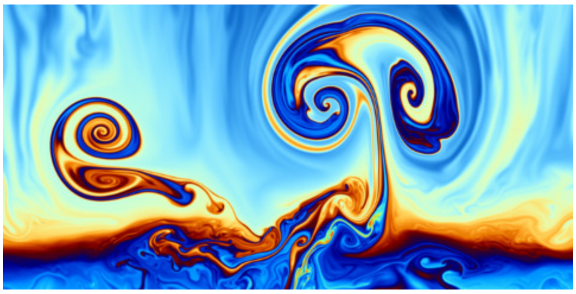
Vortices

Vortices in fluids

When a part of the fluid is rotating clockwise or anticlockwise in closed loops, we have a *vortex*.



Pairs of vortices in fluids are created when part of the fluid is set in motion.

Vortex created by the passage of an aircraft wing, revealed by coloured smoke.

The **circulation** is the line integral of the fluid velocity around a closed curve

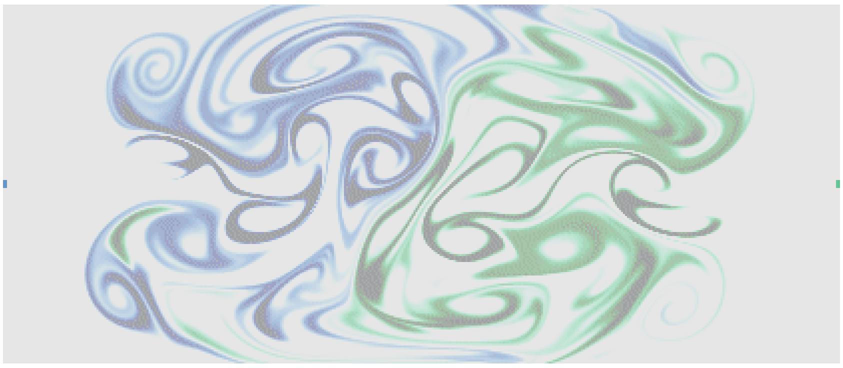
$$\Gamma = \oint_C \boldsymbol{v} \cdot d\boldsymbol{s}$$

and it measures the strength of the vortex.

Vortices are usually created in pairs with opposite circulation.

We typically call one of the two a "vortex" and the oppositely circulating one an "*antivortex*".

Pairs a vortices are traveling.



Two pairs of fluid vortices collide.

Vortices in superfluids

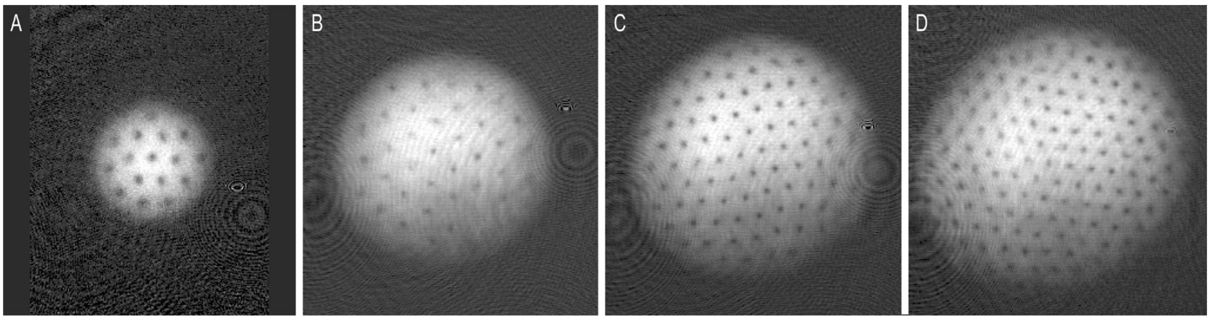
Superfluid are substances such as

- Liquid Helium in temperature $T < 2.17$ Kelvin.
- Vapours of alkali metal gases (Li, Rb, etc) in temperature $T \sim 1\,nK$.
- and many other systems.

Superfluids placed in a closed loop, can flow endlessly without friction.

Vortices were observed

- In liquid Helium. First by J. Vinen in 1958.
- In atomic gases in 1995.



Superfluid vortices, shown as dark regions, in a superfluid (trapped ultra-cold atoms).

It is believed that superfluid properties arise when we have a phase-coherent quantum mechanical macroscopic system.

