Isodynamic points of the triangle

A file of the Geometrikon gallery by Paris Pamfilos

It is in the nature of an hypothesis, when once a man has conceived it, that it assimilates every thing to itself, as proper nourishment; and, from the first moment of your begetting it, it generally grows the stronger by every thing you see, hear, read, or understand.

L. Sterne, Tristram Shandy, II, 19

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(Last update: 09-03-2021)

1 Apollonian circles of a triangle

Denote by $\{a = |BC|,\ b = |CA|,\ c = |AB|\}$ the lengths of the sides of triangle ABC. The *"Apollonian circle of the triangle relative to side BC"* is the locus of points X which satisfy

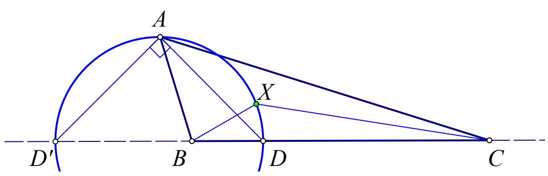

Figure 1: The Apollonian circle relative to the side BC

the condition $|XB|/|XC| = c/b$. By its definition, this circle passes through the vertex A and the traces $\{D, D'\}$ of the bisectors of $\widehat{A}$ on BC (see figure 1). Since the bisectors are orthogonal, DD' is a diameter of this circle. Analogous is the definition of the Apollonian circles on sides CA and AB. The following theorem ([Joh60, p.295]) gives another characterization of these circles in terms of **Pedal triangles**.

Theorem 1. *The Apollonian circle α, passing through A, is the locus of points X, such that their pedal triangles are isosceli relative to the vertex lying on BC (see figure 2).*

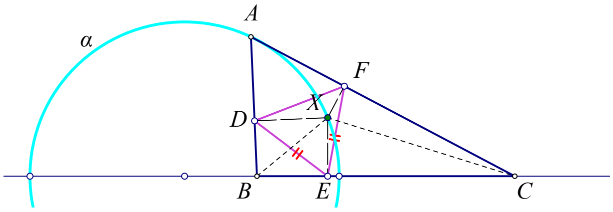

Figure 2: The pedals of points X on the Apollonian circles of the triangle are isosceli

Proof. This follows from the *sine formula*

$$\frac{|ED|}{|EF|} = \frac{|XB|\sin(\hat{B})}{|XC|\sin(\widehat{C})} = \frac{c\cdot\sin(\hat{B})}{b\cdot\sin(\widehat{C})} = 1. \qquad \square$$

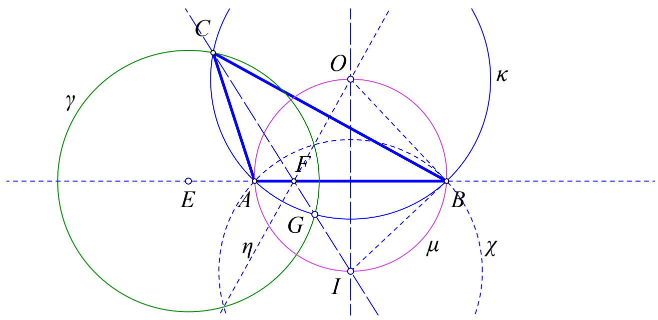

Figure 3: Points $\{A, G, I\}$ are collinear

Exercise 1. *If γ is the Apollonian circle through C of the triangle ABC intersecting again the circumcircle $\kappa(O)$ of ABC in G and I is the diametral to O on the circle $\mu = (AOB)$, show that $\{A, G, I\}$ are collinear (see figure 3). For $F = CG \cap AB$ show that $(C, G) \sim (F, I)$ are harmonic pairs and OF is the polar of I relative to γ.*

Hint: Consider the circle $\chi(I, |IB|)$ and see first that the three circles $\{\gamma, \kappa, \chi\}$ are pairwise orthogonal, which implies that I is on the radical axis of $\{\gamma, \kappa\}$.

Exercise 2. *With the definitions of the preceding exercise, show that triangles $\{ABC, AGJ, GBJ\}$ are similar, where $J = CI \cap EO$ (see figure 4).*

Hint: Angle chasing: $\widehat{AJG} = \widehat{GJB} = \widehat{AOB}/2 = \widehat{ACB}, \quad \widehat{AGJ} = \widehat{ABC}, \quad \widehat{JGB} = \widehat{CAB}.$

Exercise 3. *With the definitions of the two previous exercises, define the similarities $\{f_A, f_B, f_C\}$, where f_A has center at A, rotation angle $\widehat{A}$ and ratio $k_A = AB/AC$, the others being defined by cyclic permutation of the letters $\{A, B, C\}$. Show that the composition $f_C \circ f_B \circ f_A$ coincides with the symmetry at B.*

Hint: By the similarity of the triangles of the previous exercise, $g = f_B \circ f_A$ has $g(G) = G$. Also $g(B) = A$. Hence g is the similarity with center at G, rotation angle $\widehat{BGA}$ and ratio $GB/GA = (GB/GJ)\cdot(GJ/GA) = (AB/AC)\cdot(BC/BA)$. Its composition with f_C defines $h = f_C \circ g = f_C \circ f_B \circ f_A$ which leaves B fixed. Since the similarities build a group, the composition is a similarity with fixed point B. The angle is the sum of the angles of the similarities $\{g, f_C\}$, which is π. Finally, the ratio is the product of ratios, which turns out to be 1, thereby proving the claim.

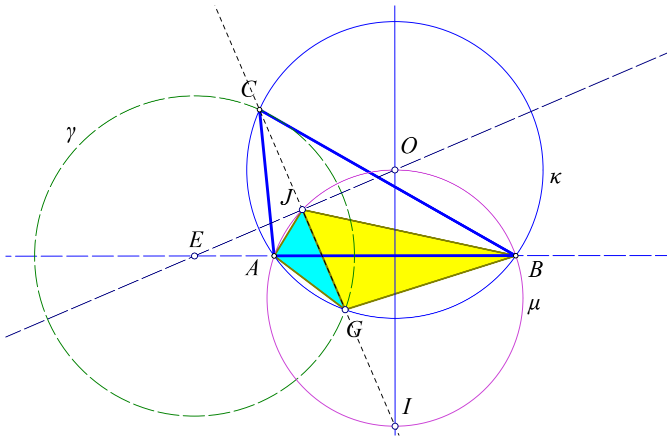

Figure 4: Three similar triangles

2 Isodynamic points, Brocard and Lemoine axes

Taking the ratios of the sides of a triangle ABC, in a given orientation, we can define three constants

