Homographic relation

A file of the Geometrikon gallery by Paris Pamfilos

> Every physicist thinks that he knows what a photon is... I spent my life to find out what a photon is and I still don't know it.
>
> *Albert Einstein, In E. Hecht Optics I,p.9*

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1 Homographic Relation

"Homographic Relation" or *"homography"* between two real variables $\{x, y\}$ is a one-to-one (invertible) function $y = f(x)$ involving algebraic operations only between the variables

$\{x, y\}$. The one-to-one and the algebraic requirements imply that the relation has the form.

$$y = \frac{ax+b}{cx+d}, \quad \text{with} \quad ad - bc \neq 0 \quad \Leftrightarrow \quad cxy - ax + dy - b = 0\ , \tag{1}$$

with an inverse function of the same kind

$$x \;=\; \frac{dy-b}{-cy+a}\ . \tag{2}$$

The graph of the function is a rectangular hyperbola (see figure 1). This is the only kind of quadratic curve that represents an invertible real function of the *"extended real line"* to itself. By *"extended real line"* I mean the real line $\mathbb{R}$, to which an additional point *"at infinity"*, denoted by ∞ has been added. In the following the terms *"number"* and *"point"* are used interchangeable to denote elements of the *extended real line*.

Besides the *"group property"* discussed in the next section, the other main properties of a homographic relation are the following.

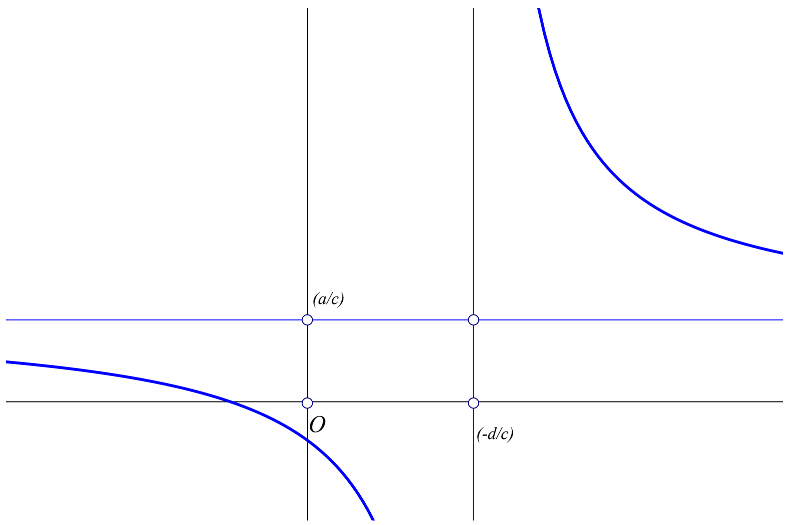

Figure 1: Function $y = (ax + b)/(cx + d)$ representing a homographic relation

1. They preserve the cross ratio $(pq; uv) = \frac{p-u}{q-u} : \frac{p-v}{q-v}$. This means that for
$$p' = f(p), q' = f(q), \ldots \quad \Rightarrow \quad (p'q'; u'v') = (pq; uv).$$

2. They are distinguished in *"involutive"* i.e. such that $f^2 = e \quad \Leftrightarrow \quad f^{-1} = f$ and *"non-involutive"* (e represents the identity function $e(x) = x$).

3. Involutive/non-involutive homographic relations are completely determined by prescribing arbitrarily the values to two/three arbitrary points.

4. If a homographic relation f interchanges two points, i.e. there are (x, y) such that $f(x) = y$ and $f(y) = x$, then f is involutive.

Nr-1 is is an easy calculation resulting to a kind of figure 1.
Nr-2 Non-involutive homographies are determined by their values at three distinguished points. This follows from *nr-1*, by solving the equality of cross ratios

$$(p'q'; u'y) = (pq; ux),$$

w.r.t. to y leading to a function $y = f(x)$ as in *nr-1*.
Nr-3+nr-4: The involutive property is equivalent to the symmetry of the hyperbola about the first diagonal (line $y = x$). This in turn is equivalent with the condition $a + d = 0$. This is equivalent also to the lying of the center O of the hyperbola on the first diagonal. *"Involutive homographies"* are determined by their values at two points. This is seen most easily from equation 1. The homographic relation obtains in this case the form.

$$Axy + B(x + y) + C = 0. \tag{3}$$

This determines the coefficients $\{A, B, C\}$, uniquely up to a multiple, by prescribing two pairs $\{(x_1, y_1), (x_2, y_2)\}$.

Theorem 1. *Any homography is the composition of two involutions.*

In fact, consider a homography $y = f(x) = (ax + b)/(cx + d)$ and for an arbitrary triple of points $\{p, q, r\}$ the images $\{p' = f(p), q' = f(q), r' = f(r)\}$, which, according to *nr-3*, completely determine f. We adopt the notation $g : (xy)(uv)$ for an involution completely determined, according to *nr-3*, by the requirement to interchange the members of the pairs $\{(x, y), (u, v)\}$ ([Cox87, p.47]). We can define, using again *nr-3*, the two involutions.

$$I_1 : (pq')(qp') \quad \text{and} \quad I_2 : (p'q')(r's) \quad \text{where} \quad s = I_1(r). \tag{4}$$

and see that $f = I_2 \circ I_1$:

$$p \overset{I_1}{\mapsto} q' \overset{I_2}{\mapsto} p', \qquad q \overset{I_1}{\mapsto} p' \overset{I_2}{\mapsto} q', \qquad r \overset{I_1}{\mapsto} s \overset{I_2}{\mapsto} r'.$$

2 The group $PGL(2,\mathbb{R})$

Two homographies of the real line $y = f(x)$ and $z = g(y)$ can be *"composed"* and produce a third homography $z = h(x) = (g \circ f)(x) = g(f(x))$. Assume that to f corresponds the matrix A and to g the matrix B. Then to their composition $h = g \circ f$ corresponds the product matrix $C = B \cdot A$. This means that if

$$y = f(x) = \frac{ax+b}{cx+d} \quad \text{with} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad z = g(y) = \frac{a'y+b'}{c'y+d'} \quad \text{with} \quad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$\text{then} \quad z = (g \circ f)(x) = g(f(x)) = \frac{a''x+b''}{c''x+d''} \quad \text{with}$$

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = C = B \cdot A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}.$$

By its definition, a matrix A and a non-zero multiple tA of the matrix define the same homography

$$y = \frac{(ta)x + (tb)}{(tc)x + (td)} = \frac{ax+b}{cx+d}.$$

Thus, it is not exactly one invertible matrix that corresponds to f but a whole *"line"* of invertible matrices $\{t \cdot A : t \in \mathbb{R}, t \neq 0\}$. We denote this set of matrices by $[A]$. The group $PGL(2,\mathbb{R})$ consists of all these elements $\{[A]\}$ for which the *"multiplication"* is the usual matrix multiplication.

It is instructive to see the formation of the inverse homography and the corresponding matrix, especially in the case of involutions $f^2 = e \quad \Leftrightarrow \quad f^{-1} = f$.

$$y = f(x) = \frac{ax+b}{cx+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{or any multiple} \quad tA, \ t \neq 0,$$

$$\text{with inverse} \quad x = f^{-1}(y) = \frac{dx - b}{-cx + a} \quad \text{or any multiple} \quad tA^{-1},\ t \neq 0.$$

In the case of involution the condition $f = f^{-1}$ translates to the matrix relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which determines a matrix different from (a multiple of) the identity only for $t = -1$ leading to the condition $a + d = 0$ characterizing the involutive homographies.

3 Fixed points of homographic relations

Of particular interest are the *"fixed points"* or *"double points"* of such relations, characterized by the equation

$$x = f(x) = (ax + b)/(cx + d). \tag{5}$$

This is equivalent to the quadratic equation

$$cx^2 + (d - a)x - b = 0 \quad \text{with discriminant} \quad D = (a + d)^2 - 4(ad - bc). \tag{6}$$

"Involutive relations" $(a + d = 0)$ have either no (when $ad - bc > 0$) or two (when $ad - bc < 0$) fixed points. This follows from the derivative

$$f'(x) = (ad - bc)/(cx + d)^2.$$

Only hyperbolas representing *"decreasing"* functions $((ad - bc) < 0)$ meet the first diagonal.