Maybe, this observation was behind the laconic remark of Onsager (1949):

> Thus the well-known invariant called the hydrodynamic circulation is quantized; the quantum of circulation is h/m.

whose ramifications keep scientists busy up to this date.

How can we obtain *quantized vortices* (as Onsager predicted)?

Instead of having a collection of particles that circulate (as in an ordinary fluid), we consider a single complex function that describes the collection of particles

$$\Psi(x,y) = n\, e^{i\Theta}, \qquad n = n(x,y),\ \Theta = \Theta(x,y).$$

Let us consider the specific complex function

$$\Psi = n\, e^{i\phi}$$

where ϕ is the angle of polar coordinates. When we go in a circle around the center of a vortex ($r = 0$), then we start from $\phi = 0$ and end at a different $\phi = 2\pi$ (this is ok because $e^{i0} = e^{i2\pi}$!). We could also choose

$$\Psi = n\, e^{2i\phi}, \quad \Psi = n\, e^{3i\phi}, \cdots$$

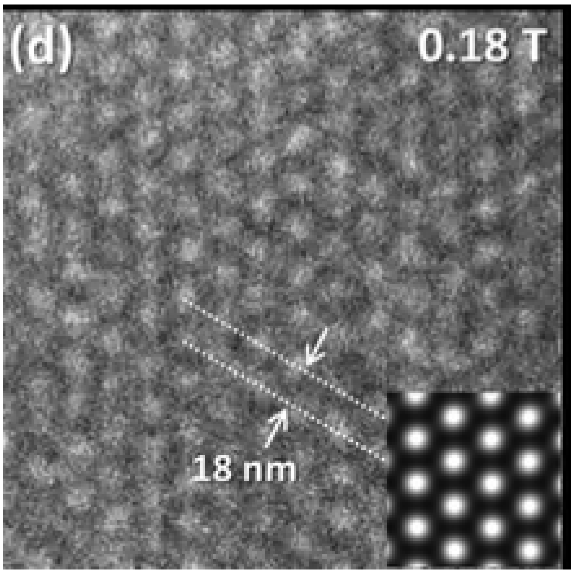
Then, going around the vortex center, we end up at $\phi = 4\pi, 6\pi, \cdots$ (again ok!).

We conclude that the construction of complex functions of the form $\Psi = n\, e^{i\kappa\pi}$ give quantized vortices with a strength equal to the integer $\kappa = 1, 2, 3, \cdots$. In fact, κ is the circulation of the vortex.

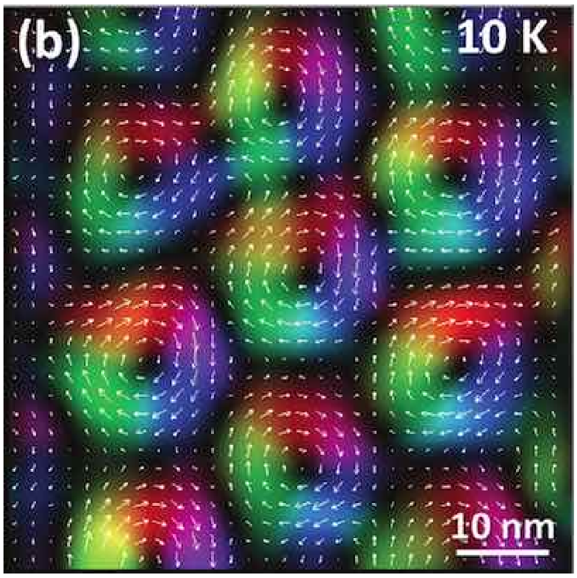
Vortices in superfluids are mathematically more tractable than ordinary vortices in fluids.

Vortices in condensed matter

In a ferromagnetic material, every atom has a magnetic moment $\vec{\mu}$. The microscopic magnetic moments organise to give a *micromagnetic* configuration. Images of magnetic configurations can be obtained experimentally.