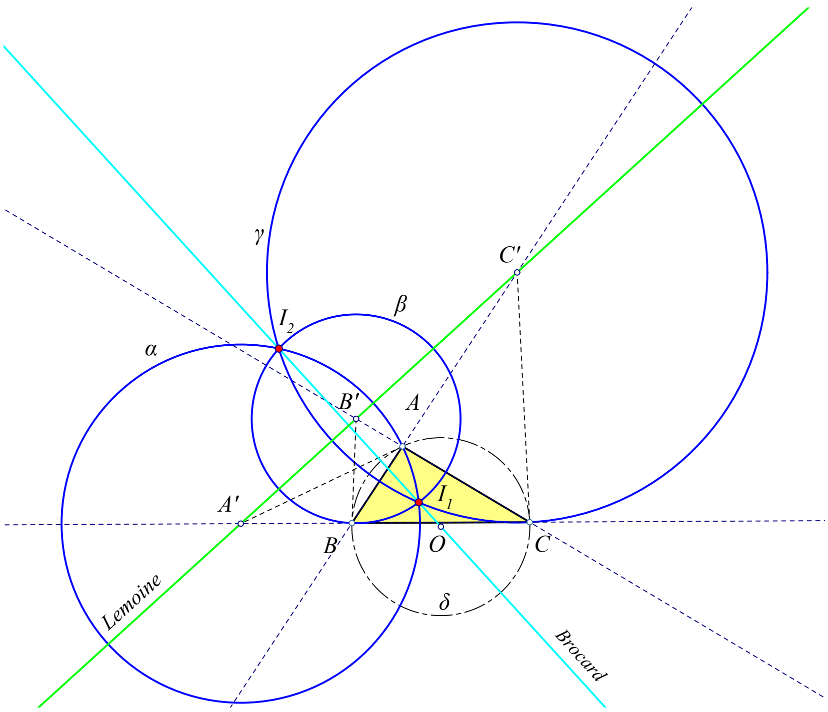

Figure 5: Isodynamic points $\{I_1, I_2\}$ of the triangle ABC

$$k_a = |AB|/|AC|, \qquad k_b = |BC|/|BA|, \qquad k_c = |CA|/|CB|,$$

And three corresponding Apollonian circles $\{\alpha, \beta, \gamma\}$ on the sides of the triangle

$$a = |BC|, \quad b = |CA|, \quad c = |AB|, \quad \text{w.r. to the constants} \quad k_a, \quad k_b, \quad k_c.$$

Theorem 2. *The three Apollonian circles $\{\alpha, \beta, \gamma\}$ of the triangle form an intersecting pencil of circles passing through two points $\{I_1, I_2\}$ (see figure 5).*

Proof. In fact, assume for the moment that I is one common point of $\{\alpha, \gamma\}$. Then, we have $|IB|/|IC| = |AB|/|AC|$ and $|IA|/|IB| = |CA|/|CB|$. Multiplying the equations side by side we obtain the relation: $|IA|/|IC| = |AB|/|CB|$. Thus I belongs also to circle β. Since the circles $\{\alpha, \beta, \gamma\}$ are orthogonal to the circumcircle δ of the triangle ABC, the inverted of I w.r. to the circumcircle δ of ABC is also a common point of the three circles.

Assuming that $|AB| < |CA| < |CB|$, implies that the two circles α, γ intersect. In fact, in that case A is inside γ and B is outside, but inside α. Thus, the two circles α, γ must intersect. In the case of isosceles triangle one of the three circles is the medial line of its base and the other two are symmetric w.r. to this line and the circles intersect again. □

The points $\{I_1, I_2\}$ are called *"isodynamic"* points of the triangle ABC. The line I_1I_2 is called *"Brocard axis"* of the triangle. The orthogonal to it, line of centers of the circles $\{\alpha, \beta, \gamma\}$ is called *"Lemoine axis"* of the triangle.

The Isodynamic points play an important role in the *"geometry of the triangle"* ([Gal13], [Yiu13]). Here are some related properties in which participate concepts studied in this branch of geometry. In these the concept of **Inversion transformation** plays a central role.

Theorem 3. *The following properties hold for every triangle ABC :*

1. *The centers $\{A', B', C'\}$ of the three Apollonian circles lie respectively on the side-lines $\{BC, CA, AB\}$ of the triangle and $\{A'A, B'B, C'C\}$ are respectively tangent to the circumcircle δ of ABC.*
2. *The Apollonian circles are orthogonal to the circumcircle of ABC.*
3. *The center O of the circumcircle δ of ABC is on the radical axis of the pencil of Apollonian circles and the isodynamic points are inverse with respect to δ.*
4. *Each one of the inversions $\{f_a, f_b, f_c\}$ w.r.t. $\{\alpha, \beta, \gamma\}$ permutes these circles, hence the three circles intersect pairwise at an angle of measure $\pi/3$.*
5. *Each center of the three circles is a "similarity center" of the two others.*
6. *The tangency of $A'A$ to δ at A implies that $A'A$ is the harmonic conjugate of the "symmedian" through A of w.r. to the sides $\{AB, AC\}$.*
7. *The previous property implies that the "Lemoine line", carrying $\{A', B', C'\}$ is the "trilinear polar" of the "symmedian point" K of the triangle ABC.*
8. *The pedal triangles of the isodynamic points are equilateral triangles (See Figure 6).*

Proof. *Nrs 1-2* follow from the orthogonality of δ to the Apollonian circles.

Nr-3 is a general property of intersecting pencils of circles and their orthogonal to it circles.

Nr-4 e.g. for circle α. The inversion relative to it maps the circle γ passing through $\{I_1, I_2, C\}$ to circle β passing through I_1, I_2, B and vice versa.

Nr-5 e.g. for circle α follows from *nr-4*, since the center of a circle whose inversion interchanges two circles is a similarity center of the two circles.

Nr-6 this is a well known property of the symmedians.

Nr-7 follows from the definition of the *"trilinear polar"* of a point w.r. to a triangle.

Nr-8 follows from theorem 2. □

Exercise 4. *The only points whose pedal triangles w.r.t $\triangle ABC$ are equilateral are the isodynamic points $\{I_1, I_2\}$ of $\triangle ABC$.*

Exercise 5. *In figure 6 show that the line pairs $\{(A'B', A''B''), (B'C', A''C''), (A'C', B''C'')\}$ form the same angle.*

Exercise 6. *The only inversions permuting the vertices of the triangle ABC are the ones relative to the circumcircle κ of the triangle and the Apollonian circles.*

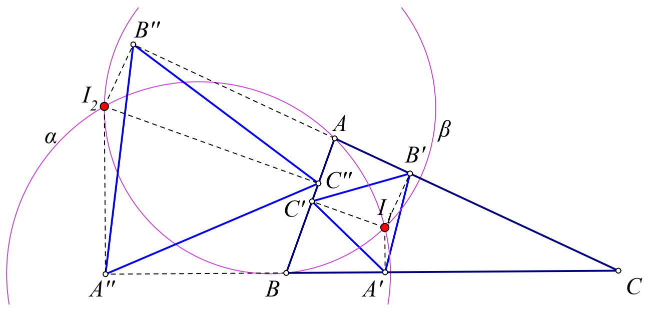

Figure 6: The two equilateral pedals of the isodynamic points of ABC

Hint: Such an inversion maps necessarily the circumcircle κ of $\triangle ABC$ to itself. If it fixes all three vertices, then it fixes κ and is the inversion relative to κ. If it interchanges two vertices, $\{B, C\}$ say, then the inversion must fix the third vertex A and coincides with the inversion relative to the Apollonian circle through A.