Exercise 1. *If the homographic relation* (5) *has two fixed points* $\{A(0), B(k)\}$ *show that the cross ratio* $(XY; AB)$ *for* $\{X(x), Y(y)\}$ *is constant and equal to* d/a.

Hint: $f(0) = 0 \Rightarrow b = 0$, and $k = f(k) \Rightarrow k = (a - d)/c$. Then, using this and (5) into

$$(XY; AB) = \frac{x}{y} : \frac{x - k}{y - k} \quad \Rightarrow \quad (XY; AB) = \frac{d}{a}.$$

4 Fixed points, the case of involution

Assume f is an involution and there are two different, real or imaginary fixed points $\{x_1, x_2\}$. Then, since f preserves the cross ratio, for any other point $x_3, x_4 = f(x_3)$ we have

$$(x_1x_2; x_3x_4) = (f(x_1)f(x_2); f(x_3)f(x_4)) = (x_1x_2; x_4x_3) \quad \Rightarrow \tag{7}$$

$$(x_4 - x_1)/(x_4 - x_2) = -(x_3 - x_1)/(x_3 - x_2) \tag{8}$$

This means that $(x_3, x_4 = f(x_3)) \sim (x_1, x_2)$ are harmonic conjugate pairs. Thus, we have the theorem.

Theorem 2. *Involutions are characterized by the existence of two (real or imaginary) points* $\{x_1, x_2\}$*, such that for every other point* x_3 *the image* $x_4 = f(x_3)$ *is the harmonic conjugate of* x_3 *with respect to* $\{x_1, x_2\}$.

5 Limit points of homographies

Given the homography $f(x) = (ax+b)/(cx+d)$ there are two distinguished values related to the point at infinity of the line on which the homography operates writing

$$f(x) = \frac{a+\frac{b}{x}}{c+\frac{d}{x}} \xrightarrow[x\to\infty]{} \frac{a}{c} = i_f = f(\infty),$$

gives the first limit point which is the image point of the point at infinity. The second limit point is the point *sent* to infinity, or using the inverse function, the point $i_{f^{-1}}$

$$j_f = i_{f^{-1}} = -\frac{d}{c}, \quad f(j_f) = \infty .$$

Using these two points and assuming $c \neq 0$ we see easily that the homographic relation $y = f(x)$ takes the form

$$(x - j_f)(y - i_f) = -\frac{ad-bc}{c^2} . \tag{9}$$

The condition $c \neq 0$ is the necessary condition for the *"converse"* of the next exercise.

Exercise 2. *Given are two points $\{X(x), Y(y)\}$ on the line ε. Show that if there exist two points $\{A(a), B(b)\}$ and a constant c, susch that the signed distances satisfy the relation*

$$AX \cdot BY = c, \tag{10}$$

then $\{x, y\}$ satisfy a homographic relation. Under certain assumptions the converse holds as well, i.e. with the assumption that $\{x, y\}$ satisfy a homographic relation, then there exist points $\{A(a), B(b)\}$, which satisfy (10).

Hint: For the converse, assume that the homographic relation has the form $xy+ax+by+c = 0$ and see that for $u = -b$, $v = -a$, $d = ab - c$, it takes the form $(x-u)(y-v) = d$.

6 Homographic relations represent projectivities

The importance of this kind of relations stems from the fact that they represent *"projectivities"* or *"homographies"* or *"homographic transformations"* of one-dimensional projective spaces or *"projective lines"* (see file **Projective Line**).

The standard model of such a line is the *"extended real line"* or *"one-point compactified real line"* of section 1.

Another model of the projective line is a *"range of points"* consisting of points of an euclidean line, in the plane, the space or a higher dimensional euclidean space, to which a point *"at infinity"* has been added.

A third example of projective line is a *"pencil of lines"* or *"range of lines"* of the euclidean plane, consisting of all lines passing from a fixed point A called the *"center"* of the pencil. For this kind of projective line I often use the symbol A^*.

In all these models their points can be described by *"classes of pairs"* of numbers (x, y) different from $(0, 0)$, two pairs considered in the same class if they define the same *"direction"*, i.e. they satisfy $(x', y') = k(x, y)$, with $k \neq 0$.

The pairs $\{(x, y)\}$ represent *"homogeneous coordinates"* for these models and a projectivity or homography is described in these coordinates by an invertible matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{defining} \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11}$$

The *"homographic relation"* results then for the corresponding quotients

$$t' := \frac{x'_1}{x'_2}, \quad t = \frac{x_1}{x_2} \quad \Rightarrow \quad t' = \frac{at+b}{ct+d}. \tag{12}$$

Homogeneous coordinates of a line ε result also from usual *"line coordinates"* on ε. Latter are defined by fixing two points $\{O, A\}$ on ε and defining as *line coordinate* the signed ratio

$$x = \frac{OX}{OA}.$$

The corresponding *"homogenization"* results by considering pairs of numbers (u, v) such that $x = u/v$. The pair (u, v) represents then the *homogeneous coordinates* of the point of the line ε represented also by x. A homographic transformation is then described by

$$x' = \frac{ax+b}{cx+d} \quad \text{w.r. to} \quad x, \quad \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au+bv \\ cu+dv \end{pmatrix} \quad \text{w.r. to} \quad (u, v).$$

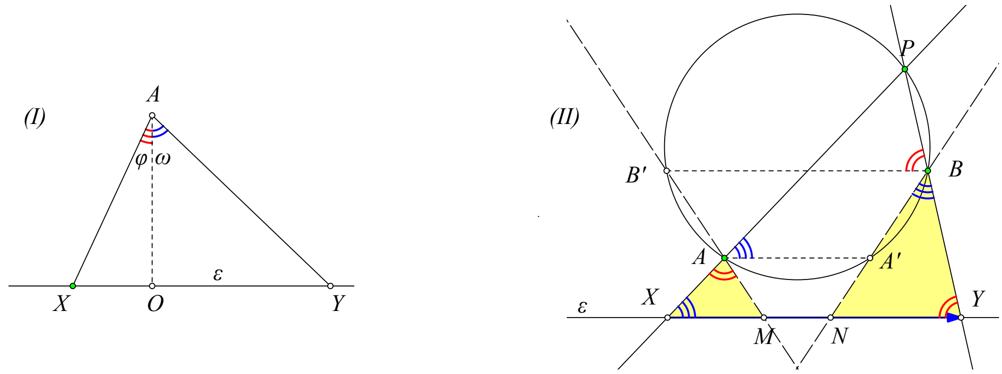

Figure 2: Homographic relation through angle... ... and through circle

Exercise 3. *The angle $\widehat{XAY}$ of fixed measure and with fixed vertex at point A rotates about A and defines the points of intersection $\{X(x), Y(y)\}$ of its sides with the fixed line ε (see figure 2-(I)). Show that $\{X, Y\}$ are related homographicaly.*

Hint: Consider a system of coordinates with origin at the projection O of A on ε. If the angle is not a right one, apply formula $\tan(\varphi+\omega) = \frac{\tan(\varphi)+\tan(\omega)}{1-\tan(\varphi)\tan(\omega)}$ and see that $\frac{y-x}{d^2+xy} = k$, where $d = |AO|$ and $k = \tan(\varphi+\omega)$, where $\varphi+\omega = \chi$ is the fixed measure of the rotating angle. If the rotating angle is right, then $x \cdot y = -d^2$ and the homographic relation is an *"involution"* (i.e. homography f satisfying $f^{-1} = f$).

Exercise 4. *Point P is moving on a fixed circle and the lines $\{PA, PB\}$, which pass through two fixed points $\{A, B\}$ of the circle, intersect line ε at points $\{X(x), Y(y)\}$ (see figure 2-(II)). Show that $\{X, Y\}$ are related homographicaly.*

Hint: ([Pap96, 24, VII]) Consider the points $\{A', B'\}$, at which the parallels of ε from $\{A, B\}$ intersect the circle a second time. Define the points $\{M, N\}$, at which $\{AB', A'B\}$ intersect ε and show that the product of the signed distances $MX \cdot NY = MA \cdot NB$ is fixed. Finally apply exercise 2.

