Minimum Chord Problem

A point P moves on the basis BC of a triangle ABC. Lines PE, PF at fixed angles $u = \angle APE$, $v = \angle APF$ to AP from P are cutting the sides AB and AC at E and F. Find the position P for which $|EF|$ becomes minimum.

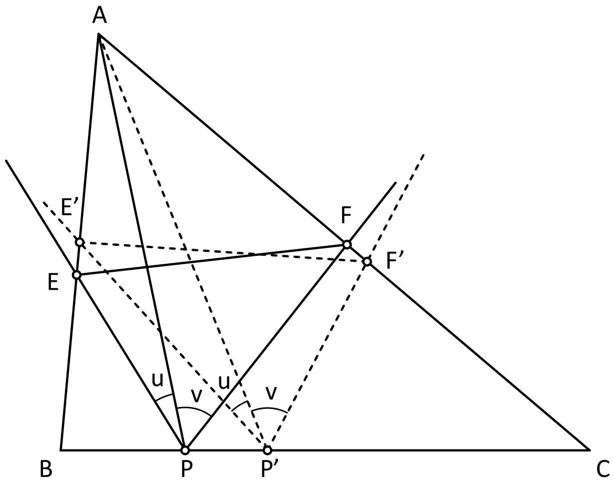

Fig.1. Minimum cord problem.

Solution

Draw the circumcircle of ΔPFE. Let it cut BC, AB and AC at G, H and J for the second time. Draw the line HJ. Draw a line parallel to AP, passing through G and cutting the circumcircle at L. Draw another line parallel to HJ, passing through L and cutting the circumcircle at K. Join K and P with a line, cutting HJ at M. Drop the perpendicular from A to HJ. Let the foot be N. Minimum $|EF|$ takes place when $|HM| = |NJ|$.

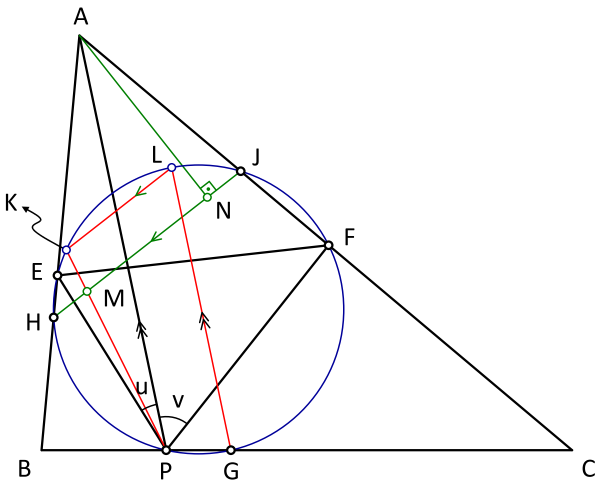

Fig.2. Geometric construction for the solution of the minimization problem.

Proof

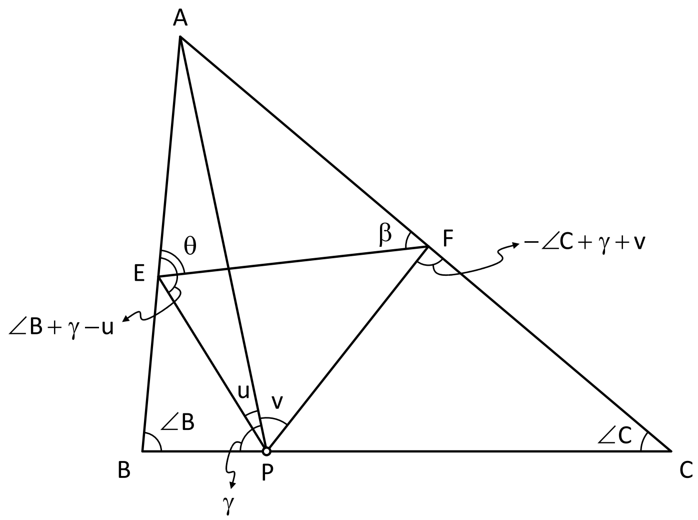

Fig.3. Angle definitions.

The angles that are going to be used at the proof are shown in Fig.3. Then the following equations can be written.

$$|EF| = |AE|\cos\theta + |AF|\cos\beta \tag{1}$$

$$|AE|\sin\theta = |AF|\sin\beta \tag{2}$$

$$\theta + \beta = \angle B + \angle C \tag{3}$$

$$|AE| = |AP|\frac{\sin u}{\sin(\angle B + \gamma - u)} \tag{4}$$

$$|AF| = |AP|\frac{\sin v}{\sin(-\angle C + \gamma - v)} \tag{5}$$

$$|AP| = \frac{h}{\sin\gamma} \tag{6}$$

h is the length of the perpendicular dropped from A to BC (not shown in Fig.3). Note that γ is the only independent variable. Then, $|EF|$ becomes a minimum when $d|EF|/d\gamma = 0$. Taking the derivative of Eqn.(1) with respect to γ:

$$\frac{d|EF|}{d\gamma} = \frac{d|AE|}{d\gamma}\cos\theta + \frac{d|AF|}{d\gamma}\cos\beta - |AE|\sin\theta\frac{d\theta}{d\gamma} - |AF|\sin\beta\frac{d\beta}{d\gamma} \tag{7}$$

