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D^+-Stable Dynamical Systems on 2-Manifolds

Konstantin Athanassopoulos
Mathematical Institute, University of Athens, 57 Solonos Str., 10679 Athens, Greece

1. Introduction

The concept of a dynamical system of characteristic 0^+ is due to T. Ura. In [1] S. Ahmad classified these dynamical systems on $\mathbb{R}^2$ and in [12] R. Knight characterized them on $\mathbb{R}^2$ in terms of their fixed point set. Because of the definition (see 2.2), it seems that the term D^+-*stable* is better than "characteristic 0^+" and we shall use it in the sequel.

In this paper we are concerned with the study of the global qualitative behavior of D^+-stable dynamical systems, in connection with the topological structure of the underlying phase spaces. More precisely, we answer the following problems:

I. Find all the 2-manifolds which can support (non-trivial) D^+-stable dynamical systems.

II. Describe the phase portraits of the D^+-stable dynamical systems on these manifolds.

III. Find how the continuous D^+-stable dynamical systems on 2-manifolds are related to the smooth ones.

It turns out that the existence of periodic orbits in a D^+-stable dynamical system on a 2-manifold M is crucial not only for its phase portrait but also for the topological structure of M. It is proved that there are only seven 2-manifolds supporting D^+-stable dynamical systems with at least one periodic orbit. Moreover, we give a rather complete description of these systems (see 3.1, 3.2). From this we deduce that the sphere S^2, the projective plane $\mathbb{P}^2$, the torus T^2 and the Klein bottle K^2 are the only compact 2-manifolds supporting (non-trivial) D^+-stable dynamical systems (see 3.4). On the contrary, there exists a (non-trivial) D^+-stable dynamical system without periodic orbits on every non-compact 2-manifold (see 4.1). Finally, using our answer to problem II and the methods of [10, 11] we prove that every continuous D^+-stable dynamical system on a 2-manifold is topologically equivalent to a smooth one (see 5.1).

2. Preliminary Results

Before proving the main theorems of the paper, we shall establish our notation and prove some preliminary results.

2.1. Let ($\mathbb{R}$, M, φ) be a dynamical system on a metric space M, i.e. a continuous action of the additive group of the real numbers $\mathbb{R}$ on M. We let $\varphi(t, x)=tx$ and if $I\subset\mathbb{R}$, $A\subset M$, then $IA=\{tx: t\in I, x\in A\}$. The orbit of the point $x\in M$ is denoted by $\mathbb{R}(x)$, the positive semiorbit by $\mathbb{R}^+(x)$ and the negative semiorbit by $\mathbb{R}^-(x)$. A point $x\in M$ is called *periodic* if there exists a $T>0$ such that $Tx=x$ and $tx\neq x$ for all $t\in(0, T)$.

Two dynamical systems ($\mathbb{R}$, M, φ) and ($\mathbb{R}$, M', φ') are called *topologically equivalent* if there is a homeomorphism $h: M\to M'$ that takes orbits onto orbits preserving their orientation.

We recall that the *positive limit set* of the point $x\in M$ is the set $L^+(x)=\{y\in M: t_n x\to y$ for some $t_n\to+\infty\}$, its *positive prolongational limit set* is the set $J^+(x)=\{y\in M: t_n x_n\to y$ for some $x_n\to x$ and $t_n\to+\infty\}$ and its *first positive prolongation* the set $D^+(x)=\mathbb{R}^+(x)\cup J^+(x)$. The sets $L^-(x)$, $J^-(x)$ and $D^-(x)$ are defined analogously. For each $A\subset M$ we let $D^+(A)=\bigcup\{D^+(x): x\in A\}$.

A set $A\subset M$ is called *stable* if every neighborhood of A contains a positively invariant neighborhood of A. The point $x\in M$ is said to be *attracted* to $A\subset M$ if for each neighborhood W of A there is a $t>0$ such that $\mathbb{R}^+(tx)\subset W$. The set $E^+(A)$ of points that are attracted to A is called the *region of attraction of A*. The set A is called *asymptotically stable* if it is stable and $E^+(A)$ is an open neigborhood of A. A stable set A is called *globally asymptotically stable* if $E^+(A)=M$.

2.2. Definition. A dynamical system ($\mathbb{R}$, M, φ) is called D^+*-stable* (or of characteristic 0^+) if $D^+(x)=\overline{\mathbb{R}^+(x)}$ for each $x\in M$.