Exercise 5. *Point P is moving on a fixed circle $\kappa(O, r)$ centered at the origin of coordinates and lines $\{PA, PB\}$, which pass through two fixed points $\{A(a, b), B(-a, b)\}$ of the circle lying on a* *parallel to the x–axis, intersect this axis at points $\{X(x), Y(y)\}$ (see figure 3). Show that $\{X, Y\}$ are related homographicaly and satisfy*

$$axy \ + \ r^2(x-y) - ar^2 \ = \ 0 \ .$$

How the more general exercise 4 can be deduced from this special case?

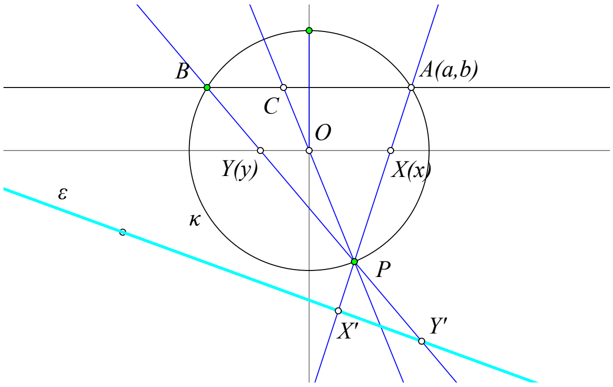

Figure 3: Homographic relation of $\{x, y\}$

Hint: The first part is a simple calculation. For the second part consider the perspectivities $\{f_1, f_2\}$ from points $\{A, B\}$ mapping the $x-$ axis to an arbitrary but fixed line ε. If $Y = f(X)$ represents the homographic relation between $\{X, Y\}$, then $Y' = (f_2 \circ f \circ f_1^{-1})(X')$ represents the homographic relation of the points $\{X', Y'\}$.

7 Orthogonality for pencils in involution

Here we handle a simple homography problem concerning a pencil of lines. The pencil consists of all lines emanating from the origin O of coordinates in the plane. Assume a Cartesian coordinate system at O and let a line through the origin intersect the line $x = 1$ at a point $(1, m)$. This line is represented by the equation $y = mx$, and $(1, m)$ are homogeneous coordinates parameterizing the pencil O^* of lines through the origin O. By the preceding discussion an involution of this pencil is defined through a function $m' = f(m)$, with $f(m) = (am + b)/(cm + d)$. By the involution condition d must be equal to $-a$ and we have the problem:

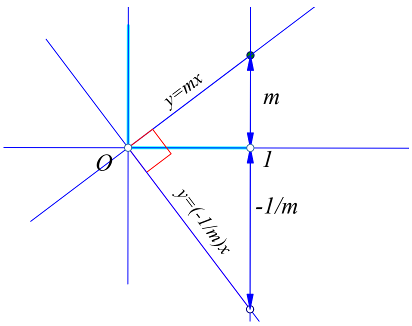

Figure 4: Involutive relation on a pencil of lines

Problem 1. To find the pairs of lines $(m, m' = f(m))$, which are orthogonal.

If the line corresponding to $m' = f(m)$ is orthogonal to the one corresponding to m, then m' must be equal to $-1/m$ (see figure 4). Thus, in order to locate pairs (m, m') which are orthogonal we must solve the equation

$$-(1/m) = (am + b)/(cm - a) \quad \Leftrightarrow \quad am^2 + (b + c)m - a = 0. \tag{13}$$

1. If a is non-zero, setting $(b + c)/a = 2d$, we arrive at equation $m^2 + 2dm - 1 = 0$, wich has always two real solutions m_1, m_2. Since their product $m_1 m_2 = -1$, the two solutions determine the same pair of orthogonals.

2. If a is zero, then the equation above becomes $(b + c)m = 0$ i.e. $b + c = 0$, and $f(m) = b/(cm) = -1/m$. Thus, in this case we have the special involution *"by orhtogonals"*, in which every line maps to its orthogonal.

The result of this short discussion is next theorem.

Theorem 3. *An involution on a pencil of lines either has exactly one pair (m, m') of orthogonal lines or all pairs $\{(m, m')\}$ are orthogonal.*

8 Involutions and pencils of circles

As noticed in equation 6 the fixed point equation of the general homographic relation has the discriminant

$$D = (a + d)^2 - 4(ad - bc) \tag{14}$$

1. If $D > 0$, then there are two real fixed points and the relation is called *"hyperbolic"*.

2. If $D < 0$, then there are two imaginary fixed points and the relation is called *" elliptic."*

3. If $D = 0$, then there are two real coincident fixed points and the relation is called *"parabolic"*.

Denoting by $\{x_1, x_2\}$ the two fixed points of the homography $x' = f(x)$, we compute the cross ratio

$$\begin{aligned}
(x_1 x_2; x x') &= \frac{x_1 - x}{x_2 - x} : \frac{x_1 - x'}{x_2 - x'} = \frac{x_1 - x}{x_2 - x} : \frac{x_1 - (ax + b)/(cx + d)}{x_2 - (ax + b)/(cx + d)} = \\
&= \frac{x_1 - x}{x_2 - x} : \frac{(cx + d)x_1 - (ax + b)}{(cx + d)x_2 - (ax + b)} = \frac{x_1 - x}{x_2 - x} : \frac{(cx_1 - a)x + (dx_1 - b)}{(cx_2 - a)x + (dx_2 - b)} = \\
&= \frac{x_1 - x}{x_2 - x} : \left(\frac{x + (dx_1 - b)/(cx_1 - a)}{x + (dx_2 - b)/(cx_2 - a)} \cdot \frac{cx_1 - a}{cx_2 - a} \right) = \\
&= \frac{x_1 - x}{x_2 - x} : \left(\left(\frac{-x + (dx_1 - b)/(-cx_1 + a)}{-x + (dx_2 - b)/(-cx_2 + a)} \right) \cdot \frac{cx_1 - a}{cx_2 - a} \right) = \\
&= \frac{x_1 - x}{x_2 - x} : \left(\left(\frac{-x + x_1}{-x + x_2} \right) \cdot \frac{cx_1 - a}{cx_2 - a} \right) = \frac{cx_2 - a}{cx_1 - a}.
\end{aligned}$$

The equality in the last line following from the fact that $y = (dx - b)/(-cx + a)$ represents the inverse f^{-1} of f, which also fixes $\{x_1, x_2\}$.

Last expression is constant, equal to k say. Thus, we have the theorem

Theorem 4. *The general homographic relation* $f(x) = (ax+b)/(cx+d)$ *is characterized by the property*

$$(x_1 x_2; x f(x)) = \frac{cx_2 - a}{cx_1 - a} = k, \quad \text{for all pairs} \quad (x, x' = f(x)), \tag{15}$$

where $\{x_1, x_2\}$ *are the fixed points of* f*.*

Conversely, the preceding equation, solving for $x' = f(x)$, implies the homographic relation:

$$x' = \frac{x(x_2 - kx_1) - (1-k)x_1 x_2}{x(1-k) - (x_1 - kx_2)}. \tag{16}$$

Note the particular value for

$$x = \frac{1}{1-k}(x_1 - kx_2) \quad \text{producing} \quad \frac{x - x_1}{x - x_2} = k \quad \text{and} \quad x' = \infty. \tag{17}$$

The particular case of *"involutions"* corresponds to the constant $k = -1$. In that case the cross ratio is

$$(x_1 x_2; x x') = -1 \quad \Leftrightarrow \quad (x_1 - x)(x_2 - x') + (x_1 - x')(x_2 - x) = 0, \tag{18}$$

which is equivalent with

$$x' = \frac{x(x_1 + x_2) - 2x_1 x_2}{2x - (x_1 + x_2)}. \tag{19}$$

Selecting the middle O with line coordinate $(x_1 + x_2)/2$ as origin, this equation becomes

$$x' = \frac{r^2}{x}, \quad \text{where} \quad r = \frac{x_1 - x_2}{2}. \tag{20}$$

This translates to the

Theorem 5. *An involution coincides with the restriction of an inversion* $I(O, r)$ *on its supporting line.*

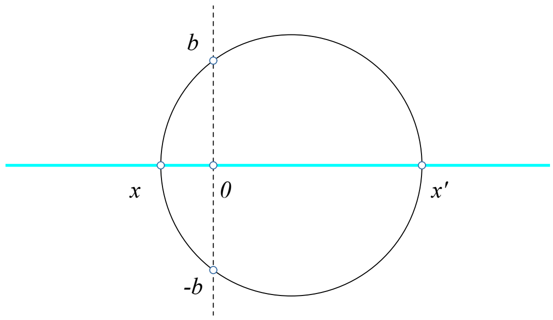

Figure 5: Elliptic involution $xx' = -b^2$ with no real fixed points

From this follows that points $\{(x, x')\}$ are intersection points of the supporting line with the members of a *"circle-pencil"*. The circle pencil being *"hyperbolic"* $(ad - bc < 0)$, *"elliptic"* $(ad - bc > 0)$ or *"parabolic"* $(ad - bc = 0)$, which is in accordance with the naming conventions at the beginning of the section. Notice that in the elliptic case, of two imaginary fixed points, $\{a \pm ib\}$, the radius $r^2 = -b^2$ and the *"inversion"* must be considered in the complex domain. Geometrically then the equation $xx' = -b^2$ has the interpretation of a map of the real line into itself produced by an *"intersecting pencil of circles"*, whose circle-members pass all through the points $a \pm ib$ (see figure 5). Notice also that the parabolic pencil cannot occur in the case of involutions, since per assumption $ad - bc \neq 0$.