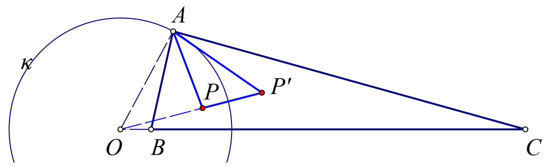

Figure 7: Common Apollonian circle of two triangles $\{ABC, PAP'\}$

Exercise 7. *Show that the Apollonian circle κ of $\triangle ABC$ through A is simultaneously the Apollonian circle through A of the triangle PAP', for every point P and its inverse relative to κ (see figure 7).*

Remark 1. Since the isodynamic points are inverse relative to the circumcircle κ of $\triangle ABC$ one is inside, conventionally denoted by I_1 and the other I_2 is outside κ. Next propositions give formulate some characteristic properties of these points.

Theorem 4. *The inversion relative to the circle $\lambda(I_2)$ which is orthogonal to the circumcircle κ of $\triangle ABC$ and has its center at the outer isodynamic point I_2, maps the vertices of $\triangle ABC$ to the vertices of an equilateral $A'B'C'$ inscribed in κ (see figure 8) and the Apollonian circles to the symmetry axes of the equilateral.*

Proof. Next equalities of ratios, resulting from similar triangles in figure 8, from properties of inversions and from properties of the isodynamic point I_2, show that $B'C' = C'A'$. Analogous equalities of ratios can be used to show also the equality of the other sides of $\triangle A'B'C'$.

$$\frac{BC}{I_2C} = \frac{C'B'}{I_2B'}, \ \frac{A'C'}{I_2C'} = \frac{AC}{I_2A} \quad \Rightarrow \quad C'B' = \frac{BC}{I_2C} I_2B', \ A'C' = \frac{AC}{I_2A} I_2C' \quad \Rightarrow$$

$$\begin{aligned} \frac{C'B'}{A'C'} &= \frac{BC}{AC} \cdot \frac{I_2A}{I_2C} \cdot \frac{I_2B'}{I_2C'} \\ &= \frac{BC}{AC} \cdot \frac{BA}{BC} \cdot \frac{I_2B'}{I_2C'} \\ &= \frac{BA}{CA} \cdot \frac{I_2B'}{I_2C'} = \frac{BA}{CA} \cdot \frac{I_2C}{I_2B} = \frac{BA}{CA} \cdot \frac{AC}{AB} = 1. \end{aligned}$$

The Apollonian circles being orthogonal to line I_1I_2 and passing through I_2 map to lines orthogonal to I_1I_2 etc. □

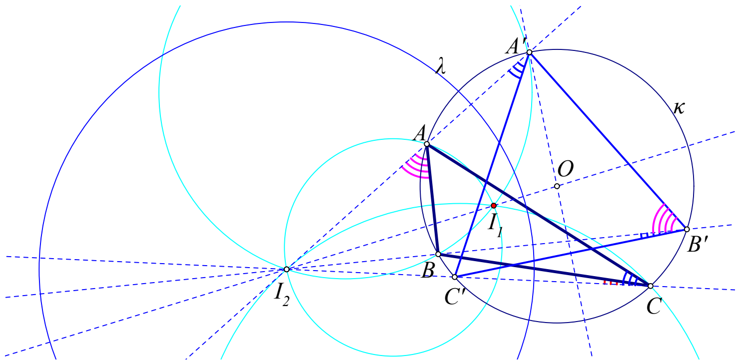

Figure 8: Inversion w.r.t. $\lambda(I_2)$ mapping $\triangle ABC$ to $\triangle A'B'C'$

Exercise 8. *The anti-inversion relative to the circle $\mu(I_1)$ with center at the inner isodynamic point I_1 and diameter the minimal chord DD' of the circumcircle κ through I_1, maps the vertices of $\triangle ABC$ to the vertices of an equilateral $A'B'C'$ inscribed in κ (see figure 9) and the Apollonian circles to the symmetry axes of the equilateral.*

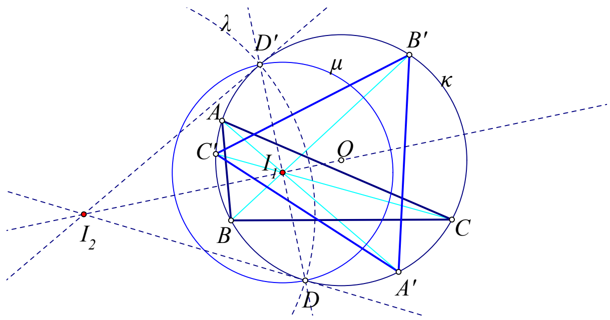

Figure 9: Antiinversion w.r.t. $\mu(I_1)$ mapping $\triangle ABC$ to the equilateral $\triangle A'B'C'$

Hint: Same reasoning as in the preceding theorem.

3 Given Apollonian circles

Next propositions lead to a sort of inverse to theorem 3, giving the description of triangles which have the same Apollonian circles, and consequently the same isodynamic points.

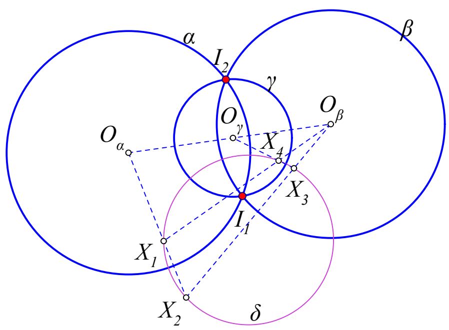

Figure 10: Inversions w.r. to the circles $\{\alpha, \beta, \gamma\}$

Theorem 5. *Given three circles $\{\alpha(O_\alpha), \beta(O_\beta), \gamma(O_\gamma)\}$ passing through the points $\{I_1, I_2\}$ and cutting each other pairwise under an angle of $\pi/3$, consider the three inversions $\{f_\alpha, f_\beta, f_\gamma\}$ relative to the corresponding circles (see figure 10). Then*

1. *Each inversion interchanges the two other circles.*
2. *The compositions $f_\beta \circ f_\alpha = f_\gamma \circ f_\beta$ are equal and the successive transforms of a point X_1, $X_2 = f_\alpha(X_1), X_3 = f_\beta(X_2), X_4 = f_\gamma(X_3)$ form a cyclic quadrangle whose circumcircle is orthogonal to the three given circles.*

Proof. *Nr-1* is equivalent with the intersection condition of the three circles by the angle of $\pi/3$.

