Affine transformations (Affinities)

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> I am among those who think that science has great beauty . . . A scientist in his laboratory is not only a technician but also a child placed in front of natural phenomena which impresses him like a fairy tale.
>
> *E. Curie, Madame Curie p.341*

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1 Definition and group property

Affinity

Considering the case of *two* dimensions, *"Affine transformations"* or *"Affinities"* are invertible transformations of the plane onto itself, which, fixing a coordinate system (not necessarily orthogonal or having equal unit-lengths on the axes), are defined by an invertible matrix $\{A,\ |A| = a_{11}a_{22} - a_{12}a_{21} \neq 0\}$ and a vector $v(v_1, v_2)$ ([Cox61, p.203]):

$$Y = f(X) = AX + v, \quad \text{with} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\left.\begin{array}{rcl} y_1 &=& a_{11}x_1 + a_{12}x_2 + v_1, \\ y_2 &=& a_{21}x_1 + a_{22}x_2 + v_2. \end{array}\right\} \tag{1}$$

Using the three dimensional extensions $X' = (x_1, x_2, 1)$ of points $X(x_1, x_2) \in \mathbb{R}^2$ and the matrix

$$A_v = \begin{pmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right), \tag{2}$$

equation (1) is equivalent with

$$Y' = A_v \cdot X' \quad \Leftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = A_v \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}. \tag{3}$$

The determinants of A_v and A, are *equal* and the inverse of A_v is of the same form:

$$A_v^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} a_{22} & -a_{12} & a_{12}v_2 - a_{22}v_1 \\ -a_{21} & a_{11} & a_{21}v_1 - a_{11}v_2 \\ 0 & 0 & |A| \end{pmatrix} = \left(\begin{array}{c|c} A^{-1} & -A^{-1} \cdot v \\ \hline 0 & 1 \end{array}\right),$$

which guarantees that this represents again an affine transformation. Also the product (composition) of two such transformations, represented by the matrices $\{A_v, B_w\}$, is of the same form:

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} B & w \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} AB & Aw + v \\ \hline 0 & 1 \end{array}\right).$$

These remarks imply easily the important property formulated as a theorem:

Group property

Theorem 1. *The affinities of the plane form a group G_A.*

affine equivalent

We say two shapes of the plane $\{A, B\}$ are *"affine equivalent"*, if there is an affinity mapping the first onto the second. The affine geometry of the plane deals with properties of shapes that remain the same (invariant) by affinities. For example, the middle of a segment. We'll see below that by an affinity a segment μ maps to a segment μ' and its middle $M \in \mu$ maps correspondingly to the middle $M' \in \mu'$. We say in short: affinities *"preserve"* the middles of segments.

Remark 1. There are successive extensions $\{G_E \subset G_A \subset G_P\}$ of the group of transformations defining the *"equality"* or *"equivalence"* of two shapes of the plane, the symbols standing respectively for the *euclidean*, the *affine* and the *projective* group of transformations. These groups can be represented by matrices in the form

$$\text{Euclidean isometries:}\quad G_E = \left\{ \begin{pmatrix} \cos(\phi) & \mp\sin(\phi) & a \\ \sin(\phi) & \pm\cos(\phi) & b \\ 0 & 0 & 1 \end{pmatrix}, \text{ for } \phi, a, b \in \mathbb{R} \right\},$$

$$\text{Affinities:}\quad G_A = \left\{ \begin{pmatrix} a & b & c \\ k & l & m \\ 0 & 0 & 1 \end{pmatrix}, \text{ for } a, b, c, k, l, m \in \mathbb{R} \right\},$$

$$\text{Projectivities:}\quad G_P = \left\{ \begin{pmatrix} a & b & c \\ k & l & m \\ p & q & r \end{pmatrix}, \text{ for } a, b, c, k, l, m, p, q, r \in \mathbb{R} \right\}.$$

In the two last groups we require also from the matrices to be invertible i.e. to have non-zero determinants. In the last group also we consider that two matrices differing by a non-zero multiplicative constant $B = \lambda \cdot A, \quad \lambda \neq 0$, define the same (projective) transformation.

Analogous to the *"affine equivalence"* is the *"congruence"* of two shapes, meaning that there is an euclidean isometry mapping one to the other. Finally, *"projective equivalent"* are called two shapes for which there is a projectivity mapping one to the other.

The fewer parameters has the group, the more entities, the group acts upon, can be distinguished as non-equivalent. Thus, the euclidean group, having only three parameters, distinguishes several kinds of hyperbolas, of ellipses and parabolas. The affine group, with six parameters, cannot distinguish that many entities. For it all hyperbolas are the same, i.e two hyperbolas, which from the euclidean aspect are different, can be transformed, one to the other, via an affine transformation $f \in G_A$ and thus, can be considered to be the same thing. Ultimately, under this aspect, there is only one hyperbola, one ellipse and one parabola. Finally, under the projective aspect, the previous three different kinds are unified and, ultimately, there is only one conic, the circle. In the following sections we confine our study to the elements of the group G_A of affine transformations and properties of shapes preserved by affinities.

2 Determination of an affinity

Given three points of the plane $\{X, Y, Z\}$, the determinant of the corresponding matrix

$$XYZ = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{pmatrix}, \quad |XYZ| := (y_1 z_2 - y_2 z_1) + (z_1 x_2 - z_2 x_1) + (x_1 y_2 - x_2 y_1),$$

if the coordinate frame is *"orthonormal"*, expresses twice the signed area of the triangle XYZ ([Str88, p.239]). For a general (oblique) coordinate frame this is a multiple of the area of the triangle. This implies that the three points are *collinear* precisely when this determinant vanishes. Also applying the affinity to $\{X, Y, Z\}$ we obtain three other points $\{U, V, W\}$ and we can describe this operation with a matrix multiplication:

$$UVV = A_v \cdot XYZ \quad \Rightarrow \quad |UVW| = |A_v| \cdot |XYZ| = |A| \cdot |XYZ|.$$

This implies that if $\{X, Y, Z\}$ are non-collinear(collinear) the same is true for $\{U, V, W\}$. In addition, it follows that the quotient of the areas of two triangles is preserved by affinities. By splitting a polygon in triangles, we conclude that affinities preserve the quotient of areas of two polygons. There is even a kind of affinities *preserving* the area. This, by the last formula, happens when the determinant of the affinity is $|A| = |A_v| = 1$. These special affinities are called *"equiaffinities"*. The following theorems formulate the previous remarks.

equiaffinity

preserve collinearity

Theorem 2. *An affine transformation maps collinear points to collinear and non-collinear to non-collinear points as well.*

lines map to lines

Corollary 1. *An affine transformation maps a line of the plane onto a line of the same.*

Corollary 2. *An affine transformation maps a genuine triangle to a genuine (non-degenerated) triangle.*

Theorem 3. *For an affine transformation the image $f(p)$ of a polygon p is a polygon with the same number of sides and the ratio of the signed areas $|f(p)|/|p|$ is a constant independent of the particular polygon.*

quotient of areas preserved

Corollary 3. *An affine transformation preserves the quotient of areas of two polygons $\{p, p'\}$, i.e. $|f(p')|/|f(p)| = |p'|/|p|$.*

The inverse of the matrix XYZ, for non-collinear points, is found to be

$$(XYZ)^{-1} = \frac{1}{|XYZ|}\begin{pmatrix} y_2 - z_2 & z_1 - y_1 & y_1 z_2 - y_2 z_1 \\ z_2 - x_2 & x_1 - z_1 & x_2 z_1 - x_1 z_2 \\ x_2 - y_2 & y_1 - x_1 & x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

Thus, given two triples of *non-collinear* points $\{(X, Y, Z),\ (U, V, W)\}$ the matrix equation

$$B \cdot XYZ = UVW \quad \Leftrightarrow \quad B = UVW \cdot (XYZ)^{-1},$$

has a unique solution and defines the matrix B, which, a short calculation shows to be of the form of equation (2), thus defining an affinity. We formulate this as a theorem.

affinity from 3 to 3 pts

Theorem 4. *Two triples of non-collinear points $\{(X, Y, Z), (U, V, W)\}$ define a unique affinity f mapping the first to the second: $\{f(X) = U, f(Y) = V, f(Z) = W\}$.*

This implies several other properties expressed by the next corollaries.

Corollary 4. *Given two triangles ABC and $A'B'C'$ there is precisely one affinity f mapping the first onto the second, in the sense $\{f(A) = A', f(B) = B', f(C) = C'\}$.*

identity: fix 3 pts

Corollary 5. *An affinity mapping the vertices of a triangle ABC onto the same vertices is the identity transformation.*

Corollary 6. *Two affinities coinciding at three non collinear points (or the vertices of a triangle) coincide at every other point of the plane and define the same transformation.*

Corollary 7. *Two lines of the plane are "affine equivalent".*

equilateral's universality

Corollary 8. *Every triangle of the plane is "affine equivalent" to the equilateral.*

In section 20 we will show that two non-degenerate conics of the same type are affine equivalent.

3 Additional fundamental properties of affinities

parallels to parallels

Corollary 9. *Two different parallel lines map by an affinity to two different parallel lines.*

Proof. In fact, assume that the parallels $\{\alpha, \beta\}$ have images $\{\alpha' = f(\alpha), \beta' = f(\beta)\}$ intersecting at the point C. Then, $f^{-1}(C)$ is a point on $\alpha \cap \beta$, which is a contradiction. □

Corollary 10. *Two arbitrary parallelograms of the plane are affine equivalent, equivalently, every parallelogram is affine equivalent to a square.*

Proof. Consider the affinity f mapping the triangle ABC onto $A'B'C'$. Then see that $f(D) = D'$ (See Figure 1). □

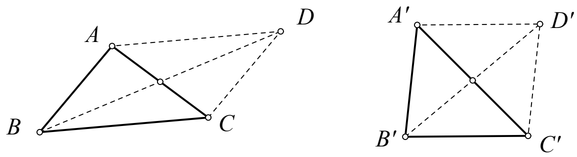

Figure 1: Parallelograms are affine equivalent

ratios preservation

Corollary 11. *If the affinity f maps the line α onto the line β, then the ratio of three points on α : CA/CB is the same with the ratio of their images on β : $f(C)f(A)/f(C)f(B) = CA/CB$.*

Proof. In fact, the point $C \in \alpha$ can be expressed as a linear combination

$$C = (1-\lambda)A + \lambda B \quad \Leftrightarrow \quad C' = (1-\lambda)A' + \lambda B',$$

the primes denoting the extended 3-dimensional points, and applying the affinity f on C means multiplying C' by a matrix A_v leading to

$$A_v \cdot C' = (1-\lambda)A_v \cdot A' + \lambda A_v \cdot B' \quad \Leftrightarrow \quad f(C) = (1-\lambda)f(A) + \lambda f(B).$$

The proof follows from this and the general relation of CA/CB with λ :

$$C = (1-\lambda)A + \lambda B \quad \Leftrightarrow \quad \frac{CA}{CB} = \frac{\lambda}{\lambda - 1}.$$

□

middles preservation

Corollary 12. *If the affinity f maps the line-segment AB onto the line segment $A'B'$, then it maps also the middle M of AB onto the middle M' of $A'B'$. If the affinity interchanges the end-points of a segment, i.e. $\{f(A) = B, f(B) = A\}$, then it fixes the middle M of AB : $f(M) = M$.*

Given n points of the plane $\{A_i, i = 1...n\}$ and n numbers $\{t_i, i = 1...n,\}$ the weighted sum

$$A_0 = \frac{1}{\sum t_i}(t_1 A_1 + \cdots + t_n A_n), \tag{4}$$

is called *"barycenter"* of the system of masses $\{t_i\}$ attached to the points $\{A_i\}$. Using the vectors $\{A'_i = (x_i, y_i, 1)\}$ instead of $\{A_i(x_i, y_i)\}$ and applying the method of corollary 11, we prove easily next property formulated as a theorem.

Theorem 5. *Affinities f preserve the barycenter of systems of weighted points of the plane, i.e.*

$$A_0 = \frac{1}{\sum t_i}(t_1 A_1 + \cdots + t_n A_n) \quad \Rightarrow \quad f(A_0) = \frac{1}{\sum t_i}(t_1 f(A_1) + \cdots + t_n f(A_n)).$$

centroid or barycenter

Remark 2. Setting $\{t_1 = t_2 = \ldots = t_n = 1\}$, point $A_0 = (A_1 + \ldots A_n)/n$ is the *"barycenter"* or *"centroid"* of the *polyon* $A_1A_2 \ldots A_n$ and the theorem means that

"Affinities preserve the barycenter of polygons".