9 Homography by polars

Fixing a conic κ and two lines $\{\varepsilon, \zeta\}$ we can define a homography using the polars of points $X \in \varepsilon$ w.r.t. the conic. Figure 6 shows the way. For each point $X \in \varepsilon$ the polar p_X

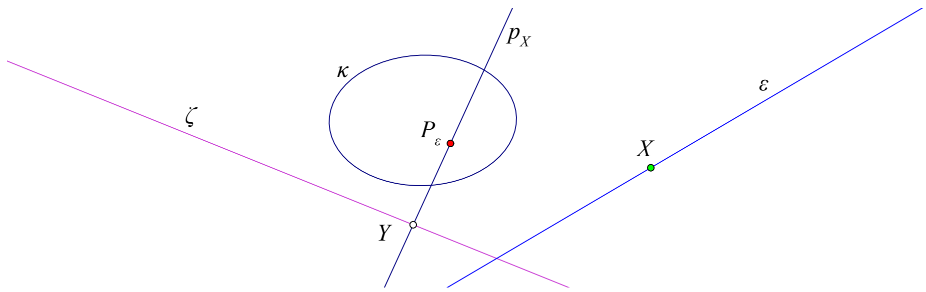

Figure 6: Homography by the polars of $X \in \varepsilon$

of X passes through a fixed point, the pole P_ε of the line ε and intersects the line ζ at a point Y.

Theorem 6. *Under the preceding assumptions the map $f : X \mapsto Y$ is a homography.*

Proof. A formal proof can be given using the representation of the conic in a projective coordinate system by a symmetric matrix A. Denoting by X the triple of coordinates of a point of the plane, the polar of this point is a line with coefficients given by the matrix multiplication $p_X = A \cdot X$. The line ε can be described by fixing two points $\{B, C\}$ of it and considering its parameterization $X = B + t \cdot C$. The polar p_X is then a linear combination of two particular polars:

$$p_X = AX = (AB) + t \cdot (AC)$$

and the triples $\{(AB), (AC)\}$ represent the coefficients of the particular polars $\{p_B, p_C\}$, intersecting at the pole P_ε of the line. If $Q = (q_1, q_2, q_3)$ are the coefficients of the line ζ, then the intersection points $\{Y = \zeta \cap p_X\}$ are given in coordinates using the vector product:

$$Q \times AX = (Q \times AB) + t \cdot (Q \times AC) = U + t \cdot V \ .$$

Thus, using the homogeneous coordinates of the lines $\{\varepsilon, \zeta\}$ defined respectively by the pairs of points $\{(B, C), (U, V)\}$, this relation between the lines is described by the identity function $t' = t$, which is a homography as claimed. □

Next exercise gives an alternative proof in the case of a circle κ using a composition of two homographies.

Exercise 6. *Given are a circle $\kappa(K, r)$ and two lines $\{\varepsilon, \varepsilon'\}$. For every point $X \in \varepsilon$ we correspond the point $Z = \varepsilon' \cap p_X$, where p_X is the "polar" of X w.r. to κ. Show that the correspondence $Z = h(X)$ from ε to ε' is a homography.*

Hint: The map $Y = f(X)$ of the line ε onto itself (see figure 7), with $Y = p_X \cap \varepsilon$ is a homography, actually an involution. All lines p_X for $X \in \varepsilon$ pass through the pol P of ε and the intersection $X' = p_X \cap XK$ is the inverse of X relative to κ satisfying $KX \cdot KX' = r^2$. The circle with diameter XY passes through X' is orthogonal to κ and intersects the line KP at a fixed point L. Then, introducing coordinates with origin at $O = \varepsilon \cap KP$ we have $OX \cdot OY = -OL^2 = k$ corresponding to the equation of coordinates $x \cdot y = k$, proving that f is an involution. Then $Z = g(Y)$ is also a *"central projection"* from P of line ε onto ε'. Hence their composition $Z = h(X) = g(f(X))$ is a homography.

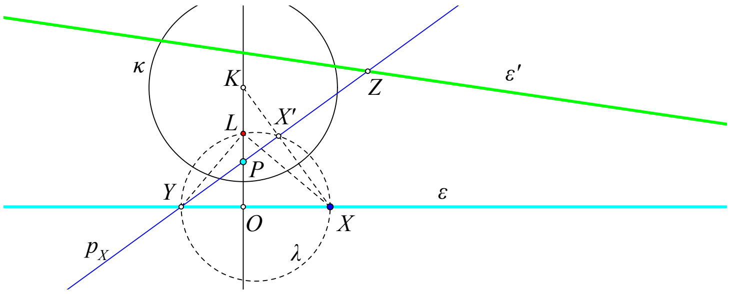

Figure 7: Homography through polars

Remark 1. This special case can be used to give an alternative proof of theorem 6. In fact, a general conic κ can be mapped by an appropriate *"projectivity"* f onto a circle. The lines $\{\varepsilon, \zeta\}$ map then to lines $\{\varepsilon', \zeta'\}$ and the correspondence $g : X \mapsto Y$ by polars of κ maps to a correspondence by polars $h : X' \mapsto Y'$ w.r.t. the circle, which is a homography. Then the correspondence by polars g is a composition $g = f^{-1} \circ h \circ f$, which is a homography too.

10 General pencil intersecting a line

A general *"pencil of circles"* intersecting a line ε defines an involution on the line. For

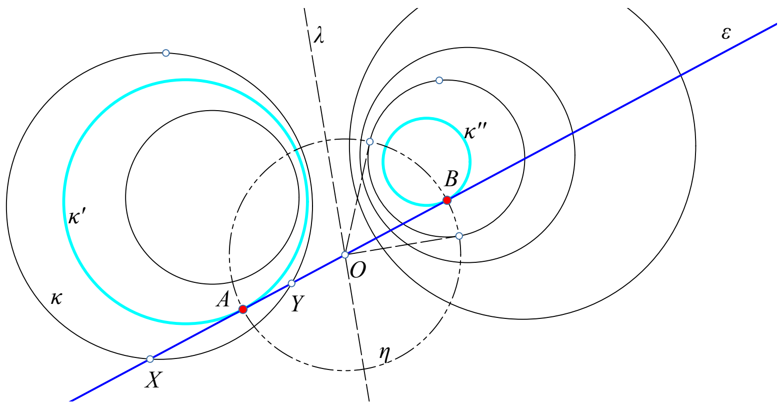

Figure 8: Homography $f : X \mapsto Y$ on line ε

this consider for an $X \in \varepsilon$ the unique circle κ of the pencil through X intersecting again the line at a second point Y. The homographic relation is $f : X \mapsto Y$ (see figure 8). The reason is very simple. Consider the intersection $O = \varepsilon \cap \mathfrak{A}$, where $\mathfrak{A}$ is the radical axis of the pencil. Then $OX \cdot OY = r^2$ is a constant independent of X. The fixed points of the involution induced on ε are the contact points $\{A, B\}$ of the two circles $\{\kappa', \kappa''\}$ of the pencil which are *tangent* to the line ε. Points $\{A, B\}$ can be located as intersections of ε with the circle η, whose center is O and its radius is the tangent from O to any circle of the pencil.