Nr-2 follows by considering the circle δ passing through $\{X_1, X_2, X_3\}$. Since $\{X_1, X_2\}$ are inverse w.r. to α the circle δ is orthogonal to α. Analogous argument shows that it is orthogonal to β and consequently to all circles of the pencil generated by $\{\alpha, \beta\}$ hence also to γ. This implies $X_4 \in \delta$ and that X_1 is the inverse of X_4 w.r. to β thereby proving the claim. □

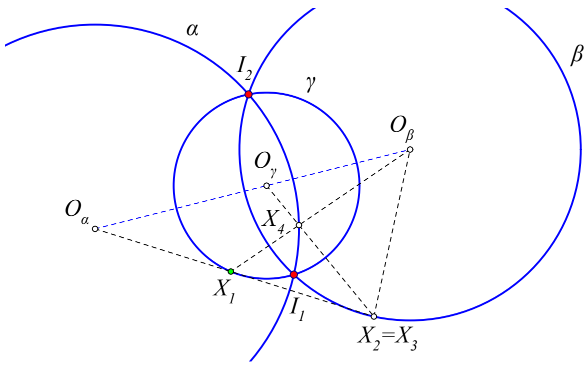

Figure 11: Constructing a triangle ABC with given isodynamic points $\{I_1, I_2\}$

Corollary 1. *With the definitions and notation of the preceding theorem, when point X_1 is on the circle γ, then $X_2 = X_3 \in \beta$ and $X_4 \in \alpha$ and the sides of the triangle $X_1X_2X_4$ pass through the centers $\{O_\alpha, O_\beta, O_\gamma\}$ of the three circles (see figure 11).*

Corollary 2. *With the definitions and conventions of the preceding corollary, the Apollonian circles of the triangle $X_1X_2X_4$ are the circles $\{\alpha, \beta, \gamma\}$. Conversely, if the triangle $X_1X_2X_4$ has $\{\alpha, \beta, \gamma\}$ as Apollonian circles, then it is constructible as in the preceding corollary.*

Proof. Consider one of these circles, α say, passing through X_4 and having its center on X_1X_2 latter being points inverse w.r. to α. Then α intersects the line X_1X_2 at diametral points $\{U, V\}$, which, by a well known property of inversion, are *"harmonic conjugate"* w.r. to $\{X_1, X_2\}$. From the orthogonality of $\{X_4U, X_4V\}$ follows that these are the bisectors of the angle $X_1\widehat{X_4}X_2$, hence the validity of the claim for the circle α. Analogous is the proof for the other two circles. The converse is trivial. □

Figure 12 displays triangles sharing the same Apollonian circles and resulting by the procedure of corollary 3, to be proved in the next section. According to this, we take equilaterals homothetic to $A_1B_1C_1$ w.r.t. point I_2 and applying to these (to their vertices) the inversion w.r.t. circle ε. The Apollonian circles are the inverses $\{\alpha, \beta, \gamma\}$ relative to ε of the lines $\{\alpha', \beta', \gamma'\}$. Next section discusses the details of this procedure.

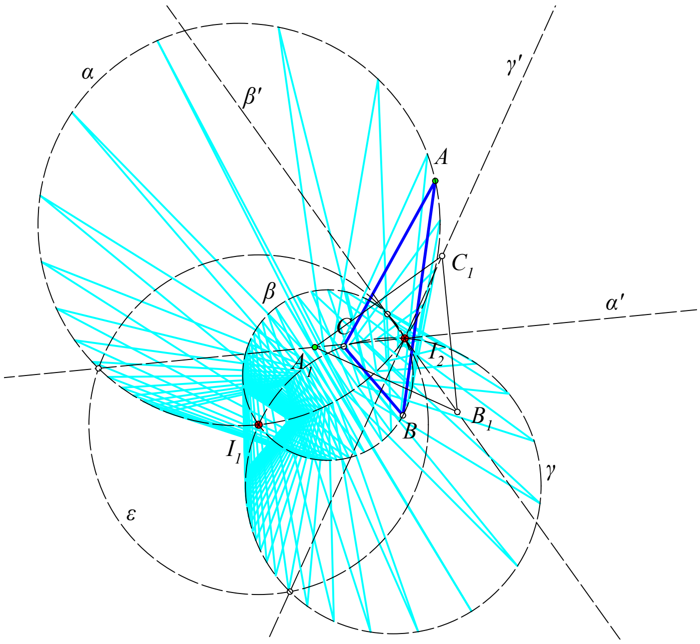

Figure 12: Triangles sharing the same Apollonian circles

4 Given isodynamic points

If we are asked to construct triangles with the given isodynamic points $\{I_1, I_2\}$ but possibly different triples of Apollonian circles, we first must notice that these triples of circles are selected among the members of the intersecting pencil $\mathcal{P}$ of all circles passing through $\{I_1, I_2\}$. From the intersection angle condition of $\pi/3$ for these triples it follows that a single circle of the triple determines completely the other two. In the following $\varepsilon(X, Y)$ will denote a circle with center X, passing through Y. Applying the inversion f_ε relative to $\varepsilon(I_1, I_2)$, the pencil $\mathcal{P}$ transforms to the pencil of lines through I_2 and the admissible triples of Apollonian circles $\{\alpha, \beta, \gamma\}$ transform to triples of lines

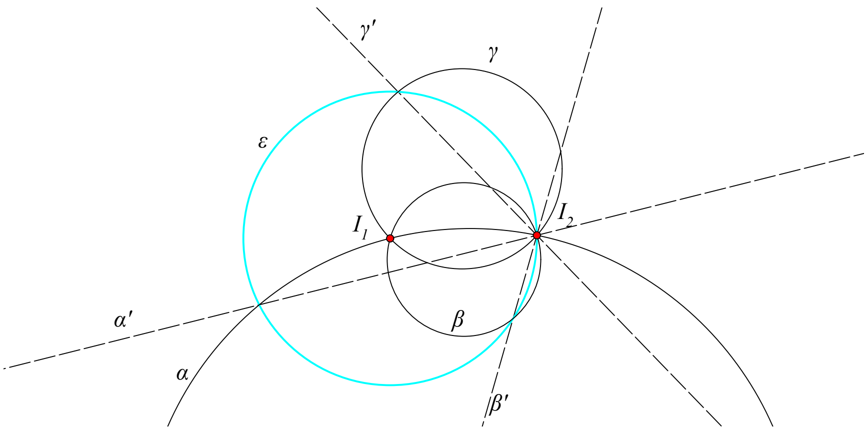

Figure 13: Admissible triples of Apollonian circles and their inverses relative to ε

$\{\alpha' = f_\varepsilon(\alpha),\ \beta' = f_\varepsilon(\beta),\ \gamma' = f_\varepsilon(\gamma)\}$ through I_2 intersecting at angles of measure $\pi/3$ (see figure 13). Next propositions show that the triangles having the triple $\{\alpha, \beta, \gamma\}$ for Apollonian circles correspond under the inversion f_ε to equilateral triangles with symmetry axes the lines α', β', γ'. To see this we use the well known property of inversions