The derivative of Eqn.(3) with respect to γ yields:

$$\frac{d\theta}{d\gamma} + \frac{d\beta}{d\gamma} = 0 \tag{8}$$

When Eqns.(2) and (8) are inserted into Eqn.(7), the last two terms cancel out and the final equation reduces to:

$$\frac{d|EF|}{d\gamma} = \frac{d|AE|}{d\gamma}\cos\theta + \frac{d|AF|}{d\gamma}\cos\beta \tag{9}$$

Inserting $|AP|$ from Eqn.(6) into Eqn.(4) and taking the derivative with respect to γ:

$$\frac{d|AE|}{d\gamma} = \frac{d}{d\gamma}\left(\frac{h}{\sin\gamma}\cdot\frac{\sin u}{\sin(\angle B+\gamma-u)}\right)$$
$$\frac{d|AE|}{d\gamma} = h\sin u\left(-\frac{\cos\gamma}{\sin^2\gamma}\cdot\frac{1}{\sin(\angle B+\gamma-u)} - \frac{1}{\sin\gamma}\cdot\frac{\cos(\angle B+\gamma-u)}{\sin^2(\angle B+\gamma-u)}\right)$$
$$\frac{d|AE|}{d\gamma} = h\sin u\frac{-\sin(\angle B+2\gamma-u)}{\sin^2\gamma\cdot\sin^2(\angle B+\gamma-u)} \tag{10}$$

Similarly, $d|AF|/d\gamma$ is calculated as:

$$\frac{d|AF|}{d\gamma} = h\sin v\frac{-\sin(-\angle C+2\gamma+v)}{\sin^2\gamma\cdot\sin^2(-\angle C+\gamma+v)} \tag{11}$$

Inserting Eqns.(10) and (11) into Eqn.(9) and equating $d|EF|/d\gamma$ to zero:

$$\frac{\sin u\cdot\sin(\angle B+2\gamma-u)}{\sin^2(\angle B+\gamma-u)}\cos\theta = \frac{\sin v\cdot\sin(\angle C-2\gamma-v)}{\sin^2(\angle C-\gamma-v)}\cos\beta \tag{12}$$

Eqn.(12) is the only requirement for $|EF|$ be minimum. This equation can further be simplified. Inserting Eqns.(4) and (5) into Eqn.(2) and rearranging the result:

$$\frac{\sin u}{\sin(\angle B+\gamma-u)}\sin\theta = \frac{\sin v}{-\sin(\angle C-\gamma-v)}\sin\beta \tag{13}$$

Dividing Eqn.(12) by (13):

$$\boxed{\frac{\sin(\angle B+2\gamma-u)}{\sin(\angle B+\gamma-u)\cdot\tan\theta} = \frac{\sin(\angle C-2\gamma-v)}{-\sin(\angle C-\gamma-v)\cdot\tan\beta}} \tag{14}$$

Now a geometric construction realizing Eqn.(14) is going to be sought.

The angles $\angle B+\gamma-u$, $\angle C-\gamma-v$, θ and β are readily available in Fig.3, but $\angle B+2\gamma-u$ and $\angle C-2\gamma-v$ do not appear in any form. To get the missing angles, first draw the circumcircle of ΔPFE. Let it cut BC, AB and AC at G, H and J for the second time (Fig.4). Then draw a line parallel to AP, passing through G and cutting the circumcircle at L. Then using the circle, $\gamma = \angle APB = \angle LGP = \angle LFP = \angle XEP$. Now the concave angle $\angle AEX = \angle B+2\gamma-u$ and $\angle LFC = -\angle C+2\gamma+v$.

After some angle algebra and using the circle, one gets; $\angle HLP = \angle HEP = \pi-(B+\gamma-u)$, $\angle HPL = \angle AEL = (B+2\gamma-u)-\pi$, $\angle JLP = \angle CFP = -\angle C+\gamma+v$, $\angle JPL = \angle JFL = \pi-(-\angle C+2\gamma+v)$. Writing the sine rule for ΔHPL and ΔPJL,