Remark 3. Barycenters can be used to give an alternative definition of affinities. According to this: *an affinity of the plane is a transformation which preserves barycenters.*

convexity preservation

Exercise 1. *Show that an affinity preserves the "convexity" property of a polygon, which means that if $\{X, Y\}$ are points inside the polygon, then the whole segment XY is also inside.*

Exercise 2. *Given n points of the plane $\{A_i, i = 1 \ldots n\}$ consider the function, which for every point of the plane P defines $m(P) = \sum_i |PA_i|^2$. Show that $m(P)$ attains its minimum value at the centroid A_0 of the polygon $A_1 \ldots A_n$ ([Rya86, p.54]).*

Corollary 13. *An affinity of the plane mapping the triangle ABC onto itself leaves the centroid G of the triangle fixed and maps each median line to another median line of the triangle.*

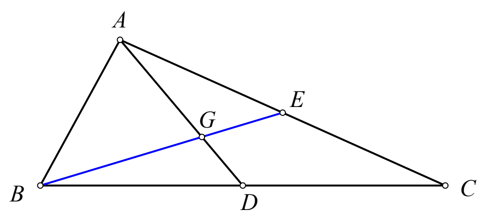

Figure 2: Fixing the centroid G of the triangle

Proof. It is obvious that an affinity leaving invariant a triangle permutes the vertices of the same. Assume now that the affinity f defines such a permutation mapping $\{f(A) = B, f(B) = C, f(C) = A\}$. Then, the middle D of BC maps to the middle $E = f(D)$ of CA and the median AD maps to the median BE (See Figure 2). Then the equality of ratios $GA/GD = GB/GE = -2$ shows that G maps onto itself. Analogous is the proof for the other permutations of the vertices of ABC introduced by f. Alternatively, the result follows from theorem 5 and the representation of the centroid:

$$G = \frac{1}{3}(A + B + C),$$

□

4 Special kinds of affinities

Using the general properties of affinities, discussed in the previous sections, we can define affinities in several ways, mainly using their determination by prescribing the images $\{f(A), f(B), f(C)\}$ of three non-collinear points, which we may consider as the vertices of a triangle. An important role in this endeavour play the *"fixed points"*, i.e. points such that $f(P) = P$ and the *"invariant lines"*, i.e. lines such that $f(\mu) = \mu$ of an affine transformation. By the previous discussion, we can have only $\{0, 1, 2\}$ fixed points (invariant lines) if the affinity $f \neq e$ is different from the identity. From the preceding discussion we deduce the following corollaries:

Corollary 14. *If the affinity has one fixed point O, then it maps every line μ through O onto a line $f(\mu) = \mu'$ through O.*

two points fixing

Corollary 15. *If the affinity f fixes two points $\{A, B\}$ of the line α then, it leaves the whole line invariant "fixing every point of the line".*

Corollary 16. *If the affinity has one invariant line μ then it maps every line μ' parallel to μ to a line μ'' also parallel to μ.*

Corollary 17. *If the affinity has two non-parallel invariant lines $\{\mu_1, \mu_2\}$, their intersection point $O = \mu_1 \cap \mu_2$ remains fixed and every parallelogram with sides parallel to these two lines maps to a parallelogram with the same property.*

Corollary 18. *If the affinity has three pairwise non-parallel invariant lines $\{\mu_1, \mu_2, \mu_3\}$, then it is the identity.*

Besides the *"equiaffinities"* alluded to in section 2, the following special kinds of affinities are often used as *"fundamental"*, to which decompose all other general affinities, in the sense that the general f is written as a composition $f = f_n \circ \ldots \circ f_1$.

1. *"Axial affinities"* characterized by having a line of fixed points ([ea83, p.80], [Tar11, p.116]).
2. *"Affine reflections"* kind of *"axial affinities"*.
3. *"Strains"* kind of *"axial affinities"* with a $B = f(A)$ and AB intersecting the axis.
4. *"Shears"* kind of *"axial affinities"* with a $B = f(A)$ and AB parallel to axis.
5. *"Dilatations"* mapping *"every line to a parallel to it"*.
6. *"Symmetries"* or *"Half-turns"* usual point-symmetries and kind of *"Dilatation"*
7. *"Translations"* kind of *"Dilatation"* without fixed points.
8. *"Homotheties"* kind of *"Dilatation"* with a fixed point.
9. *"Affine glide reflections"* product of an *"affine reflection"* and a *"translation"*.
10. *"Hyperbolic rotations"* 1 of 5 classes of *"equiaffinities"* ([Cox67, p.22]).
11. *"Crossed hyperbolic rotations"* 1 of 5 classes of *"equiaffinities"*.
12. *"Elliptic rotations"* 1 of 5 classes of *"equiaffinities"*.
13. *"Parabolic rotations"* 1 of 5 classes of *"equiaffinities"*.
14. *"Focal rotations"* 1 of 5 classes of *"equiaffinities"*.

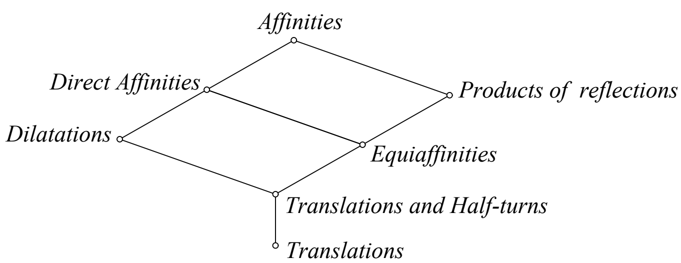

Figure 3: Inclusion diagram of main normal subgroups of the group of affinities

Figure 3 given also by Coxeter [Cox67, p.15] shows the inclusion diagram of the most important *"normal subgroups"* of the group G of affinities. These are sets G' of special affinities closed under composition, inversion and conjugation, latter meaning that:

$$g \circ f \circ g^{-1} = f' \in G' \quad \text{for every} \quad f \in G' \quad \text{and every} \quad h \in G.$$

"Direct affinities" are those that preserve the orientation of the triangles, equivalently in the description of affinities with matrices $\{A, A_v\}$, they have positive determinants $|A| = |A_v| > 0$.

5 Axial affinities or homologies, shears, strains

axial affinity

By definition, *"axial affinities"* or *"homologies"*([Tar11, p.116]) leave a single line point-wise fixed. They are usually defined by prescribing their images at four points:

$$f(A) = A, \;\; f(B) = B, \;\; (AB = \text{ the axis,}) \quad \text{and} \quad f(C) = D \neq C.$$

They could be represented by a quadrangle and its diagonals. If the diagonals intersect $AB \cap CD \neq \emptyset$, then the image $D' = f(C')$ of any other point C' can be found using the

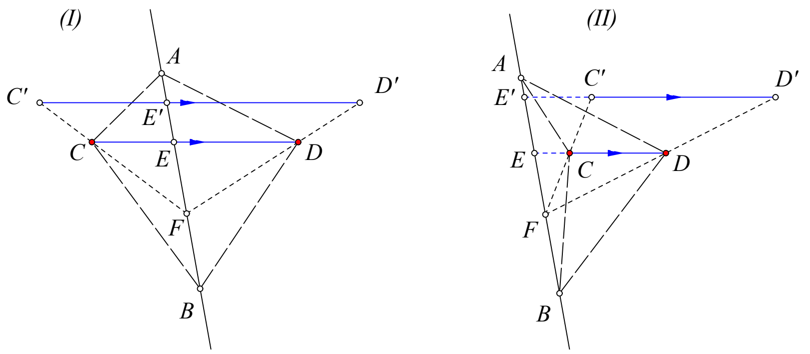

Figure 4: Strains: axial affinities with $\{A = f(A), B = f(B), D = f(C), E = CD \cap AB\}$

line CC' and its intersection $F = CC' \cap AB$ (See Figure 4). Point $D' = f(C)$ will be on line $FD = f(FC)$ and from the preservation of ratios by corollary 11, $\{C'D', CD\}$ must be parallel, giving a geometrical construction of the image point for all C' of the plane, except for those $C' \in CD$, which, by the analogous constancy of ratio are seen to map to points on the same line CD. From the similar triangles involved, follows that, introducing line coordinates with x-axis along AB and y-axis along CD the affinity f is described by a real constant k, different from 0 and 1 and the matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ ky \end{pmatrix}. \tag{5}$$

Transformations of this kind are called *"strains"*. The line AB is the *"axis"* and the constant k the *"ratio"* of the strain. In the special case $k = -1$ they are called *"affine reflections"* and the axis is often referred to as mirror. *"Euclidean reflections"* are a special case in which the mirror is orthogonal to the axis.

If the diagonals $\{AB, CD\}$ do not intersect (See Figure 5), then the same procedure leads to an analogous construction of the image $D' = f(C')$ for an arbitrary point of the plane.

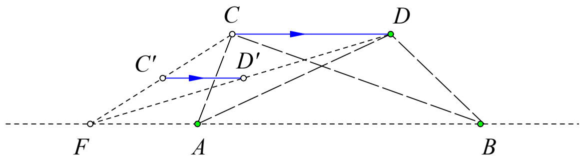

Figure 5: Shears: axial affinities with $CD \cap AB = \emptyset$

In this case introducing line coordinates with x-axis along AB and y-axis along AC, we see that the affinity is described by a constant $k \neq 0$ and the matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+ky \\ y \end{pmatrix}. \tag{6}$$

Transformations of this kind are called *shears*. Line AB is the *"axis"* and the constant k is the *"ratio"* of the shear. Next theorems summarize the discussion.

Theorem 6. *Every axial affinity is defined by giving four pairwise different arbitrary points $\{A, B, C, D\}$ with $\{C \notin AB, D \notin AB\}$ and prescribing their values:*

$$f(A) = A, \quad f(B) = B \quad \textit{and} \quad f(C) = D.$$

Theorem 7. *The only affine transformations which leave a single line ε point-wise fixed are axial affinities, i.e. shears and strains having ε for axis.*

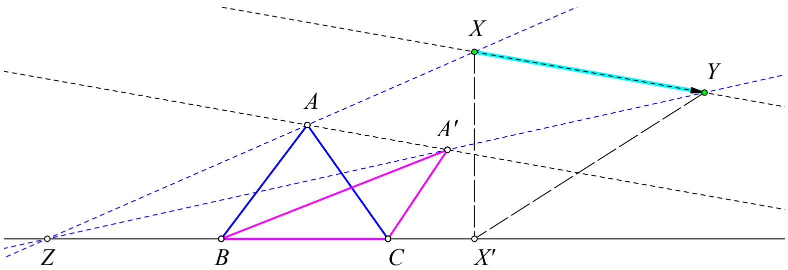

Figure 6: Axial affinity defined by two triangles with a common base

Exercise 3. *Show that two trianlges $\{ABC, A'BC\}$ with a common side define an axial symmetry f fixing point-wise the common base BC and mapping $A \mapsto A'$. Show also that for each point of the plane $X \notin BC$ the image $Y = f(X)$ defines a line XY parallel to AA' and two lines $\{AX, A'X'\}$ intersecting at a point $Z \in BC$. Show finally, that if X' is the projection of X on BC, the triangles $XX'Y$ have constant angles (See Figure* 6*).*

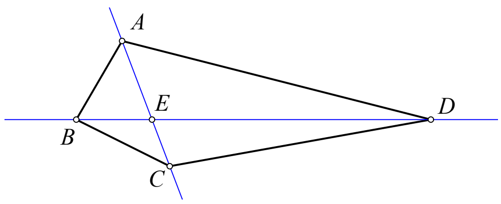

Figure 7: Strains defined by a quadrangle $ABCD$

Exercise 4. *Show that a quadrangle $ABCD$ defines through its diagonals two strains. The first strain f_1 has for axis the diagonal AC and conjugate direction that of the other diagonal BD. Its ratio is $k = EB/ED$, where $E = AC \cap BD$ (See Figure 7). Analogously is defined the other strain f_2 with axis BD and ratio $k' = EC/EA$. Find some conditions, so that these strains leave the quadrangle invariant.*