11 Homographic transformations between lines

Homographic transformations or projectivities of a line onto itself, as explained in section 6, can be generalized also to maps between two *different* lines $\{\alpha, \beta\}$. For this it suffices to adopt two homogeneous coordinate systems $\{(x, y), (x', y')\}$ on the respective lines and use an invertible matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \left\{ \text{for } t = \frac{x}{y}, t' = \frac{x'}{y'} \right\}, \quad t' = \frac{at+b}{ct+d}. \tag{21}$$

Such examples of homographies can be easily constructed using the second description, in which (t, t') represent line coordinates on two lines $\{\alpha, \beta\}$ and passing to their *"homogenization"* $\{t = u/v, t' = u'/v'\}$, as explained in section 6. Geometrically, this can be done by creating correspondences of the points of a line α to the points of a line β involving geometric constructions with intersection points (i) of two lines, (ii) of a line and a conic. Such constructions reduce to relations of the form (21) between corresponding line coordinates $\{t, t'\}$ of the two lines, thus defining homographic transformations between these lines.

A basic first example is the *"central projection"* from a point P of a line ε onto a second line ε' (see figure 9). Considering parameterizations along the lines:

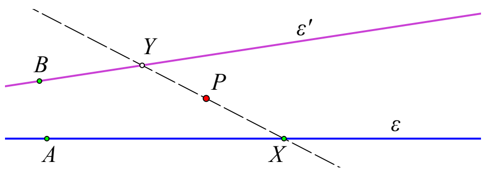

Figure 9: Central projection of ε onto ε' from P

$$\varepsilon : \quad X = A + xE, \qquad \varepsilon' : \quad Y = B + yF,$$

we find the relation between $\{x, y\}$ from the collinearity of the points $\{Y, X, P\}$:

$$\begin{vmatrix} b_1 + yf_1 & b_2 + yf_2 & 1 \\ a_1 + xe_1 & a_2 + xe_2 & 1 \\ p_1 & p_2 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad y = \frac{ax+b}{cx+d},$$

in which

$$a = E \cdot J(B - P), \qquad b = B \cdot JP + A \cdot J(B - P), \qquad c = F \cdot J(E), \qquad d = F \cdot J(A - P),$$

the dot denoting the *"inner product"* and J denoting the *"+90°-rotation"* $J(x, y) = (-y, x)$.

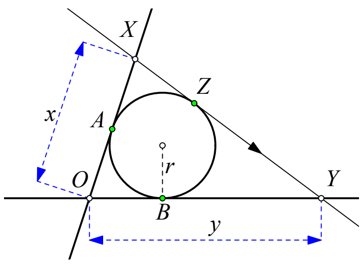

Figure 10: Homographic relation between points of different lines

Exercise 7. *Given is a circle and two tangents to it at points $\{A, B\}$, intersecting at the point O. Point Z moves on the circle and the tangent to it intersects $\{OA, OB\}$ at points $\{X, Y\}$. Show that $\{X, Y\}$ are related homographicaly (see figure 10).*

Hint: The radius r of the circle can be expressed through (see file **Tritangent circles**):

$$r^2 = \frac{|OA||AX||BY|}{|OA| + |AX| + |BY|} = \frac{a(x-a)(y-a)}{a + (x-a) + (y-a)} \quad \Leftrightarrow \quad y = (a^2 + r^2)\frac{x-a}{ax - (a^2 + r^2)},$$

where $a = |OA|$ and the systems of coordinates on $\{OA, OB\}$ have their origin at O.

The discussion in section 2 on compositions of homographies generalizes in part to homographies between different lines and the composition $g = f_2 \circ f_1$ of two homographies between different lines $\alpha \overset{f_1}{\to} \beta \overset{f_2}{\to} \gamma$ is a homography $g : \alpha \to \beta$. Also, representing $\{f_1, f_2\}$ with matrices $\{A, B\}$ we obtain the representation of g with the product matrix $C = BA$.

12 Good parameterizations of pencils and conics

Homographic relations are intimately connected with so called *"good"* parameterizations of pencils of lines (through a fixed point A) and also with *"good"* parameterizations of conics ([Ber87, II, p.173]). A good parameterization of the pencil A^* of lines through the point A is defined by fixing a line ε not containing A and considering as a parameter of a line $\zeta \in A^*$ its intersection $X = \varepsilon \cap \zeta$ (see figure 11). Using a projective coordinate

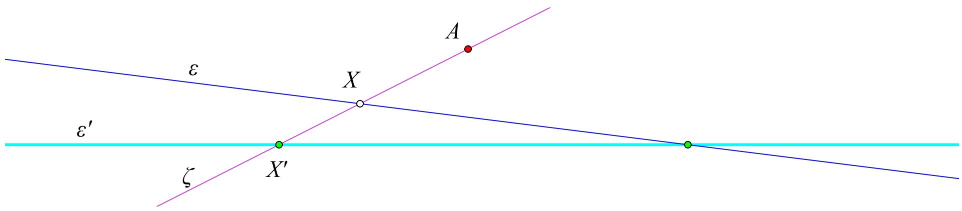

Figure 11: Good parameterization $\zeta \mapsto X = \varepsilon \cap \zeta$ of the pencil A^*

system of ε we associate then to ζ the coordinate x of X. From the discussion in section 11 follows, that if we use instead of ε another line ε', then the corresponding coordinates $\{x, x'\}$ are related by a homographic relation $x' = (ax + b)/(cx + d)$.

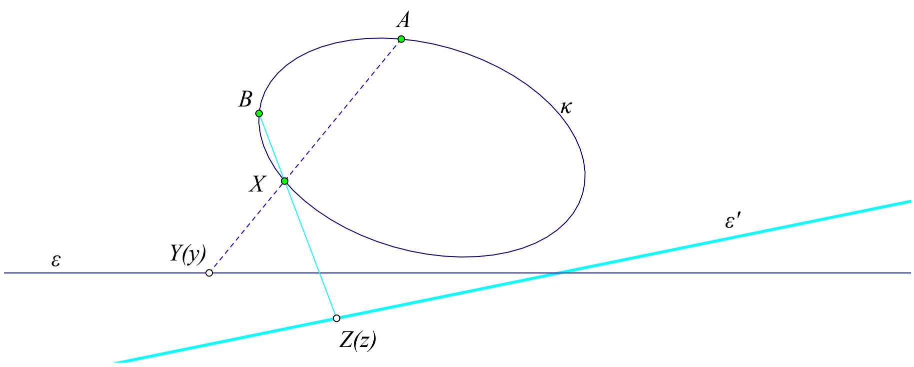

Figure 12: Good parameterization $X \mapsto Y = \varepsilon \cap AX$ of the conic κ

A *"good"* parameterization of a conic κ is reminiscent of the *"stereographic projection"* of the circle onto a line. The pattern is the same: We fix a point A on κ and a line ε not containing A. For every other point $X \in \kappa, X \neq A$ we associate the point $Y = AX \cap \varepsilon$ of the line. Fixing a projective coordinate system on ε we finally associate to X the coordinate y of Y (see figure 12). We call A the *"pole"* and ε the *axis* of the good parameterization. Changing to another pole and axis $\{B, \varepsilon'\}$ and a projective coordinate system on ε'

we obtain another good parameterization, related to the first by a homographic relation $z = (ay + b)/(cy + d)$. This is guaranteed by the theorem of Chassles Steiner characterizing the conics by just this property, i.e. that the two good parameterizations are related by such a homography ([Cox87, p.77]):

Theorem 7 (Chassles-Steiner theorem)**.** *A homographic relation $\{f : A^* \to B^*\}$ between two pencils of lines through two distinct points $\{A, B\}$ produces through the intersection points of corresponding lines $\{X = \zeta \cap f(\zeta), \zeta \in A^*\}$ a conic κ passing through $\{A, B\}$ (see figure 12). Conversely, two good parameterizations of the conic κ with poles/axes respectively $\{A, \varepsilon\}, \{B, \varepsilon'\}$ define coordinates $\{y, z\}$ of the same point $X \in \kappa$ related by homographic relation $z = (ay + b)/(cz + d)$.*

