Fundamental invariants of triangles

A file of the Geometrikon gallery by Paris Pamfilos

The greatest of faults, I should say, is to be conscious of none.

T. Carlyle, On Heroes

Contents

(Last update: 23-02-2021)

1 Fundamental invariants of triangles

The *"fundamental invariants"* of a triangle ABC are traditionally considered to be the following three quantities associated with the triangle:

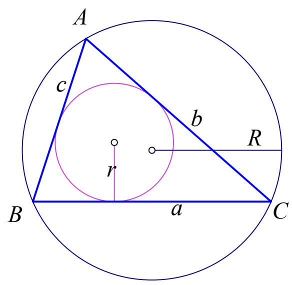

Figure 1: Fundamental invariants of the triangle: $\{s, r, R\}$

1. The *"half-perimeter"* : $s = \frac{a+b+c}{2}$,
2. The *" inradius"* r i.e. the radius of the inscribed circle,
3. The *" circumradius"* R i.e. the radius of the circumcircle of the triangle.

2 Some remarkable identities

Denoting by $\{r_a, r_b, r_c\}$ the radii of the *"excircles"* of the triangle the following identity is valid ([Joh60, p.189]):

$$r_a + r_b + r_c = 4R. \tag{1}$$

The proof relies on some other identities involving the area Δ of the triangle and the

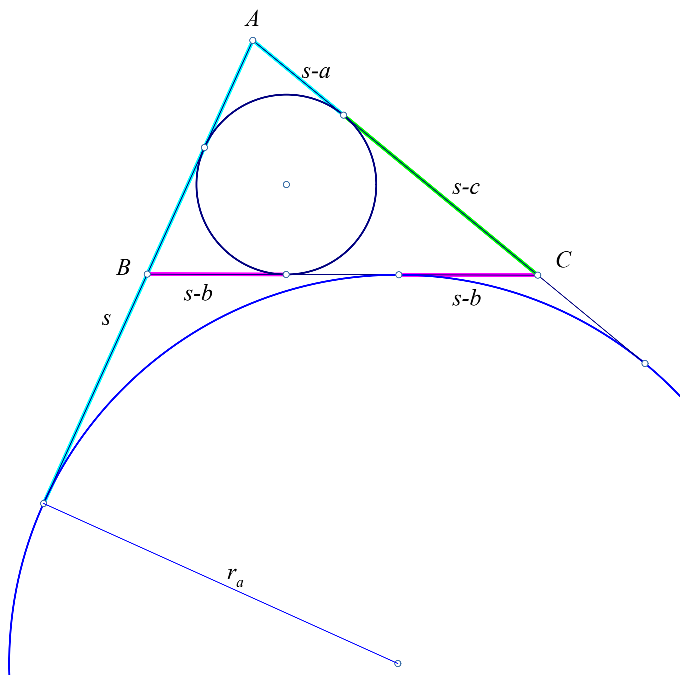

Figure 2: Distances to contact points

quantities $s, s-a, s-b, s-c$

$$r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{s-a}, \quad r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}. \tag{2}$$

$$r_a + r_b + r_c - r = \sum \left(\frac{\Delta}{s-a} - \frac{\Delta}{3s} \right) = \Delta \sum \left(\frac{1}{s-a} - \frac{1}{3s} \right) = \frac{\Delta}{3s} \sum \left(\frac{2s+a}{s-a} \right). \tag{3}$$

The first expresses the radii in terms of the area and the perimeter. The second sums over the cyclic permutations of the letters $\{a, b, c\}$:

$$\sum \left(\frac{2s+a}{s-a} \right) = \frac{1}{(s-a)(s-b)(s-c)} \sum (2s+a)(s-b)(s-c), \tag{4}$$

$$\sum (2s+a)(s-b)(s-c) = 3abc. \tag{5}$$

Last equation results by carrying out the operations (e.g. with Maxima). Then back substitution yields

$$r_a + r_b + r_c - r = \frac{\Delta}{3s} \cdot \frac{3abc}{(s-a)(s-b)(s-c)} = \frac{\Delta abc}{\Delta^2} = \frac{abc}{\Delta} = 4R.$$