Show also that in a coordinate system with (x, y)-axes the pair (ED, EA) the composition $f_2 \circ f_1$ of the two strains is described by the equations:

$$\left.\begin{array}{rcl} x' &=& k \cdot x, \\ y' &=& k' \cdot y. \end{array}\right\} \quad \Leftrightarrow \quad X' = AX \quad \text{with} \quad A = \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}. \tag{7}$$

Exercise 5. *Show that the axial affinities with the same axis α form a group. Show further that this group acts "transitively" on the plane, in the sense that for any two different points $\{A, B\}$, there is an affinity of this group mapping A to B.*

6 Affinities leaving invariant a triangle

From the affinities f, different from the identity and leaving invariant the triangle ABC, of particular importance is an affinity leaving one vertex, A say, fixed. Then it must interchange the other two: $\{f(B) = C, f(C) = B\}$ leaving also fixed the middle D of BC and, by corollary 15, fixing the whole line AD, while mapping line BC onto itself. By applying corollary 11 we see that f acts on BC as the symmetry about D. The same

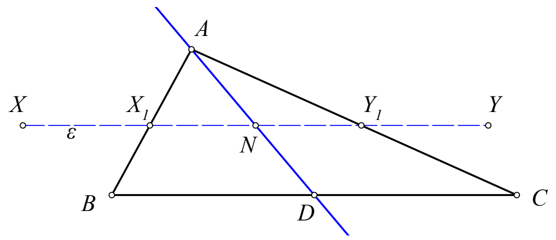

Figure 8: Affine reflection

behaviour can be seen to happen on every parallel line ε to BC. In fact, if the parallel intersects the median AD at the point N and the sides $\{AB, AC\}$ at $\{X_1, Y_1\}$, then, again by corollaries 9, 15 and 11, follows that a) X_1Y_1 is parallel to BC, and b) $f(X) = Y$ is the symmetric of X w.r. to N. Next theorem summarizes the short discussion and justifies the notation $A(BC)$ used by Coxeter for the affine reflection interchanging $\{B, C\}$ and fixing point $A \notin BC$.

reflection $A(BC)$

Theorem 8. *If an affinity, different from the identity, leaves invariant a triangle and fixes a vertex of it, then it is an affine reflection with axis the median from that vertex and conjugate direction the opposite side to that vertex.*

Exercise 6. *Show that if an affine reflection leaves invariant a quadrangle, then its axis either coincides with a diagonal or it is the line joining the middles of two opposite sides. Show also that in the first case the conjugate direction is that of the other diagonal, whose middle must lie on the axis. In the second case the quadrangle is a trapezium and the conjugate direction is that of the parallel sides (See Figure 9).*

Consider now the product (composition) of two reflections as these of the preceding section. First f_B^A then f_C^B. The first with axis the median $\varepsilon_1 = CC'$ and the second the

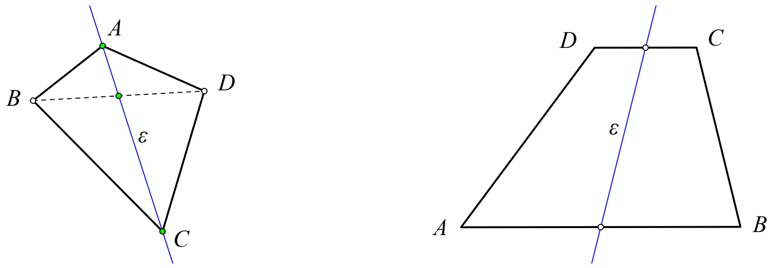

Figure 9: Affine reflections leaving invariant a quadrangle

reflection with axis the median $\varepsilon_2 = AA'$. Their product transformation $f = f_C^B \circ f_B^A$ is an affinity recycling clockwise the vertices $\{A \overset{f}{\mapsto} C \overset{f}{\mapsto} B \overset{f}{\mapsto} A\}$ (See Figure 10). Since each of the factors leaves invariant the triangle, the same happens also with their composition

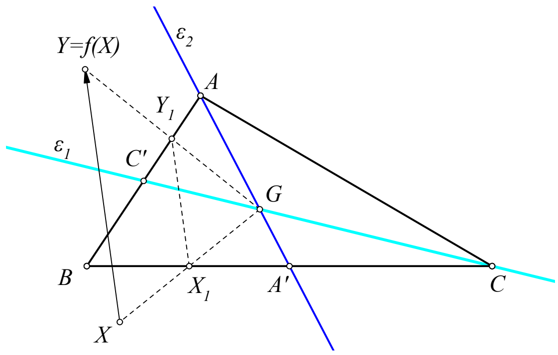

Figure 10: Composition of two reflections

f. Since the centroid G is fixed by each reflection, it is fixed also by their composition. Every other point $X \neq G$ maps to a point $Y = f(X)$ and if the line GX intersects a side, CB say at a point X_1, then this line maps to $f(GX) = GY$, which intersects the image-side BA at a point $f(X_1) = Y_1$. From the discussed properties we deduce easily the relations

$$\frac{X_1C}{X_1B} = \frac{Y_1B}{Y_1A} \quad \text{and} \quad \frac{X_1G}{X_1X} = \frac{Y_1G}{Y_1Y}.$$

From this and similar relations w.r. to the other sides of the triangle we deduce that there is no other fixed point of f.

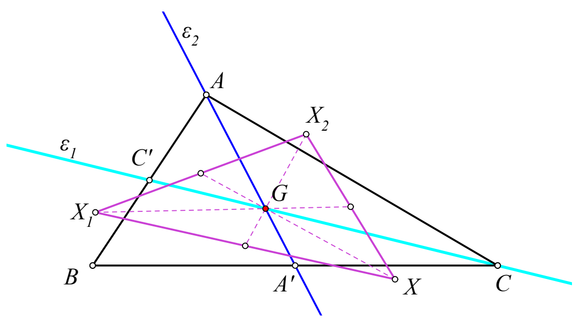

Figure 11: The centroid of triangles with vertices $\{X, X_1 = f(X), X_2 = f^2(X)\}$ is G

affine rotation

Affinities which can be represented as product of two affine reflections and have a unique fixed point, are called *"affine rotations"* and their fixed point is called *"centre"* of the rotation. Next theorem formulates what we proved so far.

Theorem 9. *The product of two affine reflections leaving invariant the triangle ABC is an affine rotation with center at the centroid G of the triangle.*

same centroid

Exercise 7. *Show explicitly that the affine rotation f, defined above, has no other fixed point except G. For the same rotation consider an arbitrary point X and show that $f^3(X) = X$. Show also that the centroid of the triangle with vertices $\{X,\ X_1 = f(X),\ X_2 = f^2(X)\}$ is always the fixed point G of f.*

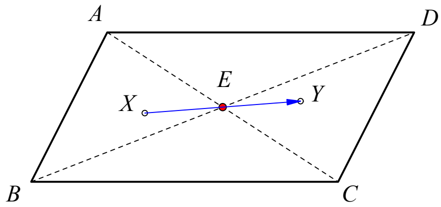

Figure 12: Symmetry at E represented as an affine rotation

symmetry

Exercise 8. *Consider the affine rotation, which is the product of reflections $f = f_2 \circ f_1$, where $\{f_1, f_2\}$ have for axes the diagonals $\{AC, BD\}$ of the parallelogram $ABCD$ and conjugate directions respectively the other diagonals $\{BD, AC\}$. Show that f coincides with the symmetry at the center E of the parallelogram (See Figure 12).*

Exercise 9. *Determine all the affinities leaving invariant a parallelogram and show that they form a group.*

Exercise 10. *Show that a pentagon has in general no affinities preserving it, i.e. mapping it onto itself. Show also that if the pentagon has each diagonal parallel to a side, then it has affinities leaving it invariant. Describe then all these affinities leaving the pentagon invariant.*

7 The period of an affinity

period

The affine reflections, defined in the previous section, have period 2. This means, they satisfy $f^2 = f \circ f = e$, where e represents the identity transformation. More general the *"period"* of a transformation is the smallest integer n, such that $f^n = e$. The transformation $f = f_C^B \circ f_B^A$ of section 6 is of period 3. This can be immediately seen by considering its effect on the vertices, which, as we already noticed, is $\{A \overset{f}{\mapsto} C \overset{f}{\mapsto} B \overset{f}{\mapsto} A\}$. Thus, f^3 leaves the three vertices fixed, hence, by corollary 5, it is the identity.

infinite period

Here we should notice, that not all affinities have a period. It is possible that no n exists, such that $f^n = e$. We say then that f has *"infinite period"*. The product of affine reflections $f = f_C^B \circ f_B^A$ of period 3 is a special one related to a triangle. In general, the product of two reflections is of infinite period. Figure 13 shows such an affine rotation of infinite period. There the affinity f is the product of the reflection f_α with axis α and conjugate direction AB and the reflection f_β with axis β and conjugate direction BC. The sequence of points is $\{X,\ X_1 = f(X),\ X_2 = f(X_1) = f^2(X),\ X_3 = f^3(X),\ \ldots,\ X_n = f^n(X),\ \ldots\}$ which never recurs to the starting point X. In section 13 we will see that all these points making the *"orbit"* of X under the composition of affine reflections $f = f_\beta \circ f_\alpha$ lie on a conic, as suggested by the figure.

period 2 = reflection or symmetry

Theorem 10. *An affinity is of period 2, if and only if it is an affine reflection or a symmetry at a point M.*

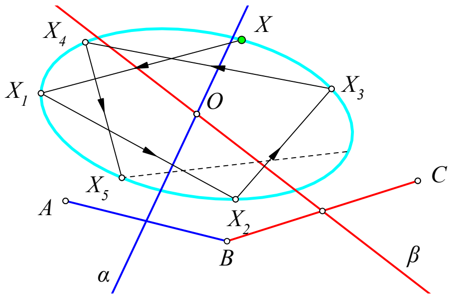

Figure 13: Composition $f = f_\beta \circ f_\alpha$ of two reflections of infinite period

Proof. That an affine reflection is of period 2, follows from the definition of the affine reflection in section 6. To show the converse, consider a point X with $Y = f(X) \neq X$ hence $f(Y) = X$. Then the line $\alpha = XY$ is invariant under f, i.e. $f(\alpha) = \alpha$ and the middle M of the segment XY is fixed by f. Take also another point X', consider again $Y' = f(X')$,

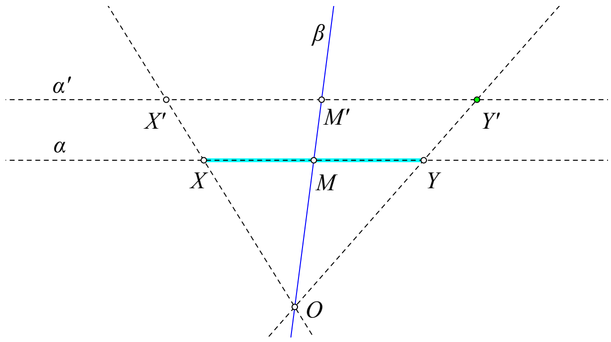

Figure 14: Affinity of period 2 is a reflection

the line $\alpha' = X'Y'$ and the middle M' of $X'Y'$, which again remains fixed under f (See Figure 14). Assume now that $M' \neq M$. Then, by corollary 9 the line $\beta = MM'$ remains fixed under f. For the intersection $O = XX' \cap MM'$ follows then that also $\{Y, Y', O\}$ are collinear and from Thales' theorem follows that lines $\{\alpha, \alpha'\}$ are parallel, thereby proving the reflective property of the transformation. The reflection's axis is line β and the conjugate direction is that of the line $\alpha = XX'$ for an arbitrary point X not lying on β.

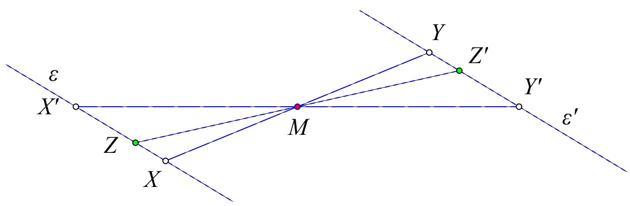

Figure 15: A point symmetry is an affinity of period 2

In the case $M = M'$ we have the situation of figure 15, in which $XX'YY'$ is a parallelogram and M is its center and line $\varepsilon = XX'$ maps under f onto $\varepsilon' = YY'$. By the preservation of ratios it is seen that points $Z \in \varepsilon$ maps to its symmetric $Z' \in \varepsilon'$. Using again the preservation of ratios, we see easily that each point on MZ maps to its symmetric w.r.t. to M. This proves that f is the symmetry at M. □

half-turns

Remark 4. Symmetries about a fixed point M are often called *"half-turns"* about M.