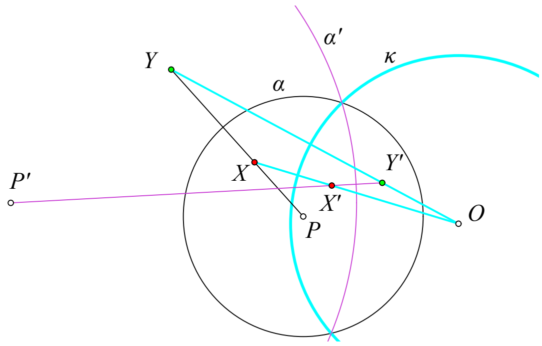

Figure 14: Inverseness is invariant under inversion

([Joh60, p.55], [Ped90, p.96]) according to which *"The property of inverseness is invariant under inversion"*. This property guarantees that, if points $\{X, Y\}$ are inverse w.r.t. to a circle α and the whole figure is inverted w.r.t. to a circle κ, then the $\kappa-$inverses $\{X', Y'\}$ are inverse w.r.t. to the $\kappa-$inverse circle α' (see figure 13). This has an interesting corollary which I formulate as a lemma. The following arguments use elementary properties of *"inversions"* and pencils of circles (coaxal circles), an account of which can be found in Johnson [Joh60, p.28], Pedoe [Ped90, p.106] and also in the file **Inversion** of this gallery.

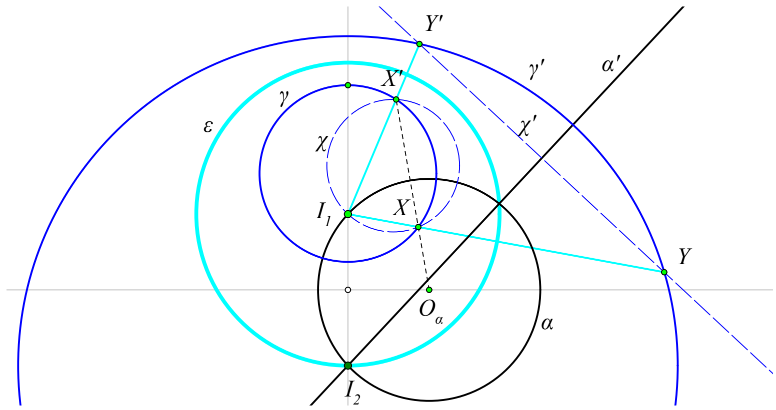

Figure 15: Inversion w.r.t. to a circle is conjugate to reflection w.r.t. to a line

Lemma 1. *For an inversion f_α w.r.t. to a circle α there is an inversion f_ε w.r.t. to a circle ε such that the conjugate transformation $f_\varepsilon \circ f_\alpha \circ f_\varepsilon$ is a reflection $r_{\alpha'}$ on the line $\alpha' = f_\varepsilon(\alpha)$.*

Proof. For the proof select two points $\{I_1, I_2\}$ on the circle $\alpha(O_\alpha)$ and consider the non-intersecting pencil $\mathcal{P}$ of (coaxal) circles $\{\gamma\}$ with limits these two points, all of them being then orthogonal to α (see figure 15). Consider also the inversion f_ε w.r.t. to the circle $\varepsilon(I_1, I_2)$ and define the line $\alpha' = f_\varepsilon(\alpha)$ passing through I_2. Since inversions leave invariant the circles orthogonal to the circle of inversion, a point X of the plane $X \neq O_\alpha$ defines a unique member $\gamma \ni X$ of the pencil $\mathcal{P}$ and $X' = f_\alpha(X) \in \gamma$ too. Then, the circle $\chi = (XX'I_1)$ maps under f_ε to a line $\chi' = YY'$ which is orthogonal to α', since χ passing through two $\alpha-$inverse points is orthogonal to α. The points $\{Y = f_\varepsilon(X), Y' = f_\varepsilon(X')\}$ then, are reflections of each other on α', as claimed. □

Theorem 6. *Let $\{I_1, I_2\}$ be the isodynamic points of the triangle ABC and f_ε the inversion relative to the circle $\varepsilon(I_1, I_2)$. The images $\{A_1 = f_\varepsilon(A),\ B_1 = f_\varepsilon(B),\ C_1 = f_\varepsilon(C)\}$ of the vertices of ABC form an equilateral triangle $A_1B_1C_1$ and the images $\{\alpha' = f_\varepsilon(\alpha),\ \beta' = f_\varepsilon(\beta),\ \gamma' = f_\varepsilon(\gamma)\}$ of the*

Apollonian circles $\{\alpha, \beta, \gamma\}$ of the triangle ABC are the symmetry axes of $A_1B_1C_1$ meeting in the center I_2 of the triangle (see figure 16).

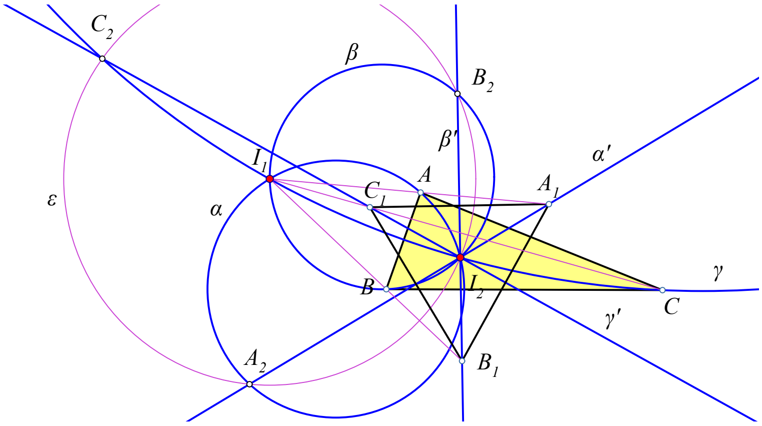

Figure 16: Inversion of vertices of $\triangle ABC$ to vertices of equilateral $A_1B_1C_1$

Proof. The proof results from the general properties of inversions and the previous discussion (see figure 16). The Apollonian circles $\{\alpha, \beta, \gamma\}$ transform respectively to lines $\{\alpha', \beta', \gamma'\}$ through I_2, intersecting pairwise at $\pi/3$. Since $\{B, C\}$ are inverse w.r. to α, their images $\{B_1 = f_\varepsilon(B), C_1 = f_\varepsilon(C)\}$, by lemma 1, are inverse w.r. to $\alpha' = f_\varepsilon(\alpha)$, hence are points reflected in α'. It follows that B_1C_1 is orthogonal to α' and analogously the other sides of $A_1B_1C_1$ have the lines $\{\alpha', \beta', \gamma'\}$ as perpendicular bisectors. □

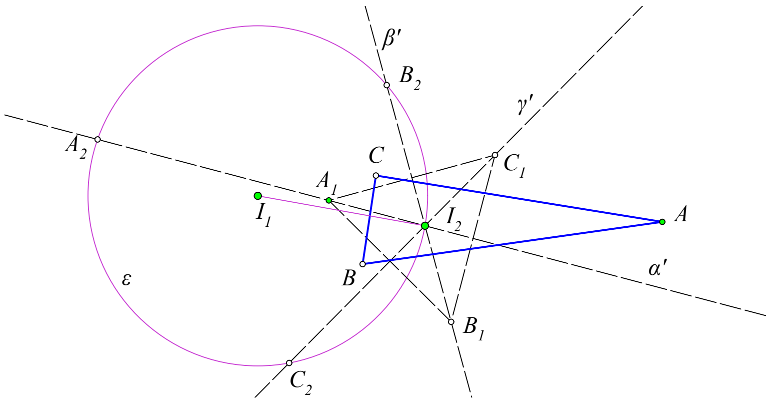

Figure 17: Generation of all triangles with the same isodynamic points

The theorem implies immediately next corollary giving a way to generate all possible triangles with given isodynamic points $\{I_1, I_2\}$ (see figure 17).