13 Homography between a conic and a line

Good parameterizations can be used to define a *"homography between a conic and a line"* $f : \kappa \to \varepsilon$. In fact, we say that f is such a homography if there is a good parameterization $\sigma : \kappa \to \eta$ such that the composition of the maps $f' = f \circ \sigma^{-1} : \eta \to \varepsilon$ (see figure 13) is a homography. If this is true for σ then it is true also for any other good parame-

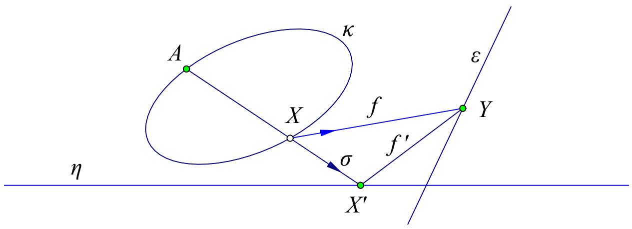

Figure 13: Homography $f : \kappa \to \varepsilon$ from a conic to a line

terization $\tau : \kappa \to \eta'$. This because $f'' = f \circ \tau^{-1} = f \circ \sigma^{-1} \circ \sigma \circ \tau^{-1} = f' \circ (\sigma \circ \tau^{-1})$ and, by theorem 7, $\sigma \circ \tau^{-1}$ is a homography from η' to η so that the composition with f' is also a homography from η' to ε.

By this definition a good parameterization $\tau : \kappa \to \varepsilon$ of κ becomes also a homography between the conic κ and the line ε. To satisfy the formal definition for it, it suffices to consider as σ appearing in the definition the same map $\sigma = \tau$, the composition f' becoming then the identity $e = \tau \circ \tau^{-1}$ on line ε.

14 Homography between two conics

Essentially the same method used in the preceding section, in order to define a homography between a conic and a line, can be used also for the definition of the *"homography between two conics"*. We define a map of this kind between two conics $f : \kappa \to \lambda$ by requiring the existence of two good parameterizations $\sigma : \kappa \to \varepsilon$ and $\tau : \lambda \to \eta$, such that the composition $f' = \tau \circ f \circ \sigma^{-1} : \varepsilon \to \eta$, called *"representation"* of f w.r.t. $\{\sigma, \tau\}$, is a homography (see figure 14) between the lines ε and η.

Again if there exist two good parameterizations $\{\sigma, \tau\}$ satisfying the above condition, then, by an argument similar to that of the preceding section, we see that the condition is valid by considering any other two parameterizations $\{\sigma', \tau'\}$ in the definition of the homography.

I am not going to develop here the whole theory of *"conic homographies"*. This is done in some detail in the file **Conic Homographies**. Here I discuss only three special examples. The first is a trivial one: the identity $e : \kappa \to \kappa, e(X) = X$, for every $X \in \kappa$.

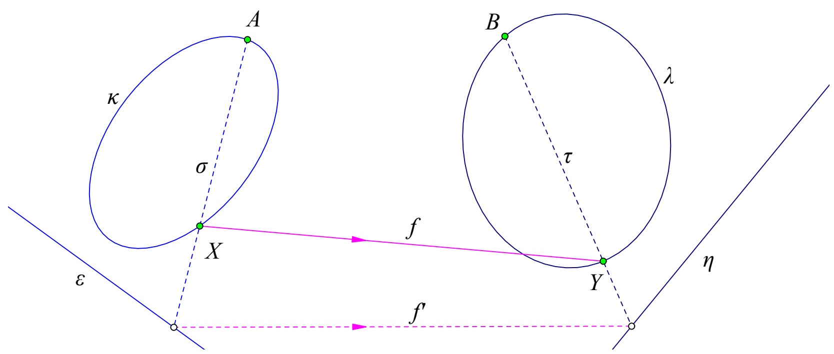

Figure 14: Homography f between two conics and its representation f'

For two good parameterizations $\{\sigma, \tau\}$ of κ the composition $e' = \tau \circ e \circ \sigma^{-1} = \tau \circ \sigma^{-1}$ is a homography between two lines according to theorem 7. Thus, the identity e of κ onto itself is a homography between conics.

The second example, perhaps the most prominent conic homography, is the *"harmonic homology"*, defined by a point or a line. In fact, given a conic κ and fixing a point $A \notin \kappa$ the harmonic homology f_A is a conic homography of κ onto itself that associates to each point $X \in \kappa$ of the conic the harmonic conjugate $Y \in \kappa$ point X w.r.t. (A, X_A), where X_A is the intersection of the line XA with the polar ε_A of A w.r.t. the conic (see figure 15). To show that $Y = f_A(X)$ is a homography fix a point $B \in \kappa$ and the line AB intersecting

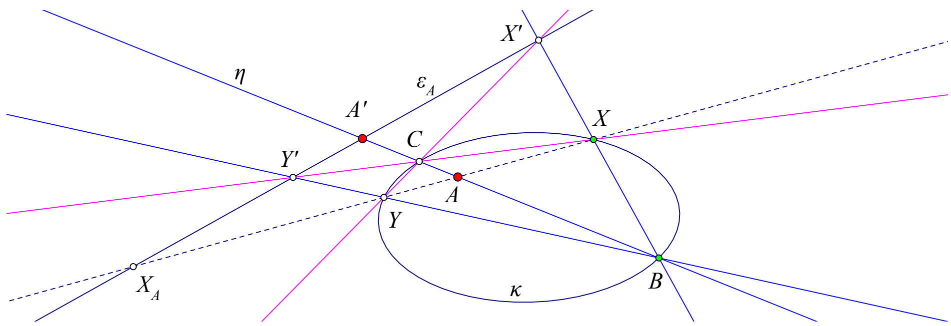

Figure 15: dd

a second time κ at C. Consider also the good parameterizations $\{\sigma_B, \sigma_C\}$ mapping each $X \in \kappa$ respectively to $X' = \varepsilon_A \cap BX$ and $X'' = \varepsilon_A \cap CX$. Due to the property of the polar to produce collinear points $\{X' = \sigma_B(X), C, Y,\}$ the harmonic homology f_A can be written $f_a = \sigma_C^{-1} \circ \sigma_B$ and can be represented using twice σ_B

$$f_A' = \sigma_B \circ f_A \circ \sigma_B^{-1} \;=\; \sigma_B \circ (\sigma_C^{-1} \circ \sigma_B) \circ \sigma_B^{-1} = \sigma_B \circ \sigma_C^{-1} \,,$$

which, according to the first example, is a line homography of the line ε_A onto itself. We notice that f_A is *"involutive"*, i.e. it coincides with its inverse $f_A^{-1} = f_A$ or equivalently $f_A(f(A(X))) = X$ for every $X \in \kappa$.

The third example is a special case of the previous one, in which the conic κ is replaced by a circle and the point A lies outside the κ. In this case the harmonic homology has two fixed points. We discuss this in the next section.

15 General pencil intersecting a circle

An intersecting or non-intersecting *"pencil of circles"* $\mathcal{P}$ defines through the intersection points $\{X, X'\}$ of its circle-members $\{\eta \in \mathcal{P}\}$ with a fixed circle κ a harmonic homology (involution) on κ. To see this, consider for an $X \in \kappa$ the unique circle $\eta \in \mathcal{P}$ of the pencil through X intersecting again the circle at a second point X'. The homographic relation is $f : X \mapsto X'$ (see figure 16).

The reason for this is very simple. Consider the intersection $F = \zeta \cap XX'$, where ζ is the radical axis of the pencil $\mathcal{P}$. Point F does not depend on the particular circle-member $\eta \in \mathcal{P}$ of the pencil. It is the same for all circles of the pencil, since taking another circle $\eta' \in \mathcal{P}$ of the pencil the radical axis of $\{\zeta, \eta'\}$ must pass through F too. Thus, drawing the two tangents $\{FX_1, FX_2\}$ to κ from F, real or imaginary, the transformation $X \mapsto X'$ coincides with the inversion f with center F and power $|FX_1|^2$. This in turn coincides with the *"harmonic homology"* of κ with fixed point F and axis the *"polar"* ʃı of F w.r.

