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REMARKS ON THE REGION OF ATTRACTION OF AN ISOLATED INVARIANT SET

BY

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Abstract. We study the complexity of the flow in the region of attraction of an isolated invariant set. More precisely, we define the instablity depth, which is an ordinal and measures how far an isolated invariant set is from being asymptotically stable within its region of attraction. We provide upper and lower bounds of the instability depth in certain cases.

1. Introduction. A fundamental problem in the theory of dynamical systems is to study the topological and dynamical structure of a compact invariant set of a continuous flow, and in particular a compact minimal set, and describe the behavior of the orbits near it. The simplest behavior occurs near an asymptotically stable compact invariant set A of a continuous flow on a separable, locally compact, metrizable space M. In this case, if W is the region of attraction of A, then the flow in $W \setminus A$ is parallelizable. If moreover M is a finite-dimensional manifold, then A has the shape of a compact polyhedron (see [7] and [8]).

The purpose of this note is to give a direction to study the region of attraction of an isolated compact invariant set A in the sense of C. C. Conley and relate its topological properties and the dynamics in it with the structure of A. We introduce the notion of the instability depth of the region of attraction W of an isolated invariant set A, which is a measure of the complexity of the flow in W. More precisely, it measures how far A is from being asymptotically stable with respect to the restricted flow in W, and is particularly informative in case W is a locally compact subspace of M.

In Section 4 we give an example of a smooth flow on $\mathbb{R}^3$ which has an isolated invariant set, consisting of a fixed point, whose region of attraction is a noncompact 3-manifold with (noncompact) boundary and which has instability depth 3. This cannot happen in $\mathbb{R}^2$. More precisely, we prove in Theorem 4.3 that if x_0 is a fixed point of a continuous flow on $\mathbb{R}^2$ (or S^2) such that $\{x_0\}$ is an isolated invariant set, then the instability depth of

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its region of attraction must be at most 2. However, this is not true on other orientable, closed 2-manifolds. We construct a smooth flow on the 2-torus having a unique fixed point, which is an isolated invariant set, whose region of attraction is the 2-torus and has instability depth 3. On higher-dimensional manifolds we show in Theorem 4.5 that in a smooth flow the region of attraction of an isolated unstable attractor of codimension at least 2 must have instability depth at least 2.

The instability depth is based on the notion of the intrinsic topology of the region of attraction W of an isolated invariant set A, originally defined in [9]. This topology is finer than the subspace topology of W inherited from the phase space M, but is nevertheless locally compact, separable and metrizable. The restricted flow on W remains continuous and A is globally asymptotically stable in W with respect to the intrinsic topology. It follows now from [1, Theorem 4.2] that if A is an isolated 1-dimensional compact minimal set and its region of attraction is an ANR with respect to the intrinsic topology, then A must be a periodic orbit. Thus it is an interesting problem to find conditions under which the region of attraction is an ANR with respect to the intrinsic topology. This could lead to Poincaré–Bendixson type theorems for flows on higher-dimensional phase spaces.

2. Isolated invariant sets and attraction. Let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a separable, locally compact, metrizable space M. We shall denote by $\phi_t(x) = tx$ the translation of the point $x \in M$ along its orbit in time $t \in \mathbb{R}$. We shall also write $IA = \{tx : t \in I,\ x \in A\}$ for $I \subset \mathbb{R}$ and $A \subset M$. The orbit of x will be denoted by $C(x)$, its positive semiorbit by $C^+(x)$ and the negative one by $C^-(x)$. The *positive limit set* of the orbit of $x \in M$ is the closed, invariant set

$$L^+(x) = \{y \in M : t_n x \to y \text{ for some } t_n \to +\infty\}.$$

A compact invariant set $A \subset M$ is called *isolated* if it has a compact neighbourhood V such that A is the maximal invariant set in V. Each such V is called an *isolating neighbourhood* of A. It is known (see [4], [5]) that every isolating neighbourhood of A contains a smaller isolating neighbourhood N of A such that there are compact sets $N^+, N^- \subset \partial N$ with the following properties:

(i) $\partial N = N^+ \cup N^-$.
(ii) For every $x \in N^+$ there exists $\varepsilon > 0$ such that $[-\varepsilon, 0)x \subset M \setminus N$ and for every $y \in N^-$ there exists $\delta > 0$ such that $(0, \delta]y \subset M \setminus N$.
(iii) For every $x \in \partial N \setminus N^+$ there exists $\varepsilon > 0$ such that $[-\varepsilon, 0)x \subset \operatorname{int} N$ and for every $y \in \partial N \setminus N^-$ there exists $\delta > 0$ such that $(0, \delta]y \subset \operatorname{int} N$.