Corollary 3. *All triangles ABC, with given isodynamic points $\{I_1, I_2\}$, result by applying the inversion f_ε relative to the circle $\varepsilon(I_1, I_2)$ to the vertices $\{A_1, B_1, C_1\}$ of equilateral triangles $A_1B_1C_1$ centered in I_2 : $\{A = f_\varepsilon(A_1),\ B = f_\varepsilon(B_1),\ C = f_\varepsilon(C_1)\}$.*

Corollary 4. *Given three points in general position $\{I_1, I_2\}$ and A, there is precisely one triangle ABC having A as vertex and $\{I_1, I_2\}$ as isodynamic points (see figure 18).*

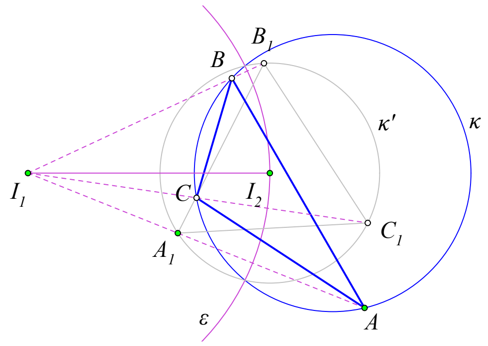

Figure 18: Triangle from the isodynamics $\{I_1, I_2\}$ and vertex A

Proof. To see this notice first that the circumcircle κ of the requested triangle is uniquely defined. This, because it is the unique member of the pencil $\mathcal{P}$ of circles of non-intersecting type with limit points the given $\{I_1, I_2\}$, passing through A. By theorem 6 the inverse $\kappa' = f_\varepsilon(\kappa)$ relative to the circle $\varepsilon(I_1, I_2)$ carries $A_1 = f_\varepsilon(A)$, which is a vertex of a unique equilateral $A_1B_1C_1$ inscribed in κ', whose centroid I_2 coincides with the center of κ'. The requested triangle ABC has vertices the inverses via f_ε of the vertices of this equilateral. □

Remark 2. The pencil $\mathcal{P}$ consisting of the circles orthogonal to the Apollonian circles is called *"Schoute pencil"* of the triangle ABC. As noticed above, the isodynamic points are its *"limit points"*[1]

Exercise 9. *Show that the equilateral of theorem 6 and that of theorem 4 are homothetic. Show that, more general, any inversion w.r.t. a circle centered at I_2 maps the vertices of $\triangle ABC$ to the vertices of an equilateral and all these equilaterals are homothetic.*

5 Trilinear coordinates of the isodynamic points

The appearance of the equilateral here offers the means to calculate easily these kind of coordinates for $\{I_1, I_2\}$. The result is

$$\begin{aligned}
(x_1 : x_2 : x_3) &= \left(\sin\left(\widehat{A} + \frac{\pi}{3}\right),\ \sin\left(\hat{B}) + \frac{\pi}{3}\right),\ \sin\left(\widehat{C} + \frac{\pi}{3}\right)\right) \\
(x_1' : x_2' : x_3') &= \left(\sin\left(\widehat{A} - \frac{\pi}{3}\right),\ \sin\left(\hat{B} - \frac{\pi}{3}\right),\ \sin\left(\widehat{C} - \frac{\pi}{3}\right)\right).
\end{aligned}$$

The formulas result from a calculation of the ratios $(x_1 : x_2 : x_3)$ of the distances of I_1 from the sides of ABC. For this we notice

$$\begin{aligned}
\widehat{BI_1C} &= \widehat{A} + \frac{\pi}{3} \quad \text{since} \\
\widehat{BI_1C} &= 2\pi - \widehat{BI_1A} - \widehat{AI_1C} \\
&= 2\pi - (\pi - \widehat{BAI_1} - \widehat{ABI_1}) - (\pi - \widehat{CAI_1} - \widehat{I_1CA}) \\
&= \widehat{A} + (\widehat{DEI_1} + \widehat{FEI_1}) = \widehat{A} + \frac{\pi}{3}.
\end{aligned}$$

[1]Thanks to Dan Reznik for the suggestion to refer here to the Schoute pencil and its limit points.

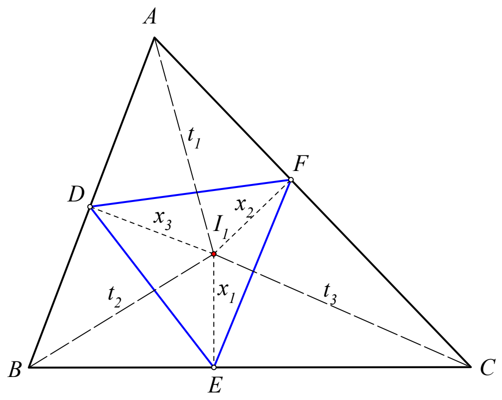

Figure 19: Trilienear coordinates $x_1 : x_2 : x_3$ of I_1

Last equality resulting from the cyclic quadrangles $DBEI_1$ and FI_1EC. Calculating the double of the area of triangles $\{I_1BC, I_1CA\}$ and dividing gives:

$$\begin{aligned}
2(I_1BC) &= x_1 \cdot BC = t_2 \cdot t_3 \cdot \sin(\widehat{A} + \pi/3), \\
2(I_1CA) &= x_2 \cdot CA = t_3 \cdot t_1 \cdot \sin(\hat{B} + \pi/3) \quad \Rightarrow \\
\frac{x_1 \cdot BC}{x_2 \cdot CA} &= \frac{t_2 \cdot \sin(\widehat{A} + \pi/3)}{t_1 \cdot \sin(\hat{B} + \pi/3)} \quad \Rightarrow \\
\frac{x_1}{x_2} &= \frac{\sin(\widehat{A} + \pi/3)}{\sin(\hat{B} + \pi/3)},
\end{aligned}$$

where the last simplification is due to the property of the isodynamic point $t_2/t_3 = AB/AC$, etc. The formulas for I_2 result by analogous reasoning.