The preceding definition is equivalent to saying that $D^+(A)=A$ for every positively invariant, closed subset A of M or that $L^+(x)=J^+(x)$ for each $x\in M$. Clearly, D^+-stability is an invariant to topological equivalence.

2.3. Proposition. *Let* ($\mathbb{R}$, M, φ) *be a* D^+*-stable dynamical system on a connected, locally compact, metric space* M.

(a) *Every positively invariant, closed subset of* M *with compact boundary is stable* (*see* [9, *Theorem* 1]).

(b) *If an invariant, closed set has compact boundary and is asymptotically stable then it is globally asymptotically stable* (*see* [1, *Proposition* 3.2]).

(c) *For each* $x\in M$ *the limit set* $L^+(x)$ *is stable, compact and minimal whenever it is non-empty and either* $L^-(x)=\varnothing$ *or* $L^-(x)=L^+(x)=\overline{\mathbb{R}(x)}$ (*see* [2, *Lemma* 4.5]).

(d) *The set* $G=\{x\in M: L^+(x)\neq\varnothing\}$ *is open, since each positive limit set is compact and stable.*

In this paper we are especially interested in the case where M is a 2-manifold. For the proof of the following theorem see [4] or [3].

2.4. Theorem. *A stable compact minimal set of a dynamical system on a 2-manifold is either a fixed point, a periodic orbit or the system is topologically equivalent to some irrational flow on the torus* T^2.

So by 2.3(c) we have:

2.5. Theorem. *Let* $(\mathbb{R}, M, \varphi)$ *be a* D^+*-stable dynamical system on a 2-manifold* M*, not topologically equivalent to any irrational flow on the torus* T^2*. Then, given a point* $x \in M$ *its positive limit set* $L^+(x)$ *is either a fixed point or a periodic orbit, whenever it is non-empty.*

2.6. *Notation.* In the remainder of this section we assume that M is a *2-manifold and* $(\mathbb{R}, M, \varphi)$ *a* D^+*-stable dynamical system, not topologically equivalent to any irrational flow on* T^2. By 2.3(c), 2.5 and [14, Corollary 1.11] we have that each point $x \in M$ is either fixed, periodic or its orbit is homeomorphic to $\mathbb{R}$. Let F denote the set of fixed points, P the set of periodic points and Q the set of points whose orbits are homeomorphic to $\mathbb{R}$. So $M = F \cup P \cup Q$. It is clear that $F \cup P = \{x \in M : x \in J^+(x)\}$. Therefore, the set $F \cup P$ is closed [14, Theorem 2.12] and Q is open in M.

2.7. Proposition. (a) $F \cup P = \{x \in M: L^-(x) \neq \varnothing\}$ *and therefore if* A *is a compact invariant subset of* M*, then* $A \subset F \cup P$.

(b) *Let* $A \subset M$ *be a closed, invariant set with compact boundary. If* A *is isolated from fixed points and periodic orbits in* $M - A$*, then it is globally asymptotically stable.*

Proof. (a) If $L^-(x) \neq \varnothing$, then $x \in L^-(x) = L^+(x)$ (2.3(c)) and by 2.5, $x \in F \cup P$.

(b) The sets A and ∂A are stable (2.3(a)). Let W be an open, relatively compact and positively invariant neighborhood of ∂A such that $\bar{W} \cap (M - A) \cap (F \cup P) = \varnothing$. For each $x \in W \cap (M - A)$ the set $L^+(x)$ is either a fixed point or a periodic orbit contained in $\bar{W} \cap \overline{M - A}$. So necessarily $\varnothing \neq L^+(x) \subset \partial A$ for each $x \in W \cap (M - A)$. This implies that $W \subset E^+(A)$, which means that A is asymptotically stable, hence globally asymptotically stable (2.3(b)).

2.8. Proposition. *The restricted dynamical system on* Q *is parallelizable and each connected component of* Q *is homeomorphic either to* $\mathbb{R}^2$ *or to* $\mathbb{R} \times S^1$*, depending upon whether the section is homeomorphic to* $\mathbb{R}$ *or to* S^1 *respectively.*

Proof. Since $L^+(x) = J^+(x) \subset F \cup P$ for all $x \in Q$, the first assertion follows from [5, Ch. IV, 2.6]. The section in a connected component C of Q is a 1-manifold [8, Ch. VII, 1.6], hence homeomorphic to $\mathbb{R}$ or to S^1. So C must be homeomorphic to $\mathbb{R}^2$ or to $\mathbb{R} \times S^1$.