This, taking into account *"Heron's formula"* for the area, and last expressing $\{a, b, c\}$ in terms of sines by the sine formula giving:

$$abc = 4R\Delta = 4Rrs. \tag{6}$$

By the occasion of this calculation I include another couple of formulas involving the two

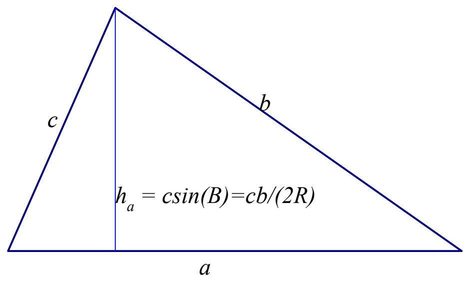

Figure 3: The formula $abc = 4\Delta R$

symmetric quadratic expressions of the sides of the triangle.

$$bc + ca + ab = s^2 + r(4R + r), \tag{7}$$
$$a^2 + b^2 + c^2 = 2(s^2 - r(4R + r)). \tag{8}$$

Denote the first sum by X and the second by Y. Obviously

$$2X + Y = (a + b + c)^2 = 4s^2.$$

On the other hand, the expression $Y - 2X$ can be written:

$$Y - 2X = \sum((b-c)^2 - a^2) = -\sum(a+c-b)(b+a-c) = -4(s-b)(s-c)(s-a)\sum\frac{1}{s-a}.$$

Replacing there $\{s-a, s-b, s-c\}$ with the expressions resulting from equation (2), we obtain:

$$\begin{aligned} Y - 2X &= -4(s-b)(s-c)(s-a)\sum\frac{1}{s_a} \\ &= -4\frac{\Delta^2}{s}\sum\frac{1}{s-a} \\ &= -4\frac{\Delta^2}{s}\sum\frac{r_a}{\Delta} = -4r\sum r_a = (-4r)(4R + r). \end{aligned}$$

Solving these equations for $\{X, Y\}$ we find the expressions in equations (7) and (8).

3 Generalizing to 3rd degree symmetric functions

The preceding method can be generalized to compute every symmetric function of $\{a, b, c\}$ in terms of the distinguished quantities $\{s, r, R\}$, the *"fundamental invariants"* of the triangle ([AA06, p.110]). As an example I examine the two basic cubic symmetric functions: $X = (a^3 + b^3 + c^3)$ and $Y = (bc(b+c) + ca(c+a) + ab(a+b))$, which satisfy:

$$(a+b+c)^3 = \sum a^3 + 3\sum ab(a+b) + 6abc \quad \Rightarrow \quad 8s^3 = X + 3Y + 6abc, \tag{9}$$
$$(a^2+b^2+c^2)(a+b+c) = \sum a^3 + \sum ab(a+b) \quad \Rightarrow \quad 4s(s^2 - r(4R+r)) = X + Y. \tag{10}$$

Solving the two equations for $\{X, Y\}$ we find the expressions for these two symmetric cubic functions:

$$a^3 + b^3 + c^3 = 2s(s^2 - 6rR - 3r^2), \tag{11}$$

$$\sum ab(a+b) = 2s(s^2 - 2rR + r^2). \tag{12}$$

Analogously we may compute the symmetric cubic functions

$$\begin{aligned} a(b-c)^2 + b(c-a)^2 + c(a-b)^2 &= \sum a(b^2+c^2) - 2\sum abc \\ &= \sum a(a^2+b^2+c^2) - \sum a^3 - 6abc \\ &= 2s(a^2+b^2+c^2) - \sum a^3 - 6abc \\ &= 2s(s^2 + r^2 - 14Rr), \end{aligned} \tag{13}$$

$$(a+b)(b+c)(c+a) = 2abc + \sum ab(a+b) = 2s(s^2 + 2rR + r^2), \tag{14}$$

$$(b+c-a)(c+a-b)(a+b-c) = 8(s-a)(s-b)(s-c) = 8sr^2. \tag{15}$$

4 Some 4th degree symmetric functions

The calculation of the higher symmetric functions has to be done gradually, since in each step we need the results of the previous. A use of these formulas is made below, in the GIO construction problem, i.e. the problem of constructing a triangle by giving the location of its three remarkable points: G(centroid), I(incenter) and O(circumcenter). As a last example I calculate the symmetric functions of fourth order:

$$\begin{aligned} (a+b+c)^4 &= \sum a^4 + 4\sum bc(b^2+c^2) + 6\sum b^2c^2 + 12abc\sum a, \\ (a^3+b^3+c^3)(a+b+c) &= \sum a^4 + \sum bc(b^2+c^2), \\ (a^2+b^2+c^2)^2 &= \sum a^4 + 2\sum b^2c^2. \end{aligned}$$