6 Same isodynamic points and same circumcircle

Figure 20 shows some members of the family $\mathcal{T}$ of triangles $\{ABC\}$ sharing the same isodynamic points $\{I_1, I_2\}$ and the same circumcircle κ but not the same Apollonian circles.

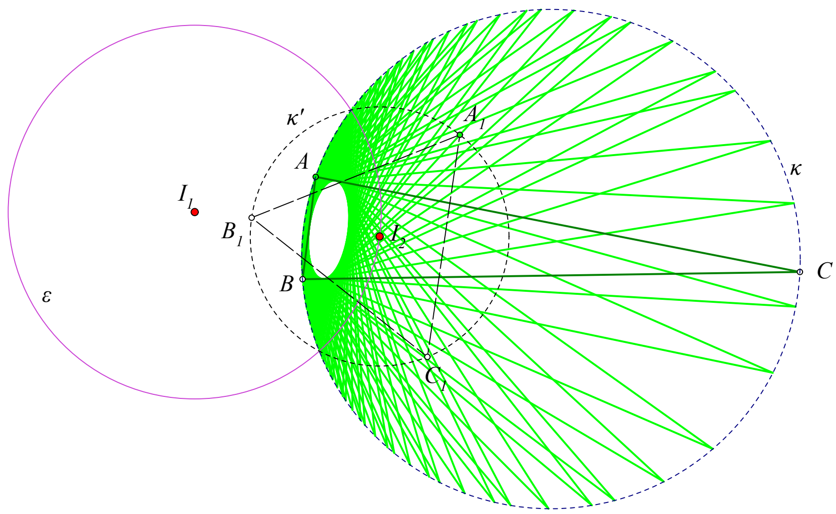

Figure 20: The same isodynamic points $\{I_1, I_2\}$ and same circumcircle

It is created by rotating the equilateral $A_1B_1C_1$ in its circumcircle about I_2 and taking from the resulting triangles $\{A'B'C'\}$ the inverses (of their vertices) relative to the circle ε (corollary 4).

On the occasion of this figure I would like to mention some facts pertaining to the *"Geometry of the triangle"* and whose proofs can be found in [Pam04].

1. As noticed in the figure the triangles of the family $\mathcal{T}$ have their sides enveloping an ellipse. This is the *"Brocard ellipse"* of the triangle ABC, inscribed in ABC and all other members of the family $\mathcal{T}$.
2. The reason of this enveloping property is simple to explain, provided one has some knowledge of elementary projective geometry. There is namely a *"projectivity"* f mapping the circle κ' onto κ and also mapping $\{A_1 \stackrel{f}{\mapsto} A,\ B_1 \stackrel{f}{\mapsto} B,\ C_1 \stackrel{f}{\mapsto} C\}$. This f maps all the equilaterals inscribed in κ' to the triangles of the family $\mathcal{T}$. The common inscribed circle of the equilaterals maps then to this *"Brocard ellipse"* also inscribed to all members of $\mathcal{T}$.

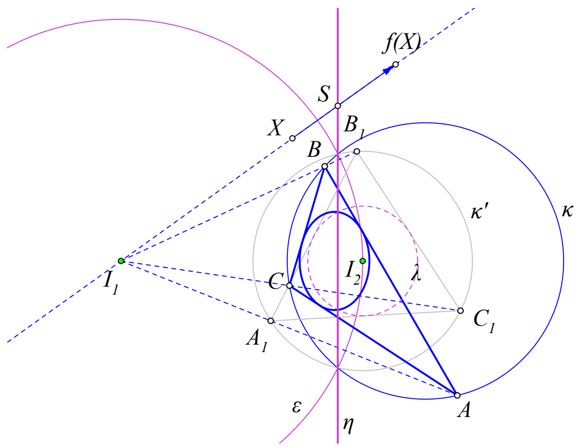

Figure 21: Projectivity f coinciding with the inversion f_ε on the points of κ'

3. The map f can be simply described as the *"homology"* with center I_1, fixed axis coinciding with the radical axis η of the circles $\{\kappa, \kappa'\}$ and a constant *"cross ratio"* $k = (Xf(X)SI_1)$, where $S = \eta \cap Xf(X)$, proved also to coincide with the inversion f_ε on the points of the circle κ' but not on the points of κ. Figure 21 shows the property of f to have the line $Xf(X)$ pass through I_1 and shows also the *"Brocard"* ellipse as image $f(\lambda)$ of the incircle of the equilateral.

4. The equilaterals are *"orbital"* triples of points resulting by repeated application of the rotation ρ by 120° about I_2 : $\{A_1,\ B_1 = \rho(A_1),\ C_1 = \rho(B_1),\ A_1 = \rho(C_1)\}$. This behavior is transferred by conjugation by f to all the triangles of the family $\mathcal{T}$. Latter are then *orbital* w.r. to the projectivity $\sigma = f \circ \rho \circ f^{-1}$, mapping κ onto itself and recycling the vertices of the triangles: $\{A,\ B = \sigma(A),\ C = \sigma(B),\ A = \sigma(C)\}$.
5. All these triangles $ABC \in \mathcal{T}$ have the same *"symmedian point"* $K = f(I_2)$, which is a fixed point of the periodic projectivity σ. The *"homography axis"* of σ coincides with the *"Lemoine axis"* of the triangle, which is also the same for all triangles $ABC \in \mathcal{T}$. Figure 22 shows two triangles $\{ABC, A'B'C'\}$ of the family $\mathcal{T}$, their common symmedian point K and also illustrates the characteristic property of the *"homography axis"* of σ which guarantees that, for two arbitrary points $\{A, B\}$ of κ and their images $\{A' = \sigma(A),\ B' = \sigma(B)\}$, the intersections $P = AB' \cap A'B$ lie on the homography axis of σ.

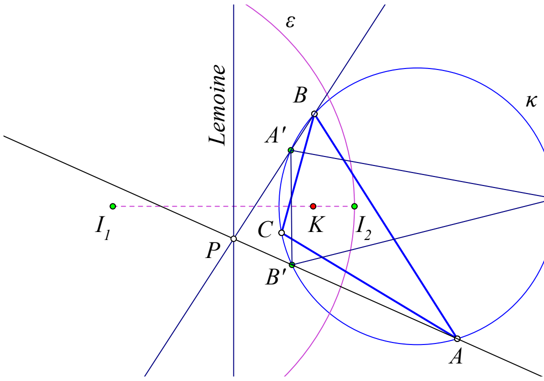

Figure 22: Intersections $\{P = AB' \cap A'B\}$ are on the homography axis

6. By the way, and because was mentioned in *nr-3*, every pair of circles $\{\kappa', \kappa\}$ defines a circle ε interchanging them through the corresponding inversion $f_{\varepsilon}: \ f_{\varepsilon}(\kappa') = \kappa$, satisfying, of course also $f_{\varepsilon}(\kappa) = \kappa'$. The two circles define also a projectivity f leaving fixed the center I_1 of ε and the radical axis η of the circles $\{\kappa', \kappa\}$ and coinciding with f on the points of κ'. Figure 23 shows the image $f(\kappa)$ of κ under f, which is a conic and never coincides with κ', except in the case the two circles are equal and f_{ε} is a reflection. Thus, while f_{ε} is always an involution satisfying $f_{\varepsilon}^2 = e$, the projectivity f for non equal circles is never an involution $f^2 \neq e$. Maps such as f, having an axis η of fixed points, an isolated fixed point I_1 and having images