The above properties hold for both orientable or non-orientable M. If M is non-orientable and $p: \tilde{M} \to M$ is its orientable double covering, then there exists a unique dynamical system $(\mathbb{R}, \tilde{M}, \tilde{\varphi})$ on $\tilde{M}$, called the *lifted* dynamical system on $\tilde{M}$, which makes p equivariant [6, Ch. I, p. 63].

2.9. Theorem. *The lifted dynamical system* $(\mathbb{R}, \tilde{M}, \tilde{\varphi})$ *of a* D^+*-stable dynamical system* $(\mathbb{R}, M, \varphi)$ *on a non-orientable 2-manifold* M *is also* D^+*-stable.*

Proof. It suffices to prove that $J^+(\tilde{x}) \subset L^+(\tilde{x})$ for each $\tilde{x} \in \tilde{M}$ with $J^+(\tilde{x}) \neq \varnothing$. Let $\tilde{y} \in J^+(\tilde{x})$ and suppose that $p(\tilde{x}) = x$, $p(\tilde{y}) = y$ and $p^{-1}(x) = \{\tilde{x}, \tilde{x}'\}$, $p^{-1}(y) = \{\tilde{y}, \tilde{y}'\}$. Then, $y \in J^+(x)$ because p is equivariant. By D^+-stability, $L^+(x) = J^+(x) \neq \varnothing$ and $L^+(x)$ is either a fixed point or a periodic orbit (2.5). Thus, $L^+(x) = \mathbb{R}(y)$ and the points $y, \tilde{y}, \tilde{y}'$ are all fixed or periodic. It is easy to see that $\mathbb{R}(\tilde{y}) \cup \mathbb{R}(\tilde{y}') = L^+(\tilde{x}) \cup L^+(\tilde{x}')$. We shall show that $\tilde{y} \notin L^+(\tilde{x})$ leads to a contradiction. If $\tilde{y} \notin L^+(\tilde{x})$, then $L^+(\tilde{x}) = \mathbb{R}(\tilde{y}')$, $L^+(\tilde{x}') = \mathbb{R}(\tilde{y})$ and $\mathbb{R}(\tilde{y}) \cap \mathbb{R}(\tilde{y}') = \varnothing$. The compact orbit $\mathbb{R}(y)$

is stable and so are $\mathbb{R}(\tilde{y})$ and $\mathbb{R}(\tilde{y}')$, because they are compact and the covering is finite. Let V be a positively invariant, open neighborhood of $\mathbb{R}(\tilde{y}')$ such that $\bar{V} \cap \mathbb{R}(\tilde{y}) = \varnothing$. Since $L^+(\tilde{x}) = \mathbb{R}(\tilde{y}')$, there is some $t > 0$ such that $\mathbb{R}^+(t\tilde{x}) \subset V$. Hence $J^+(\tilde{x}) = J^+(t\tilde{x}) \subset \bar{V}$. This implies that $\tilde{y} \notin J^+(\tilde{x})$ which is contradictory to our hypothesis at the beginning of the proof.

3. D^+-Stable Dynamical Systems on 2-Manifolds with Periodic Orbits

In this section we describe the D^+-stable dynamical systems on 2-manifolds having at least one periodic orbit and we classify the 2-manifolds supporting such systems. First we study the case where the manifold is orientable. The non-orientable case is treated then using Theorem 2.9.

3.1. Let M be an *orientable* 2-manifold and $(\mathbb{R}, M, \varphi)$ a D^+-stable dynamical system such that $P \neq \varnothing$. As M is orientable we can construct around each periodic orbit an open neighborhood V homeomorphic to $\mathbb{R} \times S^1$ with $V \cap F = \varnothing$, using local cross sections [8, Ch. VII, 2.6]. Let P_x denote the connected component of P which contains $x \in P$ and P_x^0 its interior.