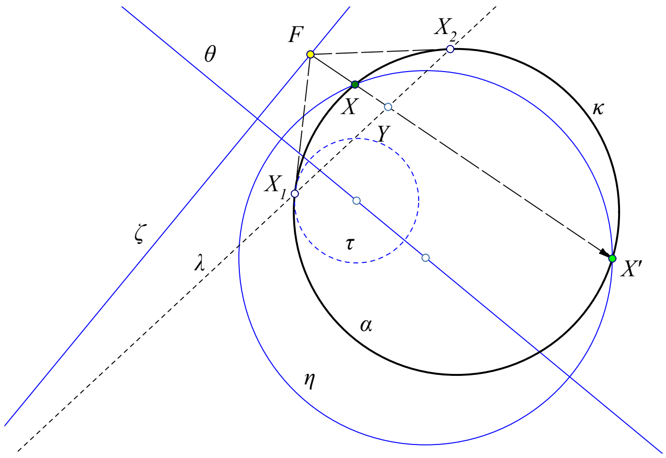

Figure 16: Involution $f : X \mapsto X'$ on the circle κ

to κ. The fixed points of this homography induced on κ by the pencil $\mathcal{P}$ are the contact points $\{X_1, X_2\}$, real or imaginary, of the two tangents.

16 Three points and a circle through one of them

Consider a circle κ passing through one vertex, A say, of the triangle ABC. Each point P of the circle defines two circles $\{\kappa_B = (BAP), \kappa_C = (CAP)\}$ (see figure 17).

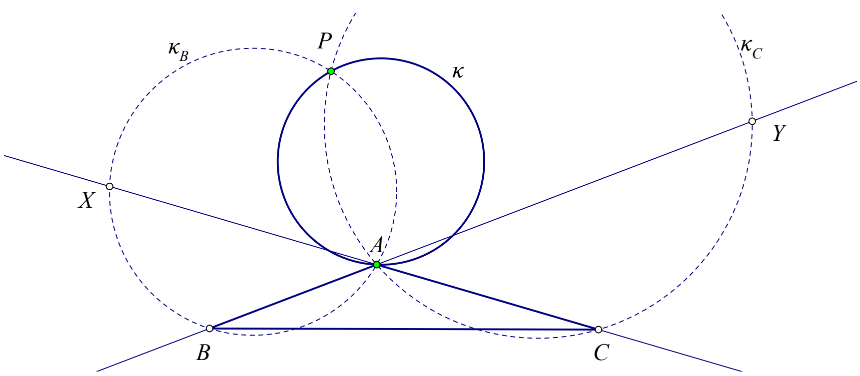

Figure 17: Homography defined by the triangle ABC and the circle κ

Theorem 8. *The second intersection points $\{X, Y\}$ of the circles respectively $\{\kappa_B, \kappa_C\}$ with the lines $\{AC, AB\}$ define points $\{X, Y\}$ homographicaly related.*

Proof. A short proof can be given by transforming the configuration through an inversion $X' = f(X)$ w.r.t. a circle $\nu(A, r)$ centered at A and with arbitrary radius r. Such an inversion transforms the circles $\{\kappa, \kappa_B, \kappa_C\}$ to corresponding lines $\{\mu, \mu_B, \mu_C\}$ and the points $\{B, C, P, X, Y\}$ respectively to points $\{B', C', P', X', Y'\}$ as shown in figure 18. In this con-

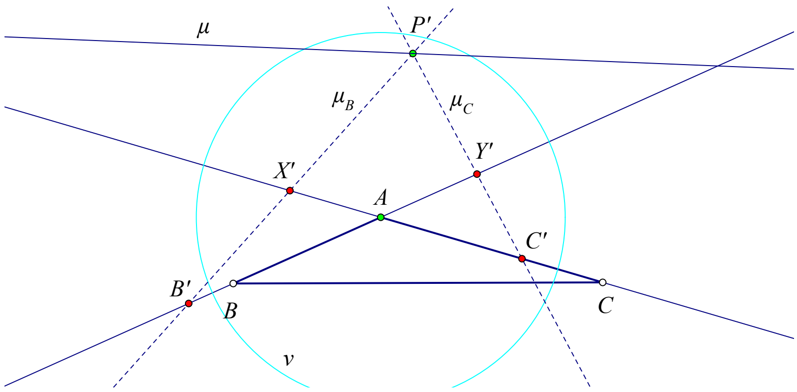

Figure 18: After the inversion w.r.t. $\nu(A, r)$

figuration $\{P'\}$ varies on μ and the variable lines $\{\mu_B, \mu_C\}$ pass through the fixed points $\{B', C'\}$ and define on AC, AB respectively the points $\{X', Y'\}$, which are homographicaly related. This because the map $X' \mapsto Y'$ is the composition $f = f_2 \circ f_1$ of the perspectivity f_1 from B' of line AC to line μ and the perspectivity f_2 from C' of line μ to line AB. Thus, considering the signed distances from A as coordinates $\{x, y\}$ on lines $\{AC, AB\}$, the coordinates of X', Y' are related by a function of the form $y' = (ax' + b)/(cx' + d)$. Expressing these by the coordinates of the inverse points $\{x' = r^2/x\ ,\ y' = r^2/y\}$ we see that $\{x, y\}$ are related by a function of the form

$$\frac{r^2}{y} = \frac{a(r^2/x) + b}{c(r^2/x) + d} \quad \Leftrightarrow \quad y = r^2 \frac{cr^2 + dx}{ar^2 + bx},$$

which is indeed a homographic relation. □

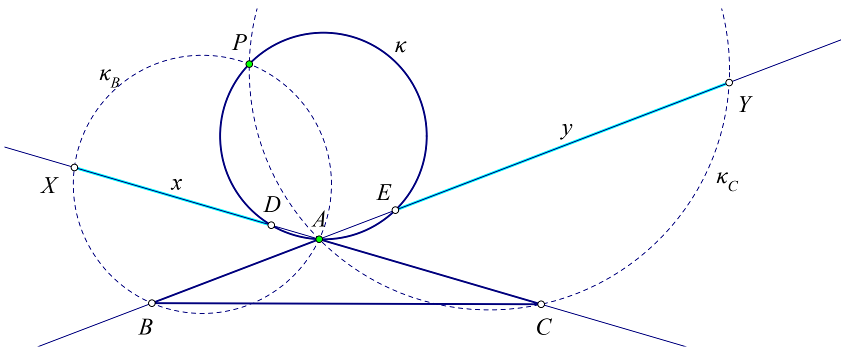

Figure 19: $\{x = DX, y = EY\}$ have constant product xy

Corollary 1. *With the notation and assumptions of this section and denoting by $\{D, E\}$ the second intersection points of the circle κ respectively with lines $\{AC, AB\}$, the signed distances $\{x = DX, y = EY\}$ satisfy $xy = k$ with k constant (see figure 19).*

Proof. In fact, changing the coordinate origins from A to D for line AC and from A to E for line AB changes the homography of theorem 8 to an other homography

$$y = (a'x + b')/(c'x + d').$$

It is then easy to see that for P tending to D the variables $\{x \to 0, y \to \infty\}$ and for P tending to E the variables $\{x \to \infty, y \to 0\}$. These conditions imply $a' = d' = 0$ and prove the corollary. □

Corollary 2. *The value of the preceding constant is $k = BE \cdot CD$.*

Proof. Consider the circle κ_0 circumscribing the triangle ABC. This intersects the circle κ besides point A also at a second point P_0. As P tends to P_0 the two circles $\{\kappa_B, \kappa_C\}$ tend both to coincide with the circle κ_0 and the points $\{X, Y\}$ tend to coincide respectively with $\{C, B\}$, thereby proving the corollary (see figure 20). □

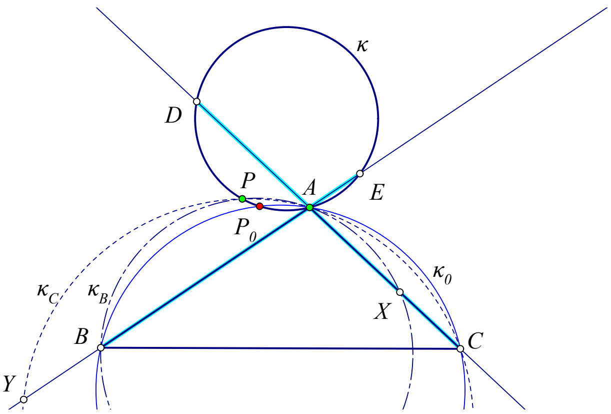

Figure 20: The value of the constant $k = BE \cdot CD$

Theorem 9. *With the notation of this section, the lines $\{XB, YC\}$ intersect at a point Q, which for P varying on the circle κ describes a line parallel to the line DE (see figure 21).*

