University of Crete

Master's Thesis

Integrability of Hamiltonian Systems by Thimm's Method

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September 2025

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Preface

A Hamiltonian system is called (completely) integrable if it admits a maximal number of independent first integrals in involution. By a classical theorem of Liouville, the Hamiltonian differential equations can then be solved by quadratures and according to Arnold's theorem the orbits on an invariant torus, which is a regular level of the collective momentum map, are the orbits of a constant vector field. The integrable Hamiltonian systems are rare, a fact that had been recognised already by H. Poincaré. The interest to find classes of integrable Hamiltonian systems had almost vanished from his time until the 70's, when is was found that certain PDEs, such as for instance the Kortweg-de Vries equation, can be considered as infinite dimensional integrable Hamiltonian systems.

This work is devoted to the detailed presentation of a method devised by A. Thimm for proving integrability of interesting Hamiltonian systems. The first such class are the geodesic flows of Riemannian manifolds. The method requires a sufficiently large Lie group of symmetries of the system and can be described briefly as follows. Suppose that we have a Poisson action (e.g. strongly Hamiltonian action) of a Lie group G on a symplectic manifold M with corresponding momentum map $\mu : M \to \mathfrak{g}^*$. If f is a G-invariant Hamiltonian, then for every smooth function h on $\mathfrak{g}^*$, the function $h \circ \mu$ is a first integral of the Hamiltonian vector field X_f (equivalently Poisson commutes with f). If h_1, h_2 are two such functions, then $h_1 \circ \mu$ and $h_2 \circ \mu$ Poisson commute. In this way one obtains a large class of first integrals in involution, which however may not be independent.

Thimm's method is based on a particular construction of first integrals by projecting the momentum map to non-degenerate subalgebras $\mathfrak{g}'$ of $\mathfrak{g}$ with respect to an Ad_G-invariant, non-degenerate, symmetric bilinear form B on $\mathfrak{g}$ and composing with Ad-invariant functions on $\mathfrak{g}'$ (identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via B). If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two such subalgebras with $[\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_2$, then the corresponding first integrals are in involution. This holds in particular if $\mathfrak{g}_1 \subset \mathfrak{g}_2$ and we are thus led to consider chains

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \mathfrak{g}_k \subset \mathfrak{g}_{k+1} = \mathfrak{g}$$

of subalgebras to obtain a family of first integrals in involution. Under certain circumstances this may lead to $\dim M/2$ independent first integrals and so to integrability of X_f. This is Thimm's method.

In the case of geodesic flows it is reasonable to consider Riemannian manifolds with large isometry groups such as Riemannian homogeneous spaces. The difficult part in order to prove integrability is to prove independence of the first integrals. As an illustrating example, we present in full detail the integrability of the geodesic flows of (real) Grassmann manifolds.

Thimm's method has been conceptualized by V. Guillemin and S. Sternberg. Using variations of the method G.P. Paternain and R.J. Spatzier found further examples of manifolds with integrable geodesic flows.

Acknowledgements

Firstly, I would like to thank my supervisor, Konstantin Athanassopoulos, for his guidance and support. Professor Athanassopoulos, with his immense knowledge, enthusiasm and patience, has a unique way of encouraging and inspiring you. His advices, feedbacks and suggestions were crucial in helping me complete this work. Through our meaningful discussions, he provided me with deep insights not only in mathematics but also in a variety of subjects.

I am also grateful to professor Grigorios Fournodavlos for granting me a scholarship under the "SINGinGR-Proj. 101078061" programm, which allowed me to focus on my gratuate studies.

I would also like to express my sincere gratitude to my parents for their support through the years. Without them, I would not be able to reach my goals and keep trying to accomplish even more. For these reasons I dedicate this thesis to them.

Last but not least, I thank my colleagues and friends for their help, the great discussions and the wonderful time we had in the University of Crete all these years.

To my family

Contents

Chapter 1

Symplectic Geometry

1.1 Symplectic manifolds

In this introductory chapter we recall some basic notions and facts of Symplectic Geometry. Details can be found in [1], [2], [4], [7], [10].

A *symplectic form* on a (real) vector space V of finite dimension is a non-degenerate, skew-symmetric, bilinear form $\omega : V \times V \to \mathbb{R}$. This means that the map $\tilde{\omega} : V \to V^*$ defined by $\tilde{\omega}(v)(w) = \omega(v, w)$ for $v,\ w \in V$, is a linear isomorphism. The pair (V, ω) is then called a *symplectic vector space*. By Cartan's lemma if V is a vector space of dimension n and ω is a skew-symmetric, bilinear form on V with $\omega \neq 0$, then the rank of $\tilde{\omega}$ is even. Moreover if $\dim\tilde{\omega}(V) = 2k$, there exists a basis $l^1, l^2, ..., l^{2k}$ of $\tilde{\omega}(V)$ such that

$$\sum_{j=1}^{k} l^{2j-1} \wedge l^{2j}.$$

For the proof see [1] at page 21. So if (V, ω) is a sympletic vector space of dimension $2n$, there exists a basis $(a_1, ..., a_n, b_1, ...b_n)$ of V^* such that

$$\omega = \sum_{k=1}^{n} a_k \wedge b_k.$$

Example 1.1.1. Let W be a vector space of dimension n. On $W \times W^*$ consider the skew-symmetric, bilinear form ω defined by

$$\omega((w, a), (w', a')) = a'(w) - a(w').$$

If now $\tilde{\omega}(w, a) = 0$, then $0 = \omega((w, a), (w', 0)) = -a(w')$ for every $w' \in W$. Thus $a = 0$. Similarly, $0 = \omega((w, a), (0, a')) = a'(w)$ for every $a' \in W^*$. Hence $w = 0$. This shows that $(W \times W^*, \omega)$ is symplectic vector space.

Let (V, ω) be a symplectic vector space. A complex structure on V is a linear automorphism $J : V \to V$ such that $J^2 = -id$. It is said to be compatible with the symplectic structure if it preserves ω and $\omega(v, J(v)) > 0$ for all non-zero $v \in V$.

Theorem 1.1.2. *On every symplectic vector space (V, ω) there exists a compatible complex structure J and a positive define inner product g given by the formula $g(u, v) = \omega(u, J(v))$ for every u, $v \in V$.*

Proof. (See [1], page 25). □

A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is a closed 2-form on M such that (T_pM, ω_p) is a symplectic vector space for $p \in M$. Necessarily then M is even dimensional and if $\dim M = 2n$, then $\frac{1}{n!}\omega^n$ is a volume $2n$-form on M.

A smooth map $f : (M, \omega) \to (M', \omega')$ between symplectic manifolds is called *symplectic* if $f^*\omega' = \omega$. If f is also a diffeomorphism, it is called *symplectomorphism*.

Example 1.1.3. For every positive integer n, the space $\mathbb{R}^{2n}$ is a symplectic manifold, by considering on each tangent space $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ the canonical symplectic vector space structure. If $dx^1, dx^2, ..., dx^n, dy^1, dy^2, ...dy^n$ are the canonical basic differential 1-forms on $\mathbb{R}^{2n}$, then the canonical symplectic manifold structure is defined by the 2-form

$$\sum_{i=1}^{n} dx^i \wedge dy^i.$$

Example 1.1.4. Another simple example is the 2-sphere with its standard area 2-form ω given by the formula $\omega_x(u, v) = \langle x, u \times v \rangle$ for u, $v \in T_xS^2$ and $x \in S^2$, where $\times$ denotes the exterior product in $\mathbb{R}^3$. With this area 2-form the total area of S^2 is 4π. More generally , let $M \subset \mathbb{R}^3$ be an oriented surface. The Gauss map $N : M \to S^2$ associates to every $x \in M$ the outward unit normal vector $N(x) \perp T_xM$. Then, as in the case of S^2, the formula $\omega_x(u, v) = \langle N(x), u \times v \rangle$ for u, $v \in T_xM$ defines a symplectic 2-form on M.

Example 1.1.5. The basic example of a symplectic manifold is the cotangent bundle T^*M of any smooth n-manifold M with the symplectic 2-form $\omega = -d\theta$, where θ is the Liouville canonical 1-form on T^*M which is defined by $\theta_a(v) = a(\pi_{*a}(v))$ for $v \in T_a(T^*M)$ and $a \in T^*M$, where $\pi_{*a} : T_a(T^*M) \to T_{\pi(a)}M$ is the derivative at a of the cotangent bundle projection $\pi : T^*M \to M$. Then locally θ is given by the formula

$$\theta|_{\text{locally}} = \sum_{i=1}^{n} p_i dq^i \quad \text{and} \quad \omega|_{\text{locally}} = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

Compare with Example 1.1.1 and 1.1.3.

Definition 1.1.6. An *almost symplectic structure* on a smooth manifold M of dimension $2n$ is a non-degenerate, smooth 2-form on M. An *almost complex structure* on M is a smooth bundle endomorphism $J : TM \to TM$ such that $J^2 = -id$.

There are vector bundle obstructions for a compact manifold to be symplectic. This is due to the following.

Proposition 1.1.7. *A smooth manifold M of dimension $2n$ has an almost complex structure if and only if it has an almost symplectic structure.*

Proof. (See [1], page 35). □

Let (M, ω) be a symplectic manifold of dimension $2n$. The tangent bundle of M can be considered as a complex vector bundle of complex dimension n to which correspond Chern classes $c_k \in H^{2k}(M; \mathbb{Z})$, $1 \leq k \leq n$. The Chern classes are related to the Pontryagin classes of the tangent bundle of M through polynomial (quadratic) equations, which can serve as obstructions to the existence of a symplectic structure on M, since not every compact, orientable, smooth $2n$-manifold has cohomology classes satisfying these equations. For instance, using these equations and Hirzebruch's Singnature Theorem, one can show that the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ cannot be a symplectic manifold.

Even though we have defined the symplectic structure in analogy to the Riemanian structure, their local behaviour differs drastically.

Theorem 1.1.8. (Darboux) *Let ω_0 and ω_1 be two symplectic 2-forms on a smooth $2n$-manifold M and $p \in M$. If $\omega_0(p) = \omega_1(p)$, there exists an open neighbourhood U of p in M and a diffeomorphism $F : U \to F(U) \subset M$, where $F(U)$ is an open neighbourhood of p, such that $F(p) = p$ and $F^*\omega_1 = \omega_0$.*

Proof. (See [1], page 41). □

Corollary 1.1.9. *Let (M, ω) be a symplectic $2n$-manifold and $p \in M$. There exists an open neighbourhood of p and a diffeomorphism $F : U \to F(U) \subset \mathbb{R}^{2n}$ such that*

$$\omega|_U = F^*(\sum_{i=1}^{n} dx^i \wedge dy^i).$$

Proof. (See [1], page 42). □

This shows that in symplectic geometry there are no local invariants, in contrast to pseudo-Riemannian geometry, where there are highly non-trivial local invariants. In other words, the study of symplectic manifolds is of global nature.

1.2 Hamiltonian systems

Let (M, ω) be a symplectic $2n$-manifold. A smooth vector field X on M is called *Hamiltonian* if there exists a smooth function $H : M \to \mathbb{R}$ such that $i_X\omega = dH$. In other words,

$$\omega_p(X_p, v_p) = v_p(H)$$

for every $v_p \in T_pM$ and $p \in M$. We usally write $X = X_H$ and obviously $X_H = \tilde{\omega}^{-1}(dH)$. Since $L_X\omega = d(i_X\omega) = 0$, the flow of X_H consists of symplectomorphisms.

If $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with the canonical symplectic 2-form

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i,$$

we have $\tilde{\omega}(\frac{\partial}{\partial q^i}) = dp_i$ and $\tilde{\omega}(\frac{\partial}{\partial p^i}) = -dq^i$. Thus,

$$X_H = \tilde{\omega}^{-1}\left(\sum_{i=1}^{n} \frac{\partial H}{\partial q^i} dq^i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i\right) = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\cdot\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\cdot\frac{\partial}{\partial p_i}\right).$$

So the integral curves of X_H are the solutions of Hamilton's differential equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad 1 \le i \le n.$$

According to Darboux's theorem, this is true locally, with respect to suitable local coordinates on every symplectic $2n$-manifold.

Two elementary properties of Hamiltonian vector fields are the following.

Proposition 1.2.1. *The smooth function $H : M \to \mathbb{R}$ is a first integral of the Hamiltonian vector field X_H.*

Proof. Indeed $X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0$. □

Proposition 1.2.2. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ is symplectic if and only if $f_*(X_{H\circ f}) = X_H$ for every open set $U \subset M_2$ and smooth function $H : U \to \mathbb{R}$.*

Proof. (See [1], page 46). □

If (M, ω) is a symplectic manifold and $F,\ G \in C^\infty(M)$, then the smooth function

$$\{F, G\} = i_{X_G} i_{X_F} \omega \in C^\infty(M)$$

is called the *Poisson bracket* of F and G. From Proposition 1.2.2 we obtain the following.

Corollary 1.2.3. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ is symplectic if and only if*

$$f^*\{F, G\} = \{f^*(F), f^*(G)\}$$

for every open set $U \subset M_2$ and $F,\ G \in C^\infty(U)$.

Proof. (See [1], page 47). □

Corollary 1.2.4. *Let X be a complete Hamiltonian vector field with flow $(\phi_t)_{t\in\mathbb{R}}$ on a symplectic manifold M. Then $\phi_t^*\{F, G\} = \{\phi_t^*(F), \phi_t^*(G)\}$ for every $F,\ G \in C^\infty(M)$. If X is not complete, the same is true on suitable open sets.* □

Corollary 1.2.5. *Let X be a complete Hamiltonian vector field with flow $(\phi_t)_{t\in\mathbb{R}}$ on a symplectic manifold (M,ω). Then*

$$X\{F,G\} = \{X(F),G\} + \{F,X(G)\}$$

for every $F,\ G \in C^\infty(M)$. If X is not complete, the same is true on suitable open sets.

Proof. (See [1], page 48). □

It is obvious that for a symplectic manifold (M,ω) the Poisson barcket

$$\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

is bilinear and skew-symmetric. From Corollary 1.2.5 follows that it satisfies the Jacobi identity. Indeed if $F,\ G,\ H \in C^\infty(M)$ then

$$\{F,G\} = (i_{X_F}\omega)(X_G) = dF(X_G) = X_G(F)$$

and thus $\{\{F,G\},H\} = X_H(\{F,G\})$. Consequently,

$$\{\{F,G\},H\} = \{X_H(F),G\} + \{F,X_H(G)\} = \{\{F,H\},G\} + \{F,\{G,H\}\}.$$

This is the Jacobi identity and so $(C^\infty(M),\{,\})$ is a Lie algebra.

There is a Leibniz formula for the product of two smooth functions with respect to the Poisson bracket, because if $F,\ G,\ H \in C^\infty(M)$, then

$$\{F\cdot G,H\} = X_H(F\cdot G) = F\cdot X_H(G) + G\cdot X_H(F) = F\cdot\{G,H\} + G\cdot\{F,H\}.$$

Proposition 1.2.6. *Let X_H be a Hamiltonian vector field with flow ϕ_t on a symplectic manifold M. Then*

$$\frac{d}{dt}(F\circ\phi_t) = \{F\circ\phi_t,H\} = \{F,H\}\circ\phi_t$$

for every $F \in C^\infty(M)$.

Proof. By the chain rule, for every $p \in M$ we have

$$\begin{aligned}\frac{d}{dt}(F\circ\phi_t)(p) &= (dF)(\phi_t(p))X_H(\phi_t(p)) = \{F,H\}(\phi_t(p))\\ &= \{F\circ\phi_t, H\circ\phi_t\}(p) = \{F\circ\phi_t,H\}(p),\end{aligned}$$

since H is a first integral of X_H. □

Corollary 1.2.7. *A smooth function $F : M \to \mathbb{R}$ on a symplectic manifold M is a first integral of a Hamiltonian vector field X_H on M if and only if $\{F,H\} = 0$.* □

Let $\mathfrak{h}(M,\omega)$ denote the linear space of the hamiltonian vector fields of the symplectic manifold (M,ω).

Proposition 1.2.8. *If* $X,\ Y \in \mathfrak{h}(M,\omega)$*, then* $[X,Y] = -X_{\omega(X,Y)}$*. In particular,* $[X_F, X_G] = -X_{\{F,G\}}$ *for every* $F,\ G \in C^\infty(M)$*.*

Proof. Indeed, $i_{[X,Y]} = [L_X, i_Y]$ and therefore

$$\begin{aligned} i_{[X,Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) &= d(i_X i_Y\omega) + i_X(d(i_Y\omega)) - 0 \\ &= d((\omega(X,Y)) + 0 = -i_{X_{\omega(X,Y)}}\omega. \end{aligned}$$

Since ω is non-degenerate the result follows. □

It follows that $\mathfrak{h}(M,\omega)$ is a Lie subalgebra of the Lie algebra of smooth vector fields of M.

1.3 The geodesic flows of pseudo-Riemannian manifolds as Hamiltonian systems

Newtonian mechanical systems with potential energy on pseudo-Riemannian manifolds are classical examples of Hamiltonian systems. A special case is that of the geodesic field where the potential energy is zero.

Let (M,g) be a n-dimensional pseudo-Riemannian manifold with metric g. There is a natural bundle isomorphism $\mathcal{L} : TM \to T^*M$, such that if $v \in T_xM$ then $\mathcal{L}(v)$ is the linear form on T_xM defined by $\mathcal{L}(v)(w) = g_x(v,w)$. The inner product g_x on T_xM is thus transfered to an inner product g_x^* on T_x^*M. If in local coordinates the matrix of g is $G = (g_{ij})$, then in the dual local coordinates the matrix of g^* is $G^{-1} = (g^{ij})$. If θ is the standard differential form of Liouville and $\omega = -d\theta$ is the standard symplectic 2-form on T^*M, then $\mathcal{L}^*\omega = -d(\mathcal{L}^*\theta)$ is a symplectic 2-form on TM.

Definition 1.3.1. A *Newtonian mechanical system* on the pseudo-Riemannian manifold (M,g) is the Hamiltonian vector field X_E on TM with Hamilitonian function of the form

$$E(v) = \frac{1}{2}g(v,v) + V(\pi(v))$$

where $V : M \to \mathbb{R}$ is a smooth function, called the *potential energy*, and $\pi : TM \to M$ is the tangent bundle projection.

We shall find Hamilton's equations of motion for a Newtonian mechanical system on a pseudo-Riemannian manifold. First we must find local expressions for $\mathcal{L}^*\theta$ and $\mathcal{L}^*\omega$. Let $(U, q^1, ..., q^n)$ be a system of local coordinates on M. Since $\mathcal{L}(x,v) = (x, g_x(v,\cdot))$, its Jacobian is

$$D\mathcal{L}(x,v) = \begin{pmatrix} I_n & 0 \\ \frac{\partial}{\partial x} g_x(v,\cdot) & g_x(\cdot,\cdot) \end{pmatrix}$$

or explicitly

$$D\mathcal{L}(x,v)\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u \\ (\frac{\partial}{\partial x} g_x(v,\cdot))u + g_x(\cdot,w) \end{pmatrix}.$$

It follows that

$$(\mathcal{L}^*\theta)_{(x,v)}(u,w) = \theta_{\mathcal{L}(x,v)}(u, (\frac{\partial}{\partial x} g_x(v,\cdot))u + g_x(\cdot, w)) = g_x(v,u).$$

This means that if $(q^1, ..., q^n, v^1, ..., v^n)$ are the corresponding local coordinates of $\pi^{-1}(U)$, then on $\pi^{-1}(U)$ we have

$$\mathcal{L}^*\theta = \sum_{i,j=1}^{n} g_{ij} v^j dq^i$$

and therefore

$$\mathcal{L}^*\omega = \sum_{i,j=1}^{n} g_{ij} dq^i \wedge dv^j + \sum_{i,j,k=1}^{n} \frac{\partial g_{ij}}{\partial q^k} \cdot v^j dq^i \wedge dq^k.$$

Note that the local coordinates on $\pi^{-1}(U)$ are not Darboux. Next we have

$$dE = \frac{1}{2} \sum_{i,j,k=1}^{n} \frac{\partial g_{ij}}{\partial q^k} v^i v^j dq^k + \sum_{i,k=1}^{n} g_{ik} v^i dv^k + \sum_{k=1}^{n} \frac{\partial V}{\partial q^k} dq^k,$$

and

$$i_{X_E}\mathcal{L}^*\omega(\frac{\partial}{\partial q^k}) = -\sum_{j=1}^{n} g_{kj} dv^j(X_E) + \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial q^k} v^j dq^i(X_E) - \sum_{k=1}^{n} \frac{\partial g_{kj}}{\partial q^l} v^j dq^l(X_E),$$

$$i_{X_E}\mathcal{L}^*\omega(\frac{\partial}{\partial v^k}) = \sum_{i=1}^{n} g_{ik} dq^i(X_E), \quad 1 \le k \le n.$$

If I is an open interval, then $(q^1(t), ..., q^n(t), v^1(t), ..., v^n(t))$, $t \in I$, is an integral curve of X_E if and only if it is a solution of the system of differential equations

$$\sum_{i=1}^{n} g_{ik}\dot{q}^i = \sum_{j=1}^{n} g_{ik} v^i$$

$$-\sum_{j=1}^{n} g_{kj}\dot{v}^j + \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial q^k} v^j \dot{q}^i - \sum_{i,j=1}^{n} \frac{\partial g_{kj}}{\partial q^i} v^j \dot{q}^i = \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial g_{ij}}{\partial q^k} v^i v^j + \frac{\partial V}{\partial q^k}, \quad 1 \le k \le n.$$

It is obvious that the first n equations are equivalent to $\dot{q}^i = v^i$, $1 \le i \le n$. The rest of them can be written

$$\sum_{j=1}^{n} g_{kj}\dot{v}^j = -\frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial g_{ij}}{\partial q^k} v^i v^j + \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial q^k} v^j v^i - \sum_{i,j=1}^{n} \frac{\partial g_{kj}}{\partial q^i} v^j v^i - \frac{\partial V}{\partial q^k}, \quad 1 \le k \le n.$$

or equivalently, since G is symmetric,

$$\dot{v}^k = \sum_{l=1}^{n} g^{kl} \left[\frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial g_{ij}}{\partial q^l} v^i v^j - \sum_{i,j=1}^{n} \frac{\partial g_{lj}}{\partial q^i} v^j v^i - \frac{\partial V}{\partial q^l} \right] = -\sum_{i,j=1}^{n} \Gamma^k_{ij} v^i v^j - \sum_{j=1}^{n} g^{kl} \frac{\partial V}{\partial q^l},$$

where Γ_{ij}^k are the Christofell symbols, because

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{jl}}{\partial q^i} + \frac{\partial g_{li}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^l}\right)$$

and thus

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j = \sum_{i,j=1}^{n}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{jl}}{\partial q^i} - \frac{1}{2}\frac{\partial g_{ij}}{\partial q^l}\right) v^i v^j.$$

So Hamilton's differential equations can be written locally

$$\dot{q}^k = v^k,$$

$$\dot{v}^k = -\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j - \sum_{i=1}^{n} g^{ki}\frac{\partial V}{\partial q^i}, \quad 1 \le k \le n,$$

which are equivalent to the system of second order differential equations

$$\ddot{q}^k + \sum_{i,j=1}^{n} \Gamma_{ij}^k \dot{q}^i \dot{q}^j = -\sum_{i=1}^{n} g^{ki}\frac{\partial V}{\partial q^i}, \quad 1 \le k \le n.$$

These calculations prove the following.

Proposition 1.3.2. *A smooth curve $\gamma : I \to M$ in a pseudo-Riemannian manifold M is the projection of an integral curve in TM of the Newtonian mechanical system with potential energy $V : M \to \mathbb{R}$ if and only if*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = -\mathrm{grad} V.$$

The mechanical system with potential energy $V = 0$ of a pseudo-Riemannian manifold M is the *geodesic vector field* of M. The metric on M is by definition *geodesically complete* if the geodesic vector field is complete on TM and so defines a flow, called *geodesic flow* of M. The projected curves on M of the integral curves of the geodesic vector field are the *geodesics*.

1.4 Coadjoint orbits

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and identity element e. The action of G on itself by conjugation, i.e. $\psi_g(h) = ghg^{-1}$, $g \in G$, fixes e and induces the *adjoint linear representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ defined by

$$\mathrm{Ad}_g(X) = (\psi_g)_{*e}(X) = \frac{d}{dt}\bigg|_{t=0} g(\exp tX)g^{-1}.$$

Example 1.4.1. The Lie group $SO(3,\mathbb{R})$ is compact, connected and its Lie algebra $\mathfrak{so}(3,\mathbb{R})$ is isomorphic to the Lie algebra of skew-symmetric linear maps of $\mathbb{R}^3$ with respect to the Lie bracket $[A,B]=AB-BA$, $A,\ B\in\mathbb{R}^{3\times 3}$.

On the other hand, the map $\widehat{\ }:\mathbb{R}^3\to\mathfrak{so}(3,\mathbb{R})$ defined by

$$\hat{v}=\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

where $v=(v_1,v_2,v_3)$, is a linear isomomprhism and $\hat{v}\cdot w=v\times w$, for every $v,\ w\in\mathbb{R}^3$. This actually characterizes $\widehat{\ }$. So we have

$$(\hat{u}\hat{v}-\hat{v}\hat{u})w=\hat{u}(v\times w)-\hat{v}(u\times w)=u\times(v\times w)-v\times(u\times w)=(u\times v)\times w=(\widehat{u\times v})w.$$

Thus, $\widehat{\ }$ is a Lie algebra isomorphism of the Lie algebra $(\mathbb{R}^3,\times)$ onto $\mathfrak{so}(3,\mathbb{R})$. Using this isomorphism one can describe the exponential map of $SO(3,\mathbb{R})$.

Let $w\in\mathbb{R}^3$, $w\neq 0$ and $\{e_1,e_2,e_3\}$ be an orthonormal basis of $\mathbb{R}^3$ such that $e_1=w/\|w\|$. The matrix of $\hat{w}$ with respect to this basis is

$$\hat{w}=\|w\|\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $t\in\mathbb{R}$ let $\gamma(t)$ be the rotation around the axis determined by w through the angle $t\|w\|$, that is

$$\gamma(t)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t\|w\| & -\sin t\|w\| \\ 0 & \sin t\|w\| & \cos t\|w\| \end{pmatrix}.$$

Then,

$$\dot{\gamma}(t)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\|w\|\sin t\|w\| & -\|w\|\cot t\|w\| \\ 0 & \|w\|\cos t\|w\| & -\|w\|\sin t\|w\| \end{pmatrix}=\gamma(t)\hat{w}=(L_{\gamma(t)})_{*I_3}(\hat{w})=X_{\hat{w}}(\gamma(t)),$$

where $L_{\gamma(t)}$ denotes the left translation on $SO(3,\mathbb{R})$ by $\gamma(t)$ and $X_{\hat{w}}$ the left invariant vector field on $SO(3,\mathbb{R})$ corresponding to $\hat{w}$. In other words, γ is an integral curve of $X_{\hat{w}}$ with $\gamma(0)=I_3$. It follows that $\exp(t\hat{w})=\gamma(t)$ for every $t\in\mathbb{R}$.

For every $A\in SO(3,\mathbb{R})$ and $v\in\mathbb{R}^3$ we have now

$$\mathrm{Ad}_A(\hat{v})=\frac{d}{dt}\bigg|_{t=0}A(\exp(t\hat{v}))A^{-1}=\frac{d}{dt}\bigg|_{t=0}A\gamma(t)A^{-1}=A\gamma(0)\hat{v}A^{-1}=A\hat{v}A^{-1}.$$

Thus,

$$\mathrm{Ad}_A(\hat{v})w=A\hat{v}(A^{-1}w)=A(v\times A^{-1}w)=Av\times w$$

for every $w\in\mathbb{R}^3$, since $\det A=1$. Hence $\mathrm{Ad}_A(\hat{v})=\widehat{Av}$ and identifying $\mathbb{R}^3$ with $\mathfrak{so}(3,\mathbb{R})$ via $\widehat{\ }$ we conclude that $\mathrm{Ad}_A=A$.

Let now $\mathrm{ad}=(\mathrm{Ad})_{*e}:\mathfrak{g}\to T_e\mathrm{Aut}(\mathfrak{g})\cong\mathrm{End}(\mathfrak{g})$, that is

$$\mathrm{ad}_X=(\mathrm{Ad})_{*e}(X)=\frac{d}{dt}\bigg|_{t=0}\mathrm{Ad}_{\exp(tX)}$$

for every $X \in \mathfrak{g}$. If we denote by X_L the left invariant vector field corresponding to X and $(\phi_t)_{t\in\mathbb{R}}$ its flow, then for every $Y \in \mathfrak{g} \cong T_0\mathfrak{g}$ we have

$$\mathrm{ad}_X(Y) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y) = \frac{d}{dt}\bigg|_{t=0} (\psi_{\exp(tX)})_{*e}(Y)$$

$$= \frac{d}{dt}\bigg|_{t=0} (R_{\exp(-tX)} \circ L_{\exp(tX)})_{*e}(Y) = \frac{d}{dt}\bigg|_{t=0} (R_{\exp(-tX)})_{*\exp(tX)} \circ (L_{\exp(tX)})_{*e}(Y)$$

$$= \frac{d}{dt}\bigg|_{t=0} (R_{\exp(-tX)})_{*\exp(tX)}(Y_L(\exp(tX)) = \frac{d}{dt}\bigg|_{t=0} (\phi_{-t})_{*\phi_t(e)}(Y_L(\phi_t(e)) = [X,Y],$$

since $\phi_t(g) = g\exp(tX) = R_{\exp(tX)} = R_{\exp(tX)}(g)$ for every $g \in G$, where R denotes right translation.

As usual, the adjoint representation induces a representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ on the dual of the Lie algebra defined by $\mathrm{Ad}^*_g(a) = a \circ \mathrm{Ad}_{g^{-1}}$, $a \in \mathfrak{g}^*$, which is called the *coadjoint representation* of G.

Example 1.4.2. Continuing from Example 1.4.1 we shall describe the coadjoint representation of $SO(3,\mathbb{R})$. The transpose of the linear isomorphism $\widehat{\ }$ induces an isomorphism from $\mathfrak{so}(3,\mathbb{R})^*$ to $(\mathbb{R}^3)^*$ and the latter can be identified naturally with $\mathbb{R}^3$ via the euclidean inner product. The composition of these two isomorphisms gives a way to identify $\mathfrak{so}(3,\mathbb{R})^*$ with $\mathbb{R}^3$ and then, for v, $w \in \mathbb{R}^3$ we have $\hat{v}^*(\hat{w}) = \langle v, w\rangle$, where $\hat{v}^*$ is the dual of $\hat{v}$ and $\langle\ ,\ \rangle$ is the euclidean inner product. Now

$$\mathrm{Ad}^*_A(\hat{v}^*)(\hat{w}) = \hat{v}^*(\mathrm{Ad}_{A^{-1}}(\hat{w})) = \left\langle v, A^{-1}w\right\rangle = \langle Av, w\rangle\,,$$

for every $A \in SO(3,\mathbb{R})$, since the transpose of A is A^{-1}. This shows that $\mathrm{Ad}^*_A = A$ via the above identification. Note that the orbit of the point $\hat{v}^* \in \mathfrak{so}(3,\mathbb{R})^* \cong \mathbb{R}^3$ is the set $\{Av \mid A \in SO(3,\mathbb{R})\}$, which is the sphere of radius $\|v\|$ centered at 0.

The orbit $\mathcal{O}_\mu$ of $\mu \in \mathfrak{g}^*$ under the coadjoint representation is an immersed submanifold of $\mathfrak{g}^*$, since the action is smooth. If G_μ is the isotropy group of μ, then the map $\mathrm{Ad}^*(\mu) : G/G_\mu \to \mathcal{O}_\mu$ taking the coset gG_μ to $\mu \circ \mathrm{Ad}_{g^{-1}}$ is a well defined, injective, smooth immersion of the homogeneous space G/G_μ onto $\mathcal{O}_\mu \subset \mathfrak{g}^*$. If the Lie group G is compact, then $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ and the above map an embedding. If however G is not compact, $\mathcal{O}_\mu$ may not be embedded.

Lemma 1.4.3. *If $\mu \in \mathfrak{g}^*$, then the tangent space of $\mathcal{O}_\mu$ is*

$$T_\mu\mathcal{O}_\mu = \{\mu \circ \mathrm{ad}_X \mid X \in \mathfrak{g}\}.$$

Proof. Let $\gamma : \mathbb{R} \to G$ be a smooth curve with $\dot{\gamma}(0) = X$. For instance, let $\gamma(t) = \exp(tX)$, in which case $\gamma(t)^{-1} = \exp(-tX)$. Then $\mu(t) = \mu \circ \mathrm{Ad}_{\gamma(t)^{-1}}$ is a smooth curve with values in $\mathcal{O}_\mu \subset \mathfrak{g}^*$ and $\mu(0) = \mu$. If $Y \in \mathfrak{g}$, then $\mu(t)(Y) = \mu(\mathrm{Ad}_{\gamma(t)^{-1}}(Y))$ for every $t \in \mathbb{R}$ and differentiating at 0 we get

$$\mu'(0)(Y) = \mu(\mathrm{ad}_{(-X)}(Y) = -\mu(\mathrm{ad}_X(Y)),$$

taking into account the natural identification $T_\mu\mathfrak{g}^* \cong \mathfrak{g}^*$. $\square$

Example 1.4.4. In the case of the Lie group $SO(3,\mathbb{R})$, for every $v,\ w \in \mathbb{R}^3 \cong \mathfrak{so}(3,\mathbb{R})$ and $\mu \in \mathbb{R}^3 \cong \mathfrak{so}(3,\mathbb{R})^*$ we have

$$\mu(\mathrm{ad}_{\hat{v}}(\hat{w})) = \langle \mu, v \times w \rangle = \langle \mu \times v, w \rangle .$$

It follows that $T_\mu \mathcal{O}_\mu = \{\mu \times v \mid v \in \mathbb{R}^3\}$, which is indeed the orthogonal plane to μ, i.e. the tangent plane of the sphere of center 0 and radius $\|\mu\|$ at μ.

The proof of Lemma 1.4.3 shows that for every $X \in \mathfrak{g}$, the fundamental vector field $X_{\mathfrak{g}^*}$ of the coadjoint action induced by X is given by the formula

$$X_{\mathfrak{g}^*}(\mu) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{\exp(tX)}(\mu) = -\mu \circ \mathrm{ad}_X .$$

Obviously, $T_\mu \mathcal{O}_\mu = \{X_{\mathfrak{g}^*}(\mu) \mid X \in \mathfrak{g}\}$. Note that if $X,\ X' \in \mathfrak{g}$ are such that $X_{\mathfrak{g}^*}(\mu) = X'_{\mathfrak{g}^*}(\mu)$, then

$$-\mu([X,Y]) = X_{\mathfrak{g}^*}(\mu)(Y) = X'_{\mathfrak{g}^*}(\mu)(Y) = -\mu([X',Y])$$

for every $Y \in \mathfrak{g}$. So there is a well defined 2-form ω^- on the coadjoint orbit $\mathcal{O} = \mathcal{O}_\mu$ such that

$$\omega^-_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)) = -\mu([X,Y])$$

for every $\mu \in \mathcal{O}$ and $X,\ Y \in \mathfrak{g}$, which is called the *Kirillov* 2-*form* on $\mathcal{O}$.

The Kirillov 2-form ω^- is non-degenerate, because if $\omega^-(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)) = 0$ for every $Y_{\mathfrak{g}^*}(\mu) \in T_\mu\mathcal{O}$, then $X_{\mathfrak{g}^*}(\mu)(Y) = -\mu([X,Y]) = 0$ for every $Y \in \mathfrak{g}$. This means $X_{\mathfrak{g}^*}(\mu) = 0$. In order to prove that ω^- is symplectic, it remains to show that it is closed.

We first note that $\mathrm{Ad}_g[X,Y] = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$ for every $X,\ Y \in \mathfrak{g}$ and $g \in G$. It is also true that $(\mathrm{Ad}_g(X))_{\mathfrak{g}^*} = \mathrm{Ad}^*_g \circ X_{\mathfrak{g}^*} \circ \mathrm{Ad}^*_{g^{-1}}$ $X \in \mathfrak{g}$ and $g \in G$. Indeed, if we let $\gamma : \mathbb{R} \to G$ be a smooth curve with $\dot{\gamma}(0) = X$, for instance $\gamma(t) = \exp(tX)$, then

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\bigg|_{t=0} g\gamma(t)g^{-1}$$

and therefore,

$$(\mathrm{Ad}_g(X))_{\mathfrak{g}^*}(\mu) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{g\gamma(t)g^{-1}}(\mu) = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_{\gamma(t)} \circ \mathrm{Ad}^*_{g^{-1}})(\mu)$$
$$= (\mathrm{Ad}^*_g \circ X_{\mathfrak{g}^*} \circ \mathrm{Ad}^*_{g^{-1}})(\mu).$$

Based on that we can see that the Kirillov 2-form is Ad^*-invariant, since if we let $\mu \in \mathfrak{g}^*$ and $\nu = \mathrm{Ad}^*_g(\mu),\ g \in G$, then $(\mathrm{Ad}_g(X))_{\mathfrak{g}^*}(\nu) = \mathrm{Ad}^*_g(X_{\mathfrak{g}^*}(\mu))$. Thus, for every $X,\ Y \in \mathfrak{g}$ we have

$$((\mathrm{Ad}^*_g)^*\omega^-)_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)) = \omega^-_\nu((\mathrm{Ad}_g(X))_{\mathfrak{g}^*}(\nu), (\mathrm{Ad}_g(Y))_{\mathfrak{g}^*}(\nu))$$
$$= -\nu([\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]) = -\nu(\mathrm{Ad}_g[X,Y]) = -\mu([X,Y]) = \omega^-_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)).$$

For every $\nu \in \mathfrak{g}^*$ we have a well defined 1-form ν_L on G such that

$$(\nu_L)_g = \nu \circ (L_{g^{-1}})_{*g} \in T^*_g G.$$

Moreover, ν_L is left invariant, because for every $h \in G$ we have

$$(L_h^*)(\nu_L)_g = (\nu_L)_{L_{h(g)}} \circ (L_h)_{*g} = \nu \circ ((L_{g^{-1}h^{-1}})_{*hg} \circ (L_h)_{*g})$$
$$= \nu \circ (L_{g^{-1}h^{-1}} \circ L_h)_{*g} = \nu_L(g).$$

Obviously, $i_{X_L}\nu_L$ is constant and equal to $\nu(X)$ for every $X \in \mathfrak{g}$.

Let $\nu \in \mathcal{O}$ and $\phi_\nu : G \to \mathcal{O}$ be the submersion $\phi_\nu(g) = \mathrm{Ad}_g^*(\nu)$. The 2-form $\sigma = \phi_\nu^*\omega^-$ on G is left invariant, beacause

$$L_g^*\sigma = (\phi_\nu \circ L_g)^*\omega^- = (\mathrm{Ad}_g^* \circ \phi_\nu)^*\omega^- = \phi_\nu^*((\mathrm{Ad}_g^*)^*\omega^-) = \phi_\nu^*\omega^- = \sigma$$

for every $g \in G$, since ω^- is Ad^*-invariant and $\phi_\nu \circ L_g = \mathrm{Ad}_g^* \circ \phi_\nu$.

Lemma 1.4.5. *For every X, $Y \in \mathfrak{g}$ we have $\sigma(X_L, Y_L) = -\nu_L([X_L, Y_L])$.*

Proof. First we observe that

$$(\phi_\nu^*\omega^-)_e(X, Y) = \omega_\nu^-((\phi_\nu)_{*e}(X), (\phi_\nu)_{*e}(Y)) = \omega^-(X_{\mathfrak{g}^*}(\nu), Y_{\mathfrak{g}^*}(\nu)) = -\nu([X, Y]).$$

Therefore,

$$\sigma(X_L, Y_L)(e) = (\phi_\nu^*\omega^-)_e(X, Y) = -\nu([X, Y]) = -\nu_L([X_L, Y_L])(e).$$

Since the smooth functions $\sigma(X_L, Y_L)$, $-\nu([X_L, Y_L]) : G \to \mathbb{R}$ are left invariant and take the same value at e, they must be identical. □

Note that

$$(d\nu_L)(X_L, Y_L) = X_L(\nu_L(Y_L)) - Y_L(\nu_L(X_L)) - \nu([X_L, Y_L]) = -\nu_L([X_L, Y_L]),$$

because the functions $\nu_L(Y_L) = i_{Y_L}\nu_L$ and $\nu_L(X_L) = i_{X_L}\nu_L$ are constant.

Lemma 1.4.6. *The 2-form σ is exact and $\sigma = d\nu_L$.*

Proof. Since σ is left invariant, for any two smooth vector fields X, Y on G we have

$$\sigma(X, Y)(g) = (L_{g^{-1}}^*\sigma)_g(X(g), Y(g)) = \sigma_e((L_{g^{-1}})_{*g}(X(g)), (L_{g^{-1}})_{*g}(Y(g)))$$
$$= \sigma({X'}_L, {Y'}_L)(e) \quad \text{(setting } X' = (L_{g^{-1}})_{*g}(X(g)) \text{ and similarly for } Y')$$
$$= (d\nu_L)(X'_L, Y'_L)(e) = (d\nu_L)_g((L_g)_{*e}(X'), (L_g)_{*e}(Y'))$$
$$= (d\nu_L)_g(X(g), Y(g)) = (d\nu_L)(X, Y)(g). \quad \square$$

Proposition 1.4.7. *The Kirillov 2-form ω^- on $\mathcal{O}$ is closed and therefore symplectic.*

Proof. By Lemma 1.4.6. $d(\phi_\nu^*\omega^-) = d\sigma = d(d\nu_L)) = 0$. Hence $\phi_\nu^*(d\omega^-) = 0$. But ϕ_ν^* is injective, since ϕ_ν is a submersion. It follows that $d\omega^- = 0$. □

In particular every orbit of the coadjoint action of a Lie group G on its dual Lie algebra $\mathfrak{g}^*$ has even dimension.

We shall end this section with an illustrating example.

Example 1.4.8. As we saw in Example 1.4.2. if $\mu \in \mathfrak{so}(3,\mathbb{R})^* \cong \mathbb{R}^3$, then $\mathcal{O}_\mu$ is the sphere centered at 0 with radius $\|\mu\|$. Let $v,\ w \in \mathfrak{so}(3,\mathbb{R}) \cong \mathbb{R}^3$. Then $v_{\mathbb{R}^3} = \mu \times v \in T_\mu\mathcal{O}_\mu$ and $w_{\mathbb{R}^3} = \mu \times w \in T_\mu\mathcal{O}_\mu$. Hence the Kirillov 2-form on $\mathcal{O}_\mu$ is given by the formula

$$\omega^-_\mu(v_{\mathbb{R}^3}, w_{\mathbb{R}^3}) = -\langle \mu, v \times w\rangle .$$

Since $\mathcal{O}_\mu$ is a sphere, its area element is given by the formula

$$dA(v,w) = \langle N, v \times w\rangle ,$$

where N is the outer unit normal vector. It follows that

$$\begin{aligned} dA(\mu\times v, \mu\times w) &= \left\langle \frac{1}{\|\mu\|}\mu, (\mu \times v) \times (\mu \times w)\right\rangle = \left\langle \frac{1}{\|\mu\|}\mu, \langle \mu, \mu \times w\rangle v - \langle \mu \times w\rangle \mu \right\rangle \\ &= -\|\mu\| \langle v, \mu \times w\rangle = -\|\mu\| \langle \mu, v \times w\rangle . \end{aligned}$$

This shows that

$$\omega^- = -\frac{1}{\|\mu\|} dA.$$

1.5 Poisson manifolds

In this section we shall describe a generalization of the symplectic structure in order to include the dual Lie algebras. A *Poisson algebra* is a triple $(\mathcal{A}, \{,\}, \cdot)$ where the pair $(\mathcal{A}, \{,\})$ is a Lie algebra, while at the same time $\mathcal{A}$ is a commutative ring with a unit element and multiplication $\cdot$, such that we have a Leibniz formula

$$\{f, g \cdot h\} = h \cdot \{f, g\} + g \cdot \{f, h\}$$

for every $f,\ g,\ h \in \mathcal{A}$. If (M,ω) is a symplectic manifold, then $(C^\infty(M), \{,\}, \cdot)$ is a Poisson algebra, where $\{,\}$ is the Poisson bracket with respect to ω and $\cdot$ is the usual multiplication of functions. A map $\phi : \mathcal{A} \to \mathcal{B}$ of Poisson algebras is called a homomorphism if it is a Lie algebra homomorphism and a homomorphism of commutative rings with unit element.

The Leibniz formula says that for every $f \in \mathcal{A}$ the linear map $\mathrm{ad}_f : \mathcal{A} \to \mathcal{A}$ with $\mathrm{ad}_f(g) = \{g, f\}$ is a derivation. It is called the *Hamiltonian derivation* defined by f. The element $f \in \mathcal{A}$ is called a *Casimir* if $\{f, g\} = 0$ for every $g \in \mathcal{A}$. For example, the unit $1\in \mathcal{A}$ is a Casimir element, since

$$\{f, 1\} = \{f, 1 \cdot 1\} = 1 \cdot \{f, 1\} + 1 \cdot \{f, 1\} = 2\{f, 1\} = 0$$

for every $f \in \mathcal{A}$. A Poisson algebra $\mathcal{A}$ is called non-degenerate if every Casimir element of $\mathcal{A}$ is of the form $t \cdot 1,\ t \in \mathbb{R}$.

A *Poisson manifold* is a smooth manifold M together with a Poisson structure on the ring of smooth functions $C^\infty(M)$. So the Poisson structure on M is completely determined by the Lie-Poisson bracket $\{,\}$ on $C^\infty(M)$. If $(U, x^1, ..., x^n)$ is a chart on M, since ad_f is a derivation of $C^\infty(M)$, it is a smooth vector field on M. So,

$$\mathrm{ad}_f|_U = \sum_{k=1}^{n} \{x^k, f\} \frac{\partial}{\partial x^k}.$$

For every $f,\ g \in C^\infty(M)$ we have

$$\{g, f\}|_U = \sum_{k=1}^{n} \{x^k, f\} \frac{\partial g}{\partial x^k} = -\sum_{k=1}^{n} \{f, x^k\} \frac{\partial g}{\partial x^k} = \sum_{j,k=1}^{n} \{x^k, x^j\} \frac{\partial f}{\partial x^j} \cdot \frac{\partial g}{\partial x^k}.$$

It follows that the Poisson structure on M is determined by a contravariant, skew-symmetric 2-tensor W, which is called the structural tensor of the Poisson structure. For every $p \in M$, the skew-symmetric, bilinear form $W_p : T_p^*M \times T^*M_p \to \mathbb{R}$ is determined by the structural matrix $(\{x^j, x^k\})_{1 \le j,k \le n}$. Its rank is called the rank of the Poisson structure at p.

Proposition 1.5.1. *The Poisson structure of a Poisson manifold M is defined by a symplectic structure on M if and only if the structural matrix is invertible at every point of M.*

Proof. (See [1], pages 78-79). □

Example 1.5.2. Let $(\mathfrak{g}, [,])$ be a (real) Lie algebra of finite dimension n and $\mathfrak{g}^*$ be its dual. Since $\mathfrak{g}$ has finite dimension, the double dual $\mathfrak{g}^{**}$ is naturally isomorphic to $\mathfrak{g}$ and so their elements can be identified. For $f,\ g \in C^\infty(\mathfrak{g}^*)$ let $\{f, g\} \in C^\infty(\mathfrak{g}^*)$ be defined by

$$\{f, g\}(\mu) = \mu[df(\mu), dg(\mu)]$$

for $\mu \in \mathfrak{g}^*$. It is obvious that the $\{,\}$ is bilinear and skew-symmetric. Moreover, the Leibniz formula holds, since it holds for d. In order to have a Poisson manifold, it remains to verify the Jacobi identity. If $\{x_1, ..., x_n\}$ is a basis of $\mathfrak{g}$, then $x_1, ..., x_n$ can be considered as (global) coordinate functions on $\mathfrak{g}^*$. If $f,\ g \in C^\infty(\mathfrak{g}^*)$, then

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j}.$$

Since $\{x_i, x_j\}(\mu) = \mu[dx_i(\mu), dx_j(\mu)] = \mu[x_i, x_j]$ for every $\mu \in \mathfrak{g}^*$, it follows from the Jacobi identity on $\mathfrak{g}$, that is also holds for $\{,\}$ on the set $\{x_1, ..., x_n\}$. In general, if $f,\ g,\ h \in C^\infty(\mathfrak{g}^*)$ we have that

$$\sum_{k=1}^{n} \{x_k, x_i\}\{\frac{\partial f}{\partial x_k}, x_j\} = \sum_{k,l=1}^{n} \{x_k, x_i\}\{x_l, x_j\} \frac{\partial^2 f}{\partial x_l \partial x_k}$$

$$= \sum_{k,l=1}^{n} \{x_l, x_j\}\{x_k, x_i\} \frac{\partial^2 f}{\partial x_k \partial x_l} = \sum_{k=1}^{n} \{x_k, x_j\}\{\frac{\partial f}{\partial x_k}, x_i\}$$

and so

$$\{\{f,g\},h\} = \sum_{i,j,k=1}^{n} \{\{x_i,x_j\},x_k\}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\frac{\partial h}{\partial x_k}$$

$$+\sum_{i,j,k=1}^{n}\{x_i,x_j\}\{\frac{\partial f}{\partial x_i},x_k\}\frac{\partial h}{\partial x_k}\cdot\frac{\partial g}{\partial x_j}+\sum_{i,j,k=1}^{n}\{x_i,x_j\}\{\frac{\partial g}{\partial x_j},x_k\}\frac{\partial h}{\partial x_k}\cdot\frac{\partial f}{\partial x_i}.$$

Cyclically,

$$\{\{g,h\},f\} = \sum_{i,j,k=1}^{n} \{\{x_j,x_k\},x_i\}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\frac{\partial h}{\partial x_k}$$

$$+\sum_{i,j,k=1}^{n}\{x_j,x_k\}\{\frac{\partial g}{\partial x_j},x_i\}\frac{\partial h}{\partial x_k}\cdot\frac{\partial f}{\partial x_i}+\sum_{i,j,k=1}^{n}\{x_j,x_k\}\{\frac{\partial h}{\partial x_k},x_i\}\frac{\partial g}{\partial x_j}\cdot\frac{\partial f}{\partial x_i}$$

and

$$\{\{h,f\},g\} = \sum_{i,j,k=1}^{n} \{\{x_k,x_i\},x_j\}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\frac{\partial h}{\partial x_k}$$

$$+\sum_{i,j,k=1}^{n}\{x_k,x_i\}\{\frac{\partial h}{\partial x_k},x_j\}\frac{\partial g}{\partial x_j}\cdot\frac{\partial f}{\partial x_i}+\sum_{i,j,k=1}^{n}\{x_k,x_i\}\{\frac{\partial f}{\partial x_i},x_j\}\frac{\partial h}{\partial x_k}\cdot\frac{\partial g}{\partial x_j}.$$

Summing up we get

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=$$

$$\sum_{i,j,k=1}^{n}(\{\{x_i,x_j\},x_k\}+\{\{x_j,x_k\},x_i\}+\{\{x_k,x_i\},x_j\})\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\frac{\partial h}{\partial x_k}=0.$$

In this way $\mathfrak{g}^*$ becomes a Poisson manifold.

If M_1 and M_2 are two Poisson manifolds, a smooth map $h : M_1 \to M_2$ is called *Poisson* if $h^* : C^\infty(M_2) \to C^\infty(M_1)$ is a homomorphism of Poisson algebras.

Let M be a Poisson manifold. For every $f \in C^\infty(M)$, the smooth vector field X_f corresponding to the Hamiltonian derivation $\mathrm{ad}_f = \{\cdot, f\}$ is called the *Hamiltonian vector field* of f. This definition agrees with the definition of section 1.2 in case M is symplectic.

Proposition 1.5.3. *Let M be a Poisson manifold and X_f be a Hamiltonian vector field on M with Hamiltonian function $f \in C^\infty(M)$. Let $\phi : D \to M$ be the flow of X_f, where $D \subset \mathbb{R} \times M$ is a open neighbourhood of $\{0\} \times M$.*

(i) *If $g \in C^\infty(M)$, then*

$$\frac{d}{dt}(g \circ \phi_t) = \{g, f\} \circ \phi_t = \{g \circ \phi_t, f\}.$$

(ii) $f \circ \phi_t = f$

(iii) *The flow of the Hamiltonian vector field X_f consists of Poisson maps.*

Proof. (See [1], page 80). □

If $h : M_1 \to M_2$ is a Poisson map of Poisson manifolds and $f \in C^\infty(M_2)$, then $h_{*p}(X_{h^*(f)}(p)) = X_f(h(p))$ for every $p \in M_1$. Therefore, h transforms integral curves of $X_{h^*(f)}$ in M_1 to integral curves of X_f in M_2.

If M is a Poisson manifold and $N \subset M$ is an immersed submanifold, then N is called a *Poisson submanifold* if the inclusion $i : N \hookrightarrow M$ is a Poisson map. On every Poisson manifold M one can define an equivalence relation $\sim$ by setting $p \sim q$ if and only if there is a piecewise smooth curve from p to q whose smooth parts are pieces of integral curves of Hamiltonian vector fields of M. The equivalence classes are called *symplectic leaves* of the Poisson structure of M. It can be proved that the symplectic leaves are immersed submanifolds and carry a unique symplectic structure so they become Poisson submanifolds of M (see [1], page 81).

Theorem 1.5.4. (Symplectic Stratification) *In a Poisson manifold M every symplectic leaf $S \subset M$ is an immersed submanifold and $T_pS = \mathrm{Im}\tilde{W}_p$ for every $p \in S$. Moreover, S has a unique symplectic structure such that S is a Poisson submanifold of M.*

Proof. (See [1], page 82). □

Example 1.5.5. Let $(\mathfrak{g}, [,])$ be the Lie algebra of a Lie group G and $\mathfrak{g}^*$ be its dual. If $f \in C^\infty(\mathfrak{g}^*)$, the Hamiltonian vector field X_f with respect to the Poisson structure on $\mathfrak{g}^*$ defined in Example 1.5.2 satisfies

$$X_f(\mu)(g) = \{g, f\}(\mu) = \mu([dg(\mu), df(\mu)]) = -(\mu \circ \mathrm{ad}_{df(\mu)})(dg(\mu))$$

for every $g \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, where we have identified $\mathfrak{g}^{**}$ with $\mathfrak{g}$. Thus, $X_f(\mu) = -(\mathrm{ad}_{df(\mu)})^*$ for every $\mu \in \mathfrak{g}^*$ and X_f is a fundamental vector field of the coadjoint representation of G. It follows that the symplectic leaves in $\mathfrak{g}^*$ are the coadjoint orbits. Moreover, the restricted Poisson structure on each coadjoint orbit is given by the Kirillov symplectic structure.

Chapter 2

Symmetries and Integrability

2.1 Hamiltonian Lie group actions

Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ be a smooth group action. If $X \in \mathfrak{g}$, the fundamental vector field $\phi_*(X) \in \mathcal{X}(M)$ of the action which corresponds to X is the infinitesimal generator of the flow $\phi_X : \mathbb{R} \times M \to M$ defined by $\phi_X(t,p) = \phi(\exp(tX), p)$. Note that for $g \in G$ the transformed vector field $(\phi_g)_*(\phi_*(X))$ is the fundamental vector field $\phi_*(\mathrm{Ad}_g(X))$, that is

$$(\phi_g)_{*p}(\phi_*(X)(p)) = (\phi_*)(\mathrm{Ad}_g(X))(\phi_g(p))$$

for every $p \in M$. Indeed,

$$\phi_*(\mathrm{Ad}_g(X))(\phi_g(p)) = \frac{d}{dt}\Big|_{t=0} \phi^{\phi_g(p)}(\exp(t\mathrm{Ad}_g(X)))$$

$$= (\phi^{\phi_g(p)})(\frac{d}{dt}\Big|_{t=0} (\exp(t\mathrm{Ad}_g(X))) = (\phi^{\phi_g(p)})_{*e}(\mathrm{Ad}_g(X))$$

$$= \frac{d}{dt}\Big|_{t=0} \phi(g\exp(tX)g^{-1}, \phi(g,p)) = \frac{d}{dt}\Big|_{t=0} \phi(g\exp(tX), p)$$

$$= \frac{d}{dt}\Big|_{t=0} (\phi^p \circ L_g)(\exp(tX)) = \frac{d}{dt}\Big|_{t=0} (\phi_g \circ \phi^p)(\exp(tX))$$

$$= (\phi_g)_{*p}((\phi^p)_{*e}(X)) = (\phi_g)_{*p}(\phi_*(X)(p)).$$

Lemma 2.1.1. *The linear map $\phi_* : \mathfrak{g} \to \mathcal{X}(M)$ is an anti-homomorphism of Lie algebras, meaning that $\phi_*([X,Y]) = -[\phi_*(X), \phi_*(Y)]$ for every $X,\ Y \in \mathfrak{g}$.*

Proof. If $p \in M$, then we compute

$$[\phi_*(X), \phi_*(Y)](p) = \frac{d}{dt}\Big|_{t=0} (\phi_{\exp(-tX)})_{*\phi_{\exp(tX)}(p)}(\phi_*(Y))(\phi_{\exp(tX)}(p)))$$

$$= \frac{d}{dt}\Big|_{t=0} \phi_*(\mathrm{Ad}_{\exp(-tX)}(Y))(p) = \phi_*(-\mathrm{ad}_X(Y))(p) = -\phi_*([X,Y]). \quad \square$$

It follows that $\phi_*(\mathfrak{g})$ is a Lie subalgebra of $\mathcal{X}(M)$ of finite dimension.

Definition 2.1.2. Let (M, ω) be a symplectic manifold and G a Lie group. A smooth group action $\phi : G \times M \to M$ is called *symplectic* if $\phi_g = \phi(g, \cdot) : M \to M$ is a symplectomorphism for every $g \in G$. The symplectic action is called *Hamiltonian* if each fundamental vector field $\phi_*(X)$, $X \in \mathfrak{g}$, is Hamiltonian.

There is no canonical way to choose a Hamiltonian for $\phi_*(X)$. If there exists a linear map $\rho : \mathfrak{g} \to C^\infty(M)$ which to each $X \in \mathfrak{g}$ assigns a Hamiltonian $\rho(X)$ for $\phi_*(X)$, there is also a smooth map $\mu : M \to \mathfrak{g}^*$ defined by $\mu(p)(X) = \rho(X)(p)$.

Examples 2.1.3. (a) Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ a smooth group action. Then, ϕ is covered by a group action $\tilde{\phi}$ of G on T^*M defined by $\tilde{\phi}(g, a) = a \circ (\phi_{g^{-1}})_{*\phi_g(\pi(a))}$, where $\pi : T^*M \to M$ is the cotangent bundle projection. Since $\pi \circ \tilde{\phi}_g = \phi_g \circ \pi$, differentiating we get

$$\pi_{*\tilde{\phi}_g(a)} \circ (\tilde{\phi}_g)_{*a} = (\phi_g)_{*\pi(a)} \circ \pi_{*a}$$

for every $a \in T^*M$ and $g \in G$. The Liouville 1-form θ on T^*M remains invariant under the action of G, because

$$((\tilde{\phi}_g)^*\theta)_a = \theta_{\tilde{\phi}_g(a)} \circ (\tilde{\phi}_g)_{*a} = a \circ (\phi_g^{-1})_{*\phi_g(\pi(a))} \circ (\phi_g)_{*\pi(a)} \circ \pi_{*a} = a \circ \pi_{*a} = \theta_a.$$

Consequently, the action of G on T^*M is symplectic with respect to the canonical symplectic structure $\omega = -d\theta$. In addition to that it is Hamiltonian, because

$$0 = L_{\tilde{\phi}_*(X)}\theta = i_{\tilde{\phi}_*(X)}(d\theta) + d(i_{\tilde{\phi}_*(X)}\theta)$$

and hence $i_{\tilde{\phi}_*(X)}\omega = d(i_{\tilde{\phi}_*(X)}\theta)$. Here we have a linear map $\rho : \mathfrak{g} \to C^\infty(T^*M)$ defined by $\rho(X) = i_{\tilde{\phi}_*(X)}\theta$ and $\mu : T^*M \to \mathfrak{g}^*$ is given by the formula

$$\mu(a)(X) = \theta_a(\tilde{\phi}_*(X)).$$

(b) Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{O}$ be a coadjoint orbit. The symplectic Kirillov 2-form ω^- is Ad^*-invariant as we saw in section 1.4 and so the natural action of G on $\mathcal{O}$ is symplectic. Recall that

$$\omega^-_\nu(X_{\mathfrak{g}^*}(\nu), Y_{\mathfrak{g}^*}(\nu)) = -\nu([X, Y]) = (\nu \circ \mathrm{ad}_Y)(X) = -Y_{\mathfrak{g}^*}(\nu)(X) = -X(Y_{\mathfrak{g}^*}(\nu))$$

for every X, $Y \in \mathfrak{g}$ and $\nu \in \mathcal{O}$, having identified $\mathfrak{g}^{**}$ with $\mathfrak{g}$. If now $\rho_X \in C^\infty(\mathfrak{g}^*)$ is the linear function defined by $\rho_X(\nu) = -\nu(X)$, then $d\rho_X(\nu) = -X$ (we identify again $\mathfrak{g}^{**}$ with $\mathfrak{g}$). It follows that $i_{X_{\mathfrak{g}^*}}\omega^- = d\rho_X$, which shows that the action of G on $\mathcal{O}$ is Hamiltonian.

Definition 2.1.4. Let M be a symplectic manifold and G be a Lie group with Lie algebra $\mathfrak{g}$. A Hamiltonian group action $\phi : G \times M \to M$ is called *Poisson* (or *strongly Hamiltonian*) if there is a lift $\rho : \mathfrak{g} \to C^\infty(M)$ which is a Lie algebra homomorphism.

Let (M, ω) be a symplectic manifold and G be a Lie group acting smoothly and symplectically on M. If G is compact (or more generally the action is proper), there exists a G-invariant Riemannian metric on M. Starting with such a Riemannian metric, one can construct a G-invariant almost complex structure J on M which is compatible with ω. The corresponding compatible Riemannian metric g on M given by the formula $g_x(u,v) = -\omega(J(u),v)$, for u, $v \in T_xM$, $x \in M$, is also G-invariant.

2.2 Momentum maps and reduction

Let (M, ω) be a connected, symplectic manifold, G be a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ be a Poisson action.

Definition 2.2.1. A *momentum map* for ϕ is a smooth map $\mu : M \to \mathfrak{g}^*$ such that $\rho : \mathfrak{g} \to C^\infty(M)$ defined by $\rho(X)(p) = \mu(p)(X)$ for $X \in \mathfrak{g}$ and $p \in M$ satisfies
(i) $\phi_*(X) = X_{\rho(X)}$,
(ii)$\{\rho(X), \rho(Y)\} = \rho([X,Y])$ for every X, $Y \in \mathfrak{g}$.

From the standpoint of dynamical systems, one important feature of momentum maps is the following. If $H : M \to \mathbb{R}$ is a G-invariant, smooth function, then μ is constant along the integral curves of the Hamiltonian vector field X_H, because, for every $X \in \mathfrak{g}$ we have

$$L_{X_H}\rho(X) = \{\rho(X), H\} = -\{H, \rho(X)\} = -L_{\phi_*(X)}H = 0.$$

Theorem 2.2.2. *If G is a connected Lie group, then a momentum map $\mu : M \to \mathfrak{g}^*$ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.*

Proof. (See [1], page 89). □

Examples 2.2.3. (a) Let $\phi : G \times M \to M$ be a smooth action of the Lie group G with Lie algebra $\mathfrak{g}$ on the smooth manifold M and $\tilde{\phi} : G \times T^*M \to T^*M$ be the lifted action on the cotangent bundle. As we saw in Example 2.1.1(a) the action of G on T^*M is Poisson and in fact the Liouville 1-form θ on T^*M is G-invariant. The momentum map $\mu : T^*M \to \mathfrak{g}^*$ is given by the formula

$$\mu(a)(X) = \theta_a(\tilde{\phi}_*(X)(a))$$

for $X \in \mathfrak{g}$ and $a \in T^*M$, and is G-equivariant, because θ is G-invariant. Indeed,

$$\mu(\tilde{\phi}_g(a))(X) = \theta_{\tilde{\phi}(a)}(\tilde{\phi}_*(X)(\tilde{\phi}_g(a))) = ((\tilde{\phi}_{g^{-1}})^*\theta)_{\tilde{\phi}(a)}(\tilde{\phi}_*(X)(\tilde{\phi}_g(a)))$$

$$= \theta_a((\tilde{\phi}_{g^{-1}})_{*\tilde{\phi}_g(a)}(\tilde{\phi}_*(X)(\tilde{\phi}_g(a)))) = \theta_a(\tilde{\phi}_*(\mathrm{Ad}_{g^{-1}}(X))(a)) = \mu(a)(\mathrm{Ad}_{g^{-1}}(X)),$$

for every $g \in G$.

In the case of the 3-dimensional euclidean space $\mathbb{R}^3$ we have $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$, where the isomorphism is defined by the euclidean inner product $\langle\ ,\ \rangle$, identifying thus $T^*\mathbb{R}^3$ with $T\mathbb{R}^3$. The Liouville 1-form is given by the formula

$$\theta_{(q,p)}(v,w) = \langle v, p\rangle\,.$$

The natural action of $SO(3,\mathbb{R})$ on $\mathbb{R}^3$ is covered by the action $\tilde{\phi}$ such that

$$\tilde{\phi}_A(q,p)(v) = \left\langle p, A^{-1}v\right\rangle = \langle Ap, v\rangle$$

for every $v \in T_q\mathbb{R}^3$ and $A \in SO(3,\mathbb{R})$. Hence, $\tilde{\phi}_A(q,p) = (Aq, Ap)$ for every $(q,p) \in T^*\mathbb{R}^3$ and $A \in SO(3,\mathbb{R})$. If now $v \in \mathbb{R}^3 \cong \mathfrak{so}(3,\mathbb{R})$, the corresponding fundamental vector field of the action satisfies

$$\tilde{\phi}_*(v)(q,p) = (\hat{v}q, \hat{v}p) = (v \times q, v \times p).$$

It follows that the momentum map satisfies

$$\mu(q,p)(v) = \langle v \times q, p\rangle = \langle q \times p, v\rangle$$

for every $v \in \mathbb{R}^3$. Consequently, the momentum map is the angular momentum

$$\mu(q,p) = q \times p.$$

Suppose now that we have a system of n particles in $\mathbb{R}^3$. The configuration space is $\mathbb{R}^{3n}$. The additive group $\mathbb{R}^3$ acts on $\mathbb{R}^{3n}$ by translations, that is

$$\phi_x(q^1, q^2, ..., q^n) = (q^1 + x, q^2 + x, ..., q^n + x)$$

foe every $x \in \mathbb{R}^3$. The lifted action on $T^*\mathbb{R}^{3n} \cong \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ is

$$\tilde{\phi}_x(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n) = (q^1 - x, q^2 - x, ..., q^n - x, p_1, p_2, ..., p_n).$$

If now $X \in \mathbb{R}^3$, the corresponding fundamental vector field of the action is

$$\tilde{\phi}_*(X)(q^1, q^2, ..., q^n, p_1, p_2, ...p_n) = (-X, -X, ..., -X, 0, 0, ..., 0).$$

Therefore, the momentum map $\mu : T^*\mathbb{R}^{3n} \to \mathbb{R}^3$ satisfies

$$\mu(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)(X) = \sum_{j=1}^{n} \langle -X, p_j\rangle = \left\langle X, -\sum_{j=1}^{n} p_j \right\rangle.$$

That is, the momentum map in this case is the total linear momentum

$$\mu(q^1, q^2, ..., q^n, p_1 p_2, ..., p_n) = -\sum_{j=1}^{n} p_j.$$

This example justifies the use of the term momentum map.

(b) Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. As we saw in Example 2.1.3(b), the transitive action of G on $\mathcal{O}$ is Poisson with momentum map $\mu : \mathcal{O} \to \mathfrak{g}^*$ given by the formula $\mu(\nu) = -\nu$ for every $\nu \in \mathcal{O}$. So, the momentum map is minus the inclusion of $\mathcal{O}$ in $\mathfrak{g}^*$, which is G-equivariant.

The significance of the momentum maps is made clear in the process of symplectic reduction.

Let (M, ω) be a connected, symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ a symplectic action. The orbit space $G\backslash M$ of the action may not in general be a smooth manifold, not even a Hausdorff space. Even in the case it is, it may not admit any symplectic structure, as for instance it may be odd dimensional. If the action is Poisson and there is a G-equivariant momentum map $\mu : M \to \mathfrak{g}^*$, there exists a well defined continuous map $\tilde{\mu} : G\backslash M \to G\backslash \mathfrak{g}^*$ in the orbit spaces. In particular cases, the level sets $\tilde{\mu}^{-1}(\mathcal{O}_a)$, $a \in \mathfrak{g}^*$, can be given a natural symplectic structure. The inclusion $j : \mu^{-1}(a) \hookrightarrow \mu^{-1}(\mathcal{O}_a)$ induces a continuous bijection $j_\# : G_a\backslash\mu^{-1}(a) \to G\backslash\mu^{-1}(\mathcal{O}_a)$. In specific instances, $j_\#$ is a homeomorphism or even a diffeomorphism of smooth manifolds. For example, if the action of G on M is free and proper and a is a regular value of μ, then $\mu^{-1}(\mathcal{O}_a)$ is a smooth submanifold of M and so are $G\backslash\mu^{-1}(\mathcal{O}_a)$ and $G_a\backslash\mu^{-1}(a)$. Additionally, in this case $j_\#$ is a diffeomorphism. Specifically, these are true if G is compact and the action is free.

Definition 2.2.4. Let P, Q be two smooth manifolds and $f : P \to Q$ be a smooth map. A point $q \in Q$ is called a *clean* (or *weakly regular*) value of f if $f^{-1}(q)$ is an embedded smooth submanifold of M and $T_p f^{-1}(q) = \mathrm{Ker} f_{*p}$ for every $p \in f^{-1}(q)$.

A regular value is always clean, while the converse is not true. For instance, $(0, 0) \in \mathbb{R}^2$ is clean, but not regular, value of the smooth function $f : \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x, y, z) = (z^2, z)$.

Theorem 2.2.5. *Let (M, ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ be a Poisoon action with G-equivariant momentum map $\mu : M \to \mathfrak{g}^*$. Let $a \in \mathfrak{g}^*$ be a clean value of μ such that the orbit space $M_a = G_a\backslash\mu^{-1}(a)$ is a smooth manifold and the quotient map $\pi_a : \mu^{-1}(a) \to M_a$ is a smooth submersion, where G_a is the isotropy group of a with respect to the coadjoint orbit. There exists a unique symplectic 2-form ω_a on M_a such that $\pi_a^*\omega_a = \omega|_{\mu^{-1}(a)}$.*

Proof. (See [1], page 95). □

Examples 2.2.6. (a) Let M be a symplectic manifold and $H \in C^\infty(M)$ be such that the Hamiltonian vector field X_H is complete. Its flow is a Poisson group action of $\mathbb{R}$ on M with momentum map H itself. Since $\mathbb{R}$ is abelian, the coadjoint action is trivial. If now $a \in \mathbb{R}$ is a clean value of H, then according to the Theorem 2.3.2 the orbit space $\mathbb{R}\backslash H^{-1}(a)$ has a natural symplectic structure.

(b) Let $SO(3, \mathbb{R})$ acts on $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ as in example 2.2.3(a). As we saw, the momentum map $\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the angular momentum

$$\mu(q, p) = q \times p.$$

The Jacobian matrix of μ at (q, p) is $(-\hat{p}, \hat{q})$, and so every non-zero $v \in \mathbb{R}^3$ is a regular value of μ. The isotropy group of v is the group of rotations of $\mathbb{R}^3$ around the axis generated by v, therefore isomorphic to S^1. Thus, the orbit space $S^1\backslash\mu^{-1}(v)$ has a symplectic structure.

2.3 Completely integrable Hamiltonian systems

Let (M,ω) be a connected, symplectic $2n$-manifold and $H_1 \in C^\infty(M)$. The Hamiltonian vector field X_{H_1} is called *completely integrable* if there are $H_2, ..., H_n \in C^\infty(M)$ such that $\{H_i, H_j\} = 0$ for every $1 \leq i, j \leq n$ and the differential 1-forms $dH_1, dH_2, ..., dH_n$ are linearly independent on a dense open set $D \subset M$. In this section we shall assume that we have such a system.

If $f = (H_1, H_2, ..., H_n) : M \to \mathbb{R}^n$, then $f|_D$ is a smooth submersion and so the connected components of the fibers $f^{-1}(y) \cap D$, $y \in \mathbb{R}^n$, are the leaves of a foliation of D and $f_{*p}(X_{H_i}(p)) = 0$ for every $1 \leq i \leq n$.

Suppose that the Hamiltonian vector fields $X_{H_1}, X_{H_2}, ..., X_{H_n}$ are complete. Since their flows commute, they define a Poisson group action $\phi : \mathbb{R}^n \times M \to M$ with fundamental vector fields $X_{H_1}, X_{H_2}, ..., X_{H_n}$ and momentum map f. Let $y \in \mathbb{R}^n$ be a regular value of f. Then $f^{-1}(y) \subset D$ is a $\mathbb{R}^n$-invariant, regular n-dimensional submanifold of M. The vector fields $X_{H_1}, X_{H_2}, ..., X_{H_n}$ are tangent to $f^{-1}(y)$, and since they are linearly independent at every point of $f^{-1}(y)$, every orbit in $f^{-1}(y)$ is an open subset of $f^{-1}(y)$. This implies that every connected component N of $f^{-1}(y)$ is an orbit of the action. Therefore, N is diffeomorphic to the homogeneous space $\mathbb{R}^n/\Gamma_p$, where Γ_p is the isotropy group of p. We notice that Γ_p does not depend on p, but only on N, since $\mathbb{R}^n$ is abelian. What is more, Γ_p is a 0-dimensional closed subgroup of $\mathbb{R}^n$ and therefore is discrete. The discrete subgroups of $\mathbb{R}^n$ are described as follows. Let $\Gamma \leq \mathbb{R}^n$ be a non-trivial discrete subgroup. Then Γ is a lattice, that is there exist $1 \leq k \leq n$ and linearly independent vectors $v_1, ..., v_k$ such that

$$\Gamma = \mathbb{Z}v_1 + ... + \mathbb{Z}v_k.$$

Consequently, there exists $1 \leq k \leq n$ such that the homogeneous space $\mathbb{R}^n/\Gamma$ is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$. If $\mathbb{R}^n/\Gamma$ is compact, then $k = n$ and $\mathbb{R}^n/\Gamma$ is diffeomorphic to the n-torus T^n. We refer to [1], page 97 for details.

Observe that if N is compact then the restrictions of the Hamiltonian vector fields $X_{H_1}, X_{H_2}, ..., X_{H_n}$ to N are automatically complete. So, we have arrived at the following famous theorem.

Theorem 2.3.1. (Arnold-Liouville) *Let $y \in \mathbb{R}^n$ be a regular value of f and N be a connected component of $f^{-1}(y)$.*

(i) If N is compact, then it is diffeomorphic to the n-torus T^n.

(ii) If N is not compact and $X_{H_1}, X_{H_2}, ..., X_{H_n}$ are complete, then N is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$. □

The flow of the Hamiltonian vector field X_{H_1} on N can be characterized as follows. Let $p \in N$ and $\tilde{\phi}^p : \mathbb{R}^n/\Gamma \to N$ be the diffeomorphism which is induced by $\phi^p = \phi(\cdot, p) : \mathbb{R}^n \to N$. Let $(\psi_t)_{t\in\mathbb{R}}$ be the flow of X_{H_1} on N and $\tilde{\psi}_t = (\tilde{\phi}^p)^{-1} \circ \psi_t \circ \tilde{\phi}^p$,

$t \in \mathbb{R}$, be the conjugate flow on $\mathbb{R}^n/\Gamma$. Then

$$\tilde{\psi}_t([t_1, ..., t_n]) = (\tilde{\phi}^p)^{-1}(\psi_t(\phi(t_1, ..., t_n), p)))$$

$$= (\tilde{\phi}^p)^{-1}(\phi((t + t_1, t_2, ..., t_n), p)) = [t + t_1, t_2, ..., t_n].$$

That is to say, $\psi_t([v]) = [v + te_1]$ for every $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Let $T(e_1) = (\nu_1, ..., \nu_n)$ and let $\chi_t = \tilde{T} \circ \tilde{\psi}_t \circ \tilde{T}^{-1}$, $t \in \mathbb{R}$, be the conjugate flow on $T^k \times \mathbb{R}^{n-k}$. Then,

$$\chi_t(e^{2\pi i t_1}, ..., e^{2\pi i t_k}, t_{k+1}, ..., t_n) = \tilde{T}(\tilde{\psi}_t([t_1 v_1 + ... + t_n v_n])) = \tilde{T}([te_1 + t_1 v_1 + ... + t_n v_n]).$$

Since

$$T(te_1 + t_1 v_1 + ... + t_n v_n) = tT(e_1) + t_1 e_1 + ... + t_n e_n = (t_1 + t\nu_1, ..., t_n + t\nu_n)$$

it follows that

$$\chi_t(e^{2\pi i t_1}, ..., e^{2\pi i t_k}, t_{k+1}, ..., t_n) = (e^{2\pi i (t_1 + t\nu_1)}, ..., e^{2\pi i (t_k + t\nu_k)}, t_{k+1} + t\nu_{k+1}, ..., t_n + t\nu_n).$$

This shows that the flow of X_{H_1} on N is smoothly conjugate to a linear flow on $T^k \times \mathbb{R}^{n-k}$. If N is compact, then $k = n$ and the real numbers, $\nu_1, ..., \nu_n$ are called the *frequences of the flow* on N. If they are linearly independent over $\mathbb{Q}$, then the flow on N is uniquely ergodic and every orbit is dense in N.

Chapter 3

Pseudo-Riemannian Homogeneous Spaces

3.1 Pseudo-Riemannian metrics on homogeneous spaces

In this paragraph we present some general facts about metrics on homogeneous spaces and we consider a special class of those spaces characterized by the property that they admit a G-invariant metric which is induced by a bi-invariant, possibly indefinite, metric on G.

Let K be a closed subgroup of a connected Lie group G and the set $\{\sigma \mathrm{K} \mid \sigma \in \mathrm{K}\}$ of left cosets be the homogeneous space $M := G/K$. Let also $\pi : G \to G/K$ denote the natural projection $\pi(\sigma) = \sigma K$. There is a natural left action $G \times M \to M$ of G on M given by $L_g(xK) = (gx)K$ for every $g, x \in G$. Since $xK = (xy^{-1})(yK)$ for every $x, y \in G$, the action is transitive so the terminology homogeneous space. Recall that the action of G on M is called $effective$ if the identity e is the only element of G for which L_g is the identity map on M i.e. $(L_g) = id_M$ implies that $g = e \in G$.

Let now K_0 denote the largest subgroup of K which is normal in G, $G^* = G/K_0$ and $K^* = K/K_0$. It is evident that K_0 is closed. Then G^*/K^* is diffeomorphic to M. In addition, G^* acts effectively on G^*/K^*. Indeed, if $g^* = gK_0$, $x^* = xK_0$ and $g^*(x^*K^*) = x^*K^*$ then $(x^{-1}gx)K_0 \in K/K_0$ and so $g \in xKx^{-1}$. But $g \in xKx^{-1}$ for every $x \in G$ implies that $g \in \bigcap\limits_{x \in G} xKx^{-1} \leq K_0$ and so $g^* = e^* \in G^*$.

We are interested in metrics on M which are invariant, that is G acts on M by isometries. If G acts effectively on M by isometries of some metric, then G may be identified with a Lie subgroup of the group of isometries of M (not always embedded).

We shall consider homogeneous spaces that possess a specific property:

Property A. *On the Lie algebra $\mathfrak{g}$ of G there exists an Ad_G-invariant, symmetric, non-degenerate billinear form B such that the restriction of B to the Lie algebra $\mathfrak{k}$ of K is non-degenerate.*

This property leads us to formulate an equation for the standard symplectic structure of T^*M. By the use of some propositions from the theory of metrics on homogeneous spaces we shall see some consequences of that property. Firstly we prove that our assumptions implies the existence of a B-orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. This is a standard result of linear algebra.

Let U be a linear subspace of the linear space V. We assume that V is equiped with a symmetric, billinear, non-degenerate form $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

The linear subspace $U^* = \{A \in V \mid \langle A, B\rangle = 0,\ \forall B \in U\}$ of V is called the orthogonal complement of U in V.

Proposition 3.1.1. $\dim U + \dim U^* = \dim V$.

Proof. Let n=dim$V$ and $\{A_1, ..., A_n\}$ be a coordinate system for V. Let F be the matrix of $\langle\cdot,\cdot\rangle$ relative to this coordinate system. We assume that $\dim U = m$ and choose a coordinate system $B_1, ..., B_m$ for U. If $B_i = b_{i1}A_1 + ... + b_{in}A_n$ for $i = 1, ..., m$ we know from linear algebra that the $m \times n$ matrix

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

has rank m. Since $\langle\cdot,\cdot\rangle$ is non-degenerate we know that $\det F \neq 0$ and hence the $m \times n$ matrix BF also has rank m. A vector $X = x_1A_1 + ... + x_nA_n$ belongs to U^* if and only if

$$(b_{i1}...b_{in})F \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

for $i = 1, ..., m$. In other words, U^* is the solution space of the system of m homogeneous equations in n-variables

$$BF \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Since the matrix BF has rank m, this solution space has dimension $n - m$. □

If now the restriction of $\langle\cdot,\cdot\rangle$ on $U \times U$ is also non-degenerate then $U \cap U^* = \{0\}$ and so $V = U \oplus U^*$. In particular there is a well defined orthogonal projection from V to U with respect to $\langle\cdot,\cdot\rangle$.

If B satisfies property (A) and $\mathfrak{m}$ is the B-orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, then $\mathfrak{m}$ can be naturally identified with $T_{\pi(e)}M$ by means of π_{*e} and $B|_{\mathfrak{m}\times\mathfrak{m}}$ is non-degenerate. There is a general description of G-invariant metrics on general homogeneous spaces M which do not necessarily satisfy property (A). We have the following useful fact.

Lemma 3.1.2. *If $\pi : G \to M$ is the quotient fibration, then $(L_k)_{*\pi(e)}\pi_{*e}(X) = \pi_{*e}(\mathrm{Ad}_k(X))$ for every $X \in T_eG = \mathfrak{g}$, $k \in K$.*

Proof. Differentiating the equation $k\exp(tX)K = k\exp(tX)k^{-1}K = \pi(k\exp(tX)k^{-1})$ with respect to t at $t=0$, from the chain rule we get that

$$(L_k)_{*\pi(e)}\pi_{*e}(X) = \left.\frac{d}{dt}\right|_{t=0} \pi(k\exp(tX)K) = \pi_{*e}(\mathrm{Ad}_k(X)). \quad \square$$

Note that the tangent space $T_{\pi(e)}M$ at $\pi(e) = K$ can be naturally identified with $\mathfrak{g}/\mathfrak{k}$, since $\pi_{*e} : \mathfrak{g} \to T_{\pi(e)}M$ induces an isomorphism of vector spaces $\pi_{*e} : \mathfrak{g}/\mathfrak{k} \to T_{\pi(e)}M$.

Proposition 3.1.3. *The set of G-invariant metrics on M is naturally identified with the set of symmetric, bilinear, non-degenerate forms $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}/\mathfrak{k}$ which are invariant under the action Ad_K on $\mathfrak{g}/\mathfrak{k}$.*

Proof. The restriction of a G-invariant metric of M to $T_{\pi(e)}M$ gives a form $\langle\cdot,\cdot\rangle$ as required by the preceding lemma. Conversely, if $\langle\cdot,\cdot\rangle$ is an Ad_K-invariant symmetric bilinear and non-degenerate form on $\mathfrak{g}/\mathfrak{k}$ we define a G-invariant metric on M as follows: for $X, Y \in T_{\pi(g)}M,\ g \in G$, we set

$$\ll X, Y \gg_{gK} = \left\langle (L_{g^{-1}})_{*gK}(X), (L_{g^{-1}})_{*gK}(Y) \right\rangle.$$

Indeed, if $k \in K$, then

$$\begin{aligned} &\left\langle (L_{(gk)^{-1}})_{*gK}(X), (L_{(gk)^{-1}})_{*gK}(Y) \right\rangle \\ &= \left\langle (L_{k^{-1}})_{*K} \circ (L_{g^{-1}})_{*gK}(X), (L_{k^{-1}})_{*K} \circ (L_{g^{-1}})_{*gK}(Y) \right\rangle \\ &= \left\langle (L_{g^{-1}})_{*gK}(X), (L_{g^{-1}})_{*gK}(Y) \right\rangle \end{aligned}$$

by Lemma 3.1.2, since $\langle\cdot,\cdot\rangle$ is assumed to be Ad_K-invariant. Obviously this metric on M is G-invariant. $\square$

Proposition 3.1.4. *If a symmetric, bilinear, non-degenerate form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}/\mathfrak{k}$ is Ad_K-invariant, then $\mathrm{ad}_{\mathfrak{k}}$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle$. If K is connected, the converse is also true.*

Proof. If $\langle\cdot,\cdot\rangle$ is Ad_K-invariant, then for $Z \in \mathfrak{k}$ and $X, Y \in \mathfrak{g}/\mathfrak{k}$ we compute

$$\begin{aligned} \langle \mathrm{ad}_Z(X), Y\rangle &= \left\langle \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tZ)}(X), Y \right\rangle = \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathrm{Ad}_{\exp(tZ)}(X), Y \right\rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \left\langle X, \mathrm{Ad}_{\exp(-t)Z}(Y) \right\rangle = -\left\langle X, \mathrm{ad}_Z(Y) \right\rangle. \end{aligned}$$

For the converse, we note that the set of all $k \in K$ such that $\langle \mathrm{Ad}_k(X), \mathrm{Ad}_k(Y)\rangle = \langle X, Y\rangle$ for every $X, Y \in \mathfrak{g}/\mathfrak{k}$ forms a closed subset K' of K. On the other hand, assuming that ad_Z is skew-symmetric for all $Z \in \mathfrak{k}$ we have:

$$\left\langle \mathrm{Ad}_{\exp(tZ)}(X), \mathrm{Ad}_{\exp(tZ)}(Y) \right\rangle = \left\langle e^{\mathrm{ad}_{tZ}}(X), e^{\mathrm{ad}_{tZ}}(Y) \right\rangle$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \left\langle e^{\mathrm{ad}_{tZ}}(X), (\mathrm{ad}_Z)^j(Y) \right\rangle = \sum_{j=0}^{\infty} \frac{t^j}{j!}(-1)^j \left\langle (\mathrm{ad}_Z)^j e^{\mathrm{ad}_{tZ}}(X), Y \right\rangle = \langle X, Y \rangle .$$

In the first equality we have used the well known commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{End}(V) & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathrm{End}(V)) \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ \mathrm{Aut}(V) & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathrm{End}(V)) \end{array}$$

(see [17], page 114). Since every element of some open neighbourhood of the identity in K is of the form $\exp tZ$, it follows that K' is also open in K. By connectedness $K = K'$. $\square$

As the proof shows if $\langle \cdot , \cdot \rangle$ is Ad_G-invariant then $\mathrm{ad}_{\mathfrak{g}}$ is skew-symmetric. That is ad_Z is skew-symmetric for all $Z \in \mathfrak{g}$.

If now we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathrm{Ad}_K(\mathfrak{m}) \subset \mathfrak{m}$, it follows from proposition 3.1.3. that the G-invariant metrics on M are in 1-1 correspondence with the Ad_K-invariant symmetric, bilinear, non-degenerate forms on $\mathfrak{m}$. The condition $\mathrm{Ad}_K(\mathfrak{m}) \subset \mathfrak{m}$ implies that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, because for $Z \in \mathfrak{k}$ and $X \in \mathfrak{k}$ we have $[Z, X] = \mathrm{ad}_Z(X) = \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp(tZ)}(X)$.

If K is connected, the converse is also true and can be proved with a similar argument as that in proof of proposition 3.1.4.

So having an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, in the Lie algebra we have that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$.

Recall that a homogeneous space M is called *reductive* if $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ for some Ad_K-invariant linear subspace $\mathfrak{m}$ of $\mathfrak{g}$. Then $\mathfrak{m}$ is an ideal in $\mathfrak{g}$.

A homogeneous space M with a G-invariant metric $\langle \cdot , \cdot \rangle$ is called *naturally reductive* if $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ for some Ad_K-invariant linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ and $\mathrm{ad}_{\mathfrak{k}}|_{\mathfrak{m}}$ is skew-symmetric with respect to the restriction of the corresponding symmetric, bilinear, non-degenerate form B to $\mathfrak{m}$, that is $B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y) = 0$, for $Z \in \mathfrak{K}$, where the subscript $\mathfrak{m}$ denotes $\mathfrak{m}$-component.

So the homogeneous spaces satisfying property (A) are naturally reductive as we can easily see: Let B be a non-degenerate, symmetric, bilinear form on the Lie algebra $\mathfrak{g}$ of G which we assume that is Ad_G-invariant. We assume further that $B|_{\mathfrak{k} \times \mathfrak{k}}$ is also non-degenerate. Let $\mathfrak{m} = \{X \in \mathfrak{g} \mid B(X, Y) = 0, \ \forall Y \in \mathfrak{k}\}$. Then $\mathrm{Ad}_K(\mathfrak{m}) \subset \mathfrak{m}$ and the restriction of B to $\mathfrak{m}$ is also non-degenerate and Ad_K-invariant. The homogeneous space $M = G/K$ is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ and the G-invariant metric corresponding to the restriction of B to $\mathfrak{m}$. Indeed, since B is Ad_G-invariant, for $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{k}$ we have:

$$0 = B([Z, X], Y) + B(X, [Z, Y])$$

$$= B([Z, X]_{\mathfrak{k}} + [Z, X]_{\mathfrak{m}}, Y) + B(X, [Z, Y]_{\mathfrak{k}} + [Z, Y]_{\mathfrak{m}})$$

$$= B([Z, X]_{\mathfrak{m}}, Y) + B(X, [Z, Y]_{\mathfrak{m}}).$$

The existence of G-invariant Riemannian metrics can be characterized also by the following propositions that we state without proof (see [5], pages 61-63), since they will not be used essentialy in the sequel.

Proposition 3.1.5. *If G acts effectively on $M = G/K$, then M admits a G-invariant Riemannian metric if and only if the closure of* Ad_K in $GL(\mathfrak{g})$ is compact. □

A (possibly indefinite) metric on G is called bi-invariant if it is invariant by left and right translations of G.

Proposition 3.1.6. *If G is a connected compact Lie group, then G admits a bi-invariant Riemannian metric.* □

Summing up, we have seen so far the existence of $\mathfrak{m}$, the B-complement of $\mathfrak{k}$ in $\mathfrak{g}$, which we identified naturally with $T_{\pi(e)}M$ by means of π_{*e}. We also saw that $B_{\mathfrak{k}\times\mathfrak{k}}$, being non-degenerate and Ad_K-invariant, defines a (possibly indefinite) G-invariant metric on M. We derived also that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$.

Next we shall investigate geodesics of invariant metrics on homogeneous spaces. Using now the fact that the homogeneous spaces satisfying property (A) are naturally reductive we shall derive the equation:

$$\mathrm{Exp}_{\pi(\mathrm{e})} = \pi \circ \exp|_{\mathfrak{m}}$$

where Exp denotes the exponential map of M. Thus we will conclude that geodesics on M are images of one-parameter subgroups of G.

To see that we first derive a formula for the Levi-Civita connection of a left invariant metric on G.

Proposition 3.1.7. *Let $\langle \cdot, \cdot \rangle$ be a left invariant metric on G and X, Y be left invariant vector fields on G. Then*

$$\nabla_X Y = \frac{1}{2}([X, Y] - \mathrm{ad}_X^*(Y) - \mathrm{ad}_Y^*(X)).$$

Proof. By left invariance of the metric, for every left invariant vector field Z we have: $X\langle Y, Z\rangle = Y\langle Z, X\rangle = Z\langle X, Y\rangle = 0$. Therefore,

$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= \frac{1}{2}(\langle [X, Y], Z\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle)\\ &= \frac{1}{2}(\langle [X, Y], Z\rangle - \langle Y, \mathrm{ad}_X(Z)\rangle - \langle X, \mathrm{ad}_Y(Z)\rangle)\\ &= \frac{1}{2}(\langle [X, Y], Z\rangle - \langle \mathrm{ad}_X^*(Y), Z\rangle - \langle \mathrm{ad}_Y^*(X), Z\rangle).\end{aligned}$$ □

If now we have a form B on G satisfying property (A) then $B([Z, X], Y) + B(X, [Z, Y]) = 0$ for all $X, Y, Z \in \mathfrak{g}$ and the above proposition 3.1.7. implies that the Levi-Civita connection of the left invariant metric on G defined by B is given by

$$\nabla_X Y = \frac{1}{2}[X,Y]$$

for $X, Y \in \mathfrak{g}$. This implies that the geodesics in G are the images of the one-parameter subgroups of G. It follows that the geodesics in M are images of one-parameter subgroups of G generated by elements of $\mathfrak{m}$. This means that

$$\mathrm{Exp}_{\pi(\mathrm{e})} = \pi \circ \exp|_{\mathfrak{m}}$$

where Exp denotes the exponential map of M and exp that of G.

3.2 The momentum map on the tangent bundle of a homogeneous space

Before the study of the symplectic structure of the tangent bundle TM of homogeneous spaces we derive a formula for the momentum map P on TM of a homogeneous space assuming it satisfies property (A) defined in the previous paragraph.

To begin with let $(M, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold and G a Lie group acting on M by diffeomorphisms which preserve the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$. Let also $\varphi : G \times M \to M$ denote the smooth group action. It can be lifted to smooth group actions $\widehat{\varphi} : G \times TM \to TM$ and $\widetilde{\varphi} : G \times T^*M \to T^*M$ which cover φ, defining $\widehat{\varphi}(g,v) = (\varphi_g)_{*\pi(v)}(v)$ for $v \in TM$, $g \in G$ and $\widetilde{\varphi}(g,a) = a \circ (\varphi_{g^{-1}})_{*\varphi_g(\pi^*(a))}$ for $a \in T^*M$, $g \in G$ where $\pi : TM \to M$ and $\widetilde{\pi} : T^*M \to M$ are the bundle maps. The pseudo-Riemannian metric gives a natural bundle isomorphism $TM \cong T^*M$ defined for $p \in M$, $v \in T_pM$ by $v \mapsto \langle\cdot,v\rangle \in T_p^*M$, which identifies TM with T^*M. This natural isomorphism transfers the action $\widehat{\varphi}$ to the action $\widetilde{\varphi}$.

$$\begin{array}{ccc} TM & \xrightarrow{\widehat{\varphi}_g} & TM \\ \downarrow\cong & & \downarrow\cong \\ T^*M & \xrightarrow{\widetilde{\varphi}_g} & T^*M \end{array}$$

Indeed, for $g \in G$ and $v \in T_pM$, $p \in M$ we have $\widetilde{\varphi}_g(\langle\cdot,v\rangle) = \left\langle v, (\varphi_{g^{-1}})_{*\varphi_g(p)}(\cdot)\right\rangle = \left\langle (\varphi_g)_{*p}(v), \cdot\right\rangle$, since φ_g preserves $\langle\cdot,\cdot\rangle$.

Let now $\mu : T^*M \to \mathfrak{g}^*$ be the G-momentum map. By Example 2.1.3(a) it is given by the formula $\mu(a)(X) = a(\widetilde{\pi}_{*a}(\varphi_*(X)))$, $a \in T^*M$, $X \in \mathfrak{g}$. If $a = \langle\cdot,v\rangle$, $v \in TM$, then $\mu(\langle v,\cdot\rangle)(X) = \langle v, \widetilde{\pi}_{*a}(\widetilde{\varphi}_*(\mathrm{X}))\rangle$ and

$$\widetilde{\pi}_{*a}(\widetilde{\varphi}_*(\mathrm{X})) = \widetilde{\pi}_{*a}(\frac{d}{dt}\Big|_{t=0}\widetilde{\varphi}(\exp(tX),a))$$

$$= \frac{d}{dt}\Big|_{t=0}(\widetilde{\pi}\circ\widetilde{\varphi})(\exp(tX),a) = \frac{d}{dt}\Big|_{t=0}\varphi(\exp(tX),\widetilde{\pi}(a)) = \varphi_*(\mathrm{X})(\widetilde{\pi}(a)).$$

Substituting we obtain the formula

$$P(v)(X) = \mu(\langle v, \cdot\rangle)(X) = \langle v, \varphi_*(\mathrm{X})(\pi(v))\rangle$$

for the transfered momentum map $P : TM \to \mathfrak{g}^*$ on TM.

We now consider the case of a pseudo-Riemannian homogeneous space satisfying property (A) and we shall conclude a more specific formula for P. We shall use the notation of the previous section. As we have seen a homogeneous space $M = G/K$ which satisfies property (A) gives a B-orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and the derivative $\pi_{*e} : \mathfrak{g} \to T_{\pi(e)}M$ identifies $T_{\pi(e)}M$ with $\mathfrak{m}$, where $\pi : G \to G/K = M$ is the quotient map. The tangent space $T_{\pi(g)}M = T_{gk}M$ consists of the tangent vectors $(L_g)_{*\pi(e)} \circ \pi_{*e}(\xi)$, where $\xi \in \mathfrak{m}$ and $L_g : M \to M$, $g \in G$ is the natural left action of G on M. We denote $g\xi = (L_g)_{*\pi(e)} \circ \pi_{*e}(\xi)$, $\xi \in \mathfrak{m}$, $g \in G$. The restriction $B|_{\mathfrak{m}\times\mathfrak{m}}$ induces the pseudo-Riemannian metric on M which is G-invariant.

Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the non-degenerate, symmetric, bilinear form B we arrive at the following conclusion.

Lemma 3.2.1. *Let M=G/K be a homogeneous space satisfying property (A). Then the momentum map $P : TM \to \mathfrak{g}$ is given by $P(g\xi) = \mathrm{Ad}_g\xi$, where $g \in G$, $g\xi \in T_{\pi(g)}M$ and $\xi \in \mathfrak{m}$.*

Proof. From the above we need to compute the fundamental vector field $\Phi_*(X)$ of the natural left action of G on M. We have

$$\Phi_*(X)(gK) = \frac{d}{dt}\bigg|_{t=0} \exp(tX)(gK) = \frac{d}{dt}\bigg|_{t=0} g(g^{-1}\exp(tX)g)K$$

$$= \frac{d}{dt}\bigg|_{t=0} (L_g \circ \pi(g^{-1}\exp(tX)g)) = (L_g)_{*\pi(e)} \circ \pi_{*e}(\mathrm{Ad}_{(g^{-1})}(X)).$$

So if $\xi \in \mathfrak{m}$ and $v = g\xi = (L_g)_{*\pi(e)} \circ \pi_{*e}(\xi)$ the G-momentum map $P : TM \to \mathfrak{g}^*$ is given by

$$\begin{aligned} P(v)(X) &= \Big\langle v, (L_g)_{*\pi(e)} \circ \pi_{*e}(\mathrm{Ad}_{(g^{-1})}(X))\Big\rangle \\ &= B(\xi, \mathrm{Ad}_{(g^{-1})}(X)) = B(\mathrm{Ad}_g(\xi), X) \end{aligned}$$

for every $X \in \mathfrak{g}$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by B, it follows (using the same symbol) that $P(g\xi) = \mathrm{Ad}_g(\xi)$. □

3.3 The symplectic structure of the tangent bundle of homogeneous spaces

In order to construct first integrals in involution for G-invariant Hamiltonian systems on $TM = T(G/K)$ we need to use explicit expressions for the Hamiltonian vector fields in order to be able to prove that these intergrals are functionally independent on an open dense set of full measure, for the specific cases we shall investigate. In order to derive a formula for the symplectic structure of TM we shall need a classical formula from the theory of Lie groups which gives the derivative of the exponential map in terms of the derivative of the adjoint representation.

Let U be an open neighbourhood of $0 \in \mathfrak{g}$ such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism and so that there exists an open neighbourhood $V \subset U \cap \mathfrak{m}$ of $0 \in \mathfrak{m}$ which is mapped by $\pi \circ \exp$ diffeomorphically onto an open neighbourhood W of $K = \pi(e) \in M = G/K$, where $\pi : G \to G/K = M$ is the quotient map. This is possible by the existence of local smooth sections (see [17], theorem 3.58). Then $(\pi \circ \exp)_* : TV \to TW$ is a parametrization of TW. Since $TG = G \times \mathfrak{g}$ we have $TTG = G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ using left translation of G for the identifications. If $\xi \in \mathfrak{m} = T_{\pi(e)}M$ then $T_\xi TM$ can be considered as a linear subspace of $\{e\} \times \{\xi\} \times \mathfrak{g} \times \mathfrak{g}$.

We shall identify this subspace as follows. Let $(v, w) \in T_{(0,\xi)}(V \times \mathfrak{m})$ and $\gamma(t) = (tv, \xi + tw)$, so that $\gamma(0) = (0, \xi)$ and $\dot\gamma(0) = (v, w)$. Then (v, w) is mapped in $\{e\} \times \{\xi\} \times \mathfrak{g} \times \mathfrak{g}$ to the velocity for $t = 0$ of the curve $(\exp(tv), (L^{-1}_{\exp(tv)} \circ \exp)_{*tv}(\xi + tw))$.

In order to calculate this derivative we shall need the following formula.

Lemma 3.3.1. *If G is a Lie group with Lie algebra $\mathfrak{g}$, then*

$$(L^{-1}_{\exp X})_{*\exp X} \circ \exp_{*X} = \frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$$

for every $X \in \mathfrak{g}$.

Proof. Let $X_0 \in \mathfrak{g}$ and $X : I \to \mathfrak{g}$ be a smooth curve, where $I \subset \mathbb{R}$ is an open interval, such that $X(0) = X_0$. We also consider $a(s) = \exp(sX_0)$, $s \in \mathbb{R}$ and $B : \mathbb{R} \to \mathfrak{g}$ be the smooth curve defined by

$$B(s) = ({L^{-1}}_{a(s)})_{*a(s)}(\frac{d}{dt}\Big|_{t=0} \exp(sX(t))) = \frac{d}{dt}\Big|_{t=0} \exp(-sX_0)\exp(sX(t)).$$

We shall compute the derivative B'. We have

$$B(s+h) - B(s)$$

$$= \frac{d}{dt}\Big|_{t=0} \exp(-sX_0 - hX_0)\exp(hX(t))\exp(sX(t))\exp\left(sX(t)\right)^{-1}\exp(sX_0)$$

$$= \frac{d}{dt}\Big|_{t=0} \exp(-(s+t)X_0)\exp(hX(t))\exp(sX_0)$$

$$= (L_{(a(s+h)^{-1})})_*(R_{a(s)})_*(\frac{d}{dt}\Big|_{t=0} \exp(hX(t)))$$

and so

$$B'(s) = \lim_{h\to 0}\frac{1}{h}(B(s+h) - B(s)) = \mathrm{Ad}_{a(s)^{-1}}(\lim_{h\to 0}\frac{1}{h}\frac{d}{dt}\Big|_{t=0} \exp(hX(t))).$$

On the other hand,

$$\lim_{h\to 0}\frac{1}{h}\frac{d}{dt}\Big|_{t=0} \exp(hX(t)) = \lim_{h\to 0}\frac{1}{h}\exp_{*hX_0}(hX'(0))$$

$$= \lim_{h\to 0}\exp_{*hX_0}(X'(0)) = X'(0).$$

Hence $B'(s) = \mathrm{Ad}_{a(s)^{-1}}(X'(0))$. Integrating,

$$\int_0^1 \mathrm{Ad}_{a(s)^{-1}} ds = \int_0^1 e^{-s(\mathrm{ad}_{X_0})} ds$$

$$= -\frac{1}{\mathrm{ad}_{X_0}} \int_0^1 e^{-s(\mathrm{ad}_{X_0})} d(-s(\mathrm{ad}_{X_0})) = -\frac{1}{\mathrm{ad}_{X_0}}(e^{-\mathrm{ad}_{X_0}} - I) = \frac{I - e^{-\mathrm{ad}_{X_0}}}{\mathrm{ad}_{X_0}}.$$

Since $B(0) = 0$, it follows that

$$B(1) = \frac{I - e^{-\mathrm{ad}_{X_0}}}{\mathrm{ad}_{\mathrm{X}_0}} X'(0).$$

But $B(1) = (L^{-1}_{\exp X_0})_{*\exp X_0}(\frac{d}{dt}|_{t=0} \exp(X(t)))$ and so

$$\left.\frac{d}{dt}\right|_{t=0} \exp(-X_0)\exp(X(t)) = \frac{I - e^{-\mathrm{ad}_{X_0}}}{\mathrm{ad}_{X_0}} X'(0).$$

Taking the particular curve $X(t) = X + tY$ for $X, Y \in \mathfrak{g}$ we get

$$(L^{-1}_{\exp X})_{*\exp X} \circ \exp_{*X}(Y) = \frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} Y$$

or

$$(L^{-1}_{\exp X})_{*\exp X} \circ \exp_{*X} = \frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$$

for every $X \in \mathfrak{g}$. $\square$

Applying Lemma 3.3.1 in our case,

$$(L^{-1}_{\exp(tv)})_{*\exp(tv)} \circ \exp_{*tv} = \frac{I - e^{-\mathrm{ad}_{tv}}}{\mathrm{ad}_{tv}}.$$

Since

$$\frac{1 - e^{-s}}{s} = \frac{1}{s}(1 - \sum_{n=0}^{\infty} \frac{(-s)^n}{n!}) = 1 - \frac{1}{2}s + O(s^2)$$

it follows that

$$(L^{-1}_{\exp(tv)})_{*\exp(tv)} \circ \exp_{*tv} = I - \frac{1}{2}t\mathrm{ad}_v + O(s^2)$$

and therefore

$$\left.\frac{d}{dt}\right|_{t=0} (L^{-1}_{\exp(tv)} \circ \exp_*)_{tv}(\xi + tw)$$

$$= \left.\frac{d}{dt}\right|_{t=0} (\xi + tw) - \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} t\mathrm{ad}_v(\xi + tw) = w - \frac{1}{2}\mathrm{ad}_v(\xi) = w - \frac{1}{2}[v, \xi].$$

We conclude that $T_\xi TM$ is identified with the linear subspace

$$\{(e, \xi, v, -\frac{1}{2}[v, \xi] + w), (v, w) \in T_{(e,\xi)}(V \times \mathfrak{m})\}$$

of $\{e\} \times \{\xi\} \times \mathfrak{g} \times \mathfrak{g}$.

Since M satisfies property (A), the (pseudo-Riemannian) exponential map of M is $\mathrm{Exp} = \pi \circ \exp|_v$ and defines normal coordinates centerd at $\pi(e)$. If we take a basis $\xi_1, ...\xi_n \in \mathfrak{m}$ and define normal local coordinates by $(x_1, ...x_n) \mapsto \mathrm{Exp} \sum_{i=1}^{n} x_i \xi_i$, then the coefficients (g_{ij}) of the metric $\langle \cdot, \cdot \rangle$ satisfy $\frac{\partial g_{ij}}{\partial x_r}(0) = 0$, $i, j, r = 1, ..., n$.

From the above follows that the symplectic 2-form on $T_\xi TM$ becomes the standard symplectic structure defined by B so that it is given by the formula

$$\omega_\xi((e, \xi, v_1, -\frac{1}{2}[v_1, \xi] + w_1), (e, \xi, v_2, -\frac{1}{2}[v_2, \xi] + w_2)) = B(v_1, w_2) - B(v_2, w_1).$$

Using the action of G on TM, which is symplectic, the symmetric 2-form at an arbitary point $g\xi \in TM$, $\xi \in \mathfrak{m}$, $g \in G$ given by the formula, that we state as a proposition.

Proposition 3.3.2. *Let M=G/K be a homogeneous space satisfying property* (A). *Identifying $T_\xi TM$, $\xi \in \mathfrak{m}$, by means of the exponential map with the subspace*

$$\{(v_1 - \frac{1}{2}[v_1, \xi] + w)_{(e,\xi)} \mid v, w \in \mathfrak{m}\}$$

of $G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, then the symplectic structure of $TM \in g\xi$ is given by

$$\omega_{g\xi}(g_{*\xi}(v_1, -\frac{1}{2}[v_1, \xi] + w_1), g_{*\xi}(v_2, -\frac{1}{2}[v_2, \xi] + w_2)) = B(v_1, w_2) - B(v_2, w_1). \quad \square$$

Remark. The horizontal subspace of $T_{g\xi}M$ is $\{g_{*\xi}(v, -\frac{1}{2}[v, \xi]) \mid v \in \mathfrak{m}\}$ and the vertical subspace is $\{g_{*\xi}(0, w) \mid w \in m\}$.

3.4 Hamiltonian systems on the tangent bundle of homogeneous spaces

As an application of the above, we shall compute the Hamiltonian vector field of a G-invariant function. If $f : TM \to \mathbb{R}$ is a G-invariant smooth function, then the smooth function $h : \mathfrak{m} \to \mathbb{R}$ with $h(\xi) = f(\pi_{*e}(\xi))$ is Ad_K-invariant, because $h(\mathrm{Ad}_k(\xi)) = f(\pi_{*e}(\mathrm{Ad}_k(\xi))) = f((L_k)_{*\pi(e)}\pi_{*e}(\xi)) = f(\pi_{*e}(\xi)) = h(\xi)$ for all $k \in K$. Conversely, if $h : \mathfrak{m} \to \mathbb{R}$ is an Ad_K-invariant smooth function then we can define the smooth function $f : TM \to \mathbb{R}$ with $f(g\xi) = h(\xi)$ for $g \in G$, $\xi \in \mathfrak{m}$, where as usual $g\xi = (L_g)_{*\pi(e)} \circ \pi_{*e}(\xi)$ which is obviously G-invariant. So G-invariant functions f on TM are in 1-1 correspondence with Ad_K-invariant functions $h : \mathfrak{m} \to \mathbb{R}$. For $h : \mathfrak{m} \to \mathbb{R}$ we consider $\mathrm{grad}h(\xi)$, $\xi \in \mathfrak{m}$ with respect to $B|_{\mathfrak{m}\times\mathfrak{m}}$.

Proposition 3.4.1. *Let $h : \mathfrak{m} \to \mathbb{R}$ be Ad_K-invariant and $f : TM \to \mathbb{R}$ the G-invariant Hamiltonian defined by h. The Hamiltonian vector field X_f of f is given by the formula*

$$X_f(g\xi) = g_{*\xi}(v_1, -\frac{1}{2}[v_1, \xi] + w_1)$$

where $v_1 = \mathrm{grad}h(\xi)$ *and* $w_1 = -\frac{1}{2}[\mathrm{grad}h(\xi), \xi]_{\mathfrak{m}}$.

If $f_1, f_2 : TM \to \mathbb{R}$ *are two invariant Hamiltonians defined by* $h_1, h_2 : \mathfrak{m} \to \mathbb{R}$ *respectively, then their Poisson bracket is*

$$\{f_1, f_2\}(g\xi) = -B([\mathrm{grad}h_1(\xi), \mathrm{grad}h_2(\xi)], \xi).$$

Proof. From the calculations of the previous section, for all $v, w \in \mathfrak{m}$ we have that

$$df(g\xi)(g_*(v, -\frac{1}{2}[v,\xi] + w)) = \frac{d}{dt}\Big|_{t=0} (f \circ (L_g)_* \circ \pi_* \circ \exp_{*tv})(\xi + tw)$$

$$= \frac{d}{dt}\Big|_{t=0} f \circ (L_{g\exp(tv)})_* \circ ({L^{-1}}_{\exp(tv)})_{*\exp(tv)} \circ \pi_* \circ \exp_{*tv}(\xi + tw)$$

$$= \frac{d}{dt}\Big|_{t=0} h(\xi + tw - \frac{1}{2}t[v,\xi]_{\mathfrak{m}} + O(t^2)) = B(\mathrm{grad}h(\xi), w) - B(-\frac{1}{2}[\mathrm{grad}h(\xi), \xi], v).$$

The gradient is considered with respect to B. The formula for the Poisson bracket follows by setting $v = \mathrm{grad}h_2(\xi)$, $w = -\frac{1}{2}[\mathrm{grad}h_2(\xi), \xi]_{\mathfrak{m}}$. □

As an example we can take $h(\xi) = \frac{1}{2}B(\xi, \xi)$ and obtain the equation of the geodesic vector field on TM: $X_f(g\xi) = g_{*\xi}(\xi, 0)$.

G-invariant Hamiltonian systems on TM have many first integrals such as all functions $f = h \circ P$ for some smooth function $h : \mathfrak{g} \to \mathbb{R}$ and $P : TM \to \mathfrak{g}$ the momentum map which as we know is given by the formula $P(g\xi) = \mathrm{Ad}_g(\xi)$ (See Corollary 4.1.3 below).

We compute the Hamiltonian vector field of $f = h \circ P$.

Proposition 3.4.2. *Let M=G/K be a homogeneous space satisfying property (A) and* $h : \mathfrak{g} \to \mathbb{R}$. *Then the Hamiltonian vector field* X_f *of* $f = h \circ P$ *is given by the formula*

$$X_f(g\xi) = g_{*\xi}(v, -\frac{1}{2}[v,\xi] + w)$$

where $v = \mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}$, $w = [\mathrm{Ad}_{g^{-1}}(\zeta), \xi]_{\mathfrak{m}} - \frac{1}{2}[\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}, \xi]_{\mathfrak{m}}$ *and* $\zeta = \mathrm{grad}h(\mathrm{Ad}_g(\xi))$.

Proof. For all $v, w \in \mathfrak{m}$ we compute

$$df(g\xi)(g_* \circ (\pi_*)_{*\xi}(v, -\frac{1}{2}[v,\xi] + w)) = \frac{d}{dt}\Big|_{t=0} h \circ \mathrm{Ad}_{g\exp(tv)}(\xi + tw - \frac{1}{2}t[v,\xi] + \mathrm{O}(t^2))$$

$$= dh(\mathrm{Ad}_g(\xi))\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g\exp(tv)}(\xi + tw - \frac{1}{2}t[v,\xi]_{\mathfrak{m}} + \mathrm{O}(t^2))$$

$$= dh(\mathrm{Ad}_g(\xi))\mathrm{Ad}_g(\mathrm{ad}_v(\xi) + w - \frac{1}{2}[v,\xi]) \text{ (by Leibniz Rule)}$$

$$= B(\mathrm{grad}h(\mathrm{Ad}_g(\xi)), \mathrm{Ad}_g([v,\xi] + w - \frac{1}{2}[v,\xi]_{\mathfrak{m}}))$$

$$= B(\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi))), w) + B(\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi))), [v,\xi] - \frac{1}{2}[v,\xi]_{\mathfrak{m}})$$

$$= B(\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi)), w) + B(\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi)))$$

$$-\frac{1}{2}\mathrm{Ad}_{g_{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi)))_{\mathfrak{m}}, [v, \xi]) = B(\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi)))_m, w)$$

$$-B([\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h(\mathrm{Ad}_g(\xi))) - \frac{1}{2}\mathrm{Ad}_{g^{-1}}(\mathrm{grad}h\mathrm{Ad}_g(\xi)))_m, \xi], v).$$

It follows that

$$X_f(g\xi) = g_{*\xi}(v, -\frac{1}{2}[v, \xi] + w),$$

where $v = \mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}$, $w = [\mathrm{Ad}_{g^{-1}}(\zeta), \xi]_{\mathfrak{m}} - \frac{1}{2}[\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}, \xi]_{\mathfrak{m}}$ and $\zeta = \mathrm{grad}h(\mathrm{Ad}_g(\xi))$.□

Note that

$$-\frac{1}{2}[v, \xi] + w = -\frac{1}{2}[\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}, \xi] + [\mathrm{Ad}_{g^{-1}}(\zeta), \xi]_{\mathfrak{m}} - \frac{1}{2}[\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}, \xi]$$

$$= -[\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}, \xi] + [\mathrm{Ad}_{g^{-1}}(\zeta), \xi] = [\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{k}}, \xi].$$

From the description of $T_{g\xi}TM$ and the above formula, it follows that $X_f(g\xi)$ is equal to the value at $g\xi$ of the fundamental vector field of the action of G on TM which corresponds to $\zeta = \mathrm{grad}h(\mathrm{Ad}_g(\xi)) \in \mathfrak{g}$.

Proposition 3.4.3. *If $g(t)$ is the solution of the ordinary differential equation*

$$\dot{g}(t) = (R_{g(t)})_{*e}(\mathrm{grad}h(P(g(t)g\xi))$$

on G with $g(0) = e$, then the integral curve of the Hamiltonian vector field X_f passing through $g\xi$ is locally $g(t)g\xi$.

Proof. Putting $\zeta = \mathrm{grad}h(P(g(t)g\xi))$ we compute

$$\frac{d}{dt}(g(t)g\xi) = (R_{g(t)})_{*e}(\zeta)g\xi = \frac{d}{ds}\Big|_{s=0}\exp(s\zeta)g(t)g\xi$$

$$= \tilde{\phi}_*(\zeta)(g(t)g\xi) = X_f(g(t)g\xi). \quad \square$$

If h is Ad_G-invariant, then $h \circ \mathrm{Ad}_g = h$ for every $g \in G$ and differentiating $\mathrm{grad}h(\mathrm{Ad}_g(\xi)) = \mathrm{Ad}_g(\mathrm{grad}h(\xi))$, since B is Ad_G-invariant. So in this case the above differential equation reduces to

$$\dot{g}(t) = (R_{g(t)})_{*e}(\mathrm{grad}h(\mathrm{Ad}_{g(t)}(\mathrm{Ad}_g(\xi)))) = (R_{g(t)})_{*e}\mathrm{Ad}_{g(t)}(\mathrm{grad}h(\mathrm{Ad}_g(\xi)))$$

$$= (R_{g(t)})_{*e}(R^{-1}_{g(t)})_{*g(t)}(L_{g(t)})_{*e}(\mathrm{grad}h(\mathrm{Ad}_g(\xi))) = (L_{g(t)})_{*e}(\mathrm{grad}h(\mathrm{Ad}_g(\xi))).$$

The solution of this equation with $g(0) = e$ is

$$g(t) = \exp(t(\mathrm{grad}h(\mathrm{Ad}_g(\xi)))), t \in \mathbb{R}$$

and so the integral curve of X_f through $g\xi$ becomes

$$g(t)g\xi = g\mathrm{exp}t(\mathrm{grad}h(\xi))\xi,$$

because $g(t) = g\exp(t\mathrm{grad}h(\xi))g^{-1}$ and so $g(t)g = g\exp(t\mathrm{grad}h(\xi))$. (See [17], page 114)

We remark that in case $\mathrm{grad}h(\xi) \in \mathfrak{m}$, then the projection of this integral curve to M is the geodesic emenating from $\pi(g)$ with initial velocity $g\mathrm{grad}h(\xi) \in T_{\pi(g)}M$.

Example 3.4.4. Let $\mathfrak{g}'$ be a non-degenerate Lie subalgebra of $\mathfrak{g}$ which is integrated to a connected Lie subgroup G' of G with corresponding B-orthogonal projection $\pi' : \mathfrak{g} \to \mathfrak{g}'$. Let $h' : \mathfrak{g}' \to \mathbb{R}$ be a smooth $\mathrm{Ad}_{G'}$-invariant function. Since π' is $\mathrm{Ad}_{G'}$-equivariant, $h^{'} \circ \pi' : \mathfrak{g} \to \mathbb{R}$ is $\mathrm{Ad}_{G'}$-invariant. Also $\mathrm{grad}(h' \circ \pi')(\xi) = \mathrm{grad}h'(\xi)$ is tangent to $\mathfrak{g}'$ for $\xi \in \mathfrak{g}'$. For $f = h' \circ \pi' \circ P : TM \to \mathbb{R}$, it follows from the above that the integral curve of X_f through $g\xi$, for $g \in G'$, $\xi \in \mathfrak{g}'$, is the image of the one-parameter subgroup $\{\exp t\zeta \mid t \in \mathbb{R}\}$ in G' with $\zeta = \mathrm{grad}h'(\pi'(\mathrm{Ad}_g)(\xi)))$.

Chapter 4

Lie subalgebras and Integrability

4.1 First integrals in involution from non-degenerate Lie subalgebras

Let G be a Lie group and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an Ad_G-invariant, non-degenerate, symmetric, bilinear form on the Lie algebra $\mathfrak{g}$ on G. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using B so that $\xi \in \mathfrak{g}$ corresponds to $B(\xi, \cdot) \in \mathfrak{g}^*$. Recall that the Poisson bracket on the Poisson manifold $\mathfrak{g}^*$ is given by

$$\{f, g\}(\mu) = \mu([df(\mu), dg(\mu)])$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ using the natural identification $\mathfrak{g}^{**} \cong \mathfrak{g}$ defined by evaluation. The Poisson bracket in $\mathfrak{g}^*$ transforms to the Poisson bracket in $\mathfrak{g}$ defined by

$$\{h_1, h_2\}(\xi) = B(\xi, [\mathrm{grad}h_1(\xi), \mathrm{grad}h_2(\xi)])$$

for $\xi \in \mathfrak{g}$ and $h_1, h_2 \in C^\infty(\mathfrak{g})$, where the gradients are considered with respect to B.

A smooth function $h : \mathfrak{g} \to \mathbb{R}$ is Ad_G-invariant by definition if $h(\mathrm{Ad}_g(\xi)) = h(\xi)$ for every $\xi \in \mathfrak{g}$ and $g \in G$. Equivalently

$$\frac{d}{dt}|_{t=0} h(\mathrm{Ad}_{\exp(tX)}(\xi)) = 0,$$

for every $X, \xi \in \mathfrak{g}$. By the chain rule and the skew symmetry of $\mathrm{ad}_\mathfrak{g}$ (with respect to B), since B is Ad_G-invariant, we have

$$\frac{d}{dt}\bigg|_{t=0} h(\mathrm{Ad}_{\exp(tX)}(\xi)) = dh(\xi)(\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tX)}(\xi)) = dh(\xi)(\mathrm{ad}_X)(\xi))$$
$$= dh(\xi)([X, \xi]) = B(X, \xi], \mathrm{grad}h(\xi)) = B(X, [\xi, \mathrm{grad}h(\xi)]).$$

So h is Ad_G-invariant if and only if $[\xi, \mathrm{grad}h(\xi)] = 0$ for all $\xi \in \mathfrak{g}$, since B is non-degenerate.

Suppose now that $\mathfrak{g}' \subset \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ so that $B|_{\mathfrak{g}' \times \mathfrak{g}'}$ is non-degenerate. Then $\mathfrak{g}'$ is a direct summand of $\mathfrak{g}$ and so there is a B-orthogonal projection $\pi' : \mathfrak{g} \to \mathfrak{g}'$. Let $G' \leq G$ be the connected Lie subgroup of G to which $\mathfrak{g}'$ integrates. Let also $\mathfrak{g}''$ be another non-degenerate with respect to B subalgebra of $\mathfrak{g}$ and let $\pi'' : \mathfrak{g} \to \mathfrak{g}''$ be the corresponding B-orthogonal projection. If $h' : \mathfrak{g}' \to \mathbb{R}$ is an $\mathrm{Ad}_{G'}$-invariant smooth function and $h'' : \mathfrak{g}'' \to \mathbb{R}$ is an $\mathrm{Ad}_{G''}$-invariant smooth function, the Poisson bracket of $h' \circ \pi'$ and $h'' \circ \pi''$ is

$$\{h' \circ \pi', h'' \circ \pi''\}(\xi) = B(\xi, [\mathrm{grad}(h' \circ \pi')(\xi), \mathrm{grad}(h'' \circ \pi'')(\xi)])$$

$$= B(\xi, [\mathrm{grad}h'(\pi'(\xi)), \mathrm{grad}h''(\pi''(\xi)]) = -B([\mathrm{grad}h'(\pi'(\xi)), \xi], \mathrm{grad}h''(\pi''(\xi)))$$

$$= B([\xi, \mathrm{grad}h'(\pi'(\xi))], \mathrm{grad}h''(\pi''(\xi))) = B([\xi_{\mathfrak{g}'^{\perp}}, \mathrm{grad}h'(\pi'(\xi))], \mathrm{grad}h''(\pi''(\xi)))$$

(since h' is $\mathrm{Ad}_{G'}$-invariant)

$$= B(\xi_{\mathfrak{g}'^{\perp}}, [\mathrm{grad}h'(\pi'(\xi)), \mathrm{grad}h''(\pi''(\xi)])$$

So if $[\mathfrak{g}', \mathfrak{g}''] \subset \mathfrak{g}'$, then $\{h' \circ \pi', h'' \circ \pi''\} = 0$. This holds in particular if $\mathfrak{g}'' \subset \mathfrak{g}'$. The above prove the following.

Proposition 4.1.1. *If $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset .... \subset \mathfrak{g}_k \subset \mathfrak{g}_{k+1} = \mathfrak{g}$ is a chain of non-degenerate (with respect to B) Lie subalgebras of $\mathfrak{g}$ and $h_i \in C^\infty(\mathfrak{g}_i)$, $1 \le i \le k+1$ are invariant functions with respect to the corresponding adjoint representations then $h_i \circ \pi_i \in C^\infty(\mathfrak{g})$, $1 \le i \le k+1$ are all in involution, where $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ is the B-orthogonal projection for every $1 \le i \le k+1$.* □

Lemma 4.1.2. *Let (M, ω) be a symplectic manifold with a Poisson action of the Lie group G on M. Let $\mu : M \to \mathfrak{g}^*$ be the corresponding momentum map. If $h_1, h_2 : \mathfrak{g}^* \to \mathbb{R}$ are smooth functions, then*

$$\{h_1 \circ \mu, h_2 \circ \mu\} = \{h_1, h_2\} \circ \mu.$$

Proof. For every smooth function $h : \mathfrak{g}^* \to \mathbb{R}$ and $p \in M$ we denote by $X_h(p) \in \mathfrak{g}$ the dual of $dh(\mu(p))$ under the natural pairing of $\mathfrak{g}^{**}$ and $\mathfrak{g}$ by evaluation which induces the natural isomorphism $\mathfrak{g}^{**} \cong \mathfrak{g}$. For $v \in T_pM$ we have

$$d(h \circ \mu)(p)(v) = dh(\mu(p)) \cdot \mu_{*p}(v) = \mu_{*p}(v)(X_h(p))$$

and on the other hand

$$d\rho(X_h(p))(q) \cdot v = \mu_{*q}(v)(X_h(p))$$

for every $q \in M$ and so

$$d\rho(X_h(p))(p) \cdot v = \mu_{*p}(v)(X_h(p)),$$

where $\rho : \mathfrak{g} \to C^\infty(M)$ is the linear lift of the action. Therefore,

$$\{h_1 \circ \mu, h_2 \circ \mu\}(p) = \{\rho(X_{h_1}(p)), \rho(X_{h_2}(p))\}(p) = \rho([X_{h_1}(p), X_{h_2}(p)])(p)$$

$$= \mu(p)([X_{h_1}(p), X_{h_2}(p)]) = \{h_1, h_2\} \circ \mu(p). \quad \square$$

Corollary 4.1.3. *Let (M, ω) be a symplectic manifold with a Poisson action of the Lie group G on M. Let $\mu : M \to \mathfrak{g}^*$ be the momentum map. If $F : M \to \mathbb{R}$ is G-invariant smooth function, then for every $h \in C^\infty(\mathfrak{g}^*)$*
(i) $h \circ \mu$ is a first integral of the Hamiltonian vector field X_F and

(ii) for every $h_1, h_2 \in C^\infty(\mathfrak{g}^*)$ such that $\{h_1, h_2\} = 0$ we also have $\{h_1 \circ \mu, h_2 \circ \mu\} = 0$. $\square$

Combining the above we arrive at the following.

Corollary 4.1.4. *Let (M, ω) be a symplectic manifold with a Poisson action of the Lie group G on M with momentum map $\mu : M \to \mathfrak{g}^*$. Suppose that there exists an Ad_G-invariant, non-degenerate, symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$ of G. If*

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_k \subset \mathfrak{g}_{k+1} = \mathfrak{g}$$

is a chain of non-degenerate (with respect to B) Lie subalgebras of $\mathfrak{g}$ and $h_i \in C^\infty(\mathfrak{g}_i)$, $1 \leq i \leq k+1$ and Ad-invariant functions, then $h_i \circ \pi_i \circ \mu$, $1 \leq i \leq k+1$ are first integrals in involution of X_F for every G-invariant Hamiltonian $F : M \to \mathbb{R}$, where $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ is the B-orthogonal projection, $1 \leq i \leq k+1$ and with respect to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ defined by B. $\square$

Let now (M, ω) be a symplectic manifold with a Poisson action $\phi : G \times M \to M$ of the Lie group G on M. Let $\mu : M \to \mathfrak{g}^*$ be the momentum map which we assume to be equivariant. That holds automatically, if G is connected. We also assume that there exists an Ad_G-invariant, non-degenerate, symplectic bilinear firm $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Then, B induces a G-equivariant natural isomorphism $\tilde{B} : \mathfrak{g} \cong \mathfrak{g}^*$ as usual by $\tilde{B}(\xi) = B(\xi, \cdot)$, $\xi \in \mathfrak{g}$. Indeed,

$$(\mathrm{Ad}_g^* \circ \tilde{B}(X))Y = \tilde{B}(X)(\mathrm{Ad}_{g^{-1}}(Y)) = B(X, \mathrm{Ad}_{g^{-1}}(Y))$$

$$= B(\mathrm{Ad}_g(X), Y) = (\tilde{B} \circ \mathrm{Ad}_g(X))Y$$

for every $X, Y \in \mathfrak{g}$ and $g \in G$. Let $\mathfrak{g}' \subset \mathfrak{g}$ be a non-degenerate (with respect to B) Lie subalgebra with orthogonal projection $\pi : \mathfrak{g} \to \mathfrak{g}'$, which is integrated to a Lie subgroup G'. If $h : \mathfrak{g}' \to \mathbb{R}$ is a smooth function which is $\mathrm{Ad}_{G'}$-invariant then obviously the composition

$$F : M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\tilde{B}^{-1}} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}' \xrightarrow{h} \mathbb{R}$$

is a G'-invariant smooth function.

4.2 Integrability of the geodesic flow of the real Grassmann manifolds

The real Grassmann manifold of p-planes in $\mathbb{R}^{n+1}$ is the homogeneous symmetric space

$$G_{p,q}(\mathbb{R}) = SO(n+1, \mathbb{R})/S(O(p, \mathbb{R}) \times O(q, \mathbb{R}))$$

$$= O(n+1, \mathbb{R})/(O(p, \mathbb{R}) \times O(q, \mathbb{R}))$$

where $p+q=n+1$, normalized by $p \leq q$. Recall (See the Appendix 5.2) that the Killing form on the Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1,\mathbb{R})$ is given by the formula

$$\langle \xi, \eta \rangle = (n-1) \cdot \mathrm{Tr}(\xi \cdot \eta)$$

and is Ad-invariant, non-degenerate, symmetric and bilinear. We denote $G = SO(n+1,\mathbb{R})$ for brevity. The orthogonal map $S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $S(u,v) = (u,-v)$ for $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$ induces the involution $\sigma : G \to G$ by conjugation, i.e. $\sigma(A) = SAS^{-1}$ with fixed point set $S(O(p,q) \times O(q,\mathbb{R}))$. Therefore

$$B(\xi,\eta) = -\frac{1}{2}\mathrm{Tr}(\xi \cdot \eta) = \frac{1}{2}\mathrm{Tr}(\xi \eta^t)$$

is an Ad_G-invariant non-degenerate (in particular positive definite), symmetric, bilinear form on $\mathfrak{g}$. Let $K = S(O(p) \times O(p))$ with corresponding Lie algebra $\mathfrak{k}$. Note that $B|_{\mathfrak{k}\times\mathfrak{k}}$ is also non-degenerate and so $\mathfrak{k}$ has a B-orthogonal complement $\mathfrak{m}$ in $\mathfrak{g}$, which is also non-degenerate with regard to B and is computed as follows. If

$$\eta = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{k},$$

then $\mathrm{Tr}(\xi \cdot \eta) = \mathrm{Tr}(AH_1) + \mathrm{Tr}(BH_4)$. So $B(\xi,\eta) = 0$ for all $\xi \in \mathfrak{so}(q,\mathbb{R})$, which implies that $H_1 = 0$ and $H_2 = 0$. Hence

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \mid X \in \mathbb{R}^{p\times q} \right\}.$$

The adjoint representation of $S(O(p,\mathbb{R}) \times O(q,\mathbb{R}))$ on $\mathfrak{m}$ is

$$\mathrm{Ad}_h \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & UXV^{-1} \\ -VX^tU^{-1} & 0 \end{pmatrix}$$

for $h = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \in S(O(p,\mathbb{R}) \times O(q,\mathbb{R})) = K$.

On $\mathfrak{g}$ we consider the polynomial functions $f_k : \mathfrak{g} \to \mathbb{R}$, $k = 1, 2, ..., p$ defined by

$$f_k(\xi) = -\frac{1}{4k}\mathrm{Tr}(\xi^{2k})$$

which are Ad_G-invariant. In particular their restriction on $\mathfrak{m}$ are Ad_K-invariant and induce G-invariant smooth (actually real analytic) functions on $TG_{p,q}(\mathbb{R})$. We shall give a formula for their gradients using the following lemmas.

Lemma 4.2.1. *Let $p, q \in \mathbb{N}$ and $A \in \mathbb{R}^{p\times q}$.The positive semidefinite symmetric matrices $A^tA \in \mathbb{R}^{q\times q}$ and $AA^t \in \mathbb{R}^{p\times p}$ have the same non-zero (e.g. positive) eigenvalues.*

Proof. Let $\lambda \geq 0$ be an eigenvalue of A^tA. There exists $v \in \mathbb{R}^q$, $v \neq 0$ such that $A^tAv = \lambda v$. Then $AA^t(Av) = \lambda Av$. If $\lambda > 0$, necessarily $Av \neq 0$ and therefore λ is an eigenvalue of AA^t. Symmetrically, if $\mu > 0$ is an eigenvalue of AA^t, there exists $u \in \mathbb{R}^p$, $u \neq 0$ such that $AA^tu = \mu u$ and therefore $A^tA(A^tu) = \mu A^tu$. Again $A^tu \neq 0$, since $\mu \neq 0$ and so μ is an eigenvalue of A^tA. ☐

Lemma 4.2.2. *Let $p, q \in \mathbb{N}$ and $A \in \mathbb{R}^{p\times q}$. Let*

$$\xi = \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \in \mathbb{R}^{(p+q)\times(p+q)}.$$

If $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of ξ, then λ is purely imaginary and $-\lambda^2$ is an eigenvalue of AA^t.

Proof. There are $u \in \mathbb{C}^p$ and $v \in \mathbb{C}^q$, not both zero, such that

$$\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow Av = \lambda u, \; -A^t u = \lambda v.$$

If $v \neq 0$, then $A^tAv = \lambda A^t u = -\lambda^2 v$ and $-\lambda^2$ is an eigenvalue of A^tA. If $u \neq 0$, then $AA^t u = -\lambda Av = -\lambda^2 u$ and $-\lambda^2$ is an eigenvalue of AA^t. Since AA^t and A^tA are positive semidefinite, symmetric, they have real non-negative eigenvalues and by Lemma 4.1.1 they have the same non-zero positive eigenvalues. Since $-\lambda^2 \in \mathbb{R}$ and $-\lambda^2 > 0$, it follows that λ is purely imaginary. $\square$

Lemma 4.2.3. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ad}_A : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be the adjoint to A linear map defined by $\mathrm{ad}_A(X) = [A, X] = AX - XA$. If $\lambda_1, \ldots\lambda_n \in \mathbb{C}$ are the eigenvalues of A then the eigenvalues of ad_A are*

$$\lambda_i - \lambda_j, \quad i, j = 1, 2, \ldots, n.$$

Proof. We recall that A and A^t have the same eigenvalues. Let $u_1, \ldots u_n \in \mathbb{C}$ be distinct eigenvectors of A and $v_1, \ldots v_n \in \mathbb{C}$ of A^t coresponding to $\lambda_1, \ldots, \lambda_n$, respectively. For any $u \in \mathbb{C}^n$ and $v \in (\mathbb{C}^n)^*$ we denote by $u \otimes v$ the element of $\mathbb{C}^{n\times n}$ defined by

$$(u \otimes v)(z) = v(z)u$$

for every $z \in \mathbb{C}^n$. Let $X_{ij} = u_i \otimes v_j$. Now we have $X_{ij} \neq 0$ and

$$AX_{ij}(z) = A((u_i \otimes v_j)(z)) = A(v_j(z)u_i) = v_j(z)A(u_i) = (A(u_i) \otimes v_j)(z) = \lambda_i X_{ij}(z)$$

and

$$X_{ij}A(z) = (u_i \otimes v_j)(A(z)) = v_j(A(z))u_i = (A^t v_j)(z)u_i = \lambda_j v_j(z)u_i = \lambda_j(X_{ij}(z)).$$

So $\mathrm{ad}_A(X_{ij}) = \lambda_i X_{ij} - \lambda_j X_{ij} = (\lambda_i - \lambda_j)X_{ij}$ for all $i, j = 1, 2, \ldots, n$. Therefore $\lambda_i - \lambda_j$, $i, j = 1, 2, \ldots, n$ are the eigenvalues of ad_A. $\square$

Recall (See [13], [12]) that the nullity of a linear map $A : \mathbb{C}^n \to \mathbb{C}^n$, i.e. an element $A \in \mathbb{C}^{n\times n}$, is the multiplicity of 0 as an eigenvalue of A. If $\mathfrak{g}$ is a complex Lie algebra and $A \in \mathfrak{g}$, the rank of $\mathfrak{g}$ is by definition

$$\mathrm{rk}(\mathfrak{g}) = \min\{\text{nullity of } \mathrm{ad}_A \mid A \in \mathfrak{g}\}.$$

The element $A \in \mathfrak{g}$ is called regular if ad_A has minimum nullity i.e. $\mathrm{rk}(\mathfrak{g})$.

Example 4.2.4. If $A \in \mathfrak{gl}(n,\mathbb{C}) = \mathbb{C}^{n\times n}$, the nullity of ad_A is at least n from the previous Lemma 4.2.3 and A is regular if and only if it is exactly n. This is equivalent to saying that A has n distinct eigenvalues. This is obviously an open and dense subset of $\mathbb{C}^{n\times n}$.

Example 4.2.5. Let $p, q \in \mathbb{N}$ with $p \geq q$, $A \in \mathbb{R}^{p\times q}$ and

$$\xi = \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \in \mathfrak{so}(p+q,\mathbb{R}).$$

Then ξ has at most $2q$ non zero (actually $\pm i\sqrt{positive\ eigenvalues\ of\ A^tA}$) eigenvalues. So ad_ξ has at most $4q^2 - 2q$ non-zero eigenvalues and the nullity of ad_ξ is greater or equal than $(p+q)^2 - 4q^2 + 2q$. Hence ξ is a regular element of $\mathfrak{so}(p+q,\mathbb{R})$ if and only if A^tA has q non-zero ($\Leftrightarrow$ positive) and distinct eigenvalues.

Proposition 4.2.6. *The polynomial functions $h_k : \mathfrak{so}(p+q,\mathbb{R}) \to \mathbb{R}$ with*

$$h_k(\xi) = \mathrm{Tr}\xi^{2k},\ \ k = 1, 2, ..., p,$$

where $p, q \in \mathbb{N}$, $p \leq q$, are $SO(p+q,\mathbb{R})$-invariant with gradients

$$\mathrm{grad} h_k(\xi) = -2k\xi^{2k-1},$$

with respect to the metric $\langle X, Y\rangle = \mathrm{Tr}(XY^t)$. Moreover at any ξ in a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}$, their gradients are tangent to $\mathfrak{a}$ and are linearly independent if ξ is a regular element of $\mathfrak{so}(p+q,\mathbb{R})$. So

$$Dh_k(\xi)X = \mathrm{Tr}(-2k\xi^{2k-1}X^t) = \mathrm{Tr}(2k\xi^{2k-1}X),\ \ X \in \mathfrak{so}(p+q,\mathbb{R}).$$

Proof. We observe first that if A, B are square matrices of the same size, then

$$\mathrm{Tr}A^k - \mathrm{Tr}B^k = \mathrm{Tr}((A-B)(A^{k-1} + A^{k-2}B + ... + B^{k-1})).$$

Indeed,

$$(A-B)(A^{k-1} + A^{k-2}B + ... + B^{k-1}) =$$
$$= A^k + A^{k-1}B + ... + AB^{k-1} - BA^{k-1} - BA^{k-2}B - ... - B^k$$

and

$$\mathrm{Tr}(A^{k-1}B) = \mathrm{Tr}(BA^{k-1}),\ \mathrm{Tr}(A^{k-2}B) = \mathrm{Tr}(BA^{k-2}B),$$

e.t.c. Hence the observation.

Applying the above to $\xi, H \in \mathfrak{so}(p+q,\mathbb{R})$ we have

$$\mathrm{Tr}(\xi+H)^k - \mathrm{Tr}\xi^k - k\mathrm{Tr}(H\xi^{k-1})$$

$$= \mathrm{Tr}(H((\xi+H)^{k-1} - \xi^{k-1})) + \mathrm{Tr}((H(\xi+H)^{k-2}\xi - \xi^{k-1})) + ... + 0.$$

For every $l = 0, 1, ..., n-1$ from the Cauchy-Schwarz inequality we have

$$\frac{\left|\mathrm{Tr}(H((\xi+H)^{k-l-1}\xi^l - \xi^{k-1})\right|}{(\mathrm{Tr}(HH^t))^{1/2}}$$

$$\leq (\mathrm{Tr}((\xi+H)^{k-l-1}\xi^l - \xi^{k-1})((\xi^t+H^t)^{k-l-1}(\xi^t)^l - (\xi^t)^{k-1})))^{\frac{1}{2}}$$

and for $H \to 0$ the right hand side tends to

$$(\mathrm{Tr}(\xi^{k-l-1}\xi^l - \xi^{k-1})((\xi^t)^{k-l-1}(\xi^t)^l - (\xi^t)^{k-1}))^{\frac{1}{2}} = 0.$$

In particular the above calculation shows that

$$\mathrm{D}h_k(\xi)X = 2k\mathrm{Tr}(X\xi^{2k-1}) = -2k\mathrm{Tr}(X^t\xi^{2k-1})$$

$$= -2k\mathrm{Tr}(\xi^{2k-1}X^t) = \left\langle -2k\xi^{2k-1}, X\right\rangle$$

and hence $\mathrm{grad}h_k(\xi) = -2k\xi^{2k-1}$.

In the particular case where

$$\xi = \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \in \mathfrak{m},\ X \in \mathbb{R}^{p\times q}$$

we have

$$\xi^2 = \begin{pmatrix} -XX^t & 0 \\ 0 & X^tX \end{pmatrix} \text{ and } \xi^{2k} = (-1)^k \begin{pmatrix} (XX^t)^k & 0 \\ 0 & (X^tX)^k \end{pmatrix}.$$

So

$$\xi^{2k-1} = (-1)^k \begin{pmatrix} 0 & (XX^t)^{k-1}X \\ -((XX^t)^{k-1}X)^t & 0 \end{pmatrix} \in \mathfrak{m},$$

because

$$(X^tX...X^tX)X^t = X^t(XX^t...XX^t) = X^t(XX^t)^{k-1} = ((XX^t)^{k-1}X)^t.$$

Actually, if $\xi \in \mathfrak{a}$, where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$, then $[\xi,\zeta] = 0$, i.e. $\xi\zeta = \zeta\xi$ and since $[\xi^{2k-1},\zeta] = \xi^{2k-1}\zeta - \zeta\xi^{2k-1} = 0$ for every $\zeta \in \mathfrak{a}$ we must necessarily have $\xi^{2k-1} \in \mathfrak{a}$, because $\mathfrak{a}$ is maximal. In other words $\mathrm{grad}h_k(\xi) \in \mathfrak{a}$ for all $\xi \in \mathfrak{a}$.

Now let $\lambda_1, .., \lambda_p$ be such that $\sum_{k=1}^{p} \lambda_k \xi^{2k-1} = 0$. Then $\sum_{k=1}^{p} \lambda_k \xi^{2k} = 0$ and so $\sum_{k=1}^{p} (-1)^k \lambda_k (X^tX)^k = 0$. If $a_1, ...a_p \geq 0$ are the eigenvalues of X^tX, there exists $R \in SO(p,\mathbb{R})$ such that

$$R^{-1}(X^tX)R = \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_p \end{pmatrix}$$

and hence $R^{-1}(X^tX)^kR = \begin{pmatrix} a_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_p^k \end{pmatrix}$. Substituting,

$$\sum_{k=1}^{p} (-1)^k \lambda_k \begin{pmatrix} a_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_p^k \end{pmatrix} = 0$$

or equivalently

$$\sum_{k=1}^{p}(-1)^k a_j^k \lambda_k = 0, \;\; j = 1, 2, ...p.$$

It follows that $\lambda_1 = ... = \lambda_p = 0$, if

$$0 \neq \begin{vmatrix} a_1 & a_1^2 & \cdots & a_1^p \\ a_2 & a_2^2 & \cdots & a_2^p \\ \vdots & \vdots & \ddots & \vdots \\ a_p & a_p^2 & \cdots & a_p^p \end{vmatrix} = a_1 a_2 ... a_p \begin{vmatrix} 1 & a_1 & \cdots & a_1^{p-1} \\ 1 & a_2 & \cdots & a_2^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_p & \cdots & a_p^{p-1} \end{vmatrix} = a_1 a_2 ... a_p \prod_{i<j}(a_j - a_i)$$

and $\xi^1, \xi^2, ..., \xi^{2p-1}$ are linearly independent. This means that if ξ is regular then $\mathrm{D}h_k(\xi)$, equivalently $\mathrm{grad}h_k(\xi)$, $k = 1, 2, ..., p$ are linearly independent. □

Remark. If on $\mathfrak{so}(p+q, \mathbb{R})$ we consider the metric $B(X, Y) = \frac{1}{2}\mathrm{Tr}(XY^t)$ and the functions $f_k(\xi) = -\frac{1}{4k}\mathrm{Tr}\xi^2 k$, $k = 1, 2, ...q$, then

$$\mathrm{D}f_k(\xi)X = \frac{1}{4k}(-2k)\mathrm{Tr}(\xi^{2k-1}X^t) = \frac{1}{2}\mathrm{Tr}(\xi^{2k-1}X^t) = B(\xi^{2k-1}, X)$$

and so $\mathrm{grad}f_k(\xi) = \xi^{2k-1}$, where the gradient now is taken with respect to B. If ξ is regular, then from the above the gradients $\mathrm{grad}f_k(\xi) = \xi^{2k-1}$, $k = 1, ..., p$ are linearly independent.

Since $f_1, ..., f_p$ are polynomials, hence real analytic functions, it follows from the identity principle for real analytic functions that their gradients $\mathrm{grad}f_1, ..., \mathrm{grad}f_p$ are linearly idependent on an open dense subset of $\mathfrak{so}(p+q, \mathbb{R})$ of full Lebesgue measure.

Now we consider the following chain

$$\mathbb{R}^1 \times \mathbb{R}^1 \subset \mathbb{R}^1 \times \mathbb{R}^2 \subset \mathbb{R}^2 \times \mathbb{R}^2 \subset ... \subset \mathbb{R}^{p-1} \times \mathbb{R}^p \subset \mathbb{R}^p \times \mathbb{R}^{p+1} \subset \mathbb{R}^p \times \mathbb{R}^{q-1} \subset \mathbb{R}^p \times \mathbb{R}^q$$

of subspaces of $\mathbb{R}^{n+1} = \mathbb{R}^p \times \mathbb{R}^q$. As usual we view $\mathbb{R}^k \subset \mathbb{R}^p$ as the vectors in $\mathbb{R}^p$ whose last $p-k$ coordinates vanish and similarly $\mathbb{R}^l \subset \mathbb{R}^q$. From this chain of vector subspaces we obtain the chain of σ- invariant Lie subgroups

$$O(1+1, \mathbb{R}) \subset O(1+2, \mathbb{R}) \subset O(2+2, \mathbb{R}) \subset ... \subset O(p-1+p, \mathbb{R}) \subset O(p+p, \mathbb{R})$$

$$\subset O(p+p+1, \mathbb{R}) \subset ... \subset O(p+q-1, \mathbb{R}) \subset O(n+1, \mathbb{R}).$$

The fixed point set of σ restricted to each $O(k+l, \mathbb{R})$ is $O(k, \mathbb{R}) \times O(l, \mathbb{R})$ and so

$$G_{k,l}(\mathbb{R}) = {}^{O(k+l, \mathbb{R})}/_{O(k, \mathbb{R}) \times O(l, \mathbb{R})} \subset G_{p,q}(\mathbb{R})$$

is a totally geodesic embedded smooth submanifold of $G_{p,q}$, since the corresponding second fundamental form must vanish at each point of $G_{k,l}(\mathbb{R})$ (see [6], Theorem 8 on page 19). The elements of $\mathfrak{so}(k+l, \mathbb{R})$ viewed as elements $\mathfrak{so}(n+1, \mathbb{R})$ have rows and columns with respective numbers $n+1, ..., p$ and $p+l+1, ..., p+q = n+1$ which vanish.

In the above chain of inclusions there are two types, namely $\mathbb{R}^{k-1} \times \mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^k$ and $\mathbb{R}^k \times \mathbb{R}^l \subset \mathbb{R}^k \times \mathbb{R}^{l+1}$ for $k \leq l$ with corresponding inclusion

$$O(k-1+k, \mathbb{R}) \subset O(k+k, \mathbb{R}) \text{ and } O(k+l, \mathbb{R}) \subset O(k+l+1, \mathbb{R}).$$

We shall consider the first case, the second being analogous. We simplify our notation setting $G_1 = O(k-1+k,\mathbb{R})$, $G_2 = O(k+k,\mathbb{R})$ and $K_1 = O(k-1,\mathbb{R}) \times O(k,\mathbb{R})$, $K_2 = O(k,\mathbb{R}) \times O(k,\mathbb{R})$. Since the fixed point set of σ restricted to G_1 is K_1 and to G_2 is K_2, we obtain orthogonal direct sum decompositions $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{m}_1$ and $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{m}_2$, where $\mathfrak{k}_j$ is the Lie algebra K_j, which is identical with the eigenspace of the eigenvalue 1 and $\mathfrak{m}_j$ is the eigenspace of the eigenvalue -1, $j = 1, 2$. The difference $\dim\mathfrak{m}_2 - \dim\mathfrak{m}_1$ is the rank of the symmetric space G_2/K_2. Using these notations we have the following.

Lemma 4.2.7. *For every regular vector $\xi_1 \in \mathfrak{m}_1$ there exists a regular vector $\xi_2 \in \mathfrak{m}_2$ with $\pi_1(\xi_2) = \xi_1$, where $\pi_1 : \mathfrak{so}(n+1,\mathbb{R}) \to \mathfrak{g}_1$ denotes the orthogonal projection, and such that the maximal abelian subalgebra $\mathfrak{a}_2$ of $\mathfrak{m}_2$ containing ξ_2 satisfies $\mathfrak{m}_1 \oplus \mathfrak{a}_2 = \mathfrak{m}_2$.*

Proof. The Lie algebra $\mathfrak{m}_2$ consists of matrices of the form

$$(X, a) = \begin{pmatrix} 0 & 0 & -X^t \\ 0 & 0 & -a^t \\ X & a & 0 \end{pmatrix}$$

with $X \in \mathbb{R}^{k\times(k-1)}$ and $a \in \mathbb{R}^{k\times 1} = \mathbb{R}^k$. In the Lie subalgebra $\mathfrak{m}_1$ of $\mathfrak{m}_2$ we have $a = 0$. We have deleted the vanishing rows and columns arising from the inclusion of $\mathfrak{m}_2$ in $\mathfrak{so}(n+1,\mathbb{R})$. There exists $h \in K_1$ such that $\mathrm{Ad}_h(\xi_1)$ is contained in the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{m}_1$ which consists of matrices $(X, 0)$ and $x_{ij} = 0$ for $i \neq j$, $1 \leq i \leq k$, $1 \leq j < k$, where $X = (x_{ij})$, $1 \leq i \leq k$, $1 \leq j < k$ (See [6], Theorem 9). So we may assume without loss of generality that $\xi_1 = (X, 0) \in \mathfrak{a}$ and it suffices to take $\xi_2 = (X, a)$ selecting a suitably in order to achieve the direct sum decomposition. The commutator of ξ_2 with any $u = (Y, 0) \in \mathfrak{m}_1$, for $Y = (y_{ij}) \in \mathbb{R}^{k\times(k-1)}$, $1 \leq i \leq k$, $1 \leq j < k$, is

$$[\xi_2, v] = \begin{pmatrix} Y^tX - X^tY & Y^ta & 0 \\ -a^tY & 0 & 0 \\ 0 & 0 & YX^t - XY^t \end{pmatrix}$$

while

$$[\xi_1, v] = \begin{pmatrix} Y^tX - X^tY & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & YX^t - XY^t \end{pmatrix}.$$

If $[\xi_1, v] = 0$, then $v \in \mathfrak{a}$ and $a^tY = (a_1y_{11}, ..., a_{k-1}y_{k-1,k-1})$, where $a^t = (a_1, ..., a_{k-1}, a_k)$. Thus, if we choose $a \in \mathbb{R}^k$ such that $a_i \neq 0$ for $1 \leq i < k$, then $[\xi_2, v] \neq 0$ for $v \neq 0$.This implies that the maximal abelian subalgebra $\mathfrak{a}_2$ in $\mathfrak{m}_2$ which contains ξ_2 satisfies $\mathfrak{a}_2 \cap \mathfrak{m}_1 = \{0\}$. □

Enumerating the groups $O(k+l,\mathbb{R})$ and $O(k,\mathbb{R})\times O(l,\mathbb{R})$ consequtively as in the chain of inclusions above by G_i and K_i we have corresponding Lie algebras $\mathfrak{g}_i$ and $\mathfrak{k}_i$ respectively, $1 \leq i \leq 2(p-1) + (q-p+1) = p+q-1$, with orthogonal direct sum decomposition $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{m}_i$. Starting with a non-zero vector $\xi_i \in \mathfrak{a}_1 = \mathfrak{m}_1 \cong \mathbb{R}$, hence regular, and applying the above Lemma 4.2.7 repeatelly we obtain a finite sequence of regular vectors $\xi_i \in \mathfrak{m}_i$ such that $\pi_i(\xi_j) = \xi_i$ for $i < j$ where $\pi_i : \mathfrak{so}(n+1,\mathbb{R}) \to \mathfrak{g}_i$

is the orthogonal projection and such that the maximal abelian subalgebra $\mathfrak{a}_i$ of $\mathfrak{m}_i$ which contains ξ satisfies $\mathfrak{m}_i = \mathfrak{a}_i \oplus \mathfrak{m}_{i-1}$. In this way we obtain a direct sum decomposition

$$\mathfrak{m}_n = \mathfrak{m}_{p+q-1} = \mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_n.$$

If now $P : TG_{p,q}(\mathbb{R}) \to \mathfrak{so}(n+1, \mathbb{R})$ denotes the representation of the momentum map induced by the metric, we have a total number of

$$2\frac{(p-1)p}{2} + p(q-p+1) = pq$$

first inegrals $F_{ij} = f_j \circ \pi_i \circ P : TG_{p,q}(\mathbb{R}) \to \mathbb{R}$ which are in involution by the results of the previous section. By Proposition 4.2.4, their gradients are linearly independent at $\xi_n = \xi_{p+q-1}$ and so are their corresponding Hamiltonian vector fields. Since P is real analytic by the formula in Lemma 3.2.1 and $f_j \circ \pi_i$ are polynomial functions, all the F_{ji} are real analytic functions. By the identity principle for real analytic functions, their gradients are linearly independent on an open dense subset of $TG_{p,q}(\mathbb{R})$ whose complement is a set of measure zero and so are their corresponding Hamiltonian vector fields. Thus we have proved the following.

Theorem 4.2.8. *The geodesic flow of the real Grassmanian $G_{p,q}(\mathbb{R})$ is completely integrable with pq real analytic functions on $TG_{p,q}(\mathbb{R})$ as a complete family of first integrals in involution.* □

Actually, the above considerations show that the conclusion of the Theorem 4.2.8 holds for the Hamiltonian vector field of any $SO(n+1, \mathbb{R})$ invariant smooth funtion on $TG_{p,q}(\mathbb{R})$.

Chapter 5

Appendices

5.1 The identity principle for real analytic functions

Theorem 5.1.1. *Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a real analytic function. If the set of zeros $\{x \in I \mid f(x) = 0\}$ of f has an accumulation point in I, then $f = 0$ on I.*

Proof. Let $x_0 \in I$ be such that $f(x_0) = 0$. There exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset I$ and $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $x \in (x_0 - \delta, x_0 + \delta)$ where $a_n = \frac{f^{(n)}(x_0)}{n!}$, $n \geq 0$. Since $f(x_0) = 0$, either $a_n = 0$ for all $n \geq 0$ and so $f|_{(x_0-\delta,x_0+\delta)} = 0$ or there exists some $m \in \mathbb{N}$ such that $f(x) = (x - x_0)^m g(x)$, $x \in (x_0 - \delta, x_0 + \delta)$ for some real analytic function $g : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$, with $g(x_0) \neq 0$. By continuity of g taking a smaller $\delta > 0$ we may assume that $g(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Consequently $f(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. So x_0 is an isolated point of $\{x \in I \mid f(x) = 0\}$. Thus if $x_0 \in I$ is an accumulation point of $\{x \in I \mid f(x) = 0\}$ then necessarily $a_n = 0$ and $f|_{(x_0-\delta,x_0+\delta)} = 0$. This argument implies that the set of the accumulation points of $\{x \in I \mid f(x) = 0\}$ in I is open and trivially closed in I. Hence if it is non-empty it must be all of I. □

Theorem 5.1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function such that $f(x) = 0$ for all $x \in E$. If E has positve Lebesgue measure, then $f = 0$ everywhere on $\mathbb{R}^n$.*

Proof. We proceed by induction on the dimension. For $n = 1$, there exists some $\rho > 0$ such that $\lambda_1(E \cap [-\rho, \rho]) > 0$, where λ_1 denotes the Lebesgue measure on $\mathbb{R}$. In particular, $E \cap [-\rho, \rho]$ is an infinite set and by the Bolzano-Weierstrass theorem, it has an accumulation point in $[-\rho, \rho]$. Hence $f = 0$ for every open interval larger than $[-\rho, \rho]$. This means that $f = 0$ everywhere on $\mathbb{R}$.

Suppose now that the conclusion is true for n and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a real analytic function such that $f(x) = 0$ for $x \in E \subset \mathbb{R}^{n+1}$ a set of $(n+1)$-dimensional Lebesgue measure $\lambda_{n+1}(E) > 0$. By Fubini's theorem we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\, d\lambda_n(x)\, d\lambda_1(t) > 0.$$

This implies that there exists a measurable set $A \subset \mathbb{R}$ such that

$$\int\limits_{\mathbb{R}^n} \chi_E(x,t) d\lambda_n(x) > 0 \text{ and } \lambda_1(A) > 0$$

for every $t \in A$. The function $g_t(x) = f(x,t)$ is real analytic on $\mathbb{R}^n$ and $g_t(x) = 0$ for $x \in E_t$, where

$$E_t = \{x \in \mathbb{R}^n \mid (x,t) \in E\}.$$

Obviously, E_t is measurable and

$$\lambda_n(E_t) = \int\limits_{\mathbb{R}^n} \chi_{E_t}(x) d\lambda_n(x) = \int\limits_{\mathbb{R}^n} \chi_{E_t}(x,t) d\lambda_n(x) > 0.$$

By the inductive hypothesis, $g_t(x) = 0$ for every $x \in \mathbb{R}^n$ and so $f(x,t) = 0$ for all $(x,t) \in A$. Now for all $x \in \mathbb{R}^n$ the function $h_x(t) = f(x,t)$ is real analytical on $\mathbb{R}$ and $h_x(t) = 0$ for all $t \in A$. Since $\lambda_1(A) > 0$ and the conclusion holds in dimension $n = 1$ we conclude that $h_x(t) = 0$ for all $t \in \mathbb{R}$, that is

$$f(x,t) = h_x(t) = 0 \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \quad \square$$

Corrolary 5.1.3. *Let $m \leq n$ and $f_1, ..., f_m : \mathbb{R}^n \to \mathbb{R}$ be real analytic functions. If there exists some point $p \in \mathbb{R}^n$ such that $\{\nabla f_1(p), ..., \nabla f_m(p)\}$ is linearly independent, then the vector fields $\nabla f_1, ..., \nabla f_m$ are linearly independent on an open dense subset of $\mathbb{R}^n$ whose complement has Lebesgue measure zero.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ denote the map $f = (f_1, ..., f_m)$. Then $\{\nabla f_1(q), ..., \nabla f_m(q)\}$ are linearly independent if and only if $\mathrm{D}f(q)$ has rank m. Obviously, this holds in an open subset of $\mathbb{R}^n$. Recall that $\mathrm{D}f(q)$ has rank m if and only if the determinant of some $m \times m$ submatrix of $\mathrm{D}f(q)$ is non-zero. By our assumption, there exists such a determinant D with $\mathrm{D}(p) \neq 0$. The function $\mathrm{D} : \mathbb{R}^n \to \mathbb{R}$ is real analytic, as a polynomial of the real analytic functions $\frac{\partial f_i}{\partial x_j}$, $1 \leq i, j \leq n$. By the identity principle for real analytic functions, the set

$$E = \{q \in \mathbb{R}^n \mid \mathrm{D}(q) = 0\}$$

has Lebesgue measure zero and in particular it does not contain any open subset of $\mathbb{R}^n$, that is it is nowhere dense. In other words $\{\nabla f_1(q), ..., \nabla f_m(q)\}$ is linearly independent for $q \in \mathbb{R}^n \setminus E$ and $\mathbb{R}^n \setminus E$ is dense in $\mathbb{R}^n$. $\square$

5.2 Killing forms

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the symmetric, bilinear form defined by

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$

If σ is a Lie algebra automorphism of $\mathfrak{g}$, we have

$$\mathrm{ad}_{\sigma(X)}(Z) = [\sigma(X), Z] = \sigma[X, \sigma^{-1}(Z)]$$

for every $X,\ Y \in \mathfrak{g}$ and therefore $\mathrm{ad}_{\sigma(X)} = \sigma\circ\mathrm{ad}_X\circ\sigma^{-1}$. It follows that $B(\sigma(X), \sigma(Y)) = B(X, Y)$ for every $X,\ Y \in \mathfrak{g}$. In particular, $B(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)) = B(X, Y)$ for every $X,\ Y \in \mathfrak{g}$ and $g \in G$. In other words, B is Ad-invariant. The Ad-invariant, symmetric, bilinear form B is called the *Killing form* of G (and $\mathfrak{g}$). If B is non-degenerate, then G (and $\mathfrak{g}$) is called *semi-simple* Lie group.

Example 5.2.1. If G is abelian, then Ad is trivial and ad$=0$. Hence $B = 0$. So G cannot be semi-simple. As special cases we get that $\mathbb{R}^n$ and S^1 are not semi-simple.

Example 5.2.2. We shall find the Killing form of the general linear group $GL(n,\mathbb{R})$, $n > 1$. Recall that $\mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n}$ and the adjoint representation in this case is just conjugation, that is $\mathrm{Ad}_A(X) = AXA^{-1}$ for $X \in \mathbb{R}^{n\times n}$ and $A \in GL(n,\mathbb{R})$. Also $[X, Y] = \mathrm{ad}_X(Y) = XY - YX$ for every $X,\ Y \in \mathbb{R}^{n\times n}$, because

$$\frac{(I_n + X)Y(I_n + X)^{-1} - Y - (XY - YX)}{\|X\|}$$

$$= \frac{(I_n + X)Y - Y(I_n + X) - (XY - YX)(I_n + X)}{\|X\|}(I_n + X)^{-1}$$

$$= (YX - XY)\frac{X}{\|X\|}(X + I_n)^{-1},$$

whose norm is $\leq \|YX - XY\| \cdot \|(I_n + X)^{-1}\| \to 0$, when $X \to 0$. Since $\mathrm{ad}_{I_n} = 0$ it follows that the Killing form of $GL(n,\mathbb{R})$ is degenerate, and so $GL(n,\mathbb{R})$ is not semi-simple.

Note that if G is any Lie group with Lie algebra $\mathfrak{g}$, it follows from the Jacobi identity that

$$\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y]$$

for every $X,\ Y \in \mathfrak{g}$, which means that ad: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g},\mathfrak{g})$ is a Lie algebra homomorphism.

In order to calculate the Killing form of $GL(n,\mathbb{R})$, we consider the cannonical basis $\{E_{ij} \mid 1 \leq i, j \leq n\}$ of $\mathbb{R}^{n\times n}$. If $X = (x_{ij})_{1\leq i,j\leq n} = \sum\limits_{i,j=1}^{n} x_{ij}E_{ij}$, then

$$\mathrm{ad}_X(E_{ij}) = [X, E_{ij}] = \sum_{k=1}^{n} x_{ki}E_{kj} - \sum_{k=1}^{n} x_{jk}E_{ik}.$$

If $Y = (y_{ij})_{1\leq i,j\leq n}$, then

$$(\mathrm{ad}_X \circ \mathrm{ad}_Y)(E_{ij}) = \sum_{k=1}^{n} y_{ki}\mathrm{ad}_X(E_{kj}) - \sum_{k=1}^{n} y_{jk}\mathrm{ad}_X(E_{ik})$$

$$= \sum_{k=1}^{n} y_{ki}(\sum_{l=1}^{n} x_{lk}E_{lj} - \sum_{l=1}^{n} x_{jl}E_{kl}) - \sum_{k=1}^{n} y_{jk}(\sum_{l=1}^{n} x_{li}E_{lk} - \sum_{l=1}^{n} x_{kl}E_{il})$$

$$= \sum_{k,l=1}^{n} x_{lk}y_{ki}E_{lj} - \sum_{k,l=1}^{n} x_{jl}y_{ki}E_{kl} - \sum_{k,l=1}^{n} x_{li}y_{jk}E_{lk} + \sum_{k,l=1}^{n} x_{kl}y_{jk}E_{il}$$

$$= \sum_{k,l=1}^{n} x_{lk}y_{ki}E_{lj} - \sum_{k,l=1}^{n} (x_{jl}y_{ki} + x_{kl}y_{jl})E_{kl} + \sum_{k,l=1}^{n} x_{kl}y_{jk}E_{il}.$$

The coefficient of E_{ij} is

$$\sum_{k=1}^{n} x_{ik}y_{ki} - x_{jj}y_{ii} - x_{ii}y_{jj} + \sum_{k=1}^{n} x_{kj}y_{jk}$$

and so

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = \sum_{i,j,k=1}^{n} x_{ik}y_{ki} - \sum_{i,j=1}^{n} (x_{jj}y_{ii} + x_{ii}y_{jj}) + \sum_{i,j,k=1}^{n} x_{kj}y_{jk}$$

$$= n\mathrm{Tr}(XY) - 2\mathrm{Tr}X \cdot \mathrm{Tr}Y + n\mathrm{Tr}(XY) = 2n\mathrm{Tr}(XY) - 2\mathrm{Tr}X \cdot \mathrm{Tr}Y.$$

Example 5.2.3. We shall compute the Killing form of the special orthogonal group $SO(n,\mathbb{R})$ for $n > 2$. We recall that $\mathfrak{so}(n,\mathbb{R}) = \{H \in \mathbb{R}^{n\times n} \mid H + H^t = 0\}$ is the space of skew-symmetric elements of $\mathbb{R}^{n\times n}$. We consider the basis $\{E_{ij} - E_{ji} \mid 1 \leq i < j \leq n\}$ of $\mathfrak{so}(n,\mathbb{R})$. If $X = (x_{ij})_{1\leq i,j\leq n}$ and $Y = (y_{ij})_{1\leq i,j\leq n}$ are two elements of $\mathfrak{so}(n,\mathbb{R})$, then $x_{ij} = -x_{ji}$ and $y_{ij} = -y_{ji}$ and

$$\mathrm{ad}_Y(E_{ij} - E_{ji}) = \sum_{k=1}^{n} y_{ki}(E_{kj} - E_{jk}) - \sum_{k=1}^{n} y_{jk}(E_{ik} - E_{ki}).$$

Therefore,

$$(\mathrm{ad}_X \circ \mathrm{ad}_Y)(E_{ij} - E_{ji}) = \sum_{k=1}^{n} y_{ki}\mathrm{ad}_X(E_{kj} - E_{jk}) - \sum_{k=1}^{n} y_{jk}\mathrm{ad}_X(E_{ik} - E_{ki})$$

$$= \sum_{k=1}^{n} y_{ki} \sum_{l=1}^{n} x_{lk}(E_{lj} - E_{jl}) - \sum_{k=1}^{n} y_{ki} \sum_{l=1}^{n} x_{jl}(E_{kl} - E_{lk})$$

$$- \sum_{k=1}^{n} y_{jk} \sum_{l=1}^{n} x_{li}(E_{lk} - E_{kl}) + \sum_{k=1}^{n} y_{jk} \sum_{l=1}^{n} x_{kl}(E_{il} - E_{li}).$$

The coefficient of $E_{ij} - E_{ji}$ is

$$\sum_{k=1}^{n} x_{ik}y_{ki} - 2x_{ij}y_{ji} + \sum_{k=1}^{n} x_{kj}y_{jk} \quad \text{and so}$$

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = \sum_{1\leq i<j\leq n} \sum_{k=1}^{n} x_{ik}y_{ki} + \sum_{1\leq i<j\leq n} \sum_{k=1}^{n} x_{kj}y_{jk} - 2\sum_{1\leq i<j\leq n} x_{kj}y_{ji}$$

$$= \sum_{1\leq i<j\leq n} \sum_{k=1}^{n} x_{ik}y_{ki} + \sum_{1\leq i<j\leq n} \sum_{k=1}^{n} x_{jk}y_{kj} - \sum_{1\leq i<j\leq n} x_{ij}y_{ji} - \sum_{1\leq i<j\leq n} x_{ji}y_{ij}$$

$$= \sum_{1\le i<j\le n} (XY)_{ii} + \sum_{1\le i<j\le n} (XY)_{jj} - \sum_{i,j=1}^{n} x_{ij}y_{ji}$$
$$= (n-1)\mathrm{Tr}(XY) - \mathrm{Tr}(XY) = (n-2)\mathrm{Tr}(XY).$$

This formula for the Killing form B implies that B is non-degenerate and so $SO(n,\mathbb{R})$ is semi-simple. Indeed, if $X \in \mathfrak{so}(n,\mathbb{R})$ and $X = (x_{ij})_{1\le i,j\le n}$, then for $1 \le i < j \le n$ we have

$$\mathrm{Tr}(X \cdot (E_{ij} - E_{ji})) = \mathrm{Tr}(\sum_{k=1}^{n} x_{ki}E_{kj} - \sum_{k=1}^{n} x_{kj}E_{ki})$$
$$= \sum_{k=1}^{n} x_{ki}\mathrm{Tr}E_{kj} - \sum_{k=1}^{n} x_{kj}\mathrm{Tr}E_{ki} = x_{ji} - x_{ij} = -2x_{ij}.$$

So if $\mathrm{Tr}(XY) = 0$ for every $Y \in \mathfrak{so}(n,\mathbb{R})$, then $X = 0$.

In case $n = 3$ we have the Lie algebra isomorphism $\widehat{\ }: (\mathbb{R}^3, \times) \to \mathfrak{so}(n,\mathbb{R})$ with

$$\hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \quad \text{for} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3. \text{ If } \quad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

and then a simple calculation shows that $\mathrm{Tr}(\hat{u}\hat{v}) = -2\langle u, v\rangle$, where $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbb{R}^3$. In other words B is negative defined and $-\frac{1}{2}B$ corresponds to the Euclidean inner product under the isomorphism $\widehat{\ }$.

Bibliography

[1] K. Athanassopoulos, Notes on Symplectic Geometry, Course Notes.

[2] V.I. Arnold, Mathematical methods of Classical Mechanics, Springer-Verlag, New York/Berlin, 1984.

[3] W. Ballman, Complete integrability of Hamiltonian systems after Anton Thimm, preliminary seminar notes.

[4] A. Canas da Silva, Lectures on Symplectic Geometry, Lecture Notes in Math. vol 1764, Springer-Verlag, Berlin, 2001.

[5] J. Cheeger and D.G. Ebin, Comparison theorems in Riemannian Geometry, North-Holland, Amsterdam, 1975.

[6] J.-H. Eschenburg, Lecture notes on symmetric spaces, Course notes.

[7] V. Guillemin and S. Sternberg, The momentum map and collective motion, *Annals of Physics* 127 (1980), 220-253.

[8] V. Guillemin and S. Sternberg, On collective complete integrability according to the method of Thimm, *Ergodic Theory & Dynamical Systems* 3 (1983), 219-230.

[9] V. Guillemin and S. Sternberg, Multiplicity-free spaces, *J. Diff. Geom.* 19 (1984), 31-56.

[10] V. Guillemin and S. Sternberg, Symplectic techniques in Physics, Cambridge Univ. Press, Cambridge, 1984.

[11] S. Helgason, Differential Geometry, Lie groups and symmetric spaces, Academic Press, New York, 1962.

[12] J. Hilgert and K.-H. Neeb, Structure and Geometry of Lie Groups, Springer Science and Bussiness Media, 2012.

[13] A. Kirillov Jr., An introduction to Lie groups and Lie algebras, Cambridge Univ. Press, Cambridge, 2010.

[14] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vols. I and II, Interscience, New York, 1963 and 1969.

[15] G.P. Paternain and R.J. Spatzier, New examples of manifolds with completely integrable geodesic flows, *Advances in Math.* 108 (1994), 346-366.

[16] A. Thimm, Integrable geodesic flows on homogeneous spaces, *Ergodic Theory & Dynamical Systems* 1 (1981), 495-517.

[17] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer, New York/Berlin, 1983.