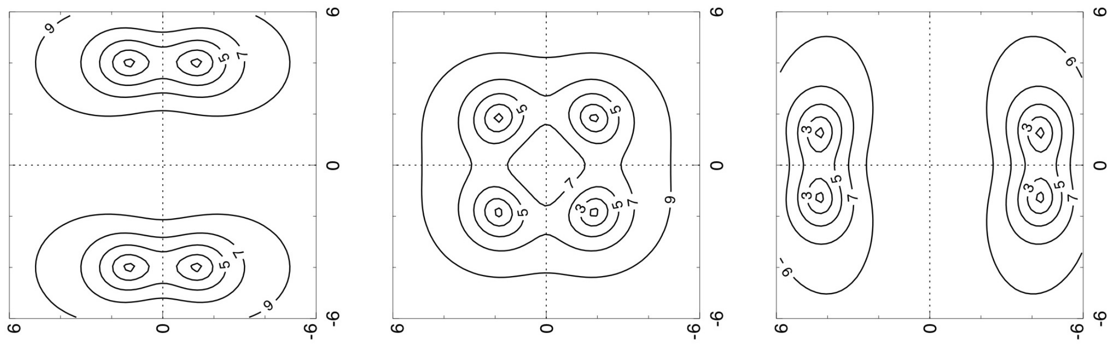


A ferromagnetic film with a periodic micromagnetic configuration.



Micromagnetic configuration. Vortices (called skyrmions) are observed. [Source. Tonomura et al.]

Vortex pairs can travel and two vortex pair can collide. (See simulations)



The collisions of vortex pairs gives results similar as in the collisions of ordinary vortices.

Dynamics of point vortices

We consider a fluid flowing in two-dimensions. Helmholtz considered the case that the vorticity of the fluid is confined in some areas (think of small circular areas that are cylinders if we extend this in depth of the fluid). We call these vortex filaments. When these areas are not overlapping, then we have the approximation of point vortices.

Helmholtz (1858) derived the equations for interacting point vortices.

We consider vortices with circulation γ and we would like to describe the dynamics of the position (x, y) of each vortex on the plane.

The equations of motion for two interacting vortices are [Kirchhoff, 1876]

$$\dot{x}_1 = -\gamma_2 \, \frac{y_1 - y_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}, \qquad \dot{x}_2 = -\gamma_1 \, \frac{y_2 - y_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2},$$
$$\dot{y}_1 = \gamma_2 \, \frac{x_1 - x_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}, \qquad \dot{y}_2 = \gamma_1 \, \frac{x_2 - x_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2},$$

where $\boldsymbol{r}_1 = (x_1, y_1), \boldsymbol{r}_2 = (x_2, y_2)$ are the positions of the vortices and γ_1, γ_2 are their circulations (strengths).

Also

$$|\boldsymbol{r}_1 - \boldsymbol{r}_2| \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

is the distance between the vortices.

[The interaction between the vortices is inversely proportional to the distance between them.]

A vortex and an antivortex with equal strengths.

We consider two vortices with $\gamma_1 = -\gamma_2 = \gamma$. Let us write the equations of motion

$$\dot{x}_1 = \gamma \, \frac{y_1 - y_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}, \qquad \dot{x}_2 = -\gamma \, \frac{y_2 - y_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2},$$
$$\dot{y}_1 = -\gamma \, \frac{x_1 - x_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}, \qquad \dot{y}_2 = \gamma \, \frac{x_2 - x_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}.$$

A solution of these equations is found is we assume $y_1 - y_2 = d$ (constant) and $x_1 = x_2$. Then,

$$|\boldsymbol{r}_1 - \boldsymbol{r}_2| \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d$$

and the equations reduce to

$$\dot{x}_1 = \frac{\gamma}{d}, \qquad \dot{x}_2 = \frac{\gamma}{d},$$
$$\dot{y}_1 = 0, \qquad \dot{y}_2 = 0.$$

That is, we only need to solve one equation

$$\dot{x}_{1,2} = \frac{\gamma}{d} \Rightarrow x_{1,2} = \frac{\gamma}{d} \, t.$$

The pair of vortices is traveling along the horizontal direction with a velocity $v = \gamma/d$ while the distance d between them remains constant.

Exercise. In order develop a systematic method for solving the system of equation, first prove that the following are conserved quantities

$$I_x = x_1 - x_2, \qquad I_y = y_1 - y_2.$$

Then, find the general solution for the system of equations.

Three vortices

See Groebli 1878(?).

The system is completely integrable.