Theorem 11. *An affinity is of period 3, if and only if it is an affine rotation with axes two medians of a triangle and the corresponding sides as conjugate directions.*

Proof. That an affinity with the above properties is an affine rotation has been established in section 6. To show the converse, consider a point A with $C = f(A) \neq A$ and $f(C) = B$, having necessarily $f(B) = A$. Assume for the moment that $\{A, B, C\}$ are not collinear and form a genuine triangle ABC. In this triangle consider the reflections $\{f_B^A, f_C^B\}$ and their product $f' = f_C^B \circ f_B^A$, defined as in the preceding section. The two affinities f and f' recycle the vertices in the same way $\{A \mapsto C \mapsto B \mapsto A\}$ (See Figure 16). Thus, by corol-

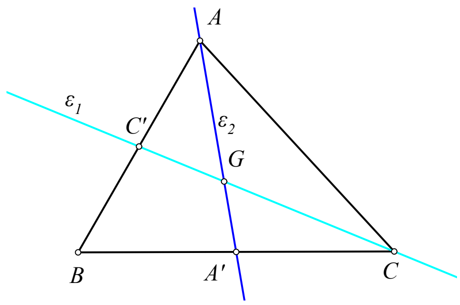

Figure 16: Affinity of period 3 is a rotation

lary 6, they coincide everywhere, thereby proving the theorem.

It remains to show that there is a point A, such that $\{A, C = f(A), B = f(C)\}$ are not collinear. To see this we work with the matrices F_v representing the affinity f as explained in the first two sections. The collinearity of $\{A, f(A), f^2(A)\}$ means for the corresponding matrix representation, the existence of a number λ such that:

$$\begin{aligned}
F_v^2 \cdot A \quad &= \lambda A + (1-\lambda)F_v \cdot A \quad \Rightarrow \\
A = F_v^3 \cdot A \quad &= \lambda F_v \cdot A + (1-\lambda)F_v^2 \cdot A \\
&= \lambda F_v \cdot A + (1-\lambda)[\lambda A + (1-\lambda)F_v \cdot A] \quad \Rightarrow \\
(\lambda^2 - \lambda + 1)A &= (\lambda^2 - \lambda + 1)F_v \cdot A.
\end{aligned}$$

Since the factor never vanishes, last equation is equivalent with $F_v \cdot A = A$ and if this is valid for every A, then F_v is the identity matrix, which is contrary to the assumption. □

8 Translations

Theorem 12. *The product $f = f_2 \circ f_1$ of two reflections $\{f_1, f_2\}$ with parallel axes $\{\varepsilon_1, \varepsilon_2\}$ and common conjugate direction β is a "translation" i.e. a map defined by a fixed vector v, which to every point P corresponds the point $P' = P + v$.*

Proof. The proof follows from figure 17. For an arbitrary point A and $f_1(A) = B, f_2(B) = C$ the segment AC is the double of DE in the common conjugate direction β. This defines a vector of constant direction and constant length, hence proves the claim. □

Remark 5. Notice that the relevant ingredient in the representation of the translation as a product of reflections is the *"conjugate direction"* β. In fact we can *"turn"* the two parallels$\{\varepsilon_1, \varepsilon_2\}$ to a new position taking care only that their distance in the β-direction remains the same and equal to half the measure of v. Then the translation can be again represented as a product of the two reflections w.r. to the two parallel lines in their new position. Thus, turning v_1 about D in the new position ε_1' and turning also ε_2 about E

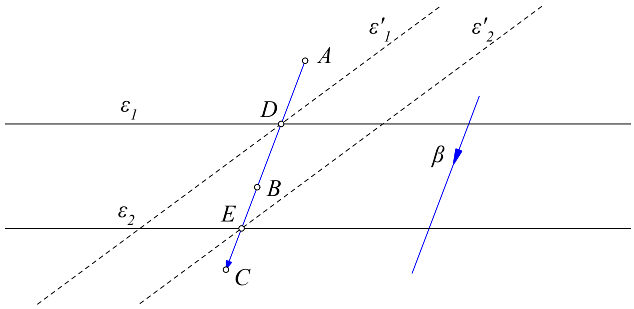

Figure 17: Translation as a product of two affine reflections

in the new position ε'_2 so that it is parallel to ε'_1, we obtain two other affine reflections $\{f_{\varepsilon'_1,\beta}, f_{\varepsilon'_2,\beta}\}$, whose composition is the same translation $f = f_{\varepsilon'_2,\beta} \circ f_{\varepsilon'_1,\beta}$ by the double v of the vector DE.

Remark 6. Translations build a *normal* subgroup of the group of all affinities of the plane. The composition $f = f_2 \circ f_1$ of two translations by the vectors $\{v_1, v_2\}$ is the translation by the vector $v_1 + v_2$. This is not so for the set of reflections. The product of two reflections is not a reflection. The previous representation of a translation as a product of two reflections gives a counterexample.

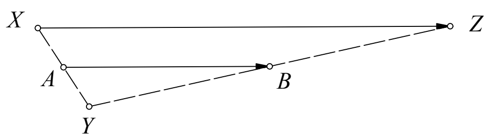

Figure 18: Translation as composition of two symmetries

Exercise 11. *Show that the composition $S_B \circ S_A$ of two symmetries w.r.t. to the points $\{A, B\}$ is the translation by the vector $v = 2AB$ (See Figure 18).*

9 Dilatations

dilatation

central dilatation

Translations are a particular case of *"dilatations"*. Latter are affinities characterized by the property to *map every line α to a line α' parallel to α*. A dilatation leaving at least one point fixed is called a *"central dilatation"*. The *symmetry* or *half-turn* about a point O is obviously a central dilatation.

Theorem 13. *A central dilatation f, different from the identity, has precisely one fixed point O, called "center" and there is a constant k, called "ratio" of the dilatation, such that for every other point $P \neq O$ it is*

$$f(P) = O + k \cdot (P - O).$$

Proof. Assume first that besides the fixed point O there is a second one O' remaining also fixed under f. Consider also a line ε through O. Since, per definition, ε maps to a parallel $\varepsilon' = f(\varepsilon)$ containing also O, line ε remains invariant under f : $f(\varepsilon) = \varepsilon$. If a point $P \in \varepsilon$ were fixed by f, then we would have three fixed points and f would be the identity transformation, contrary to the hypothesis. Thus, $P' = f(P) \neq P$ and the

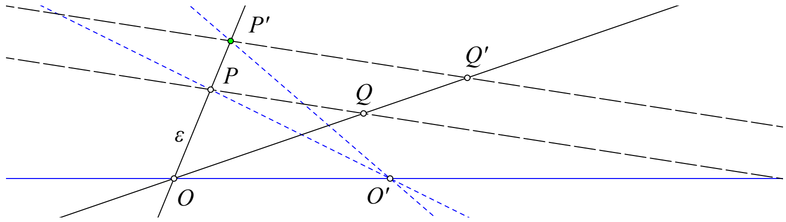

Figure 19: Central dilatation

lines $\{O'P, O'P'\}$ have the property $f(O'P) = O'P'$ without to be parallel, contrary to the hypothesis for f.

Having that, the claim about the constant k is verified by selecting another point Q and its image $Q' = f(Q)$, which, by the previous reasoning is on line OQ. By assumption the line PQ maps then to a parallel $P'Q'$ and by Thales' theorem $OP'/OP = OQ'/OQ$, completing the proof. □

Exercise 12. *If f is a dilatation and for a point $X, Y = f(X)$, show that the line $\varepsilon = XY$ is invariant under f i.e. $f(\varepsilon) \subset \varepsilon$.*

Remark 7. Central dilatations coincide with usual euclidean *"homotheties"* and half-turns or symmetries about a point O of the plane are dilatations with ratio $k = -1$.

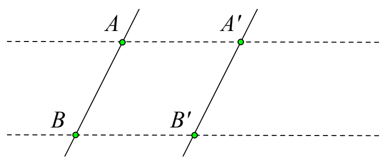

Figure 20: Non-central dilatation is a translation

Exercise 13. *Show that a non-central dilatation maps two points $\{A, B\}$ to $\{A', B'\}$, such that $ABB'A'$ is a parallelogram and therefore is a translation and has no fixed points.*

Exercise 14. *Show that a dilatation is completely determined by prescribing the images $\{f(A), f(B)\}$ of two points $\{A, B\}$.*

Exercise 15. *Show that the dilatations build a normal subgroup of the group of all affinities of the plane. Show also that the translations build a normal subgroup of the group of all dilatations. If T_v denotes the translation by the vector v and g is a dilatation, what kind of affinity is $g^{-1} \circ T_v \circ g$?*

10 Shears

shear

Shears are products $s = g \circ f$ of two affine reflexions $\{f, g\}$ whose *"mirrors"* are identical. Figure 21 shows the image $s(P) = g(f(P))$ of a point P under such a composition of transformations. Line ε is the common mirror of the two reflections. The segment AB defines the conjugate direction of f and segment BC defines the conjugate direction of the affine reflection g. A fundamental property of this transformation is that the points of the mirror ε are left fixed by the shear and that the triangle $PP'P''$ with vertices $\{P, P' = f(P), P'' = g(P')\}$ has fixed angles. This implies that all these triangles resulting for various positions of P are similar to each other. Thus, also the direction of the median MP'' is fixed and all triangles PMP'' are also similar to each other. This enables a quick construction of P'' once the direction of the median MP″ has been determined:

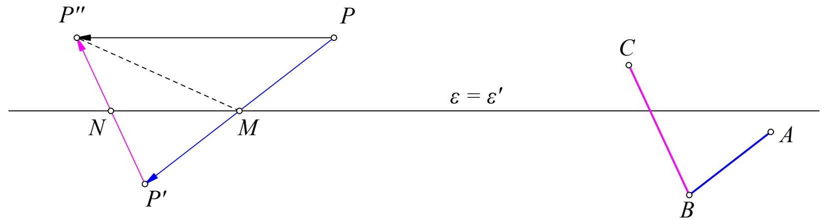

Figure 21: Shear as a composition of two reflections

1. Project P on the first mirror L along the conjugate direction L′ of f to the point M.
2. Draw from M parallel to the direction of the median and find its intersection P″ with the parallel from P to the mirror L.

This implies immediately that each parallel to ε is left invariant by the shear which, restricted there, coincides with a translation by a vector of fixed length, depending only of the distance of the parallel from ε. Thus, moving P on the parallel to ε through that point gives a point $P'' = s(P)$ such that PP'' is parallel to ε and has a constant signed measure, which is the double of that of MN.

Notice that in the product representing the shear $s = g \circ f$ the order plays a role and $f \circ g$ is different from s, latter defining the *"inverse"* shear $s^{-1} = f \circ g$.