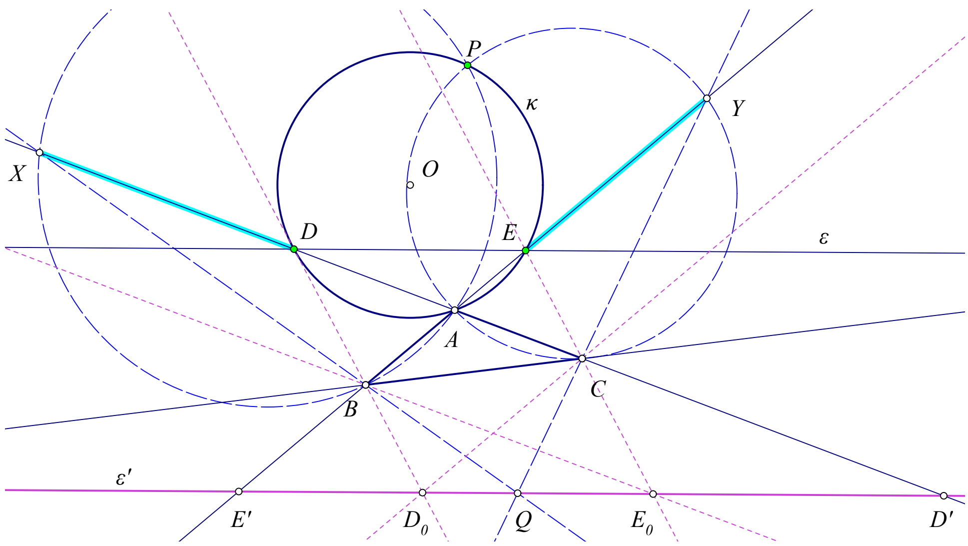

Figure 21: The line ε' locus of intersections $\{Q = XB \cap YC\}$ for $P \in \kappa$

Proof. The proof results by showing that the locus ε' of Q is one of the lines of a degenerate conic consisting of two intersecting lines. For this we notice first that theorem 8 implies that the two pencils of lines $\{B^*, C^*\}$ are homographicaly related. Hence by theorem 7 the intersection point $Q = XB \cap YC$ of corresponding lines describes a conic passing through the centers $\{B, C\}$ of the two pencils. We show that this conic passes through four collinear points $\{E', D_0, E_0, D'\}$ obtained for special positions of P on κ and defined as follows (see figure 21).

1. E' is obtained for $X = A \quad \Rightarrow \quad Q = Y \in AB$ and $k = DA \cdot EE'$.
2. D' is obtained for $Y = A \quad \Rightarrow \quad Q = X \in AC$ and $k = EA \cdot DD'$.
3. D_0 is obtained for $P = X = D$ implying that Y goes to infinity on AB and CD_0 becomes parallel to AB.
4. E_0 is obtained for $P = Y = E$ implying that X goes to infinity on AC and BE_0 becomes parallel to AC.

From the two first conditions we have $DA \cdot EE' = EA \cdot DD'$ showing that $\{DE, D'E'\}$ are parallel. From (3) we have $\frac{BD_0}{BD} = \frac{AC}{AD} \quad \Rightarrow \quad \frac{AC+AD}{AD} = \frac{CD}{AD}$ (*). From the parallelism of $\{DE, D'E'\}$ we have $\frac{BE'+BE}{BE} = \frac{EE'}{BE} = \frac{k}{BE \cdot DA}$ (**). But (*) = (**) since this equation simplifies to $k = CD \cdot BE$ which is true. This implies $\frac{BD_0}{BD} = \frac{BE'}{BE}$ showing that D_0E' is parallel to DE. Analogously we show that E_0D' is parallel to ED, which completes the proof that the four points are collinear. This implies that the geometric locus of Q is a subset of a degenerate conic consisting of the two lines $\{\varepsilon', BC\}$, point Q varying on line ε'. □

The sequence of the conclusions of the preceding discussion can be reversed and for a given triangle ABC and a selected vertex, A say, show a means to associate to each line ε of the plane a circle κ_ε passing through this vertex.

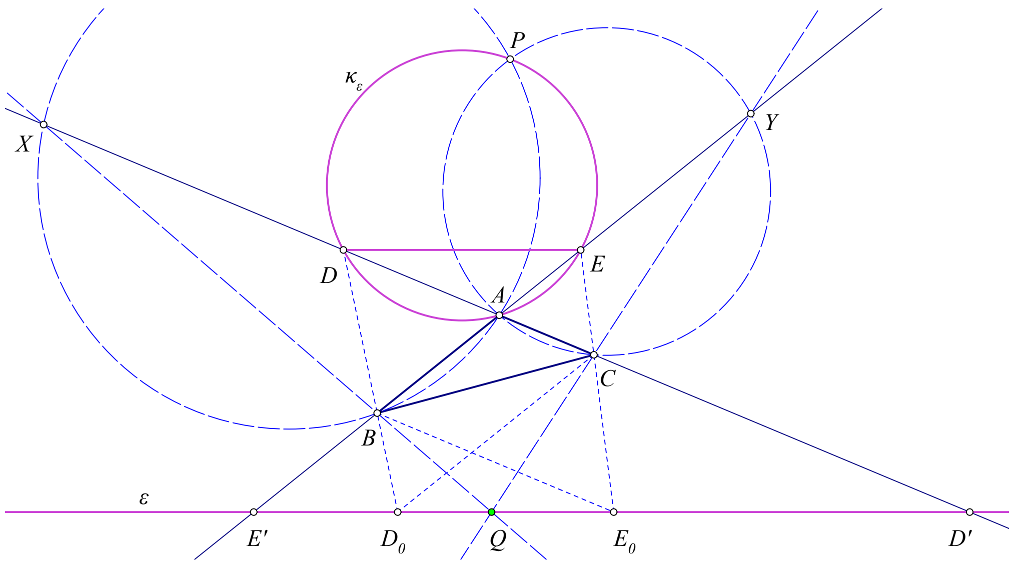

Figure 22: Associating to a line ε a circle κ_ε through A

Theorem 10. *Given a triangle $\tau = ABC$ and a line ε we consider for each point $Q \in \varepsilon$ the intersections $\{X = BQ \cap AC\,,\ Y = CQ \cap AB\}$. The circles $\{(ABX), (ACY)\}$ intersect a second time at a point P describing a circle κ_ε (see figure 22).*

Proof. We start with the homographic relation $f : X \mapsto Y$ of the line AC onto AB. This is the composition $f = f_2 \circ f_1$ of two central projections: f_1 of the line AC to ε with center

B and f_2 of the line ε to AB with center C. The signed distances $\{x = AX, y = AY\}$ are seen again, as in the preceding theorem to satisfy the relation $x \cdot y = k$ with $k = CD \cdot BE$. The meaning of the points $\{D, E, E', D', D_0, E_0\}$ seen in figure 22 and the related to them properties together with their proofs are the same with the corresponding labels and proofs used in theorem 9. The circle κ_ε is now defined as the circumcircle of triangle DAE. Applying now the results of theorem 9 we find that for each point P of this circle the corresponding point Q constructed by that theorem is on the line ε. But the dependence of Q from P is precisely the inverse of the dependence of P from Q, thereby showing that P is on the circle κ_ε. □

Remark-2 In [Pam21] we show that the map associating $Q \mapsto P$ is the restriction on the line ε of a Moebius transformation with limit points $\{B, C\}$.

Bibliography

[Ber87] Marcel Berger. *Geometry vols I, II*. Springer Verlag, Heidelberg, 1987.

[Cox87] H Coxeter. *Projective Geometry*. Springer, New York, 1987.

[Pam21] P. Pamfilos. Geometric aspects of Moebius transformations. *Inernational Journal of Geometry*, 10:115–122, 2021.

[Pap96] Georges Papelier. *Exercices de Geometrie moderne*. Editions Jacques Gabay, 1996.

Related material

1. **Cross Ratio**
2. **Projective line**
3. **Projective plane**
4. **Tritangent circles**

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