The triad (N, N^+, N^-) is called an *isolating block* of A. The set N^+ is the *entrance set* and N^- is the *exit set* of the isolating block. The sets $A^\pm = \{x \in N : C^\pm(x) \subset N\}$ and $\alpha^\pm = \partial N \cap A^\pm$ are compact and $A = A^+ \cap A^-$. Moreover, $\emptyset \neq L^+(x) \subset A$ for every $x \in A^+$, and $\alpha^+ \subset N^+ \setminus N^-$.

If M is a smooth n-manifold and the flow is smooth, then every neighbourhood of an isolated invariant set A contains a smooth isolating block (N, N^+, N^-) of A. This means that N is a smooth compact n-manifold with boundary $\partial N = N^+ \cup N^-$, the sets N^+ and N^- being smooth compact $(n-1)$-manifolds with common boundary $N^+ \cap N^-$, which is a smooth compact $(n-2)$-manifold (without boundary) and on which the flow is externally tangent to N. Moreover, the flow is transverse to $N^+ \setminus N^-$ when entering into N and transverse to $N^- \setminus N^+$ when going out of N (see [6]).

If $A \subset M$ is a compact invariant set, then the invariant set

$$W^+(A) = \{x \in M : \emptyset \neq L^+(x) \subset A\}$$

is the *region of attraction* (or *stable manifold*) of A. If $W^+(A)$ is an open neighbourhood of A, then A is called an *attractor*.

A compact invariant set $A \subset M$ is called *stable* (in the sense of Lyapunov) if every neighbourhood of A contains a positively invariant neighbourhood of A. A stable attractor A is also called *asymptotically stable*. Note that these notions make sense also in the case where the phase space M is not locally compact.

If A is asymptotically stable, it is an isolated invariant set and there exists an isolating block (N, N^+, N^-) such that $N^- = \emptyset$ and $\partial N = N^+ = \alpha^+$. Moreover, the restricted flow in $W^+(A) \setminus A$ is parallelizable with a compact global section (see [3, p. 83]). A compact invariant set is called an *isolated unstable attractor* if it is an isolated invariant set, an attractor, and is not asymptotically stable.

3. The region of attraction of an isolated invariant set. Let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a separable, locally compact, metrizable space M. Let $A \subset M$ be an isolated compact invariant set and let (N, N^+, N^-) be an isolating block of A. The *final entrance time function* $f : W^+(A) \to [-\infty, +\infty)$ defined by

$$f(x) = \begin{cases} \sup\{t \in \mathbb{R} : tx \in M \setminus N\} & \text{if } x \in W^+(A) \setminus A, \\ -\infty & \text{if } x \in A, \end{cases}$$

is lower semicontinuous. This follows immediately from the definition and the continuity of the flow. Obviously, $f(x)x \in \alpha^+$ and $f(tx) = f(x) - t$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. The final entrance time function f is discontinuous at $x \in W^+(A) \setminus A$ if and only if there are $x_n \to x$ such that $f(x_n) \to +\infty$ (see [2, Lemma 3.1]). It follows that f is continuous on $\mathbb{R}\operatorname{int}_{\partial N}\alpha^+$ and so the flow maps $\mathbb{R}\times\operatorname{int}_{\partial N}\alpha^+$ homeomorphically onto $\mathbb{R}\operatorname{int}_{\partial N}\alpha^+$ (see [2, Proposition 3.2]). If the flow is smooth and the isolating bock is smooth, then f is smooth on $\mathbb{R}\operatorname{int}_{\partial N}\alpha^+$.

It is clear from the above that if A is an isolated compact invariant set, then A is not necessarily asymptotically stable with respect to the restricted flow in $W^+(A)$. However, it is possible to define a new topology in $W^+(A)$, which is finer than the subspace topology inherited from M, with respect to which the flow remains continuous and A becomes asymptotically stable. Roughly speaking, this new topology is obtained by cutting $W^+(A)$ along the discontinuity set of the final entrance time function with respect to any isolating block of A. It was first defined in [9, Section 7] (see also [10]).

Let $(X_t,p_{st})_{s,t\in\mathbb{R}}$ be the following inverse system of compacta. For every $t\in\mathbb{R}$ we let $X_t=N/N^+$ and for $s\le t$ the map $p_{st}:X_t\to X_s$ is defined by

$$p_{st}(x)=\begin{cases}(s-t)x & \text{if } [s-t,0]x\subset N\setminus N^+,\\ [N^+] & \text{otherwise.}\end{cases}$$

Obviously, $p_{st}([N^+])=[N^+]$ for every $s\le t$.

Let $X_N=\varprojlim(X_t,p_{st})$ and let $*$ denote the point of X all of whose coordinates are equal to $[N^+]$. Clearly, $X_N\setminus\{*\}$ is a locally compact, separable, metrizable space. If $(x_t)_{t\in\mathbb{R}}\in X_N\setminus\{*\}$ and $x_{t_0}=[N^+]$, then $x_t=[N^+]$ for every $t\le t_0$. Moreover, there exists $\tau\in\mathbb{R}$ such that $x_\tau\ne[N^+]$ and so $x_t\in N\setminus N^+$ and $x_\tau=(\tau-t)x_t$ for every $t\ge\tau$. This implies that we have a well defined function $h_N:X_N\setminus\{*\}\to M$ with $h_N((x_t)_{t\in\mathbb{R}})=(-\tau)x_\tau$, where $\tau\in\mathbb{R}$ is any such that $x_\tau\ne[N^+]$. Of course, since $(t-\tau)x_\tau=x_t\in N\setminus N^+$ for every $t\ge\tau$, we have $C^+(x_\tau)\subset N$, and so $x_\tau\in W^+(A)$, because N is an isolating neighbourhood of A. Thus, $h_N(X_N\setminus\{*\})\subset W^+(A)$. Conversely, if $x\in W^+(A)$, we let

$$x_t=\begin{cases}[N^+] & \text{if } t\le f(x),\\ tx & \text{if } t>f(x).\end{cases}$$

Then $h_N((x_t)_{t\in\mathbb{R}})=x$. This shows that $h_N(X_N\setminus\{*\})=W^+(A)$. Moreover, h_N maps injectively and continuously $X_N\setminus\{*\}$ onto $W^+(A)$. Note that

$$(h_N)^{-1}(A)=\{(tx)_{t\in\mathbb{R}}:x\in A\}=\varprojlim(A,p_{st}|A),$$

which is homeomorphic to A, since $(\phi_s|A)\circ(p_{st}|A)=\phi_t|A$ for every $s\le t$, hence compact. So, h_N maps $(h_N)^{-1}(A)$ homeomorphically onto A. Similarly, h_N maps $(h_N)^{-1}(\alpha^+)$ homeomorphically onto α^+.

It is immediate from the above formula giving $(h_N)^{-1}(x)$ that $(h_N)^{-1}$ is discontinuous at $x\in W^+(A)$ if and only if the final entrance time function f is discontinuous at x. Since f is continuous on $\mathbb{R}\operatorname{int}_{\partial N}\alpha^+$, so is $(h_N)^{-1}$ on that set.

If $(\Lambda, \Lambda^+, \Lambda^-)$ is another isolating block of A, then $(h_N)^{-1} \circ h_\Lambda : X_\Lambda \setminus \{*\} \to X_N \setminus \{*\}$ is a homeomorphism (see [10, Theorem 7.3]). It follows that there is a topology on $W^+(A)$ which is finer than the subspace topology inherited from M, and which makes h_N a homeomorphism and does not depend on the chosen isolating block (N, N^+, N^-). This topology is called the *intrinsic topology* of the region of attraction of A. We shall denote by $W_{\mathrm{i}}^+(A)$ the region of attraction of A equiped with the intrinsic topology.

LEMMA 3.1. *The final entrance time function $f : W_{\mathrm{i}}^+(A) \to [-\infty, +\infty)$ is continuous for any isolating block (N, N^+, N^-) of A.*

Proof. Using the previous notation, we have to prove the continuity of the function $g = f \circ h_N : X_N \setminus \{*\} \to [-\infty, +\infty)$. First observe that $g((x_t)_{t\in\mathbb{R}}) = \inf\{t \in \mathbb{R} : x_t \neq [N^+]\}$. Suppose that $g((x_t)_{t\in\mathbb{R}}) = t_0$ and $a < t_0 < b$ for some $a, b \in \mathbb{R}$. There exists a point $x \in \alpha^+$ such that $x_t = (t - t_0)x$ for every $t > t_0$. Since α^+ is a compact subset of $N^+ \setminus N^-$, there is an open neighbourhood V of x such that $V \cap N^- = \emptyset$, and $(-\varepsilon)V \subset M \setminus N$ and $\varepsilon V \subset \operatorname{int} N$ for some $\varepsilon > 0$ such that $a < t_0 - \varepsilon < t_0 < t_0 + \varepsilon < b$. The set

$$C = \Big((\varepsilon V) \times \prod_{t \neq t_0 + \varepsilon} N/N^+\Big) \cap (X_N \setminus \{*\})$$

is an open neighbourhood of $(x_t)_{t\in\mathbb{R}}$ in $X_N \setminus \{*\}$. If $(y_t)_{t\in\mathbb{R}} \in C$, then $y_{t_0+\varepsilon} \in \varepsilon V \subset \operatorname{int} N$ and so $g((y_t)_{t\in\mathbb{R}}) < t_0 + \varepsilon < b$. On the other hand, there exists a unique $-2\varepsilon < s \leq 0$ such that $s y_{t_0+\varepsilon} \in \alpha^+$, because $(-\varepsilon)V \subset M \setminus N$, and so $g((y_t)_{t\in\mathbb{R}}) = t_0 + \varepsilon + s > t_0 - \varepsilon > a$. This shows the continuity in case $t_0 \in \mathbb{R}$. If $t_0 = -\infty$, there exists some $x \in A$ such that $x_t = tx$ for every $t \in \mathbb{R}$ and we proceed similarly considering an open neighbourhood V of x such that $\varepsilon V \subset \operatorname{int} N$ for some $\varepsilon > 0$. In this case it suffices to take

$$C = \Big((\varepsilon V) \times \prod_{t \neq \varepsilon} N/N^+\Big) \cap (X_N \setminus \{*\}).$$

This completes the proof.

At this point observe that for every $x \in W^+(A)$ and $s \in \mathbb{R}$, if $(h_N)^{-1}(sx) = (y_t)_{t\in\mathbb{R}}$, then

$$y_t = \begin{cases} [N^+] & \text{if } s + t \leq f(x), \\ tx & \text{if } s + t > f(x). \end{cases}$$

Hence $(h_N)^{-1}$ transforms the restricted flow on $W^+(A)$ to the left shift on $X_N \setminus \{*\}$, which is a continuous flow. This implies that the flow remains continuous with respect to the intrinsic topology and A remains a compact invariant set in $W_{\mathrm{i}}^+(A)$. The following proposition with a different proof can also be found in [10, Theorem 3].

PROPOSITION 3.2. *A is a globally asymptotically stable compact invariant set in $W_{\mathrm{i}}^{+}(A)$.*

Proof. Let $F : W_{\mathrm{i}}^{+}(A) \to \mathbb{R}^{+}$ be the function defined by

$$F(x) = \begin{cases} e^{f(x)} & \text{if } x \in W_{\mathrm{i}}^{+}(A) \setminus A, \\ 0 & \text{if } x \in A, \end{cases}$$

where f is the final entrance time function with respect to any isolating block (N, N^{+}, N^{-}) of A. From Lemma 3.1 we see that F is continuous. It is also immediate from the definition that $A = F^{-1}(0)$ and $F(tx) = e^{-t}F(x)$ for every $t \in \mathbb{R}$ and $x \in W_{\mathrm{i}}^{+}(A) \setminus A$. Thus F is a strictly decreasing Lyapunov function for A along the orbits in $W_{\mathrm{i}}^{+}(A) \setminus A$. This shows that A is globally asymptotically stable in $W_{\mathrm{i}}^{+}(A)$ (see [3, Ch. V, Theorem 2.2]).

Recall that the restricted flow on $W_{\mathrm{i}}^{+}(A) \setminus A$ is parallelizable and each level set $F^{-1}(a)$, $a > 0$, is a compact global section. In particular the set $\alpha^{+} = F^{-1}(1)$ is a global section to the flow on $W_{\mathrm{i}}^{+}(A) \setminus A$ and thus $W_{\mathrm{i}}^{+}(A) \setminus A$ is homeomorphic to $\mathbb{R} \times \alpha^{+}$.

EXAMPLE 3.3. Consider the smooth flow on $\mathbb{R}^2$ defined by the system of differential equations (in polar coordinates)

$$r' = r(1 - r), \qquad \theta' = \sin^2(\theta/2).$$

Then $\{(1,0)\}$ is an isolated unstable attractor with $W^{+}(1,0) = \mathbb{R}^2 \setminus \{(0,0)\}$. The closed disc of radius $1/2$ centred at $(1,0)$ is an isolating block N and α^{+} is the southern semicircle on ∂N, hence homeomorphic to the closed interval $[0,1]$. The final entrance time function f is discontinuous at $(s,0)$, or equivalently the identity $\mathrm{id} : W^{+}(1,0) \to W_{\mathrm{i}}^{+}(1,0)$ is not continuous at $(s,0)$. Now $W_{\mathrm{i}}^{+}(1,0) \setminus \{(1,0)\}$ is homeomorphic to $\mathbb{R} \times [0,1]$ and $W_{\mathrm{i}}^{+}(1,0)$ is homeomorphic to $\mathbb{R} \times \mathbb{R}^{+}$.

4. The instability depth of the region of attraction. Let $(\phi_t)_{t \in \mathbb{R}}$ be a continuous flow on a separable, locally compact, metrizable space M and let $A \subset M$ be an isolated compact invariant set. Since $W_{\mathrm{i}}^{+}(A)$ is locally compact, $\mathrm{id} : W^{+}(A) \to W_{\mathrm{i}}^{+}(A)$ is continuous on an invariant, open subset (in the relative topology) of $W^{+}(A)$ by [11, Lemma 1]. Note that $W^{+}(A)$ is not always locally compact, but only an F_σ-set in M. If $W^{+}(A)$ is not locally compact, then $\mathrm{id} : W^{+}(A) \to W_{\mathrm{i}}^{+}(A)$ may be nowhere continuous. For example, let ξ be the infinitesimal generator of an irrational flow on the 2-torus $S^1 \times S^1$. Let $x_0 \in S^1 \times S^1$ and $g : S^1 \times S^1 \to [0,1]$ be a smooth function such that $g^{-1}(0) = \{x_0\}$. The flow of the smooth vector field $g \cdot \xi$ on $S^1 \times S^1$ has the unique fixed point x_0 and $\{x_0\}$ is an isolated invariant set. Obviously, there exists $x \in S^1 \times S^1$ such that $W^{+}(x_0) = \{x_0\} \cup C(x)$ and $C^{-}(x)$ is dense in $S^1 \times S^1$. It follows that $\mathrm{id} : W^{+}(x_0) \to W_{\mathrm{i}}^{+}(x_0)$ is nowhere continuous.

Proposition 4.1. *If $W^+(A)$ is locally compact, then:*

(a) *A is asymptotically stable with respect to the restricted flow in $W^+(A)$ if and only if* $\mathrm{id} : W^+(A) \to W^+_{\mathrm{i}}(A)$ *is continuous at every point of A.*

(b) $\mathrm{id} : W^+(A) \to W^+_{\mathrm{i}}(A)$ *is continuous on an invariant, dense, open subset (in the relative topology) of $W^+(A) \setminus A$.*

Proof. (a) Suppose that $\mathrm{id} : W^+(A) \to W^+_{\mathrm{i}}(A)$ is continuous at every point of A. This is equivalent to saying that the final entrance time function $f : W^+(A) \to [-\infty, +\infty)$, with respect to any isolating block of A, is continuous at every point of A. If A is not stable in $W^+(A)$, then, since $W^+(A)$ is locally compact, there exist points $x \in A$, $y \in W^+(A) \setminus A$, $x_n \in W^+(A) \setminus A$, $n \in \mathbb{N}$, and times $t_n \to +\infty$ such that $x_n \to x$ and $t_n x_n \to y$. Since f is continuous at x, we have $f(x_n) \to -\infty$, and so $f(t_n x_n) = f(x_n) - t_n \to -\infty$. This however contradicts the fact that f is lower semicontinuous, because we get

$$-\infty < f(y) \leq \liminf_{n\to\infty} f(t_n x_n) = -\infty.$$

The converse is trivial.

(b) The identity from $W^+(A)$ to $W^+_{\mathrm{i}}(A)$ is discontinuous at a point $x \in W^+(A) \setminus A$ if and only if the final entrance time function $f : W^+(A) \setminus A \to \mathbb{R}$ is discontinuous at x. Since f is lower semicontinuous the set of points of $W^+(A) \setminus A$ at which f is continuous, is a countable intersection of open sets dense in $W^+(A) \setminus A$. If $W^+(A)$ is locally compact, so is $W^+(A) \setminus A$, and from the Baire Category Theorem the continuity set of f is dense in $W^+(A) \setminus A$.

If (N, N^+, N^-) is an isolating block of A, then the invariant, open, dense subset of $W^+(A) \setminus A$ provided by Proposition 4.1(b) contains $\mathbb{R}\,\mathrm{int}_{\partial N}\, \alpha^+$. It is precisely $\mathbb{R}\,\mathrm{int}_{\partial N}\, \alpha^+$ in case A is an isolated unstable attractor.

Let now $A \subset M$ be an isolated compact invariant set which is not asymptotically stable with respect to the restricted flow in $W^+(A)$. The identity from $W^+(A)$ to $W^+_{\mathrm{i}}(A)$ is continuous on an open, invariant subset G_0 of $W^+(A) \setminus A$. If $W^+(A)$ is locally compact, G_0 is also dense in $W^+(A) \setminus A$, and the invariant set $W^+_1(A) = W^+(A) \setminus G_0$ is locally compact. Inductively, if α is an ordinal, and $W^+_\alpha(A) \subset W^+(A)$ has been defined, then the identity from $W^+_\alpha(A)$ to $W^+_{\alpha\mathrm{i}}(A)$, which is the same set with the intrinsic topology, is continuous on an open, invariant subset G_α of $W^+_\alpha(A) \setminus A$, and G_α is dense in $W^+_\alpha(A) \setminus A$, in case $W^+_\alpha(A)$ is locally compact. We then define $W^+_{\alpha+1}(A) = W^+_\alpha(A) \setminus G_\alpha$. If α is a limit ordinal, we put

$$W^+_\alpha(A) = \bigcap_{\beta<\alpha} W^+_\beta(A).$$

Since M has a countable basis, there exists an ordinal δ smaller than the first uncountable ordinal such that $W^+_\alpha(A) = W^+_\delta(A)$ for all $\alpha > \delta$. We call the least such δ the *instability depth* of $W^+(A)$. It is a measure of the complexity of the flow in $W^+(A)$, and measures how far A is from being asymptotically stable with respect to the restricted flow in $W^+(A)$. Note that if $W^+(A)$ is locally compact, then at every step $W^+_\alpha(A)$ is locally compact, and $G_\alpha \neq \emptyset$, since it is dense in $W^+_\alpha(A)$. So in this case the instability depth is δ if and only if δ is the least ordinal such that $W^+_\delta(A) = A$.

EXAMPLES 4.2. (a) Let $g : S^1 \to [0,1]$ be a smooth function such that $g^{-1}(0) = \{1\}$. Then $\{1\}$ is an isolated invariant set with respect to the flow of the smooth vector field $g \cdot \frac{\partial}{\partial t}$ on S^1 and $W^+(1) = S^1$. Any closed interval in S^1 which contains 1 in its interior is an isolating block of $\{1\}$. Here we have $W^+_{\mathrm{i}}(1) = (-\infty, 1]$ and the identity $\mathrm{id} : W^+(1) \setminus \{1\} \to W^+_{\mathrm{i}}(1) \setminus \{1\}$ is continuous. Therefore, $W^+_1(1) = \{1\}$ and the instability depth is 1.

(b) The instability depth of the region of attraction of the isolated invariant set $\{(1,0)\}$ in Example 3.3 is 2, since $W^+_1(1,0) = (0,+\infty) \times \{0\}$ and $W^+_2(1,0) = \{(1,0)\}$. More generally, let x_0 be a fixed point of a flow on a connected 2-manifold M (without boundary) such that $\{x_0\}$ is an isolated invariant set. According to [2, Lemma 4.1], every neighbourhood of x_0 contains an isolating block (N, N^+, N^-) such that N is a compact 2-manifold with boundary of genus zero. Moreover, the set $\partial_{\partial N}\alpha^+$ is finite, by [2, Proposition 4.2]. Since $W^+_1(x_0) \subset \{x_0\} \cup \mathbb{R}\partial_{\partial N}\alpha^+$, it follows that the instability depth of $W^+(x_0)$ is finite.

(c) The smooth flow on $\mathbb{R}^3$ defined by the system of differential equations (in cylindrical coordinates)

$$r' = r(1-r), \qquad \theta' = \sin^2(\theta/2), \qquad z' = z^2$$

has the two fixed points $(0,0,0)$ and $(1,0,0)$. Let $A = \{(1,0,0)\}$. The flow has the following features:

(i) The cylinder $r = 1$, the vertical halfplane $\theta = 0$ and the horizontal plane $z = 0$ are invariant. The vertical line $r = 0$ is also invariant and consists of three orbits.

(ii) If $r > 0$ and $z \leq 0$, then $L^+(r,\theta,z) = A$, and if $z > 0$, then $L^+(r,\theta,z) = \emptyset$.

(iii) A is an isolated invariant set and

$$W^+(A) = \{(r,\theta,z) : r > 0,\ 0 \leq \theta < 2\pi \text{ and } z \leq 0\}.$$

Note that $W^+(A)$ is a locally compact subspace of $\mathbb{R}^3$. The closed ball

$$N = \{(r,\theta,z) : r^2 - 2r\cos\theta + 1 + z^2 \leq 1/4\}$$

of radius 1/2 centred at $(1,0,0)$ is an isolating block of A and

$$\alpha^+ = \{(r,\theta,z) : r^2 - 2r\cos\theta + 1 + z^2 = 1/4,\ \sin\theta \leq 0,\ z \leq 0\}.$$

So $\partial_{\partial N}\alpha^+ = \{(r,\theta,z) \in \alpha^+ : \theta = 0 \text{ or } z = 0\}$. Here we have

$$\begin{aligned} W_1^+(A) &= A \cup \mathbb{R}\partial_{\partial N}\alpha^+ \\ &= \{(r,0,z) : r > 0 \text{ and } z \le 0\} \cup \{(r,\theta,0) : r > 0 \text{ and } 0 \le \theta < 2\pi\}, \end{aligned}$$

$W_2^+(A) = \{(r,0,0) : r > 0\}$ and $W_3^+(A) = A$. Hence the instability depth of $W^+(A)$ is 3.

The last example shows that assertion (ii) in [2, Proposition 3.2] is not correct, as in the present terminology it states that the instability depth is always at most 2. Of course it is correct if we consider on $\mathbb{R}\partial_{\partial N}\alpha^+$ the intrinsic topology. This mistake does not affect the rest of the content of [2]. One case where the instability depth is always at most 2 is given by the following.

THEOREM 4.3. *Let x_0 be a fixed point of a continuous flow on $\mathbb{R}^2$ or S^2 such that $\{x_0\}$ is an isolated invariant set. Then the instability depth of $W^+(x_0)$ is at most* 2*.*

Proof. Let (N, N^+, N^-) be an isolating block of $\{x_0\}$ such that N is a compact 2-manifold with boundary (see [2, Lemma 4.1]). There are finitely many $x_1, \ldots, x_m \in \partial N$ such that $\partial_{\partial N}\alpha^+ = \{x_1, \ldots, x_m\}$ (see [2, Proposition 4.2]). We shall prove by contradiction that $W_2^+(x_0) = \{x_0\}$. If this is not true, the restriction of the final entrance time function to

$$W^+\{x_0\} \subset \{x_0\} \cup C(x_1) \cup \cdots \cup C(x_m)$$

is discontinuous at x_k for some $1 \le k \le m$. This implies that there is some $1 \le l \le m$ such that $x_k \in L^-(x_l)$. Since $x_k \in \alpha^+$ and N is an isolating block, there exists an open interval $I \subset N^+ \setminus N^-$ such that I is a local section to the flow of some extent $\varepsilon > 0$ and $x_k \in I$. There are $t_n \to -\infty$ such that $t_n x_l \in I$ and if $[x_k, t_n x_l]$ denotes the subinterval of I with endpoints x_k and $t_n x_l$, then $[x_k, t_{n+1}x_l] \subset [x_k, t_n x_l]$ for every $n \in \mathbb{N}$. Obviously, $t_{n+1} - t_n < 2\varepsilon$. The set

$$C_n = \{x_0\} \cup C^+(x_k) \cup [x_k, t_n x_l] \cup C^+(t_n x_l)$$

is a simple closed curve, which bounds a topological closed disc D_n by the Jordan–Schönflies Theorem. Since I is a local section, D_n is positively invariant and $(-\varepsilon, 0)[x_k, t_n x_l] \cap D_n = \emptyset$, while $[0,\varepsilon)[x_k, t_n x_l] \subset D_n$. Since $t_{n+1}x_l \in [x_k, t_n x_l]$, it follows that $C^+(t_{n+1}x_l) \subset D_n$, which contradicts the fact that $(t_n - \varepsilon, t_n)x_l \cap D_n = \emptyset$.

The question now arises whether Theorem 4.3 is true on any orientable, closed 2-manifold. The answer to this question is negative even on the 2-torus.

EXAMPLE 4.4. Let $\psi : \mathbb{R} \to [0,1]$ be a smooth function such that $\psi^{-1}(0) = (-\infty, 3/2]$ and $\psi^{-1}(1) = [7/4, +\infty)$, and let $g : \mathbb{R}^2 \to \mathbb{R}^+$ be the smooth function defined in polar coordinates by

$$g(r,\theta) = \sin^2(\theta/2) + \psi(r)\cos^2(\theta/2).$$

Let also $h : \mathbb{R}^2 \to [0,1]$ be a smooth function such that $h^{-1}(0) = \{(2,0)\}$ and $h^{-1}(1) = \{(r,\theta) : 1 \le r \le 3/2,\ 0 \le \theta < 2\pi\}$. The closed annulus

$$[1,2] \times S^1 = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta < 2\pi\}$$

is invariant under the flow of the system of differential equations (in polar coordinates)

$$r' = (r-1)(r-2), \quad \theta' = h(r,\theta)g(r,\theta).$$

The flow in $[1,2] \times S^1$ has only the two fixed points $(1,0)$ and $(2,0)$. For $1 < r < 2$ we have $L^+(r,\theta) = \{(1,0)\}$ and $L^-(r,\theta) = \{2\} \times S^1$, while $L^+(1,\theta) = L^-(1,\theta) = \{(1,0)\}$ and $L^+(2,\theta) = L^-(2,\theta) = \{(2,0)\}$.

We can identify $\{1\} \times S^1$ with $\{2\} \times S^1$ suitably and get a smooth flow on the 2-torus $S^1 \times S^1$ having only one fixed point $x_0 = p(1,0) = p(2,0)$, such that $\{x_0\}$ is an isolated invariant set and $W^+(x_0) = S^1 \times S^1$, where $p : [1,2] \times S^1 \to S^1 \times S^1$ is the identification map. If $Y = \{(r,\theta) \in [1,2] \times S^1 : r^2 - 2r\cos\theta + 1 < 1/25$ or $r^2 - 4r\cos\theta + 4 < 1/25\}$, then $N = p(Y)$ is an isolating block of $\{x_0\}$ such that

$$\begin{aligned}\alpha^+ = p(\{(r,\theta) \in [1,2] \times S^1 : r^2 - 2r\cos\theta + 1 = 1/25 \\ \text{and } 49/50 \le \cos\theta \le 1,\ \sin\theta \le 0\})\end{aligned}$$

and therefore $W_{\mathrm{i}}^+(x_0)$ is homeomorphic to $\mathbb{R} \times \mathbb{R}^+$.

If $x_1 = p(6/5,0)$ and $x_1 = p(1,\theta_0)$, where $\cos\theta_0 = 49/50$ and $\sin\theta_0 < 0$, then $\partial_{\partial N}\alpha^+ = \{x_1, x_2\}$. By the properties of the flow in $[1,2] \times S^1$ we have $L^+(x_2) = L^-(x_2) = \{x_0\}$ and $L^-(x_1) = C(x_2) \cup \{x_0\}$. It follows that $W_1^+(x_0) = C(x_1) \cup C(x_2) \cup \{x_0\}$, $W_2^+(x_0) = C(x_2) \cup \{x_0\}$ and $W_3^+(x_0) = \{x_0\}$. So the instability depth is 3.

In higher dimensions we have the following lower bound.

THEOREM 4.5. *Let ξ be a smooth vector field on a connected, smooth n-manifold M and $A \subset M$ be an invariant continuum of dimension at most $n-2$. If A is an isolated unstable attractor, then the instability depth of $W^+(A)$ is at least* 2.

Proof. Suppose that the instability depth of $W^+(A)$ is 1. This means that the identity maps $W_{\mathrm{i}}^+(A) \setminus A$ homeomorphically onto $W^+(A) \setminus A$. If (N, N^+, N^-) is a smooth isolating block of A, then the flow on $W^+(A) \setminus A$ is parallelizable with section α^+. It follows that α^+ is a union of connected components of ∂N, thus being a compact, $(n-1)$-dimensional, smooth submanifold of M without boundary. Since the dimension of A is at most $n-2$ and $W^+(A)$ is connected and open, $W^+(A) \setminus A$ is connected. It now follows from [2, Theorem 3.4] that M is compact and $M = W^+(A)$. Moreover, A is an isolated unstable attractor with respect to $-\xi$ whose region of attraction (with respect to $-\xi$) has instability depth 1. This implies that $\partial N = \alpha^+ \cup \alpha^-$ and $N \setminus A = \mathbb{R}^+\alpha^+ \cup \mathbb{R}^-\alpha^-$, where these two sets are nonempty, disjoint and open in $N \setminus A$. This contradicts our assumption that A has dimension at most $n-2$, since $A \subset \operatorname{int} N$.

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