From the similarity of all triangles $PP'P''$ and all triangles PMP'' follows that the translation vector effecting the shear along the parallel to ε from P is a constant multiple of the distance of P from ε. This implies that in a coordinate system with x-axis identical with ε, the shear is described by a constant k and a matrix in the form:

$$\left.\begin{array}{lcl} x' & = & x + k \cdot y, \\ y' & = & y. \end{array}\right\} \quad \Leftrightarrow \quad X' = AX \quad \text{with} \quad A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Exercise 16. *Show that two reflections with axes intersecting at a point P and the same conjugate direction u define a shear with axis parallel to u and passing through P.*

Exercise 17. *Given a shear f, write it as a composition $f = h \circ g$ of two reflections. How many possibilities are there for this?*

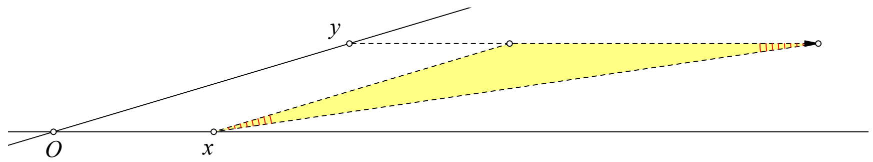

Figure 22: Shear's *"canonical"* representation

Exercise 18. *Given a shear f, show that selecting for x-axis its axis and an appropriate transversal to it y-axis (See Figure 22), the transformation can be represented by the matrix*

$$f \ : \ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

11 Strains

strain

A *"strain"* is a transformation f somewhat more general than an affine reflection. It can be defined by describing three points and their images (theorem 4). For the first two points $\{A, B\}$ we assume to remain fixed by f, so that the whole line $\varepsilon = AB$ remains also fixed by f (See Figure 23). For the third point and its image $\{P, P' = f(P)\}$ we assume that their line is not parallel to $\varepsilon = AB$. To describe the effect of f on any other point X

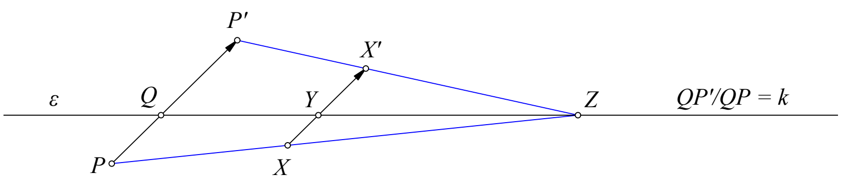

Figure 23: Strain's axis ε and ratio k

of the plane, consider the intersection point Q of line PP' with ε. Notice first that, by the preservation of ratios, line PP' is invariant by f, i.e. it maps onto itself. Then, for any point X and its image $X' = f(X)$ draw the parallel XY to PP' intersecting ε at Y. By the preservation of parallels by affinities and the constancy of Y follows that line XY is also left invariant by f. Hence X' is on XY. Using Thales' theorem is then easy to see that PX and $P'X'$ intersect at a point Z of ε and $X'Y/XY = QP'/QP$ is constant. Thus the image X' is found in this case by a simple recipe:

1. Draw from X a parallel to PP'.
2. Take X' on this parallel so that $YX'/YX = QP'/QP$, where Y is the intersection point of XX' with ε.

12 Invariant pencils

pencils

pencils invariant under affinities

A *"pencil"* of lines consists of all the lines passing through a point P, real or at infinity, called *"center"* of the pencil. If P is at infinity, then the pencil consists of all the lines parallel to a given direction. As remarked by Coxeter [Cox67, p.17] in his marvellous talk, for every affinity which is a *shear, strain, translation* or *central dilatation* there is a pencil of lines, the members of which remain invariant under the affinity. A shear leaves invariant all lines parallel to its axis. A strain leaves invariant all lines parallel to its conjugate axis. A translation leaves invariant all lines parallel to the translation vector and a central dilatation leaves invariant all lines through its center. Next theorem uses this property to characterize axial symmetries and dilatations.

Theorem 14. *An affinity f, for which there is a pencil of lines $\mathcal{P}$, such that each member-line ε of $\mathcal{P}$ remains invariant, is an axial affinity or a dilatation and vice versa.*

Proof. We distinguish three cases: 1) f has no real fixed points. 2) f has precisely one fixed point. 3) f has at least two fixed points.
1) If f has no real fixed point, then two invariant lines $\{PP', QQ'\}$ cannot intersect, since their common point would be a fixed point for f. Thus, pencil $\mathcal{P}$ consists of parallel lines. Take then points $\{P, Q\}$ (See Figure 24) on two such lines and their images $\{P', Q'\}$ contained in the same lines. Then, $\{PQ, P'Q'\}$ cannot intersect at a finite point S. This, because then, by Thales' theorem, the ratios $QS/QP = Q'S/Q'P'$ would imply that S is a fixed point of f, contrary to the hypothesis. Thus, in this case $PP'Q'Q$ is a parallelogram and the transformation is a translation.

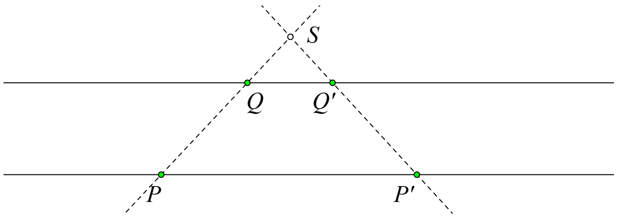

Figure 24: No fixed point for translation

2) If f has one only fixed point, then the lines of the invariant pencil $\mathcal{P}$ must go through that point. This because if they didn't, then they would be parallel, since the intersection of two invariant lines is a fixed point contained in both lines. Assuming then that the

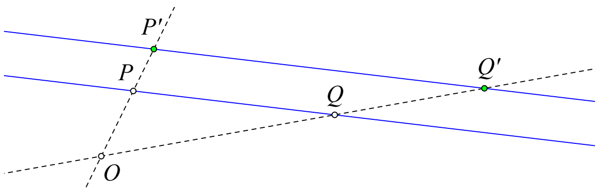

Figure 25: One only fixed point for central dilatation

invariant lines are parallel and do not pass through the fixed point, take points $\{P, P'\}$ on two of them (See Figure 25) and their images $\{Q, Q'\}$ contained each in the same invariant line. Then applying again Thales' theorem, we see that the intersection $O = PP' \cap QQ'$ is a real fixed point, except the case in which O is at infinity. Later though cannot happen, since then, as is easily seen, f would be a translation, without real fixed points. Thus O must be a real point and varying the locations of $\{P, P'\}$ on the same lines we obtain infinite many fixed points $\{O\}$, contrary to the hypothesis. Thus, in any case, assuming that the invariant lines do not pass through the fixed point we land to contradiction. Now assuming that the invariant lines pass through the same real point O and arguing as in theorem 13, we see that f is a central dilatation.
3) If f has at least two real fixed points, then the line of these points is fixed and the result follows from theorem 7. □

13 Equiaffinities as products of reflections

Equiaffinities are affinities preserving the area of triangles. By theorem 14, an equiaffinity preserving a pencil of lines must be a shear or half-turn or a translation, which, by the discussion so far, are affinities representable as a product of two affine reflections. Next theorem, whose proof I take from Coxeter's lecture [Cox67, p. 17], guarantees this property for *every* equiaffinity.

Theorem 15. *Every equiaffinity is the product of two affine reflections.*

Proof. For equiaffinities f preserving a pencil of lines, this results from the previous remarks. If f does not preserve any pencil of lines, then there is a triangle ABC with $\{B = f(A), C = f(B)\}$ (See Figure 26). For point $D = f(C)$ it follows that the triangles $\{ABC, BCD\}$ have the same area and a common base, hence their vertices $\{A, D\}$ are on a parallel to BC. If $\{M, N\}$ are the middles of $\{AD, BC\}$ consider the affine reflections

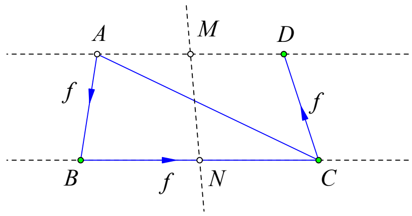

Figure 26: Equiaffinities are products of affine reflections

$\{g = M(BC),\ h = C(BD)\}$ (see section 6 for the notation):

$$A \overset{g}{\mapsto} D \overset{h}{\mapsto} B, \qquad B \overset{g}{\mapsto} C \overset{h}{\mapsto} C, \qquad C \overset{g}{\mapsto} B \overset{h}{\mapsto} D.$$

Thus, the equiaffinity f coincides with $h \circ g$ at three non collinear points, hence it coincides with it everywhere and $f = h \circ g$, thereby proving the theorem. □

Since the composition of an even number of affine reflections is an equiaffinity, it can be refactored to a product of two reflections and this proves next corollary.

Corollary 19. *A product of n affine reflections can be refactored to a product of two, if n is even, or three affine reflections, if n is odd.*

Exercise 19. *Show that an affinity of finite period $n > 2$ is necessarily an equiaffinity.*

14 Conjugacy for affinities with fixed points

"Conjugate" are called the affinities $\{f, f'\}$ for which there is a third affinity g, such that

$$g \circ f \circ g^{-1} = f'.$$

Conjugate affinities have the same geometric properties and, converesly, affinities with the same geometric properties are conjugate. Writing the relation with corresponding matrices, as in equation 2,

$$g : \left(\begin{array}{c|c} G & h \\ \hline 0 & 1 \end{array}\right), \quad f : \left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right), \quad f' : \left(\begin{array}{c|c} A' & v' \\ \hline 0 & 1 \end{array}\right),$$

the conjugacy condition takes the form

$$\left(\begin{array}{c|c} G & h \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} G^{-1} & -G^{-1}h \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A' & v' \\ \hline 0 & 1 \end{array}\right) \quad \Leftrightarrow$$

$$\left(\begin{array}{c|c} GAG^{-1} & -GAG^{-1}h + Gv + h \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A' & v' \\ \hline 0 & 1 \end{array}\right) \quad \Leftrightarrow$$

$$(1)\ GAG^{-1} = A' \quad \text{and} \quad (2)\ GAG^{-1}h - h = Gv - v'. \tag{9}$$

Thus, if the affinities are conjugate, then the matrix parts are also conjugate, but the converse is not valid in general, since the second condition is not necessarily a consequence of the first. Anyway, if the first condition holds, then the second can be written:

$$A'h - h = Gv - v'. \tag{10}$$

If the affinity f has no fixed point, then selecting a point X and its image $Y = f(X)$ and composing $f' = T_v \circ f$ with the translation by the vector $v = YX$ we get the affinity f', which has X as fixed point. Taking the fixed point as origin of coordinates, the affinity reduces to a representation in the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right),$$

and the respective conjugacy classes correspond to the conjugacy classes of the matrix A. Thus, the *"classification"* of affinities fixing a point results from the conjugate classes of the matrices A. These, in turn, are distinguished by the corresponding eigenvalues and their multiplicities. If the eigenvalues are (conjugate) complex, i.e. the characteristic polynomial of A

$$p(x) \;=\; x^2 + bx + c,$$

has $b^2 < 4c$ and *"complex roots"*. Then, selecting an arbitrary vector $u \neq 0$ and taking the basis of $\{u, u' = Au\}$, we have

$$Au \;=\; u', \quad Au' = A^2 u = -bAu - cu,$$

since A satisfies its characteristic polynomial $A^2 + bA + cI \;=\; 0$. Hence, w.r. to the basis $\{u, u'\}$ the matrix A is conjugate to

$$A' \;=\; \begin{pmatrix} 0 & -c \\ 1 & -b \end{pmatrix}.$$

If the eigenvalues are real and distinct $\{a, b\}$, then, changing to the basis of two eigenvectors, the matrix A is conjugate to the diagonal one

$$A' \;=\; \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;=\; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$

representing a product of *"strains"* for $a, b \notin \{0, 1\}$.

If the characteristic polynomial has a double root, then in the case of one only independent eigenvector, there is a basis such that A is conjugate to

$$A' \;=\; \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \;=\; a\begin{pmatrix} 1 & 1/a \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad a \neq 0.$$

This, since then there is a vector $v : (A - aI)v = v' \neq 0$. This, since $(A - aI)^2 v = 0$ for every vector v, implies that in the basis of $\{v', v\}$ the matrix has the previous representation. In this case the affinity is the product of a shear and a homothety if $a \notin \{1, 0\}$ and a simple shear if $a = 1$.

In the case of multiplicity two, with two independent eigenvectors, A is conjugate to the diagonal matrix representing a *"homothety"* for $a \neq 1$.

$$A' \;=\; \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

15 Elliptic affinities

"Elliptic" are called the affinities f fixing a point and, by the discussion in section 14, having the representation resulting from a characteristic polynomial with *complex roots* $\chi_A(x) \;=\; x^2 + bx + c$, with $b^2 < 4c$:

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right) \quad \text{with} \quad A \;=\; \begin{pmatrix} 0 & -c \\ 1 & -b \end{pmatrix}.$$

The eigenvalues of A are in this case complex-conjugate:

$$\alpha = \mu + i\mu', \quad \beta = \mu - i\mu' \quad \text{with} \quad \mu = -\frac{b}{2} \quad \text{and} \quad \mu' = \frac{\sqrt{4c - b^2}}{2}$$

leading to two complex conjugate eigenvectors $\{u_\alpha = v + iw, \quad u_\beta = v - iw\}$ and

$$Au_\alpha = \alpha \cdot u_\alpha, \quad Au_\beta = \beta \cdot u_\beta \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} Av &=& \mu \cdot v \; - \; \mu' \cdot w, \\ Aw &=& \mu' \cdot v \; + \; \mu \cdot w. \end{array} \right\}$$

Taking into account $\mu^2 + \mu'^2 = d^2 = c > 0$ and setting

$$\frac{\mu}{d} = \cos(\phi), \quad \frac{\mu'}{d} = \sin(\phi) \quad \Rightarrow \quad \left\{ \begin{array}{lcl} Av &=& d(\cos(\phi) \cdot v \; - \; \sin(\phi) \cdot w), \\ Aw &=& d(\sin(\phi) \cdot v \; + \; \cos(\phi) \cdot w), \end{array} \right\}$$

we see that the affinity in this case is the composition $f = h \circ g$ of an *equiaffinity* g and a *homothety* h. These two affinities are represented in the frame of $\{v, w\}$ by matrices $\{C, B\}$ and their composition by the product of matrices

$$A \;=\; B \cdot C \;=\; \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}.$$

Denoting by U the affine transformation fixing the origin and mapping the standard basis $\{e_1, e_2\}$ to $\{v, w\}$, we see that the equiaffinity g is conjugate to a rotation R_ϕ, since $\{U, R_\phi\}$ can be identified with the matrices

$$U \;=\; \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}, \quad R_\phi = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

and g is represented w.r. to the standard base $\{e_1, e_2\}$ by the matrix

$$G \;=\; U \cdot R_\phi \cdot U^{-1}.$$

It is easy to see that G leaves invariant the homothetic w.r. to the origin ellipses $\{U(\kappa_r)\}$,

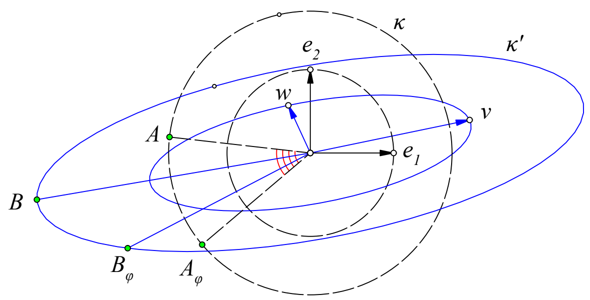

Figure 27: Equiaffinity leaving invariant a family of homothetic ellipses

where κ_r denotes the circle $x^2 + y^2 = r^2$ (See Figure 27), mapping the orthogonal directions $\{e_1, e_2\}$ to two conjugate directions $\{v, w\}$ w.r. to the ellipses. The figure shows also the images on the ellipse $U(\kappa)$

$$B = U(A) \quad \text{and} \quad B_\phi \;=\; U(A_\phi) \;=\; UR_\phi(A) \;=\; UR_\phi U^{-1}(B) \;=\; G(B).$$

of two points $\{A, A_\phi = R_\phi(A)\}$ on the circle κ.

Remark 8. The claim about the conjugate diameters is a simple consequence of the preservation of middles of segments by affinities. Since in the circle two orthogonal directions define diameters in the one direction whose middles are on a diameter parallel to the second direction, the same property will hold with the images of these two families of diameters by the affinities.

16 Orbits of points under equiaffinities

In this and the following sections I study Coxeter's ([Cox67, p.18]) classification of equiaffinities, which he started with the following comment:

> *"Until two weeks ago, I thought the complete classification of equiaffinities might be a rather awkward problem, but then (while sitting on a bench in Leicester Square, London) I saw something very simple which I should have thought of long ago, ... "*

equiaffinity invariant $\lambda = \frac{AD}{BC}$

What Coxeter saw is the fact, used in section 13, that for every affinity f, not possessing a pencil of invariant lines (section 12), there is a point A, such that $\{A,\ B = f(A),\ C = f(B)\}$ are non-collinear, and in the case of *"equiaffinities"* the point $D = f(C)$ is then on the parallel ε to line BC from A (See Figure 28). Then the equiaffinity is characterized by

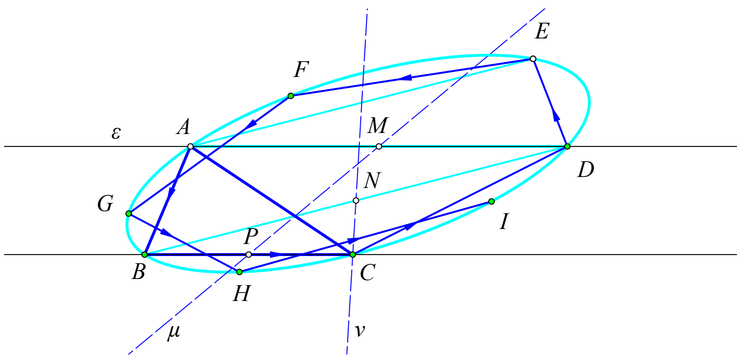

Figure 28: The orbit of point A lies on a conic

the ratio

$$\lambda = \frac{AD}{BC} \tag{11}$$

orbit

and the *"orbit"* under the action of f of a point A of the plane:

$$A,\ \ B = f(A),\ \ C = f^2(A),\ \ D = f^3(A),\ \ E = f^4(A),\ \ldots \tag{12}$$

lies on a conic, which, depending on the value of λ, can be an ellipse, parabola, hyperbola or a degenerate one. In the non-degenerate cases, the equiaffinity is correspondingly named *"elliptic, parabolic, hyperbolic, crossed-hyperbolic"*, the last class relating to orbits that have their points alternatively on a hyperbola and its conjugate. In the degenerate cases the equiaffinity is called *"focal"* and the points of the corresponding orbit, as will be seen below, alternate on two parallel lines.

In section 15, using analytic arguments we saw an *"elliptic"* case. The geometric argument applicable to any case and used by Coxeter is based on the fact that the orbit $ABCD$..., which, in general, is an infinite polygon, is invariant not only under f, but also under the affine reflections $\{g = M(BC), h = C(BD)\}$ whose axes are the lines $\{\mu, \nu\}$ and which, by theorem 15, represent $f = h \circ g$. This is inductively seen by first noticing that $\{h(E) = A \Leftrightarrow h(A) = E\}$, which follows from (See Figure 28):

$$h(A) = E \quad \Leftrightarrow \quad h(A) = f(D) = h(g(D)) \quad \Leftrightarrow \quad A = g(D),$$

implying that $\{g, h\}$ map the first four points of the orbit to points of the same:

$$\begin{aligned} g(A) &= D = f(C), & h(A) &= E = f(D), \\ g(B) &= C = f(B), & h(B) &= D = f(C), \\ g(C) &= B = f(A), & h(C) &= C, \\ g(D) &= A, & h(D) &= B. \end{aligned}$$

Then, the first six points of the orbit map under $\{g, h\}$ to points of the same:

$$g(E) = g(h(A)) = f^{-1}(A) \quad \Rightarrow \quad g(F) = g(f(E)) = g(h(g(E)) = g(h(f^{-1}(A)) = f^{-2}(A),$$
$$\Rightarrow \quad h(F) = h(g(f^{-2}(A)) = f(f^{-2}(A)) = f^{-1}(A), \ \ldots \quad \text{and so on.}$$

To show that the orbit is contained in a conic, consider the conic κ passing through $\{A, B, C\}$ and having for diameters the lines $\{\mu, \nu\}$ (See Figure 28). By the assumed invariance of κ w.r. to the affine reflections $\{g, h\}$, the conic can be seen inductively to contain also all other points of the orbit:

$$D = g(A), \ E = h(A), \ F = f(E) = h(g(E)) = h(g(h(A)), \ \ldots \quad \text{and so on.}$$

We summarize these facts in the form of a theorem:

Theorem 16. *The orbit $\{f^n(A),\ n \in \mathbb{Z}\}$, under the equiaffinity f, for every point A of the plane, is invariant under f and also invariant under the affine reflections $\{g, h\}$ representing $f = h \circ g$. It is also contained in a conic κ, invariant under f.*

17 Classification of equiaffinities

Continuing the discussion of the previous section, we examine how the kind of an orbit $\{f^n(A),\ n \in \mathbb{Z}\}$, of a point under an equiaffinity and the kind of the conic κ containing such an orbit is influenced by the value of the signed-ratio invariant $\lambda = AD/BC$. Drawing the parallels $\{MQ, DD'\}$ to AB from $\{M, D\}$ we see that the ratio

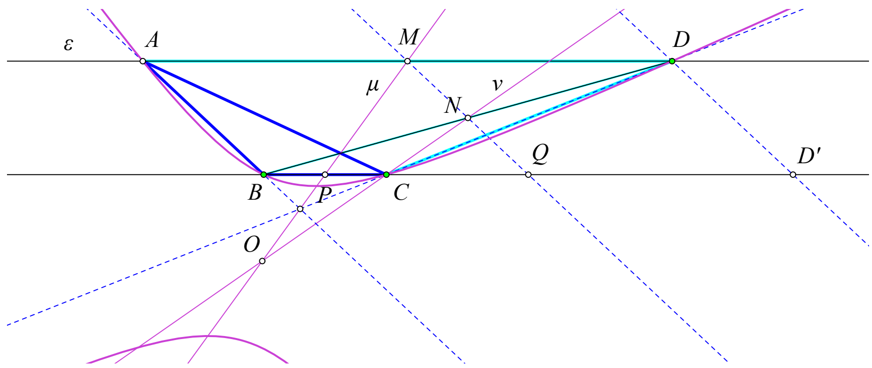

Figure 29: Center O of the conic carrying the orbit. $\lambda > 3$: hyperbolic rotation

$$\lambda = \frac{AD}{BC} = \frac{BD'}{BC} = \frac{2BQ}{2BP} = \frac{BQ}{BP}.$$

Since $\{M, N\}$ are *diameters* of the conic κ containing the orbit of A, the intersection point $O = \mu \cap \nu$ is the center of κ. The location of O depends on λ and figure 29 shows the

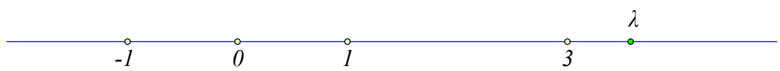

Figure 30: Location of the invariant λ

hyperbolic rotation

case $\lambda > 3$. In this the center O is on the other than A side of BC, κ is a hyperbola containing the entire orbit in one of its branches and the equiaffinity is called *"hyperbolic*

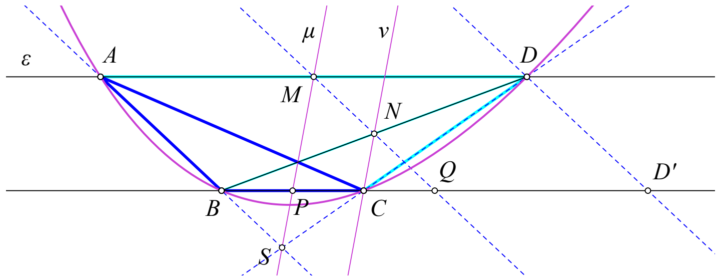

Figure 31: Center O of κ at infinity. $\lambda = 3$: parabolic rotation

rotation". In the case $\lambda = 3$ it is easily seen that the diameters $\{\mu, \nu\}$ are parallel, hence the conic carrying the orbit is a parabola and the equiaffinity is called *"parabolic rotation"* (See Figure 31).

parabolic rotation

Exercise 20. *Show that the orbits of hyperbolic and parabolic rotations have infinite many points.*

In the case $-1 < \lambda < 3$ the two diameters $\{\mu, \nu\}$ intersect on the side of BC containing the points $\{A, D\}$, the conic is an ellipse and the affinity is called *"elliptic rotation"* (See Figure 32).

elliptic rotation

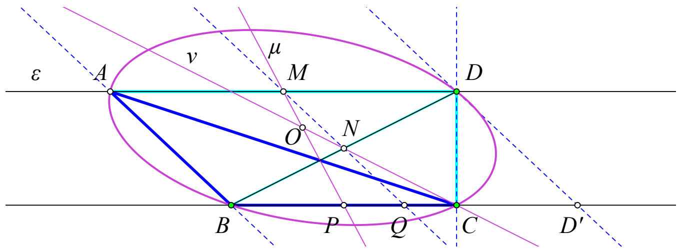

Figure 32: Invariant $-1 < \lambda < 3$: elliptic rotation

Remark 9. Notice that the affinities of finite period are *elliptic rotations*, since they are equiaffinities (see exercise 19) and their orbit is finite (see exercise 20).