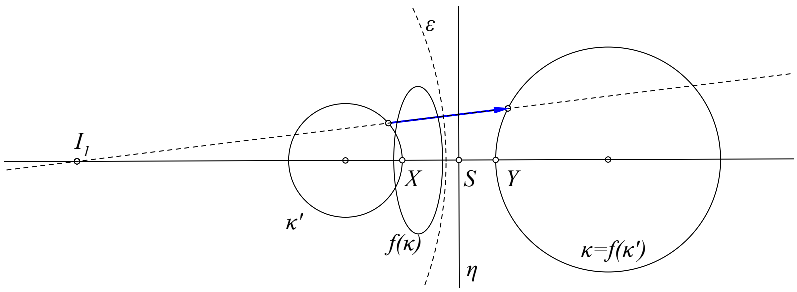

Figure 23: Projectivity f coinciding with the inversion f_{ε} on the points of κ'

$Y = f(X)$ such that the lines $XY \ni I_1$ and the cross ratio $(XYSI_1) = k$ is constant, where $S = XY \cap \eta$, are called *"homologies"* and play a fundamental role in projective geometry ([VY10, I,p.72]).

7 A matter of uniqueness

A matter of uniqueness, concerning the reduction to equilaterals, arises from the possibility to interchange the roles of $\{I_1, I_2\}$. In fact, the same procedure can be applied using the circle $\varepsilon'(I_2, I_1)$ instead of $\varepsilon(I_1, I_2)$ and defining the equilateral $A''B''C''$ inscribed in κ'', which this time is the inverse of κ relative to ε' (see figure 24). It is though easy to see that the two equilaterals produce the same triangle ABC :

$$A = f_{\varepsilon}(A') = f_{\varepsilon'}(A''), \quad B = f_{\varepsilon}(B') = f_{\varepsilon'}(B''), \quad C = f_{\varepsilon}(C') = f_{\varepsilon'}(C'').$$

This is proved by the following lemma, which uses the definitions and conventions adopted so far.

Lemma 2. *The points* $\{A' \in \kappa', A'' \in \kappa''\}$ *are respectively the inverses* $\{A' = f_\varepsilon(A), A = f_{\varepsilon'}(A)\}$ *of the same point* $A \in \kappa$, *if and only if, the lines* $\{I_1A'', I_2A'\}$ *intersect at a point* D *on the perpendicular bisector of the segment* I_1I_2.

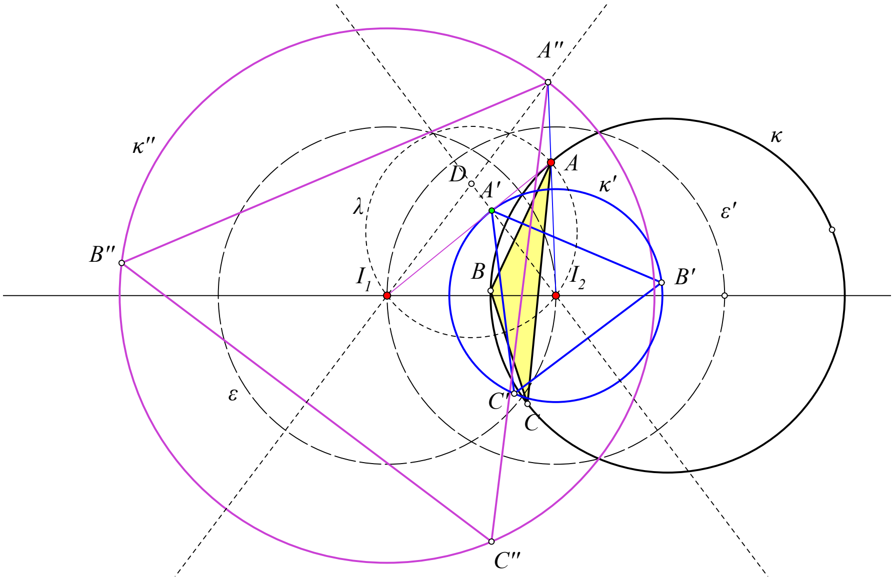

Figure 24: $A = f_\varepsilon(A') = f_{\varepsilon'}(A'')$ ...

If we trust this lemma, the claim follows at once, since starting from the isosceles I_1DI_2 resulting from the lines $\{I_1A'', I_2A'\}$ we obtain the other vertices of the equilaterals by rotating these lines in opposite orientation and by the same angle of $120°$, respectively $240°$, implying that they intersect again in a point of the perpendicular bisector.

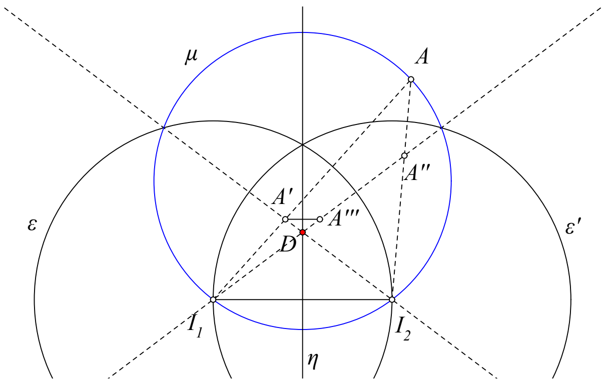

Figure 25: $f_\varepsilon(A') = f_{\varepsilon'}(A'') = A$

The lemma in turn, is a corollary of the more general following one (see figure 25).

Lemma 3. *Given the equal circles* $\{\varepsilon, \varepsilon'\}$ *centered respectively in* $\{I_1, I_2\}$ *with radius* $|I_1I_2|$, *the inversions* $\{f_\varepsilon, f_{\varepsilon'}\}$ *relative to these circles, map the points* $\{A', A''\}$ *respectively to the same point* A, *if and only if, the lines* $\{I_1A'', I_2A'\}$ *intersect in a point* D *on the perpendicular bisector of the segment* I_1I_2.

Proof. The lemma follows by applying lemma 1. By it inverting $\{A, A'' = f_{\varepsilon'}(A)\}$ relative to ε we obtain two points $\{A', A'''\}$, which are inverse w.r. to the inverse of ε' relative to ε. But this inverse $f_\varepsilon(\varepsilon')$ is the perpendicular bisector of I_1I_2 and $\{A', A'''\}$ are then reflected w.r. to this line. The argument can be reversed to show also the converse. □

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Related material

1. **Barycentric coordinates**
2. **Cross Ratio**
3. **Inversion transformation**
4. **Pedal triangles**
5. **Projectivities**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr