

On the existence of absolutely continuous automorphic measures

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Abstract. Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection. We present sufficient conditions which imply the existence of absolutely continuous automorphic measures for T with respect to a given ergodic T -invariant Borel probability measure.

1. Introduction.

Let X be a compact metric space, $T : X \rightarrow X$ a continuous surjection and let $f : X \rightarrow \mathbb{R}$ be a continuous function. A Borel probability measure ν on X is called an e^f -automorphic measure for T if ν is equivalent to $T_*\nu$ and $\frac{d\nu}{d(T_*\nu)} = e^f$. This kind of measure has been used without a particular name in [4].

In this note we study the existence of absolutely continuous automorphic measures with respect to a given ergodic T -invariant Borel probability measure. In Corollary 3.5 we present a sufficient condition for the existence of an absolutely continuous automorphic measure for a continuous surjection. In Theorem 4.1 we present a weaker sufficient condition in the special case of uniquely ergodic homeomorphisms using a different method, namely the Schauder-Tychonoff fixed point theorem applied to the dual Perron-Frobenius operator on an appropriate convex set.

The problem of the existence of an e^f -automorphic measure ν for a homeomorphism T which is absolutely continuous with respect to an ergodic T -invariant Borel probability measure μ is closely related to the existence and regularity properties of solutions of the cohomological equation $f = u - u \circ T$. The relation is explained with details in subsection 2.3. If there exists a continuous solution u , then f is called a continuous coboundary. According to the classical Gottschalk-Hedlund theorem (see page 102 in [3]), if T is minimal, then f is a continuous coboundary if and only if there exists $x_0 \in X$ such that

$$\sup\left\{\left|\sum_{k=0}^{n-1} f(T^k(x_0))\right| : n \in \mathbb{N}\right\} < +\infty.$$

In Corollary 3.7 we give a necessary and sufficient condition for a continuous function $f : X \rightarrow \mathbb{R}$ such that $\int_X f d\mu = 0$ to be a continuous coboundary, without the minimality assumption for T . This condition is similar to the one given in Corollary 3.5, and the result is obtained by investigating the regularity of the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

The example that led to this note was Denjoy C^1 diffeomorphisms of the circle. Let $T : S^1 \rightarrow S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number which is not topologically conjugate to a rotation. It is well known that T has a

unique Cantor minimal set K and a unique invariant Borel probability measure μ whose support is precisely K . These maps are usually called Denjoy C^1 diffeomorphisms. Our original motivation was to examine whether μ is in some sense geometric. We interpret this to the question whether $\log T'$ is a continuous coboundary on K . This is closely related to a problem stated in [1]. A. Denjoy proved that $\int_K (\log T') d\mu = 0$ (see page 71 in [2]). However one can show that $\log T'$ is never a continuous coboundary on S^1 . As a corollary of the results of section 3 we obtain sufficient conditions for $\log T'$ to be a measurable or continuous coboundary on K .

2. Automorphic measures.

Let $T : X \rightarrow X$ be a continuous surjection of a compact metric space X and let $f : X \rightarrow \mathbb{R}$ be a continuous function. An e^f -automorphic measure for T is a Borel probability measure ν on X such that

$$\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$. Evidently, an e^f -automorphic measure for T is T -quasi-invariant. In case T is a homeomorphism, a e^f -automorphic measure for T is an $e^{-f \circ T^{-1}}$ -automorphic measure for T^{-1} .

It is easy to see that if $h : X \rightarrow X$ is a homeomorphism and $S = h \circ T \circ h^{-1}$, then $h_*\nu$ is an $e^{f \circ h^{-1}}$ -automorphic measure for S for every e^f -automorphic measure ν for T .

In the case of a homeomorphism T the construction of an automorphic measure can be described as follows. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and let $c = \limsup_{n \rightarrow +\infty} \frac{a_n}{n}$.

The series $\sum_{n=1}^{\infty} e^{a_n - ns}$ converges for $s > c$, diverges for $s < c$ and we cannot tell for $s = c$, by the root test. There exists a sequence of positive real numbers $(b_n)_{n \in \mathbb{N}}$ such that

$\lim_{n \rightarrow +\infty} \frac{b_n}{b_{n+1}} = 1$ and the series $\sum_{n=1}^{\infty} b_n e^{a_n - ns}$ converges for $s > c$ and diverges for $s \leq c$.