Exercise 21. *Show that an equiaffinity f is of period 3 (theorem 11) precisely when $\lambda = 0$.*

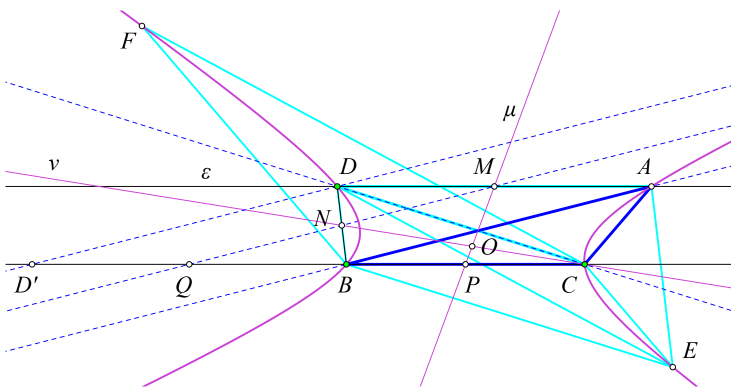

Figure 33: Invariant $\lambda < -1$: crossed hyperbolic rotation

In the case $\lambda < -1$ the segments $\{BC, AD\}$ are inversely oriented and, by the discussion in the previous section, $ADBE$ is a trapezium and using the value of λ we see that

$\{ACE, BD\}$ are a triangle, respectively a segment, inscribed in the conic with C lying in the interior of the trapezium. This can happen only if the conic is a hyperbola and the two shapes are on different branches. Inductively, using the value of λ by means of the analogous pairs $\{f^n(ACE), f^n(BD), n \in \mathbb{Z}\}$, we see that the points of the orbit lie alternatively on the two branches of a hyperbola (See Figure 33). The equiaffinity in this case is called *"crossed hyperbolic rotation"*.

crossed hyperbolic rotation

The case left $\lambda = -1$, leads to a degenerate conic of two parallel lines. In fact, it is then easy to see that the orbit points lie alternatively on the two side-lines $\{BD, AC\}$ of the parallelogram $ADBC$ (See Figure 34). The centers $\{O, P\}$ of the parallelogram, respectively of side BC are interchanged by f which is called *"focal rotation"*. On each line the orbit points are evenly distributed and successive points on each are at distance $2|OP|$

focal rotation

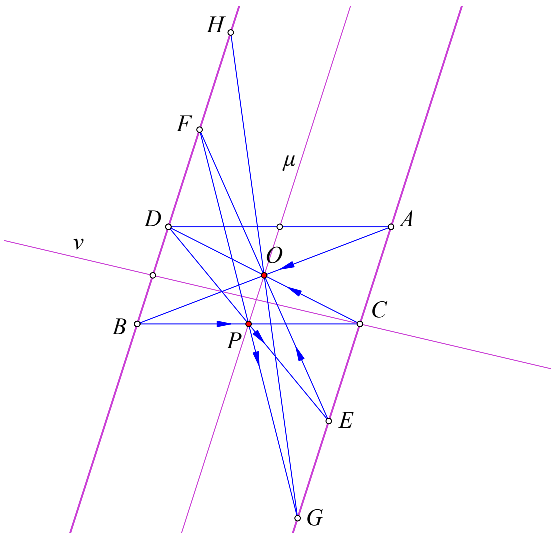

Figure 34: Invariant $\lambda = -1$: focal rotation

proceeding in opposite directions.

Thus, the equiaffinities, which do not preserve a pencil of lines (see section 12), quoting Coxeter ([Cox67, p.23]):

> "To sum up, the equiaffinities, other than translations, half-turns, and shears, may be classified into the following types:"

$\lambda < -1$	$\lambda = -1$	$-1 < \lambda < 3$	$\lambda = 3$	$\lambda > 3$
crossed hyperbolic	*focal*	*elliptic*	*Parabolic*	*Hyperbolic*

Figure 35: Classification of equiaffinities not preserving a pencil of lines

18 Analytic description of orbits of equiaffinities

For the analytic description of the orbits we can use the fact that affinities do not distinguish the conics of the *"same kind"* (see section 20). Thus, all ellipses are *affinely equivalent* to the circle represented in appropriate coordinates by $x^2 + y^2 = 1$, all hyperbolas are affinely equivalent to the hyperbola $xy = 1$, and all parabolas are equivalent to the one represented by $y = x^2$. Using such an affinity the various *"affine rotations"* are equivalent, i.e. conjugate $f = g \circ f' \circ g^{-1}$ by an affinity g, to standard affinities described by:

elliptic rotation

1. *"Elliptic rotations"* corresponding to f' : rotation by an angle ϕ

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{13}$$

the orbit consisting of the points

$$(1,0)\ ,\ (\cos\phi, \sin\phi)\ ,\ (\cos 2\phi, \sin 2\phi)\ ,\ (\cos 3\phi, \sin 3\phi)\,,\ \ldots$$

the ratio λ being given, as expected $-1 < \lambda < 3$, since by the figure 36 and the well known identity $\sin(3\phi) = 3\sin(\phi) - 4\sin^3\phi$ ([Hob18, p.52]):

$$\lambda = \frac{AD}{BC} = \frac{\sin 3\phi}{\sin 2\phi - \sin\phi} = \frac{3 - 4\sin^2\phi}{2\cos\phi - 1} = 2\cos\phi + 1.$$

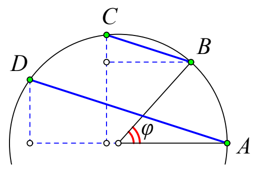

Figure 36: $\lambda = \frac{AD}{BC}$

hyperbolic rotation

2. *"Hyperbolic rotations"* corresponding to f' : hyperbolic rotation by t

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{14}$$

the orbit consisting of the points

$$(1,1)\ ,\ (e^t, e^{-t})\ ,\ (e^{2t}, e^{-2t})\ ,\ (e^{3t}, e^{-3t})\,,\ \ldots$$

and λ having the value

$$\lambda = \frac{AD}{BC} = \frac{e^{3t} - 1}{e^{2t} - e^t} = \frac{e^{2t} + e^t + 1}{e^t} = 2\cosh t + 1 > 3.$$

parabolic rotation

3. *"Parabolic rotations"* corresponding to f' given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x + 1 \\ 2x + y + 1 \end{pmatrix}, \tag{15}$$

the orbit consisting of the points

$$(0,0)\ ,\ (1,1)\ ,\ (2,4)\ ,\ (3,9)\,,\ \ldots$$

and λ having the value

$$\lambda = \frac{9 - 0}{4 - 1} = 3.$$

crossed hyperbolic rotation

4. *"Crossed hyperbolic rotations"* corresponding to f' given by the commutative product of the hyperbolic rotation and the half-turn $(x', y') = (-x, -y)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -e^t & 0 \\ 0 & -e^{-t} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{16}$$

the orbit consisting of the points

$$(1,1)\ ,\ (-e^t,-e^{-t})\ ,\ (e^{2t},e^{-2t})\ ,\ (-e^{3t},-e^{-3t}),\ \ldots$$

and λ having the value

$$\lambda = \frac{AD}{BC} = \frac{-e^{3t}-1}{e^{2t}+e^t} = -\frac{e^{2t}-e^t+1}{e^t} = 1-2\cosh t < -1.$$

focal rotation

5. *"Focal rotations"* corresponding to f' given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{17}$$

with focal points $\{P, O = (\pm\frac{1}{2}, 0)\}$ and orbit consisting of the points

$$(0,1)\ ,\ (-1,-1)\ ,\ (2,1)\ ,\ (-3,-1),\ \ldots.$$

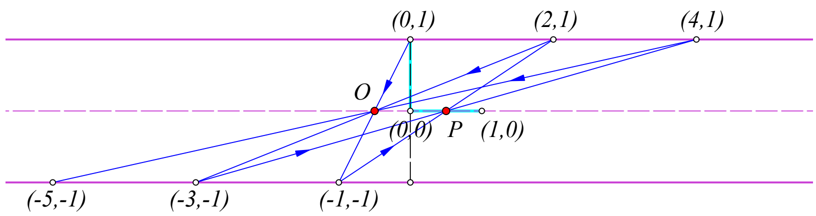

Figure 37: Orbit of a focal rotation

19 Remarks and exercises on orbits of equiaffinities

Here I collect some properties of affinities related directly or indirectly to the much cited talk by Coxeter [Cox67]. First, we should notice that equiaffinities f preserving a non-degenerate conic κ are intimately related to orbits of points by equiaffinities. This follows by considering a point $A \in \kappa$ and its orbit under f, i.e. the iterates $\{f^n(A), n \in \mathbb{Z}\}$. The whole orbit must be a subset of the conic and, if the orbit has more than 4 points in general position, this determines uniquely κ.

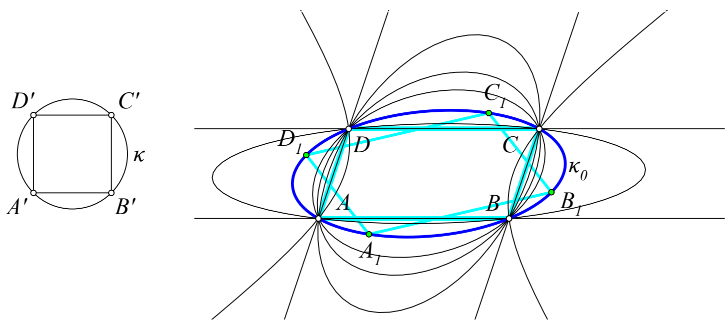

Figure 38: Orbit $\{A, B, C, D\}$ of equiaffinity contained in many conics

Figure 38 shows a simple example of four points which are the vertices of a parallelogram and represent also the orbit $\{A, B = f(A), C = f(B), D = f(C)\}$ of an equiaffinity f. There is a *"pencil of conics"* passing through these four points, but from its infinite many non-degenerate members, it can be shown that only one conic κ_0 is *invariant* under f. This is the image-conic $\kappa_0 = g(\kappa)$ of the circumcircle κ of a square $A'B'C'D'$ transformed to $ABCD$ by the affinity g mapping the $\{A', B', C'\}$ respectively onto $\{A, B, C\}$. The figure shows also a second orbit $\{A_1B_1C_1D_1\}$ of f for a point A_1 on κ_0. The conjugate affinity $f' = g \circ f \circ g^{-1}$ is the rotation by $\pi/2$ about the center of the circle κ.

Second, we should notice also that a conic κ has plenty of affine reflections leaving it invariant. Every diameter α of the conic and its conjugate direction u define such an affine reflection.

Third, we should notice also that the two reflections $\{g, h\}$ to which decomposes an equiaffinity (theorem 15) $f = h \circ g$, which preserves the conic κ, preserve this conic too. This, for the case of affinities of infinite period follows from the discussion in section 16. For orbits of affinities of *finite period* this follows from the fact that they occur only in the *elliptic* type. But ellipses are affinely equivalent to circles and there the equiaffinities are ordinary *rotations*, represented by two ordinary reflections, whose axes (mirrors) are at angle half the angle of rotation.

These remarks lead to the next two exercises.

Exercise 22. *An affinity leaving invariant a non-degenerate conic κ is either an equiaffinity or a product of an equiaffinity and an affine reflection.*

Hint: This can be quickly seen using matrices and representing the conic in cartesian homogeneous coordinates by an invertible matrix B and the corresponding quadratic equation:

$$\kappa \ : \ X^t \cdot B \cdot X \ = \ 0.$$

An affine transformation f expressed through a matrix A_v (notation of section 1) leaves invariant the conic if the first condition below is valid, implying the second one of respective determinants:

$$A_v^t \cdot B \cdot A_v = B \quad \Rightarrow \quad |A_v^t||B||A_v| = |B| \quad \Rightarrow \quad |A_v|^2 = 1.$$

Thus, if $|A_v| = 1$ the transformation f is an equiaffinity, otherwise composing with an affine reflection g preserving the conic, the transformation $g \circ f$ is an equiaffinity, thereby proving the claim.

affinity preserving a conic

Exercise 23. *Every affinity f preserving a conic κ is a product of 1, 2 or 3 affine reflections, each of them preserving the conic too.*

Exercise 24. *Assume that the orbits $\{X_n = f^n(X), n \in \mathbb{Z}\}$ and $\{Y_n = f^n(Y), n \in \mathbb{Z}\}$ are contained in the same non-degenerate conic κ. Show that there is an affinity g preserving κ and mapping the first orbit onto the second.*

Hint: Use again matrices, A_v for f and C_w for the affinity g mapping $\{X_0, X_1, X_2\}$ respectively to $\{Y_0, Y_1, Y_2\}$. Since A_v satisfies its characteristic polynomial, there are constants $\{a, b, c\}$ such that, with the unit 3×3 matrix I_3 :

$$A_v^3 \ = \ aI_3 \ + \ bA_v \ + \ cA_v^2.$$

By the assumption

$$C_w X_i = Y_i \quad \Leftrightarrow \quad C_w A_v^i X_0 = A_v^i Y_0 \quad \text{for} \quad i = 0, 1, 2$$

we have then

$$C_w \cdot X_3 = C_w \cdot (A_v^3 X_0) = C_w \cdot (aX_0 + bA_v X_0 + cA_v^2 X_0) = aY_0 + bA_v Y_0 + cA_v^2 Y_0 = A_v^3 Y_0 = Y_3.$$

In the same way, using $A_v^4 = aA_v + bA_v^2 + cA_v^3$ we show $C_w X_4 = Y_4$ and inductively the equivalent to the claimed condition $C_w X_n = Y_n$ for all n. The preservation of the conic κ under g follows immediately if the orbits contain more than 4 points and by reduction to the circle in the other cases.

area swept

Exercise 25. *Show that for a central conic κ and the orbit $\{X_n = f^n(X), n \in \mathbb{Z}\}$ in it by an affinity f preserving κ the areas of the triangles OX_iX_{i+1} are equal, O being the center of the conic. Show also that the areas of the corresponding sectors of the conic have the same area and*

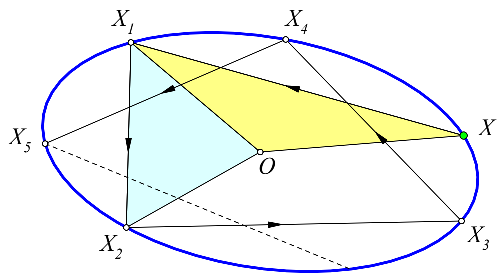

Figure 39: Areas swept by the radius from the center

also the corresponding "conic segments" enclosed by the chord X_iX_{i+1} and the corresponding arc of the conic are equal.

affinities preserving a circle

Exercise 26. *Show that an affine reflection preserving the circle $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ is an ordinary euclidean reflection (with conjugate axis orthogonal to the axis). Show that the product of three reflections preserving S^1 is a reflection too. Conclude that the group of affinities preserving S^1 consists of "rotations" about its center and reflections on its diameters.*

Exercise 27. *Show that an affinity f can be represented as a composition $f = s \circ r_2 \circ r_1$, with two affine reflections $\{r_1, r_2\}$ and a shear or strain s. Conclude that every affinity is the product, $f = s \circ g$, of an equiaffinity $g = r_2 \circ r_1$ and a shear or strain s. In some particular cases one or more of these factors of f can be the identity. Investigate these cases and see when they happen.*

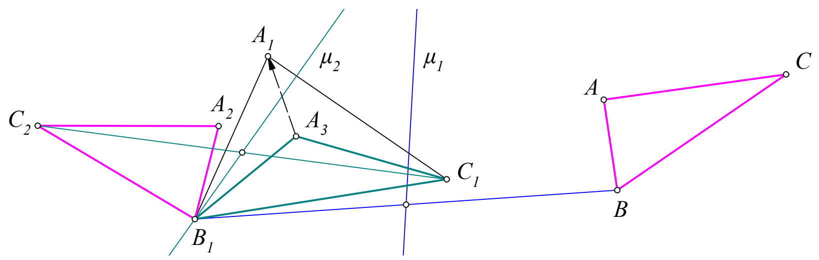

Figure 40: General affinity decomposition

Hint: Consider a triangle ABC and its image $A_1B_1C_1$ under f. Define an affine reflection r_1 with conjugate axis BB_1 and axis μ_1 arbitrary (See Figure 40). This maps ABC onto a triangle $A_2B_1C_2$. Then, define the affine reflection r_2 with conjugate axis C_1C_2 and axis μ_2 the median of $B_1C_1C_2$ from B_1. This maps $A_2B_1C_1$ onto $A_3B_1C_1$. Then, the two triangles with common base B_1C_1 define a shear or strain fixing the line B_1C_1 and mapping A_3 to A_1.

20 Affine equivalence of conics

Analogously to *"homotheties = central dilatations"* centerred at O, represented in cartesian coordinates by $\{x' = \lambda x, \;\; y' = \lambda y\}$, which map central conics κ to central conics κ' with the *same axes and the same eccentricity*, the use of *affinities* and in particular *strains* cre-

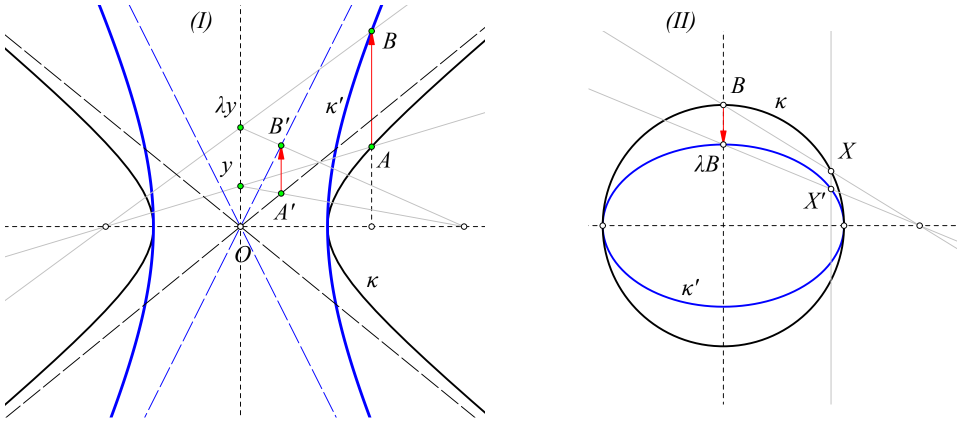

Figure 41: Central conics κ' resulting via a strain from conic κ

ates maps between conics of the same type and of different eccentricities. Figure 41-I illustrates such a case of a strain applied to hyperbola κ, which maps to hyperbola κ' by *"bending"* the original hyperbola κ while maintaining fixed its vertices, i.e. preserving the transverse axis a and changing only the conjugate axis b. It shows also how the point A of κ and A', on the asymptote of κ, map correspondingly to points B of κ' and B' of the asymptote of κ'. By varying the parameter λ of the strain, we can obtain from a hyperbola with eccentricity e a hyperbola with any other value e' of eccentricity we wish. Equivalently, from a hyperbola κ with angle between the asymptotes ω, a hyperbola κ' with arbitrary angle ω' between its asymptotes. Figure 41-II shows the analogous procedure mapping a circle κ to an ellipse.

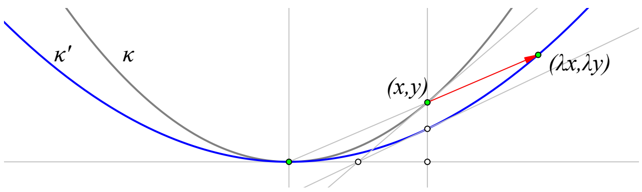

Figure 42: All parabolas are similar

The analogous property for parabolas is trivial, since all of them are pairwise similar and similarities are special affinities. Figure 42 shows the parabola $\kappa' : \;\; y = \frac{1}{\lambda}x^2$, resulting from $\kappa : \;\; y = x^2$ by the homothety $x' = \lambda x, y' = \lambda y$.

21 Affinities preserving a non-degenerate conic

By exercise 23, every affinity preserving a non-degenerate conic is the composition of at most three affine reflections. The first property to show is the following concerning a single reflection.

Exercise 28. *An affine reflection f preserving a non-degenerate conic has necessarily its axis μ*

coinciding with a diameter of the conic and its conjugate direction u *coinciding with the conjugate direction of* μ.

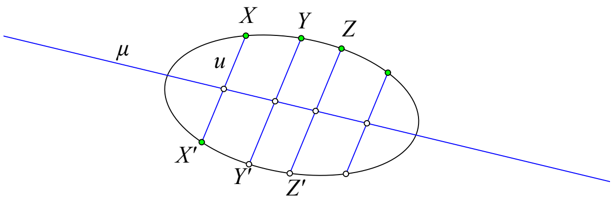

Figure 43: Affine reflection preserving a conic

Hint: For all points X on the conic $X' = f(X)$ defines chords XX' parallel to u whose middles lie on the conjugate diameter μ to u.

Exercise 29. *The product* f *of two affine reflections preserving an ellipse/parabola/hyperbola is respectively an elliptic/parabolic/hyperbolic or crossed hyperbolic rotation.*

Hint: Take a point X on the conic and apply repeatedly f to create the orbit $\{f^n X\}$ of points of the conic (section 17).

Next exercise can be given a solution valid for all three kinds of conics. This depends however on the possibility to rewrite an affine rotation f as a product $f = f_2 \circ f_1$ of two reflections whose axes and respective conjugate axes are conveniently chosen. For non-euclidean rotations this is not trivial, but this sort of argument can be avoided by examining the cases separately. For the elliptic case this is already handled in exercise 26. Thus, the solution can be given by examining the remaining cases: (i) of the parabola $y = x^2$ and (ii) the case of the hyperbola $xy = 1$.

Exercise 30. *The product* f *of three affine reflections preserving a non-degenerate conic is an affine reflection preserving the conic.*

Hint: Show that this is equivalent with the following geometric property of quadrangles inscribed in conics.

Exercise 31. *Given three directions* $\{u_1, u_2, u_3\}$, *a conic* κ *and an arbitrary point* A *on it, construct the inscribed in it quadrangle* $ABCD$, *such that* $\{AB, BC, CD\}$ *are respectively parallel to* $\{u_1, u_2, u_3\}$. *Show that the fourth side* DA *is then parallel to a fixed direction* u_4.

Hint: In the case of the parabola $y = x^2$ project the vertices on the x-axis to the points respectively $\{A', B', C', D'\}$. Use the description of the symmetry of the x-axis about its point

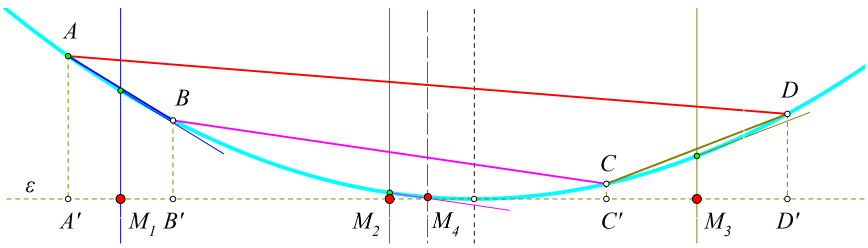

Figure 44: Inscribed quadrangles with fixed directions of three sides

m, given by $x' = 2m - x$ and show that the coordinates $\{\alpha, \delta\}$ respectively of $\{A', D'\}$ are related by the equation (See Figure 44)

$$\delta = 2(m_3 - m_2 + m_1) - \alpha,$$

$\{m_1, m_2, m_3\}$ being respectively the (fixed) coordinates of the middles of $\{AB, BC, CD\}$. In the case of hyperbola and a chord AB of it such that $AB = (\alpha, \beta)$ and coordinates $\{A(x_1, y_1), B(x_2, y_2)\}$ show first that for fixed direction $\lambda = \alpha/\beta$ of the chords AB

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda \\ -\frac{1}{\lambda} & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

describes the affine reflection with conjugate direction the one of AB. Calculate the composition of three analogous reflections and show that the resulting transformation is an

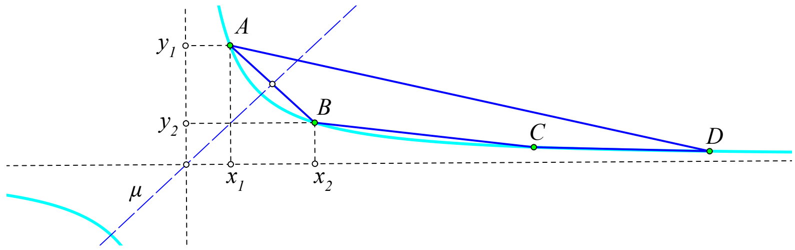

Figure 45: Affine reflection with conjugate direction defined by AB

affine reflection (See Figure 45).

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