3.1.1. Proposition. *If there exists a point $x \in P$ such that $P_x^0 = \varnothing$, then M is homeomorphic to $\mathbb{R} \times S^1$ and the periodic orbit $\mathbb{R}(x)$ is globally asymptotically stable.*

Proof. Let V be an open neighborhood of $\mathbb{R}(x)$ homeomorphic to $\mathbb{R} \times S^1$ with $V \cap F = \varnothing$. There is a connected, positively invariant, open neighborhood W of $\mathbb{R}(x)$ such that $\bar{W} \subset V$, because $\mathbb{R}(x)$ is stable (2.3(a)). We shall show $W \cap (P - \mathbb{R}(x)) = \varnothing$. Suppose there exists some $z \in W \cap (P - \mathbb{R}(x))$. Then $\mathbb{R}(z) \subset W \cap (P - \mathbb{R}(x))$ and since $V \cap F = \varnothing$, the periodic orbit $\mathbb{R}(z)$ is not nullhomotopic in V [7, Proposition 1.7], [5, Ch. V, 3.8]. This implies that $\mathbb{R}(x)$, $\mathbb{R}(z)$ are the boundary curves of an invariant annulus $A \subset V$. Since A is a connected, compact, invariant set containing no fixed points and $x \in A$, we have $A \subset P_x$ (2.7(a)). Hence $P_x^0 \neq \varnothing$, a contradiction to the hypothesis.

Thus, the periodic orbit $\mathbb{R}(x)$ is isolated from fixed points and other periodic orbits. Hence it is globally asymptotically stable (2.7(b)) and $M = \mathbb{R}V$.

The restricted dynamical system on $M - \mathbb{R}(x)$ is parallelizable and has a compact section S whose connected components are homeomorphic to S^1. We may also choose S to be contained in V. The set $V - \mathbb{R}(x)$ has two connected components V_1 and V_2. There are exactly two connected components S_1 and S_2 of S contained in V_1 and V_2 respectively. The simple closed curves S_1, S_2 are the boundary curves of a positively invariant annulus which contains $\mathbb{R}(x)$ in its interior. Since $\mathbb{R}(x)$ is globally asymptotically stable, each orbit in $M - \mathbb{R}(x)$ intersects $S_1 \cup S_2$. It follows that $S = S_1 \cup S_2$ and $M - \mathbb{R}(x)$ has exactly two connected components, namely $\mathbb{R}S_1$ and $\mathbb{R}S_2$, both homeomorphic to $\mathbb{R} \times S^1$ with common boundary $\mathbb{R}(x)$. Therefore M is homeomorphic to $\mathbb{R} \times S^1$.

3.1.2. Lemma. *Let $x \in P$ be such that $P_x^0 \neq \varnothing$. Then, $\overline{P_x} = \overline{P_x^0}$ and one of the following holds.*

(a) *P_x^0 is homeomorphic to $\mathbb{R} \times S^1$ so that the factor $\mathbb{R}$ corresponds to a section of the restricted dynamical system in P_x^0 and the factor S^1 to the periodic orbits.*

(b) *The dynamical system on* M *is topologically equivalent to the rational flow on the torus* T^2.

Proof. Let $z \in \overline{P_x}$. Since $F \cup P$ is closed, $z \in F \cup P$. If $z \in F$, $\{z\}$ is stable and therefore every disk D with $z \in D^0$ contains a periodic orbit $\mathbb{R}(y) \subset P_x$ bounding a disk $D_y \subset D$. By 2.7(a), $D_y \subset F \cup P$. Hence, there is a connected, open neighborhood U of $\mathbb{R}(y)$ such that $U \cap D_y^0$ consists entirely of periodic orbits. Thus, $\emptyset \neq D \cap U \subset D \cap P_x^0$. So $z \in \overline{P_x^0}$.

In case $z \in P$ we take an open neighborhood V of $\mathbb{R}(z)$ homeomorphic to $\mathbb{R} \times S^1$ with $V \cap F = \emptyset$. Let $y' \in P_x$ such that $\mathbb{R}(y') \subset V - \mathbb{R}(x)$, which exists because $P_x^0 \neq \emptyset$. Then, $\mathbb{R}(z)$ and $\mathbb{R}(y')$ bound an invariant annulus K in V. By 2.7(a) we have $K \subset F \cup P$, i.e. $K^0 \subset P_x^0$. Therefore $z \in \overline{P_x^0}$.

Next we show that the P_x^0 is connected. If A, B are non-empty, open sets such that $P_x^0 = A \cup B$, then the closures of A, B relative to $M - F$ have non-empty intersection because $\overline{P_x} = \overline{P_x^0}$ and P is closed in $M - F$. Let z be a point in the intersection of the closures of A, B in $M - F$, V an open neighborhood of $\mathbb{R}(z)$ homeomorphic to $\mathbb{R} \times S^1$ such that $V \cap F = \emptyset$ and $z_1 \in A$, $z_2 \in B$ such that $\mathbb{R}(z_1) \cup \mathbb{R}(z_2) \subset V$. Then, $\mathbb{R}(z_1)$, $\mathbb{R}(z_2)$ bound an annulus $K \subset P_x^0 \cap V$. Thus, $K = (K \cap A) \cup K \cap B)$ from which follows that $A \cap B \neq \emptyset$. Hence P_x^0 is connected.

The rest of the assertion follows from [16, Proposition 4.5].

3.1.3. Lemma. *The set* P *is closed in* M *if and only if there are no nullhomotopic periodic orbits.*

Proof. If $z \in \bar{P} - P \subset F$, then any disk D with $z \in D^0$ contains a nullhomotopic periodic orbit.

If $\mathbb{R}(x)$ is a nullhomotopic periodic orbit, then by [7, Proposition 1.7] it bounds a disk $D \subset F \cup P$. The closed set $F \cap D$ is not empty [5, Ch. V, 3.8] and the connectivity of D implies that $F \cap \bar{P} \neq \emptyset$.

3.1.4. Theorem. *Suppose that* P *is (non-empty and) non-closed. Then one of the following holds.*

(a) M *is homeomorphic to the sphere* S^2, $M = F \cup P$ *and* F *consists of two centers.*

(b) M *is homeomorphic to* $\mathbb{R}^2$ *and* F *is a singleton* $\{s\}$ *which is either a global center or a local one. In the last case there is a globally asymptotically stable disk consisting of* s *and periodic orbits surrounding it.*

Proof. Since P is non-closed, there exists a nullhomotopic periodic orbit $\mathbb{R}(x)$ bounding a disk $D_x \subset F \cup P$ (3.1.3, 2.7(a)). By [1, Theorem 4.2], $D_x \cap F = \{s\}$ for some $s \in F$ which is a global center with respect to the restricted dynamical system in D_x^0. By 3.1.2, for each $z \in P_x^0$ the periodic orbit $\mathbb{R}(z)$ bounds a disk D_z such that $F \cap D_z = \{s\}$ and either $D_{z_1} \subset D_{z_2}$ or $D_{z_2} \subset D_{z_1}$ whenever $z_1, z_2 \in P_x^0$. The set $E = P_x^0 \cup \{s\}$ is invariant, open and homeomorphic to $\mathbb{R}^2$. Furthermore, $\bar{E} \subset F \cup P$ and so either $F \cap \partial E \neq \emptyset$ or $\partial E \subset P$.

Let $F \cap \partial E \neq \emptyset$ and $y \in F \cap \partial E$. Let D be a disk such that $y \in D^0$ and $s \in M - D$. There is a $z \in P_x^0$ such that $\mathbb{R}(z) \subset D$. The periodic orbit $\mathbb{R}(z)$ bounds a disk $U_z \subset D$ and $D_z \cap U_z = \mathbb{R}(z)$ because $D_z^0 \cap U_z^0$ is open and closed in D_z^0 and $s \notin D$.

This implies that $D_z \cup U_z$ is homeomorphic to the sphere S^2 and $M = D_z \cup U_z$. By [1, Theorem 4.2], $F \cap U_z = \{s'\}$ for some $s' \in F$ which is a global center with respect to the restricted dynamical system in U_z. Hence case (a) follows.

Now let $\partial E \subset P$. If $\partial E = \varnothing$, then $M = E$ and s is a global center. Suppose that $\varnothing \neq \partial E \subset P$, $y \in \partial E$ and V be an open neighborhood of $\mathbb{R}(y)$ homeomorphic to $\mathbb{R} \times S^1$ such that $V \cap F = \varnothing$. There is a point $z \in E$ such that $\mathbb{R}(z) \subset V$. The periodic orbits $\mathbb{R}(y)$, $\mathbb{R}(z)$ bound an annulus $A \subset P_x \cap V$ such that $A \cap D_z = \mathbb{R}(z)$. Hence $D_y = D_z \cup A$ is a disk such that $\partial D_y = \mathbb{R}(y)$. It is easy to see that D_y^0 is open and closed in E. Therefore, $D_y^0 = E$, $D_y = \bar{E} = \overline{P_x}$ and $P_x = D_y - \{s\}$. The disk D_y is isolated from fixed points and periodic orbits in $M - D_y$, for if there is a periodic orbit in $(M - D_y) \cap V$ then it bounds an invariant disk containing D_y which means that $y \in P_x^0$, a contradiction. By 2.7(b), D_y is globally asymptotically stable and $P = D_y - \{s\}$, $F \cup P = D_y$.

The set $V - \mathbb{R}(y)$ has two connected components V_1 and V_2 contained in D_y^0 and $M - D_y$ respectively. Since every orbit in $M - D_y$ intersects V_2, the set $M - D_y$ is connected. The restricted dynamical system on $M - D_y$ is parallelizable and has a compact section S homeomorphic to S^1. The simple closed curve S bounds with $\mathbb{R}(y)$ a positively invariant annulus, because it is not nullhomotopic in V. Therefore, S bounds a positively invariant disk U containing D_y. On the other hand $M - U^0$ is homeomorphic to $\mathbb{R}^- \times S^1$. It follows that M is homeomorphic to $\mathbb{R}^2$ and (b) holds.

3.1.5. Theorem. *Let P be closed and $P^0 \neq \varnothing$. Then, either the dynamical system is topologically equivalent to the rational flow on the torus T^2 or M is homeomorphic to $\mathbb{R} \times S^1$ and one of the following holds.*

(a) $M = P$.

(b) *P corresponds to $\mathbb{R}^- \times S^1$ and is globally asymptotically stable.*

(c) *P corresponds to $[-1, 1] \times S^1$ and is globally asymptotically stable.*

Proof. Since P is closed and $P^0 \neq \varnothing$, we have $P_x = \overline{P_x^0}$ for each $x \in P$ (3.1.1, 3.1.2). Suppose that the system is not topologically equivalent to the rational flow on the torus T^2. Then, P_x^0 is homeomorphic to $\mathbb{R} \times S^1$ for each $x \in P$ (3.1.2). If $\partial P_x = \varnothing$ for some $x \in P$, then case (a) occurs. So, in the remainder of the proof we assume that $\partial P_x \neq \varnothing$ for each $x \in P$.

Let $x \in P$, $y \in \partial P_x$, V be an open neighborhood of $\mathbb{R}(y)$ homeomorphic to $\mathbb{R} \times S^1$ such that $V \cap F = \varnothing$ and $z \in P_x^0$ with $\mathbb{R}(z) \subset V$. The periodic orbits $\mathbb{R}(y)$, $\mathbb{R}(z)$ bound an annulus $A \subset V \cap P_x$.

Suppose that $\partial P_x = \mathbb{R}(y)$. The open set $P_x^0 - \mathbb{R}(z)$ has two connected components X_1, X_2 say with $X_1 \cap A^0 = \varnothing$, $X_2 \cap A^0 \neq \varnothing$. The set $X_2 \cap A^0$ is open and closed in X_2. Hence, $X_2 = A^0$. Since $\partial P_x = \mathbb{R}(y)$, $P_x = X_1 \cup A$ is homeomorphic to $\mathbb{R}^- \times S^1$ and has compact boundary. It is easy to verify that P_x is isolated from fixed points and periodic orbits in $M - P_x$. Hence, P_x is globally asymptotically stable (2.7(b)). Working as in the last part of the proof of 3.1.4, we can prove that M is homeomorphic to $\mathbb{R} \times S^1$ and case (b) occurs.

Now let $\partial P_x \neq \mathbb{R}(y)$ and $y' \in \partial P_x$ be such that $\mathbb{R}(y') \neq \mathbb{R}(y)$. The periodic orbits $\mathbb{R}(y)$, $\mathbb{R}(y')$ are the boundary curves of an invariant annulus $K \subset P_x$ which is isolated from fixed points and periodic orbits in $M - K$. Hence, K is globally asymptotically stable and $K=P$. The open set $M-P$ has two connected components, each homeomorphic to $\mathbb{R}\times S^1$. Again we can prove that M is homeomorphic to $\mathbb{R}\times S^1$ and case (c) occurs.

3.2. Theorem. *Let M be a non-orientable 2-manifold and $(\mathbb{R}, M, \varphi)$ a D^+-stable dynamical system on M such that $P\neq\varnothing$. Then, one of the following holds.*

(a) *M is homeomorphic to the projective plane $\mathbb{P}^2$, $M=F\cup P$ and F is a singleton.*

(b) *M is homeomorphic to the open Möbius strip M^2 and either* (i) *$M=P$,* (ii) *there is a globally asymptotically stable periodic orbit or* (iii) *P is a globally asymptotically stable closed Möbius strip.*

(c) *M is homeomorphic to the Klein bottle K^2 and $M=P$.*

Proof. Let $(\mathbb{R}, \tilde{M}, \tilde{\varphi})$ be the lifted dynamical system on the orientable double covering space $\tilde{M}$ of M, which is D^+-stable (2.9) and has at least one periodic orbit. Therefore, applying the results of 3.1, $\tilde{M}$ is homeomorphic either to $\mathbb{R}^2$, S^2, $\mathbb{R}\times S^1$ or T^2. Note that if $(\mathbb{R}, M, \varphi)$ has at least one fixed point then $(\mathbb{R}, \tilde{M}, \tilde{\varphi})$ has at least two fixed points. The case $\tilde{M}=\mathbb{R}^2$ is thus excluded by 3.1.4(b) and we are left with the last three cases. If $\tilde{M}=S^2$, then the lifted dynamical system is described by 3.1.4(a) and we have case (a). If $\tilde{M}=\mathbb{R}\times S^1$ or T^2 then M is homeomorphic to M^2 or K^2 respectively and the lifted dynamical system is described by 3.1.1 or 3.1.5. However, the dynamical system described by 3.1.5(b) is not compatible with the covering map of $\mathbb{R}\times S^1$ onto M^2. So, we have (b) and (c).

3.3. Corollary. *The only 2-manifolds which can support a D^+-stable dynamical system with at least one periodic orbit are the euclidean plane $\mathbb{R}^2$, the sphere S^2, the cylinder $\mathbb{R}\times S^1$, the torus T^2, the projective plane $\mathbb{P}^2$, the open Möbius strip M^2 and the Klein bottle K^2.*

3.4. Corollary. *The sphere S^2, the projective plane $\mathbb{P}^2$, the torus T^2 and the Klein bottle K^2 are the only compact 2-manifolds which can support* (*non-trivial*) *D^+-stable dynamical systems.*

Proof. Let M be a compact 2-manifold and $(\mathbb{R}, M, \varphi)$ a D^+-stable dynamical system. If the system is non-trivial and not topologically equivalent to any irrational flow on the torus T^2, then $M=F\cup P$ and $P\neq\varnothing$. So, the assertion is a consequence of 3.3.

4. D^+-Stable Dynamical Systems on 2-Manifolds Without Periodic Orbits

Let M be a 2-manifold supporting a (non-trivial) D^+-stable dynamical system without periodic orbits, not topologically equivalent to any irrational flow on the torus T^2. Then, M is necessarily non-compact (2.7(a)). In this section we consider *D^+-stable dynamical systems on non-compact 2-manifolds without periodic orbits.*

4.1. Example. Every non-compact 2-manifold M supports a (non-trivial) D^+-stable dynamical system without periodic orbits. As a result of the classifica-

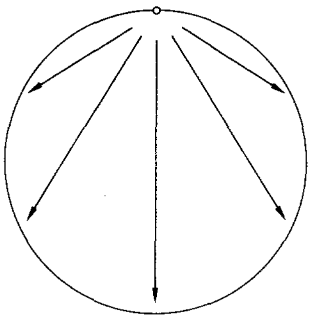

Fig. 1

tion of non-compact 2-manifolds given in [13], if e is an end of M, there is a disk D in the end point compactification M^+ of M (see [15]) such that $D \cap (M^+ - M) = \{e\}$. On M we consider the dynamical system which fixes the points of M outside the interior of D and in the interior of D is as illustrated in Fig. 1. This dynamical system on M is D^+-stable and has no periodic orbit.

4.2. Proposition. *Let* $(\mathbb{R}, M, \varphi)$ *be a* D^+*-stable dynamical system on the non-compact 2-manifold* M *with* $P = \varnothing$.

(a) *The restricted dynamical system on* $M - F$ *is parallelizable and each connected component of* $M - F$ *is homeomorphic to* $\mathbb{R}^2$ *or* $\mathbb{R} \times S^1$.

(b) *Let* $G = \{x \in M \colon L^+(x) \neq \varnothing\}$. *The map* $g \colon G \to F$ *with* $L^+(x) = \{g(x)\}$ *is continuous.*

(c) *The set* F *is locally connected, asymptotically stable and* $E^+(F) = G$.

(d) *For each* $s \in \partial F$ *there is at least one* $x \in G - F$ *such that* $L^+(x) = \{s\}$.

Proof. Assertion (a) is a restatement of 2.8, while (b) follows from the stability of fixed points. It is also evident that F is asymptotically stable and $E^+(F) = G$.

We show that F is locally connected. Let $s \in F$ and V be a positively invariant, open neighborhood of s such that $\bar{V} \subset G$ (2.3(d)). The non-empty set $A = \{x \in V \colon L^+(x) \subset V\}$ is a positively invariant, open neighborhood of s by (b). Let W be the connected component of A containing s. Then, W is an open neighborhood of s and $\overline{\mathbb{R}^+(x)} \subset W$ for all $x \in W$. It suffices to prove that $W \cap F$ is connected. Let F_1, F_2 be non-empty, closed sets in $W \cap F$ such that $W \cap F = F_1 \cup F_2$. Set $W_1 = \{x \in W \colon L^+(x) \subset F_i\}$, $i = 1, 2$. The sets W_1, W_2 are non-empty and $W = W_1 \cup W_2$. By (b) they are also closed in W. Hence, $W_1 \cap W_2 \neq \varnothing$. Therefore, $F_1 \cap F_2 \neq \varnothing$ and $W \cap F$ is connected. For (d) see the remark at the bottom of p. 569 in [1].

The following proposition can be proved in the same way as Theorem 4.8 in [1].

4.3. Proposition. (a) *Each connected component* K *of* F *is contained in a connected component* C *of* G *and* $K = C \cap F$. *Furthermore,* K *is asymptotically stable and* $E^+(K) = C$.

(b) *If* F *has a connected component* K *with compact boundary, then* $K = F$ *and* F *is globally asymptotically stable.*

(c) *F has a countable number of connected components.*

4.4. Remark. The study of D^+-stable dynamical systems without periodic orbits on non-compact 2-manifolds is continued in [15] with the study of the "behavior at infinity" of the orbits with empty limit sets. This is done by extending the system to a (possibly non D^+-stable) dynamical system on the end point compactification M^+ of M and studying the extended system near $M^+ - M$ [15].

5. Smoothing D^+-Stable Dynamical Systems on 2-Manifolds

In this last section we combine the results obtained in the preceding sections and the ideas of [10, 11] in order to smooth D^+-stable dynamical systems on 2-manifolds. Since the method of proof has already been presented, we do not provide full details.

5.1. Theorem. *Every D^+-stable dynamical system on a 2-manifold is topologically equivalent to a smooth D^+-stable dynamical system.*

Proof. Let $(\mathbb{R}, M, \varphi)$ be a D^+-stable dynamical system on a 2-manifold M. If it is topologically equivalent to some irrational flow on the torus T^2, then it is of course smoothable. If not, by 2.8, 3.1, 3.2 and using flow boxes as charts, we can construct a C^∞ structure $\mathscr{B}$ on $M-F$ with respect to which the restricted dynamical system in $M-F$ is smooth. The C^∞ structure $\mathscr{B}$ induces on $M-F$ the given topology of M. Hence it is diffeomorphic to the original C^∞ structure $\mathscr{A}$ of M restricted on $M-F$, by a smooth diffeomorphism $h: (M-F, \mathscr{B}) \to (M-F, \mathscr{A})$, because M is 2-dimensional. Moreover, we may choose h so that it can be extended to a homeomorphism H of M onto itself that fixes each point of F. Let $\mu(t, x) = h(\varphi(t, h^{-1}(x)))$ for $t \in \mathbb{R}$, $x \in M-F$. The triple $(\mathbb{R}, M-F, \mu)$ is a smooth dynamical system on $M-F$ (with respect to $\mathscr{A}$). Let η be the infinitesimal generator of this system. Using standard techniques one can construct a smooth function $f: M \to \mathbb{R}^+$ with $F = f^{-1}(0)$ such that the vector field $f\eta$ can be smoothly extended to all of M leaving the points of F fixed. The flow of the extended vector field on M is a smooth D^+-stable dynamical system topologically equivalent to $(\mathbb{R}, M, \varphi)$ under H.

Remark. As was pointed out by the referee, a much more powerful smoothing result concerning dynamical systems on *compact* 2-manifolds is proved in Gutierrez, C.: Smoothing continuous flows on two-manifolds and recurrence. Ergodic Th., Dynamical Systems **6**, 17–44 (1986).

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