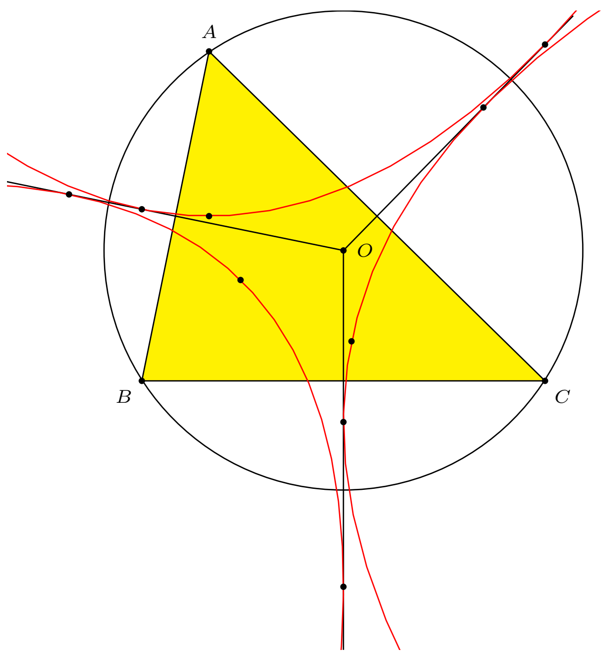
The division points on AC are

$$(1-t:0:t) = \left(2b - s \mp \sqrt{s^2-2bc} : 0 : s \pm \sqrt{s^2-2bc}\right).$$

12.3 Parabolas with vertices of a triangle as foci and sides as directrices

Given triangle ABC, consider the three parabolas each with one vertex as focus and the opposite side as directrix, and call these the $a-$, $b-$, and $c-$parabolas respectively. The vertices are clearly the midpoints of the altitudes. No two of these parabolas intersect. Each pair of them, however, has a unique common tangent, which is the perpendicular bisector of a side of the triangle. The three common tangents therefore intersect at the circumcenter.

The points of tangency of the perpendicular bisector BC with the $b-$ and $c-$parabolas are inverse with respect to the circumcircle, for they are at distances $\frac{bR}{c}$ and $\frac{cR}{b}$ from the circumcenter O. These points of tangency can be easily constructed as follows. Let H be the orthocenter of triangle ABC, H_a its reflection in the side BC. It is well known that H_a lies on the circumcircle. The intersections of BH_a and CH_a with the perpendicular bisector of BC are the points of tangency with the $b-$ and $c-$parabolas respectively.



Exercises

1. Find the equation of the a-parabola. [6]

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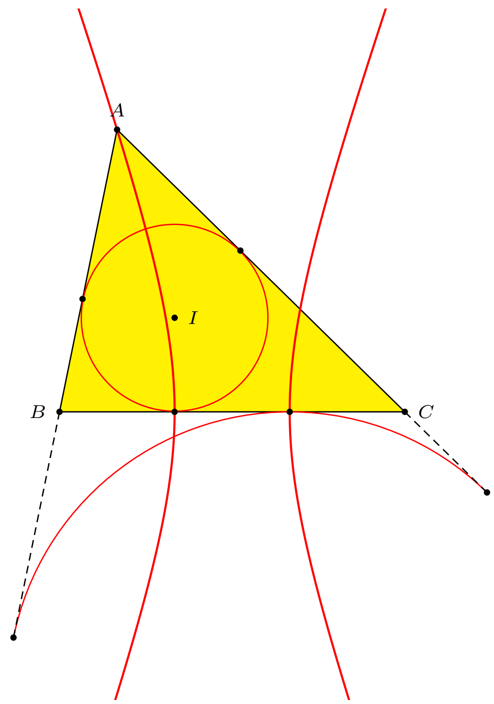
[6] $-S^2x^2 + a^2(c^2y^2 + 2S_Ayz + b^2z^2) = 0.$