This leads to the following system with obvious meaning of the symbols:

$$\begin{aligned} (a+b+c)^4 &= X + 4Y + 6Z + 12abc\sum a, \\ (a^3+b^3+c^3)(a+b+c) &= X + Y, \\ (a^2+b^2+c^2)^2 &= X + 2Z. \end{aligned}$$

This is a linear system of equations, in which the right side is known from the previous steps:

$$\begin{aligned} \begin{pmatrix} 1 & 4 & 6 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} &= \begin{pmatrix} (a+b+c)^4 - 12abc\sum a \\ (a^3+b^3+c^3)(a+b+c) \\ (a^2+b^2+c^2)^2 \end{pmatrix} \quad \Rightarrow \\ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} &= \frac{1}{12} \begin{pmatrix} -2 & 8 & 6 \\ 2 & 4 & -6 \\ 1 & -4 & 3 \end{pmatrix} \begin{pmatrix} (a+b+c)^4 - 12abc\sum a \\ (a^3+b^3+c^3)(a+b+c) \\ (a^2+b^2+c^2)^2 \end{pmatrix} \\ &= \frac{1}{12} \begin{pmatrix} -2 & 8 & 6 \\ 2 & 4 & -6 \\ 1 & -4 & 3 \end{pmatrix} \begin{pmatrix} (2s^4) - 12(4Rrs)(2s) \\ (2s(s^2 - 6rR - 3r^2))(2s) \\ (2(s^2 - r(4R+r)))^2 \end{pmatrix} \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} X = \sum a^4 &= 2([4rR - s^2 + r^2]^2 - [2rs]^2), \\ Y = \sum bc(b^2+c^2) &= -2(16r^2R^2 + 4rs^2R + 8r^3R - s^4 + r^4), \\ Z = \sum b^2c^2 &= 16r^2R^2 - 8rs^2R + 8r^3R + s^4 + 2r^2s^2 + r^4. \end{aligned}$$

5 Relations involving the tritangent circles

The *"tritangent circles"* of the triangle ABC (see file **Tritangent circles**) are its *"incircle"* and the three *"excircles"*, which are the four circles tangent to all sides of the triangle ([Cou80, p.72]). Using figure 4, it is not difficult to show the relations

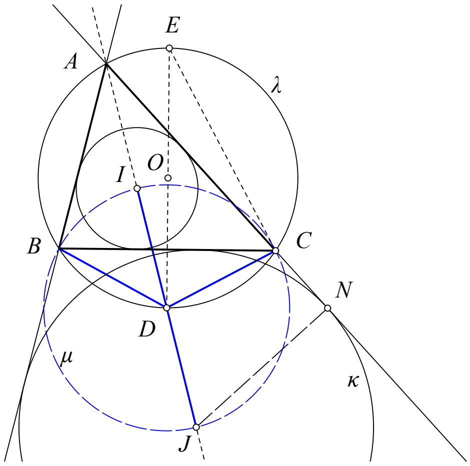

Figure 4: Relations connected with the tritangent circles

$$|CD| = \frac{a}{2\cos\left(\frac{\alpha}{2}\right)} = 2R\sin\left(\frac{\alpha}{2}\right), \qquad |IB| = |IJ|\sin\left(\frac{\gamma}{2}\right) = 4R\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right),$$
$$r = 4R\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right), \qquad s-a = |AI|\cos\left(\frac{\alpha}{2}\right) = 4R\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\alpha}{2}\right),$$
$$\cos\left(\frac{\alpha}{2}\right) = \frac{s}{|AJ|}, \quad \widehat{ADE} = \frac{|\beta-\gamma|}{2}, \qquad |AD| = 2R\cos\left(\frac{|\beta-\gamma|}{2}\right), \quad |AE| = 2R\sin\left(\frac{|\beta-\gamma|}{2}\right),$$
$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}, \quad \cot\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