Two pairs of vortices

For N interacting vortices, the equations of motion are

$$\dot{x}_i = -\sum_{j=1, j\neq i}^{N} \gamma_j \frac{y_i - y_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2}, \qquad \dot{y}_i = \sum_{j=1, j\neq i}^{N} \gamma_j \frac{x_i - x_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2}.$$

Let us consider two same vortex-anitvortex pairs, with $(x_1, y_1; \gamma)$, $(x_2, y_2; -\gamma)$ and $(x_3, y_3; \gamma)$, $(x_4, y_4; -\gamma)$. The equations of motion are

$$
\begin{aligned}
\dot{x}_1 &= \gamma\,\frac{y_1 - y_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2} - \gamma\,\frac{y_1 - y_3}{|\boldsymbol{r}_1 - \boldsymbol{r}_3|^2} + \gamma\,\frac{y_1 - y_4}{|\boldsymbol{r}_1 - \boldsymbol{r}_4|^2},\\
\dot{x}_2 &= -\gamma\,\frac{y_2 - y_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^2} - \gamma\,\frac{y_2 - y_3}{|\boldsymbol{r}_2 - \boldsymbol{r}_3|^2} + \gamma\,\frac{y_2 - y_4}{|\boldsymbol{r}_2 - \boldsymbol{r}_4|^2},\\
\dot{y}_1 &= -\gamma\,\frac{x_1 - x_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2} + \gamma\,\frac{x_1 - x_3}{|\boldsymbol{r}_1 - \boldsymbol{r}_3|^2} - \gamma\,\frac{x_1 - x_4}{|\boldsymbol{r}_1 - \boldsymbol{r}_4|^2},\\
\dot{y}_2 &= \gamma\,\frac{x_2 - x_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^2} + \gamma\,\frac{x_2 - x_3}{|\boldsymbol{r}_2 - \boldsymbol{r}_3|^2} - \gamma\,\frac{x_2 - x_4}{|\boldsymbol{r}_2 - \boldsymbol{r}_4|^2}\\
\dot{x}_3 &= \cdots
\end{aligned}
$$

We may guess a solution based on the experience from the problem of a single pair. We assume

$$x_1 = x_2,\; x_3 = x_4, \quad y_2 - y_1 = y_3 - y_4 = d_y(t)$$

and

$$y_1 = y_4,\; x_2 = x_3, \quad x_4 - x_1 = x_3 - x_2 = d_y(t).$$

Thus

$$|\boldsymbol{r}_2 - \boldsymbol{r}_1| = |\boldsymbol{r}_3 - \boldsymbol{r}_4| = d_y, \qquad |\boldsymbol{r}_4 - \boldsymbol{r}_1| = |\boldsymbol{r}_3 - \boldsymbol{r}_2| = d_x$$

The four equations reduce to

$$
\begin{aligned}
\dot{x}_1 &= \gamma\,\frac{-1}{d_y} - \gamma\,\frac{-d_y}{d_x^2 + d_y^2},\\
\dot{x}_2 &= -\gamma\,\frac{1}{d_y} + \gamma\,\frac{d_y}{d_x^2 + d_y^2},\\
\dot{y}_1 &= \gamma\,\frac{-d_x}{d_x^2 + d_y^2} - \gamma\,\frac{-1}{d_x},\\
\dot{y}_2 &= \gamma\,\frac{-1}{d_x} - \gamma\,\frac{-d_x}{d_x^2 + d_y^2}\\
\dot{x}_3 &= -\gamma\frac{d_y}{d_x^2 + d_y^2} + \gamma\frac{1}{d_y}, \quad \dot{x}_4 = \cdots.
\end{aligned}
$$

Note that $\dot{x}_1 = \dot{x}_2$ and this is consistent with the assumption $x_1 = x_2$. Combining equations we have

$$\dot{d}_y = \dot{y}_2 - \dot{y}_1 = 2\gamma\left(\frac{d_x}{d_x^2 + d_y^2} - \frac{1}{d_x}\right)$$

and

$$\dot{d}_x = \dot{x}_3 - \dot{x}_2 = 2\gamma\left(\frac{1}{d_y} - \frac{d_y}{d_x^2 + d_y^2}\right).$$

Exercise. Solve the system of equations for $\dot{d}_x, \dot{d}_y$.

In the asymptotic regimes, we have

- For $d_x \gg d_y$, we have $\dot{d}_x \sim \frac{1}{d_y},\ \dot{d}_y \to 0.$
- For $d_y \gg d_x$, we have $\dot{d}_y \sim -\frac{1}{d_x},\ \dot{d}_x \to 0.$

Vortices with a 3D structure

They come under the names of vortex rings and Hopfions.