Let now $x \in X$ be such that the limit

$$c = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n f(T^{-k}(x))$$

exists in \mathbb{R} . It is well known that the set consisting of these points has measure 1 with respect to every T -invariant Borel probability measure, and is therefore non-empty. Let

$a_n = nc - \sum_{k=1}^n f(T^{-k}(x))$ and $M_s = \sum_{n=1}^{\infty} b_n e^{a_n - ns}$, where $(b_n)_{n \in \mathbb{N}}$ is the corresponding

sequence as above. Each accumulation point in the weak* topology as $s \downarrow 0$ of the directed family

$$\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \delta_{T^{-n}(x)}, \quad s > 0$$

is an e^f -automorphic measure for T .

There is a relation between e^f -automorphic measures for a continuous surjection $T : X \rightarrow X$ of a compact metric space and solvability of the cohomological equation $f = u - u \circ T$, where $f : X \rightarrow \mathbb{R}$ is continuous.

Let μ be any T -invariant Borel probability measure. If there exists a measurable solution u of the above cohomological equation defined μ -almost everywhere such that $e^{-u} \in L^1(\mu)$, then there exists an e^f -automorphic measure ν for T equivalent to μ with density

$$\frac{d\nu}{d\mu} = \frac{e^{-u}}{\int_X e^{-u} d\mu}.$$

Thus, if there exists a continuous solution u , then for every T -invariant Borel probability measure we get an equivalent e^f -automorphic measure for T . Moreover, in this case, every e^f -automorphic measure ν for T is obtained in this way. Indeed, we have

$$\int_X \phi e^u d\nu = \int_X (\phi \circ T) e^u d\nu$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$, and so the equivalent measure μ to ν with density

$$\frac{d\mu}{d\nu} = \frac{e^u}{\int_X e^u d\nu}$$

is T -invariant. Consequently, if f is a continuous coboundary, then the e^f -automorphic measures for T are in one-to-one correspondence with the T -invariant Borel probability measures and each e^f -automorphic measure for T is equivalent to its corresponding T -invariant measure.

Conversely, suppose that μ is an ergodic T -invariant Borel probability measure and $f : X \rightarrow \mathbb{R}$ is a continuous function such that $\int_X f d\mu = 0$. Suppose further that there exists an e^f -automorphic measure $\nu \in \mathcal{M}(X)$ for T which is absolutely continuous with respect to μ and $g = \frac{d\nu}{d\mu}$. For every measurable set $A \subset X$ we have

$$\int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T) e^f d\nu = \int_X (\chi_A \circ T) e^f g d\mu$$

and therefore

$$\int_{T^{-1}(A)} [ge^f - (g \circ T)] d\mu = 0.$$

Since μ is T -invariant, it follows that $g \circ T = ge^f$ μ -almost everywhere. The ergodicity of μ implies now that $g > 0$ μ -almost everywhere. So, $u = -\log g$ is a measurable solution of the cohomological equation $f = u - u \circ T$. If $\log g \in L^\infty(\mu)$ and T is a minimal homeomorphism or T is a locally eventually onto local homeomorphism and μ has full support, then there exists some continuous function $u : X \rightarrow \mathbb{R}$ such that $f = u - u \circ T$, by Proposition 4.2 on page 46 in [2] and Theorem 6 in [6], respectively.

3. Absolutely continuous automorphic measures.

Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. The set

$$E_\mu = \{\nu \in \mathcal{M}(X) : \nu \ll \mu\}$$

is not empty, since it contains μ , and is convex. In general, E_μ is not a closed subset of $\mathcal{M}(X)$ with respect to the weak* topology. For example, if we let μ be the Lebesgue measure on the unit interval $[0, 1]$ and for $0 < \epsilon < 1$ we let μ_ϵ denote the Borel probability measure on $[0, 1]$ with density $\frac{1}{\epsilon}\chi_{[0, \epsilon]}$, then $\lim_{\epsilon \rightarrow 0} \mu_\epsilon$ is the Dirac point measure at 0.

Lemma 3.1. *Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence in E_μ converging weakly* to some $\nu \in \mathcal{M}(X)$ and let $f_n = \frac{d\nu_n}{d\mu}$, $n \in \mathbb{N}$. If there exist non-negative $h, g \in L^1(\mu)$ such that $h \leq f_n \leq g$ for every $n \in \mathbb{N}$, then $\nu \in E_\mu$ and $h \leq \frac{d\nu}{d\mu} \leq g$.*

Proof. Since ν is a finite measure, there exists a (countable) basis \mathcal{U} of the topology of X such that $\nu(\partial U) = 0$ for every $U \in \mathcal{U}$. So \mathcal{U} is contained in the algebra

$$\mathcal{C}(\nu) = \{A \mid A \subset X \text{ Borel and } \nu(\partial A) = 0\}$$

and since it generates the Borel σ -algebra of X , so does $\mathcal{C}(\nu)$. Let now $A \subset X$ be a Borel set with $\mu(A) = 0$ and $\epsilon > 0$. There exists $0 < \delta < \epsilon$ such that $\int_B g d\mu < \epsilon$ for every Borel set $B \subset X$ with $\mu(B) < \delta$, because $g \in L^1(\mu)$. There exists some $A_0 \in \mathcal{C}(\nu)$ such that $\mu(A \Delta A_0) < \delta$ and $\nu(A \Delta A_0) < \delta$. Thus $\mu(A_0) < \delta$ and $|\nu(A) - \nu(A_0)| < \delta$. By weak* convergence, $\nu(A_0) = \lim_{n \rightarrow +\infty} \nu_n(A_0)$ and so there exists some $n_0 \in \mathbb{N}$ such that $|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \geq n_0$. Therefore,

$$\nu(A) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq \int_{A_0} g d\mu + \epsilon < 2\epsilon.$$

It follows that $0 \leq \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in E_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(\mathcal{P}_n)_{n \in \mathbb{N}}$ of X such that \mathcal{P}_{n+1} is a refinement of \mathcal{P}_n , the Borel σ -algebra of X is generated by $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ and $\mu(\partial B) = 0$ for every $B \in \mathcal{P}_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of X such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively \mathcal{P}_n to be the finite family consisting of Borel sets with positive μ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in \mathcal{P}_{n-1}$, taking $\mathcal{P}_0 = \{X\}$.

Let $\mathcal{P}_n(x)$ denote the element of \mathcal{P}_n which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n \rightarrow +\infty} \frac{\nu(\mathcal{P}_n(x))}{\mu(\mathcal{P}_n(x))},$$

μ -almost everywhere on X and in $L^1(\mu)$ (see page 8 in [5]). On the other hand, by the weak* convergence and since $\nu \in E_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k} \mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \leq \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} < \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \leq \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} g d\mu.$$

Since

$$\lim_{k \rightarrow +\infty} \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} g d\mu = g(x)$$

μ -almost everywhere on X and in $L^1(\mu)$, it follows that $0 \leq \frac{d\nu}{d\mu}(x) \leq g(x)$ μ -almost everywhere on X .

Similarly, from

$$\frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} > -\frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \geq -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} h d\mu$$

follows that $h(x) \leq \frac{d\nu}{d\mu}(x)$ μ -almost everywhere on X . \square

Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection. For any continuous function $f : X \rightarrow \mathbb{R}$ we put $S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k$ and $E_n(f) = e^{S_n(f)}$. Let $M_n = \sup\{S_n(f)(x) : x \in X\}$ and $L_n = \inf\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{N}$. Since $S_n(f) \circ T = S_{n+1}(f) + f$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T)e^{-f} - g_n = E_n(f) - e^{-f}.$$

Let now $\mu \in \mathcal{M}(X)$ be T -invariant and suppose that $\int_X f d\mu = 0$. So, $L_n \leq 0 \leq M_n$ for every $n \in \mathbb{N}$. Putting $h_n = \frac{g_n}{\int_X g_n d\mu}$, we get

$$(h_n \circ T) - h_n e^f = \frac{e^f - e^{-S_n(f)}}{e^{-S_n(f)} \int_X g_n d\mu},$$

for every $n \in \mathbb{N}$.

Suppose that there exists a positive $h \in L^1(\mu)$ such that $E_n(f) \leq h \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$. Then also $0 \leq h_n \leq h$ for $n \in \mathbb{N}$. If ν_n denotes the element of E_μ with $h_n = \frac{d\nu_n}{d\mu}$, then $\{\nu_n : n \in \mathbb{N}\} \subset E_\mu$, by Lemma 3.1.

Proposition 3.2. *Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be T -invariant and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that*

(i) *there exists a positive $h \in L^1(\mu)$ such that $E_n(f) \leq h \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and*

(ii) the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}$, $n \in \mathbb{N}$, is unbounded.

Then there exists an e^f -automorphic measure for T which is absolutely continuous with respect to μ .

Proof. Using the above notations, it suffices to prove that there exists a sequence of positive integers $n_j \rightarrow +\infty$ such that $\lim_{j \rightarrow +\infty} ((h_{n_j} \circ T) - h_{n_j} e^f) = 0$ μ -almost everywhere on X . Indeed, passing to a subsequence if necessary, there exists $\nu \in E_\mu$ such that $\nu = \lim_{j \rightarrow +\infty} \nu_{n_j}$, by Lemma 3.1. Since μ is T -invariant, for every continuous function $\phi : X \rightarrow \mathbb{R}$ we have

$$\int_X (\phi - (\phi \circ T) e^f) d\nu = \lim_{j \rightarrow +\infty} \int_X (\phi \circ T) ((h_{n_j} \circ T) - h_{n_j} e^f) d\mu = 0,$$

by dominated convergence, because

$$|(\phi \circ T) ((h_n \circ T) - h_n e^f)| \leq \|\phi\| ((h \circ T) + h e^f) \in L^1(\mu).$$

Since

$$|(h_n \circ T) - h_n e^f| = e^f \frac{|E_n(f) - e^{-f}|}{\int_X g_n d\mu},$$

we need only prove that there exist $n_j \rightarrow +\infty$ such that

$$\lim_{j \rightarrow +\infty} \mu(\{x \in X : |E_{n_j}(f)(x) - e^{-f(x)}| \geq \delta \int_X g_{n_j} d\mu\}) = 0$$

for every $\delta > 0$. Let

$$A_{n,\delta} = \{x \in X : E_n(f)(x) \geq e^{-f(x)} + \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x)\}, \quad \text{and}$$

$$A'_{n,\delta} = \{x \in X : E_n(f)(x) \leq e^{-f(x)} - \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x)\}.$$

Our assumption (i) implies that it suffices to prove the existence of a sequence of positive integers $n_j \rightarrow +\infty$ such that $\lim_{j \rightarrow +\infty} \mu(A_{n_j,\delta}) = \lim_{j \rightarrow +\infty} \mu(A'_{n_j,\delta}) = 0$ for every $\delta > 0$.

For every $x \in A_{n,\delta}$ we have

$$\frac{h(x)}{\delta} > e^{-M_n} \sum_{k=0}^{n-1} E_k(f)(x)$$

and integrating over $A_{n,\delta}$ we obtain

$$\frac{1}{\delta} \int_X h d\mu \geq \mu(A_{n,\delta}) e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}.$$

Similarly, for every $x \in A'_{n,\delta}$ we have

$$\sum_{k=0}^{n-1} E_k(f)(x) < \frac{h(x)}{\delta} e^{-f(x)}$$

and integrating over $A'_{n,\delta}$ we get

$$\mu(A'_{n,\delta}) \sum_{k=0}^{n-1} e^{L_k} \leq \frac{1}{\delta} \int_X h e^{-f} d\mu.$$

Our assumption (ii) means that there exist $n_j \rightarrow +\infty$ such that $e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{L_k} \rightarrow +\infty$,

and therefore we also have $\sum_{k=0}^{n_j-1} e^{L_k} \rightarrow +\infty$, because $L_n \leq 0 \leq M_n$. Consequently,
 $\lim_{j \rightarrow +\infty} \mu(A_{n_j,\delta}) = \lim_{j \rightarrow +\infty} \mu(A'_{n_j,\delta}) = 0$. \square

In the next proposition we make a more restrictive assumption (i) and a weaker assumption (ii).

Proposition 3.3. *Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be T -invariant and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that*

(i) *there exists a constant $c \geq 1$ such that $E_n(f) \leq c \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and*

(ii) *the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}$, $n \in \mathbb{N}$, is unbounded,*

Then there exists an e^f -automorphic measure for T which is absolutely continuous with respect to μ .

Proof. Our assumption (ii) means that there exists a sequence of positive integers $n_j \rightarrow +\infty$ such that $e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{M_k} \rightarrow +\infty$, as $j \rightarrow +\infty$. Using the same notations as

above we have $\int_X g_{n_j} d\mu \rightarrow +\infty$ and

$$e^{-S_{n_j}} \int_X g_{n_j} d\mu \geq \frac{1}{c} \cdot e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{M_k} \rightarrow +\infty,$$

as $j \rightarrow +\infty$, by our assumptions. Therefore, $\lim_{j \rightarrow +\infty} ((h_{n_j} \circ T) - h_{n_j} e^f) = 0$ uniformly on X and as in the proof of Proposition 3.2, every $\nu \in \overline{\{\nu_{n_j} : j \in \mathbb{N}\}}$ is e^f -automorphic measure for T that is absolutely continuous with respect to μ . \square

As the following Lemma shows, if in Proposition 3.3 the T -invariant measure $\mu \in \mathcal{M}(X)$ is ergodic, then condition (ii) is implied by condition (i).

Lemma 3.4. *Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T -invariant measure and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$.

(a) If $A_n = \{x \in X : S_n(x) > M_n - \log c - 1\}$, $n \in \mathbb{N}$, then $\mu(A_n) \geq \frac{e-1}{ec-1}$ for $n \in \mathbb{N}$.

(b) For every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$.

(c) The sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}$, $n \in \mathbb{N}$, is unbounded.

Proof. (a) From our assumption we have

$$e^{M_n - \log c} \leq \int_X E_n(f) d\mu \leq e^{M_n} \mu(A_n) + e^{M_n - \log c - 1} \mu(X \setminus A_n),$$

from which the required inequality follows.

(b) We proceed to prove the assertion by contradiction assuming that there exists some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $1 \leq j_n \leq N$ such that $M_{n+j_n} > M_n + 1$. Inductively, if we put $n_k = 1 + j_1 + \dots + j_k$, then $M_{n_k} > M_1 + k$ and $1 + k \leq n_k \leq 1 + kN$ for every $k \in \mathbb{N}$. Therefore,

$$\frac{M_{n_k}}{n_k} > \frac{1}{N+1}$$

for every $k \in \mathbb{N}$. If now $k_0 \in \mathbb{N}$ is such that $\left| \frac{\log c - 1}{n_k} \right| < \frac{1}{2(N+1)}$ for $k \geq k_0$, then for $x \in A_n$ we have

$$\frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N+1)}$$

and by (a) we get

$$\mu(\{x \in X : \frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N+1)}\}) \geq \frac{e-1}{ec-1} > 0$$

for every $k \geq k_0$. Hence the sequence $(\frac{1}{n} S_n)_{n \in \mathbb{N}}$ does not converge in measure to zero. This contradicts the Ergodic Theorem of Birkhoff, since we assume that μ is an ergodic T -invariant Borel probability measure.

(c) Suppose on the contrary that there exists a real number $a > 0$ such that $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k} \leq a$, for every $n \in \mathbb{N}$. By (b), for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$, and so

$$\sum_{j=0}^N \sum_{k=0}^{n+j-1} e^{M_k} \leq a \sum_{j=0}^N e^{M_n+j} \leq ea e^{M_n} + a \left(\sum_{k=0}^{n+N-1} e^{M_k} - \sum_{k=0}^{n-1} e^{M_k} \right)$$

$$\leq ea(1+a)e^{M_n} - a \sum_{k=0}^{n-1} e^{M_k}.$$

Substituting

$$\sum_{j=0}^N \sum_{k=0}^{n+j-1} e^{M_k} = (N+1) \sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N-i)e^{M_{n+i}},$$

we arrive at

$$(N+1+a) \sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N-i)e^{M_{n+i}} \leq ea(1+a)e^{M_n}$$

and therefore $N \leq ea(1+a)$ for every $N \in \mathbb{N}$, contradiction.

Corollary 3.5. *Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T -invariant measure and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, then there exists an e^f -automorphic measure ν for T which is absolutely continuous with respect to μ . Moreover, $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ and $-\log(\frac{d\nu}{d\mu})$ is a measurable solution of the cohomological equation $f = u - u \circ T$. \square

Corollary 3.6. *Let X be a compact metric space and $T : X \rightarrow X$ a continuous surjection which is a locally eventually onto local homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T -invariant measure and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, then there exists an e^f -automorphic measure ν for T which is absolutely continuous with respect to μ . Moreover, $-\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu)$ and in case μ has full support the cohomological equation $f = u - u \circ T$ has a continuous solution. \square

If X is a compact metric space and $T : X \rightarrow X$ is a homeomorphism, for any continuous function $f : X \rightarrow \mathbb{R}$ we put

$$E_n(f) = \begin{cases} \exp \sum_{k=1}^n f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp \left(- \sum_{k=0}^{|n|-1} f \circ T^k \right), & \text{if } n < 0. \end{cases}$$

As before we also put $S_n(f) = \log E_n(f)$ and $M_n = \sup\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{Z}$.

Let now $\mu \in \mathcal{M}(X)$ be T -invariant and suppose that $\int_X f d\mu = 0$. Then, $M_n \geq 0$ for every $n \in \mathbb{Z}$. Since $S_n(f) \circ T^{-1} = S_{n+1}(f) - f \circ T^{-1}$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T^{-1})e^{f \circ T^{-1}} - g_n = E_n(f) - 1.$$

Putting $h_n = \frac{g_n}{\int_X g_n d\mu}$, we get

$$(h_n \circ T^{-1})e^{f \circ T^{-1}} - h_n = \frac{1 - e^{-S_n(f)}}{e^{-S_n(f)} \int_X g_n d\mu},$$

for every $n \in \mathbb{N}$. So the same reasoning as above and Lemma 3.1 give the following.

Theorem 3.7. *Let X be a compact metric space and $T : X \rightarrow X$ a homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T -invariant measure and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$.*

(a) *If there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then there exists an e^f -automorphic measure ν for T which is equivalent to μ such that $\frac{d\nu}{d\mu} \in L^\infty(\mu)$.

(b) *Moreover, if*

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then $\log\left(\frac{d\nu}{d\mu}\right) \in L^\infty(\mu)$. \square

Combining Theorem 3.7 with section 2 we get the following.

Corollary 3.8. *Let X be a compact metric space and $T : X \rightarrow X$ a minimal homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T -invariant measure and let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Then the following assertions are equivalent.*

(i) *f is a continuous coboundary.*

(ii) *There exists a constant $c \geq 1$ such that*

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$). \square

4. Absolutely continuous automorphic measures for uniquely ergodic homeomorphisms.

In the special case of a uniquely ergodic homeomorphism $T : X \rightarrow X$ automorphic measures can be obtained as fixed points of the dual Perron-Frobenius operator. Let μ be the unique T -invariant Borel probability measure and $c = \int_X f d\mu$, where as before $f : X \rightarrow \mathbb{R}$ is a continuous function. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X equipped with the weak* topology. The dual Perron-Frobenius operator is the continuous map $W : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ defined by

$$W(\nu)(\phi) = \frac{1}{\int_X e^f d\nu} \cdot \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$ (see page 185 in [4]). Since T is a homeomorphism, W is a homeomorphism and its inverse is given by the formula

$$W^{-1}(\nu)(\phi) = \frac{1}{\int_X e^{-f \circ T^{-1}} d\nu} \cdot \int_X (\phi \circ T^{-1}) e^{-f \circ T^{-1}} d\nu.$$

It follows from the Schauder-Tychonoff theorem that W has a fixed point in $\mathcal{M}(X)$. If ν is a fixed point of W , then $\int_X e^f d\nu = e^c$, and therefore ν is an e^{f-c} -automorphic measure for T . Indeed, if ν is a fixed point of W , then for every $n \in \mathbb{N}$ we have

$$\left(\int_X e^f d\nu \right)^n = \int_X \exp\left(\sum_{k=0}^{n-1} f \circ T^k\right) d\nu,$$

as one easily verifies by induction. It follows that

$$n \left| \log\left(\int_X e^{f-c} d\nu\right) \right| = \left| \log\left(\int_X \exp\left(-nc + \sum_{k=0}^{n-1} f \circ T^k\right) d\nu\right) \right| \leq \left\| -nc + \sum_{k=0}^{n-1} f \circ T^k \right\|,$$

and therefore

$$\left| \log\left(\int_X e^{f-c} d\nu\right) \right| \leq \left\| \lim_{n \rightarrow +\infty} \left(-c + \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k\right) \right\| = 0.$$

Alternatively, we can use the Schauder-Tychonoff theorem to get the following result for the existence of absolutely continuous automorphic measures in the case of uniquely ergodic homeomorphisms.

Theorem 4.1. *Let X be a compact metric space and $T : X \rightarrow X$ a uniquely ergodic homeomorphism with unique invariant Borel probability measure μ . Let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a non-negative $g \in L^1(\mu)$ and an integer $m \geq 0$ such that*

$$E_n(f) \leq g \int_X E_n(f) d\mu$$

for every $n \geq m$, then there exists an e^f -automorphic measure for T equivalent to μ and f is a measurable coboundary.

Proof. Let $W : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ be the dual Perron-Frobenius operator. One can prove by induction that

$$W^n(\nu)(\phi) = \frac{1}{\int_X (\exp \sum_{k=0}^{n-1} f \circ T^k) d\nu} \cdot \int_X (\phi \circ T^n) (\exp \sum_{k=0}^{n-1} f \circ T^k) d\nu$$

and

$$W^{-n}(\nu)(\phi) = \frac{1}{\int_X (\exp(-\sum_{k=1}^n f \circ T^{-k})) d\nu} \cdot \int_X (\phi \circ T^{-n}) (\exp(-\sum_{k=1}^n f \circ T^{-k})) d\nu$$

for every $n \in \mathbb{N}$. From the invariance of μ we get

$$W^n(\mu)(\phi) = \frac{1}{\int_X E_n(f) d\mu} \cdot \int_X \phi E_n(f) d\mu$$

for every $n \in \mathbb{Z}$. Note also that

$$\int_X (\phi \circ T) e^f E_n(f) d\mu = \int_X \phi E_{n+1}(f) d\mu$$

for every continuous $\phi : X \rightarrow \mathbb{R}$.

If now $A \subset X$ is a measurable set, it follows from regularity that

$$W^n(\mu)(A) = \frac{1}{\int_X E_n(f) d\mu} \cdot \int_A E_n(f) d\mu$$

which implies that $W^n(\mu) \in E_\mu$ and

$$\frac{dW^n(\mu)}{d\mu} = \frac{E_n(f)}{\int_X E_n(f) d\mu}$$

for every $n \in \mathbb{Z}$. If C_m is the convex hull of $\{W^n(\mu) : n \geq m\}$, then $W(C_m) \subset C_m$. Indeed, let

$$t_n = \frac{\int_X E_{n+1}(f) d\mu}{\int_X E_n(f) d\mu}$$

for all $n \in \mathbb{Z}$. If $a_1, \dots, a_n \geq 0$ are such that $a_1 + \dots + a_n = 1$ and $j_1, \dots, j_n \in \mathbb{Z}$, then

$$W\left(\sum_{k=1}^n a_k W^{j_k}(\mu)\right) = \sum_{k=1}^n \frac{a_k t_{j_k}}{a_1 t_{j_1} + \dots + a_n t_{j_n}} \cdot W^{j_k+1}(\mu).$$

This shows that $W(C_m) \subset C_m$ and by continuity $W(\overline{C_m}) \subset \overline{C_m}$. Since $\overline{C_m}$ is a compact convex subset of $\mathcal{M}(X)$, it follows from the Schauder-Tychonoff theorem that W has a fixed point in $\overline{C_m}$ or in other words there is an e^f -automorphic measure for T in $\overline{C_m}$. Moreover, our assumption and Lemma 3.1 imply that $\overline{C_m} \subset E_\mu$. This proves the conclusion. \square

The conclusion of Theorem 4.1 remains true under the assumption that there exists an integer $m \leq 0$ such that

$$E_n(f) \leq g \int_X E_n(f) d\mu$$

for every $n \leq m$, by considering W^{-1} .

One of the most important examples of uniquely ergodic homeomorphisms are the Denjoy homeomorphisms of the circle. Here we are interested in Denjoy C^1 diffeomorphisms which were the original motivation for this work.

Let $T : S^1 \rightarrow S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number $\rho(T)$. It is well known [2] that T is uniquely ergodic and there exists a unique minimal set $K \subset S^1$ which is the support of the unique T -invariant Borel probability measure μ , and either $K = S^1$, in which case T is topologically conjugate to the rotation by the angle $2\pi\rho(T)$ or K is a Cantor set and T is only topologically semi-conjugate to the rotation by the angle $2\pi\rho(T)$. In the latter case T is a Denjoy C^1 diffeomorphism and the semi-conjugation is never C^1 . In both cases, K is the non-wandering set of T and

$$\int_{S^1} \log(T^n)' d\mu = 0$$

for every $n \in \mathbb{Z}$.

A T' -automorphic measure ν for T on K will be called automorphic for T and is a Borel probability measure on K such that

$$\int_K \phi d\nu = \int_K (\phi \circ T) T' d\nu$$

for every continuous function $\phi : K \rightarrow \mathbb{R}$. If $\log T'$ is a continuous coboundary on K , then there exists a unique automorphic measure for T absolutely continuous with respect to μ , by section 2, since T is uniquely ergodic. We note that $\log T'$ is never a continuous coboundary on S^1 , by an argument due to M. Herman [2]. Indeed, since K is the non-wandering set of T , there exists a non-trivial closed interval $I \subset S^1 \setminus K$ such that $(T^n(I))_{n \in \mathbb{Z}}$ is a family of mutually disjoint closed intervals. So,

$$\sum_{n=1}^{\infty} \int_I (T^n)' dz = \sum_{n=1}^{\infty} \int_{T^n(I)} dz = \lambda\left(\bigcup_{n=1}^{\infty} T^n(I)\right) \leq 2\pi,$$

where λ denotes the Lebesgue measure on S^1 . It follows that the series $\sum_{n=1}^{\infty} (T^n)'(z)$ converges for λ -almost all $z \in I$ and therefore $\lim_{n \rightarrow +\infty} (T^n)'(z) = 0$, λ -almost everywhere on I . Since $\lambda(I) > 0$, there exists a point $z \in S^1 \setminus K$ such that $\lim_{n \rightarrow +\infty} (T^n)'(z) = 0$. It follows that $\log T'$ is not a continuous coboundary on S^1 . The question whether $\log T'$ is a continuous coboundary on K is related to a problem stated in [1].

Let now $h : S^1 \rightarrow S^1$ be an orientation preserving C^1 diffeomorphism and $S = h \circ T \circ h^{-1}$. Then S is a Denjoy C^1 diffeomorphism with unique minimal set $h(K)$ and unique S -invariant Borel probability measure $h_*\mu$. If ν is an automorphic measure for T , then

$$\nu' = \frac{h'}{\int_K h' d\nu} \cdot h_*\nu$$

is automorphic for S . It follows that if $\nu \ll \mu$, then $\nu' \ll h_*\mu$.

From Theorem 4.1 and the chain rule we obtain a sufficient condition for a Denjoy C^1 diffeomorphism to have an automorphic measure which is equivalent to its invariant Borel probability measure.

Corollary 4.2. *Let $T : S^1 \rightarrow S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K and unique T -invariant Borel probability measure μ . If there exists a positive $g \in L^1(\mu)$ and an integer $m \geq 0$ such that*

$$(T^n)' \leq g \int_K (T^n)' d\mu$$

on K for $n \geq m$, then there exists an automorphic measure for T which is equivalent to μ and $\log T'$ is a measurable coboundary on K . \square

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