6 The cubic equation satisfied by $\{a, b, c\}$

The converse problem, that of the existence of a triangle with given $\{s, r, R\}$, occupied Euler, in a slight variation ([San15, p.7]) and led him to the third degree equation of next theorem:

Theorem 1. *The side-lengths $\{a, b, c\}$ of the triangle ABC satisfy the cubic equation:*

$$x^3 - 2sx^2 + (s^2 + r^2 + 4Rr)x - 4sRr = 0. \qquad (16)$$

Proof. Replacing in $\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) = 1$ the corresponding expressions of the previous section and using equation (6), we find the relations:

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{ar}{4R(s-a)}, \quad \cos^2\left(\frac{\alpha}{2}\right) = \frac{a(s-a)}{4Rr} \quad \Rightarrow \quad \frac{ar}{4R(s-a)} + \frac{a(s-a)}{4Rr} = 1.$$

Last equation for $x = a$ is equivalent to the mentioned in the theorem and will hold also for $\{x = b, x = c\}$, since the coefficients of the relation are independent of a. □

7 Blundon's inequalities

Every equation of degree 3 can be written in the form

$$x^3 + Ax^2 + Bx + C \;=\; 0,$$

and making the substitution $x = y - A/3$ this reduces to

$$y^3 + Py + Q = 0, \quad \text{with} \quad P = B - \frac{A^2}{3} \quad \text{and} \quad Q = C - \frac{A}{3}\left(B - \frac{2}{9}A^2\right).$$

In order for the roots of equation (16) to be real, the known inequality ([Bur86, p.71])

$$\frac{Q^2}{4} + \frac{P^3}{27} \;<\; 0 \quad \text{must be satisfied.}$$

This is a condition, which in the present case reduces to

$$s^4 \;+\; 2s^2[r^2 - 10Rr - 2R^2] + r(r + 4R)^3 \;\leq\; 0\,. \tag{17}$$

Besides that one, in order for a triangle to exist with the given data, certain additional conditions must be satisfied, like for example the deduced from the well known *"Euler's relation"* (see file **Tritangent circles**) inequality $R > 2r$, as well as, the deduced from equation (8) inequality, $s > r$. If such a triangle exists, then the lengths of its sides are determined fully through the roots of the polynomial. However the construction of the triangle with these data using only a ruler and compass is not possible in general. Note that the inequality (17), considered with respect to s^2 is quadratic and is satisfied when s^2 is between the roots of the corresponding trinomial, whose discriminant is

$$D \;=\; [4(R - 2r)]^2 R(R - 2r)\;.$$

This leads to the double inequality of Blundon, [Bir15]

$$2R(R + 5r) - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq 2R(R + 5r) - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

8 The GIO triangle

Here we use *"barycentric coordinates"* (see file **Barycentric coordinates**) to determine the sides of the triangle with vertices $\{G, I, O\}$ (orthocenter, incenter, circumcenter). For this, in the case of $|GI|$ we apply the formula for the distance of two points expressed in *absolute barycentrics*:

$$|UU'|^2 = S_A(u' - u)^2 + S_B(v' - v)^2 + S_C(w' - w)^2,$$

where

$$S_A = (b^2 + c^2 - a^2)/2, \;\; S_B = (c^2 + a^2 - b^2)/2, \;\; S_C = (a^2 + b^2 - c^2)/2,$$

are the **Conway triangle symbols** and $\{U = (u, v, w)^t, U' = (u', v', w')\}$ are the barycentrics of the two points.

From the fact that the absolute barycentrics of the the three points are respectively given by:

$$G = (1,1,1)/3, \tag{18}$$
$$I \;= (a, b, c)/(2s), \tag{19}$$
$$O = (a^2 S_A, b^2 S_B, c^2 S_C)/(2S^2), \tag{20}$$

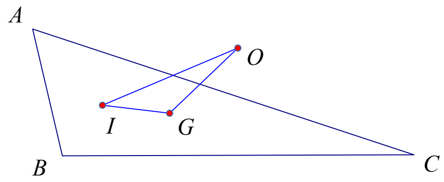

Figure 5: The GIO triangle

where S is twice the area of the triangle ABC, we deduce:

$$|GI|^2 = \sum S_A \left(\frac{a}{2s} - \frac{1}{3}\right)^2 = \dots$$
$$= \frac{2\sum ab(a+b) - \sum a^3 - 9abc}{9(2s)} = \dots \quad \Rightarrow$$
$$|GI|^2 = \frac{s^2 + 5r^2 - 16Rr}{9}. \tag{21}$$

Here the sums are over the cyclic permutations of $\{a, b, c\}$ and the dots mean calculations, taking into account equations (6), (11), (12).

For the other sides of the triangle GIO is computationally more favorable to use the euclidean norm with origin at the circumcenter expressed in barycentrics (see file **barycentric coordinates**):

$$|OP|^2 = R^2 - (a^2vw + b^2wu + c^2uv), \quad \text{for} \quad P = (u, v, w) \quad \text{in absolute barycentrics.} \quad \Rightarrow$$

$$|OG|^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2) = R^2 - \frac{2}{9}(s^2 - r(4R + r)). \tag{22}$$

$$|OI|^2 = R^2 - \frac{1}{4s^2}(a^2bc + b^2ca + c^2ab) = \cdots = R(R - 2r), \tag{23}$$

latter being the *"Euler's relation"* for the circumradius and inradius of the triangle ABC. Using equations (21), (22) and (23), we can express the *fundamental invariants* $\{r, R, s\}$ in terms of the side-lengths $\{|GI|, |IO|, |OG|\}$ of the triangle GIO. In fact, solving equation (23) w.r. to $2rR$ and replacing into the two other equations, we obtain the system of equations:

$$5r^2 + \quad s^2 = 8(R^2 - OI^2) + 9IG^2,$$
$$-2r^2 + 2s^2 = 4(R^2 - OI^2) + 9R^2 - 9OG^2.$$

Eliminating s^2 from these equations and using again equation (23) to express the radius $r = (R^2 - OI^2)/(2R)$, we obtain, after some easy calculation:

$$R^2 = \frac{OI^4}{6IG^2 + 3OG^2 - 2OI^2}. \tag{24}$$

Replacing into the previous equation, we find:

$$r^2 = \frac{9(OI^2 - OG^2 - 2IG^2)^2}{4(6IG^2 + 3OG^2 - 2OI^2)}. \tag{25}$$

$$s^2 = \frac{3OI^2(17OI^2 - 2OG^2 - 28IG^2) - 9(OG^2 + 2IG^2)(5OG^2 - 2IG^2)}{4(6IG^2 + 3OG^2 - 2OI^2)}. \tag{26}$$

9 The orthocentroidal circle

The *"orthocentroidal"* circle of the triangle ABC is the circle with diameter GH, where H is the *orthocenter* of the triangle (See Figure 6). This circle is of importance because of the

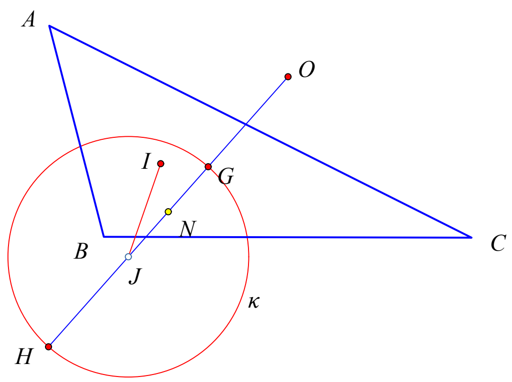

Figure 6: The orthocentroidal circle κ of ABC

next theorem.

Theorem 2. *Given the points $\{G, I, O\}$ there is a triangle ABC having these points respectively as centroid, incenter and circumcenter, if and only if the incenter I is inside the orthocentroidal circle with diameter GH.*

Proof. The necessity of the condition follows from equation (23), by which $R^2 > OI^2$. Replacing in this inequality R^2 from equation (24) and doing some calculation, we see that it is equivalent with:

$$\begin{aligned} 2IG^2 + OG^2 - OI^2 < 0 \quad &\Leftrightarrow \quad 2IG^2 + 2OG^2 - OI^2 < OG^2 \\ &\Leftrightarrow \quad JI^2 < OG^2 = JG^2. \end{aligned} \tag{27}$$

Here we applied *"Stewart's theorem"*, implying $JI^2 = 2(IG^2 + OG^2) - OI^2$ and the fact that $GH = 2OG$.