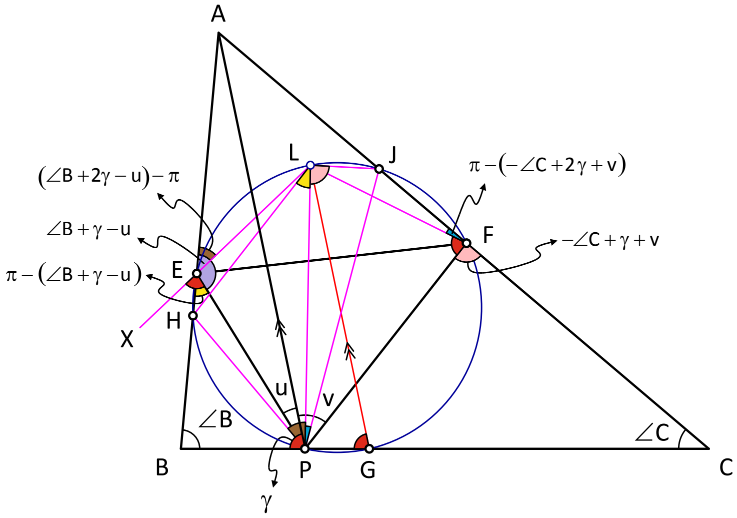

Fig.4. New angle definitions.

$$\frac{|HL|}{-\sin(\angle B+2\gamma-u)}=\frac{|HP|}{\sin(\angle B+\gamma-u)} \tag{15}$$

$$\frac{|LJ|}{-\sin(\angle C-2\gamma-v)}=\frac{|PJ|}{-\sin(\angle C-\gamma-v)} \tag{16}$$

Dividing Eqn.(15) by (16):

$$-\frac{\sin(\angle B+2\gamma-u)\cdot\sin(\angle C-\gamma-v)}{\sin(\angle C-2\gamma-v)\cdot\sin(\angle B+\gamma-u)}=\frac{|HL|\cdot|PJ|}{|LJ|\cdot|HP|} \tag{17}$$

Comparing Eqn.(17) with (14), one gets,

$$\frac{|HL|\cdot|PJ|}{|LJ|\cdot|HP|}=\frac{\tan\theta}{\tan\beta} \tag{18}$$

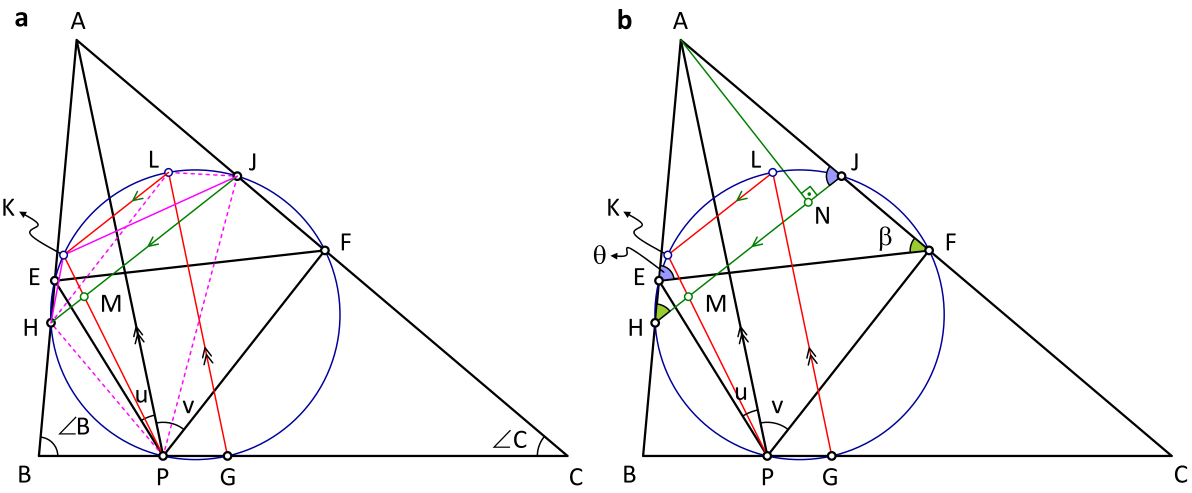

Fig.5. Final touches.

The left side of Eqn.(18) can be reduced into a single ratio. Draw a line parallel to HJ, passing through L and cutting the circumcircle at K (Fig.5a). Now $|KJ| = |HL|$ and $|HK| = |LJ|$. Noting that $\angle KJP$ and $\angle KHP$ are complementary, the ratio of areas of ΔKJP and ΔKHP is calculated as,

$$\frac{A(\Delta KJP)}{A(\Delta KHP)} = \frac{|KJ| \cdot |PJ| \cdot \sin(\angle KJP)}{|HK| \cdot |HP| \cdot \sin(\angle KHP)} = \frac{|HL| \cdot |PJ|}{|LJ| \cdot |HP|} = \frac{|MJ|}{|HM|} \tag{19}$$

Drop the perpendicular from A to HJ. Let the foot be N (Fig.5b). Using the circle; $\angle AEF = \angle AJH = \theta$ and $\angle AFE = \angle AHJ = \beta$. By using the definition of tangent, one gets,

$$\frac{|HN|}{|NJ|} = \frac{\tan\theta}{\tan\beta} \tag{20}$$

Inserting Eqns.(19) and (20) into Eqn.(18),

$$\frac{|MJ|}{|HM|} = \frac{|HN|}{|NJ|} \tag{21}$$

Eqn.(21) implies $|HM| = |NJ|$.