12.4 The Soddy hyperbolas and Soddy circles

12.4.1 The Soddy hyperbolas

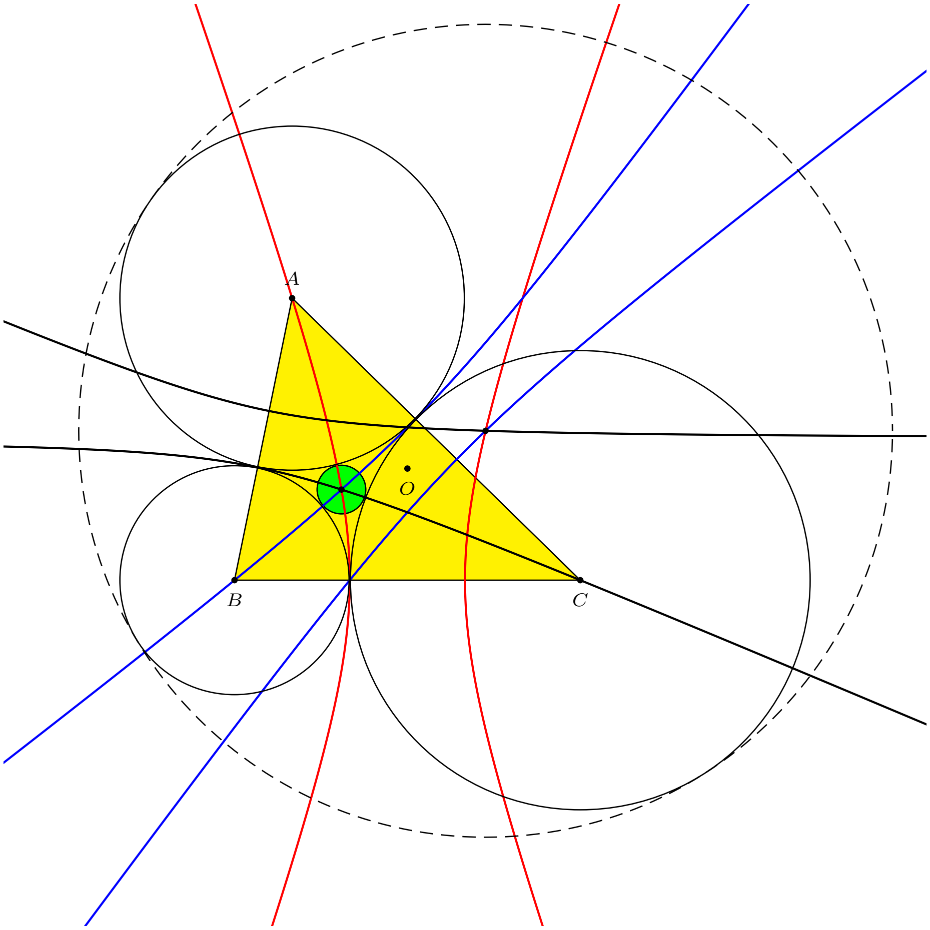
Given triangle ABC, consider the hyperbola passing through A, and with foci at B and C. We shall call this the a-Soddy hyperbola of the triangle, since this and related hyperbolas lead to the construction of the famous Soddy circle. The reflections of A in the side BC and its perpendicular bisector are clearly points on the same hyperbola, so is the symmetric of A with respect to the midpoint of BC. The vertices of the hyperbola on the transverse axis BC are the points $(0 : s-b : s-c)$, and $(0 : s-c : s-b)$, the points of tangency of the side BC with the incircle and the A-excircle.



Likewise, we speak of the B- and C-Soddy hyperbolas of the same triangle, and locate obvious points on these hyperbolas.

12.4.2 The Soddy circles

Given triangle ABC, there are three circles centered at the vertices and mutually tangent to each other externally. These are the circles $A(s-a)$, $B(s-b)$, and $C(s-c)$. The *Soddy circles* of triangle ABC are the two circles each tangent to these three circles, all externally or all internally. The centers of the Soddy circles clearly are the intersections of the three Soddy hyperbolas.



Exercises

1. Show that the equation of A-Soddy hyperbola is

$$\begin{aligned} F_a &= (c+a-b)(a+b-c)(y^2+z^2) \\ &\quad -2(a^2+(b-c)^2)yz - 4b(b-c)zx + 4(b-c)cxy = 0. \end{aligned}$$

12.5 Appendix: Constructions with conics

Given 5 points A, B, C, D, E, no three of which are collinear, and no four concyclic, the conic $\mathcal{C}$. Through these 5 points is either an ellipse, a parabola, or a hyperbola.

12.5.1 The tangent at a point on $\mathcal{C}$

(1) $P := AC \cap BD$;
(2) $Q := AD \cap CE$;
(3) $R := PQ \cap BE$.
AR is the tangent at A.

12.5.2 The second intersection of $\mathcal{C}$ and a line ℓ through A

(1) $P := AC \cap BE$;
(2) $Q := \ell \cap BD$;
(3) $R := PQ \cap CD$;
(4) $A' := \ell \cap ER$.
A' is the second intersection of $\mathcal{C}$ and ℓ.

12.5.3 The center of $\mathcal{C}$

(1) $B' :=$ the second intersection of $\mathcal{C}$ with the parallel through B to AC;
(2) $\ell_b :=$ the line joining the midpoints of BB' and AC;
(3) $C' :=$ the second intersection of $\mathcal{C}$ with the parallel through C to AB;
(4) $\ell_c :=$ the line joining the midpoints of CC' and AB;
(5) $O := \ell_b \cap \ell_c$ is the center of the conic $\mathcal{C}$.

12.5.4 Principal axes of $\mathcal{C}$

(1) $K(O) :=$ any circle through the center O of the conic $\mathcal{C}$.

(2) Let M be the midpoint of AB. Construct (i) OM and (ii) the parallel through O to AB each to intersect the circle at a point. Join these two points to form a line ℓ.

(3) Repeat (2) for another chord AC, to form a line ℓ'.

(4) $P := \ell \cap \ell'$.

(5) Let KP intersect the circle $K(O)$ at X and Y.

Then the lines OX and OY are the principal axes of the conic $\mathcal{C}$.

12.5.5 Vertices of $\mathcal{C}$

(1) Construct the tangent at A to intersect to the axes OX and OY at P and Q respectively.

(2) Construct the perpendicular feet P' and Q' of A on the axes OX and OY.

(3) Construct a tangent OT to the circle with diameter PP'. The intersections of the line OX with the circle $O(T)$ are the vertices on this axis.

(4) Repeat (3) for the circle with diameter QQ'.

12.5.6 Intersection of $\mathcal{C}$ with a line $\mathcal{L}$

Let F be a focus, ℓ a directrix, and e = the eccentricity.

(1) Let $H = \mathcal{L} \cap \ell$.

(2) Take an arbitrary point P with pedal Q on the directrix.

(3) Construct a circle, center P, radius $e \cdot PQ$.

(4) Through P construct the parallel to $\mathcal{L}$, intersecting the directrix at O.

(5) Through O construct the parallel to FH, intersecting the circle above in X and Y.

(6) The parallels through F to PX and PY intersect the given line $\mathcal{L}$ at two points on the conic.