The sufficiency proof is more involved and can be seen in [Gui84]. Point I though must be different from the middle N of OH, which is the center of the Euler circle ([Ste07], [Yiu13]). □

10 Euler's construction problem

Euler solved the problem of constructing a triangle from the points $\{I, H, O\}$, which is equivalent to the problem of constructing the triangle from $\{G, I, O\}$, since each triple determines the other. The method can be described as follows.

1. Find $\{s, r, R\}$, the *fundamental invariants* of the under construction triangle as in section 8.
2. Consider the cubic equation (16), whose roots are the side-lengths $\{a, b, c\}$ of the trianlge.
3. Solve the cubic to find these lengths and construct the triangle. See section 6 for the resulting cubic equation. See also section 7 for the restrictions satisfied by $\{s, r, R\}$.

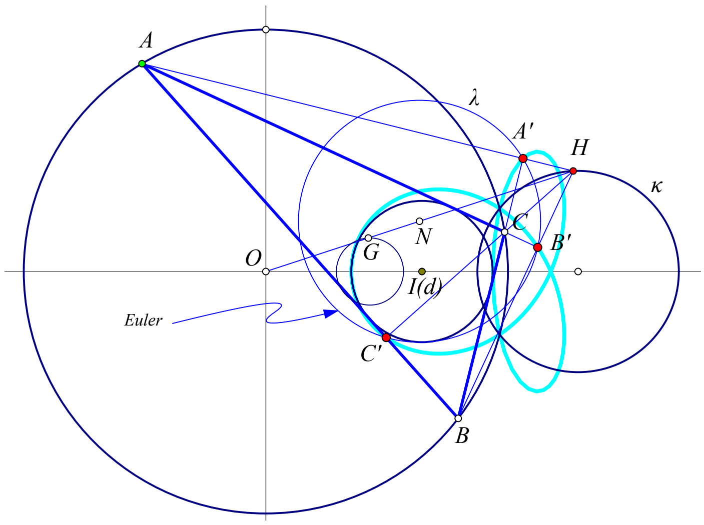

Figure 7: The locus of the feet of the altitudes

Figure 7 shows a curve related to the determination of this triangle. The curve contains the feet $\{A', B', C'\}$ of the altitudes of the triangles $\{ABC\}$ inscribed in a circle $\kappa_1(O, R)$ and circumscribed to a circle $\kappa_2(I, r)$, the respective radii satisfying the Euler relation $d^2 = |OI|^2 = R(R - 2r)$. These three points lie on a curve resembling an *"hypotroichoid"*, given in parametric form by the equations:

$$x(t) = \frac{4Rd(R^2 - d^2)\cos^2(t) + (d^4 - R^4 + 4R^2d^2)\cos(t) - 4dR^3}{2R(2Rd\cos(t) - (d^2 + R^2))}, \tag{28}$$

$$y(t) = \frac{4Rd(R^2 - d^2)\cos(t)\sin(t) + (d^4 - R^4)\sin(t)}{2R(2Rd\cos(t) - (d^2 + R^2))} \tag{29}$$

Its derivation goes back to a related computation by Odehnal of the *"poristic"* triangle ABC, i.e. triangle varying but with fixed incircle and fixed circumcircle ([Ode11]). The requested triangle ABC, with given points $\{O, I, H\}$ and from them resulting $\{r, R\}$, has its altitude feet $\{A', B', C'\}$ on the intersection of the Euler circle $\lambda(E, R/2)$ and this curve. These points determine the *"orthic"* triangle $A'B'C'$ of ABC. In the aforementioned reference is proved that the orthocenters of the *poristic* triangles $\{ABC\}$ lie on a circle κ, as seen in the figure.

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Related topics

1. **Barycentric coordinates**
2. **Conway triangle symbols**
3. **Tritangent circles**

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