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Chain recurrence in flows on the Klein bottle

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1. *Introduction*

In the classical Poincaré–Bendixson theory the object of study are the limit sets of a continuous flow on the 2-sphere S^2 and the behaviour of the orbits near them (see [**7**, **9**]). In [**2**] the second author proved that an assertion similar to the Poincaré–Bendixson theorem is true in the wider class of the 1-dimensional invariant (internally) chain recurrent continua of flows on S^2. On the other hand, it is known that among the closed 2-manifolds, the 2-sphere S^2, the projective plane $\mathbf{R}P^2$ and the Klein bottle K^2 are the only ones for which the Poincaré–Bendixson theorem is true (see [**1**, **8**, **11**]).

The motivation of the present paper was to examine to what extent the main results of [**2**] carry over to flows on $\mathbf{R}P^2$ and K^2. A first attempt to study chain recurrent sets of flows on closed 2-manifolds other than the 2-sphere was [**3**]. As one expects, the results of [**2**] carry over easily to $\mathbf{R}P^2$, since chain recurrence behaves well with respect to regular covering maps of compact manifolds, as we show in Section 3. The situation with K^2 is quite different, since it is doubly covered by the 2-torus T^2, where we have no Poincaré–Bendixson theorem. Actually, the Poincaré–Bendixson theorem for 1-dimensional invariant chain recurrent continua of flows on K^2 is not true. For example, identifying suitably the boundary periodic orbits of a 2-dimensional Reeb flow on a closed annulus (see [**7**, chapter III, 2·6]) we get a flow on K^2 with a 1-dimensional invariant chain recurrent continuum consisting of the unique periodic orbit and another orbit, which spirals against it in positive and negative time. As we prove in Theorem 4·4, this situation, or concatenations of it, is the only one where the Poincaré–Bendixson theorem for 1-dimensional invariant chain recurrent continua of flows on K^2 is not true. Then, we are concerned with the topological structure of the 1-dimensional chain components of a flow on K^2 with finitely many singularities. In Proposition 4·6 we find when such a set consists of finitely many orbits and is homeomorphic to a finite graph. An example shows that the hypothesis of Proposition 4·6 is essential. Finally, in Theorem 4·9 we give a description of the structure of the 1-dimensional chain components of a flow on K^2 with finitely many singular points.

2. *Chain recurrence and flows on surfaces*

Let X be a compact metrizable space, d a compatible metric and $\phi\colon \mathbf{R}\times X \to X$ a flow. We shall write $\phi(t,x) = tx$ and $\phi(I\times A) = IA$, if $I \subset \mathbf{R}$ and $A \subset X$. The orbit of the point $x \in X$ will be denoted by $C(x)$, the positive semiorbit by $C^+(x)$ and the negative by $C^-(x)$. The positive limit set of x will be denoted by $L^+(x)$ and the negative by $L^-(x)$.

Given ϵ, $T > 0$, an (ϵ, T)-*chain* from x to y is a pair of finite sets of points and times, denoted by $(x_0, ..., x_{q+1}|t_0, ..., t_q)$, such that $x = x_0$, $y = x_{q+1}$, $t_j \geqslant T$ and $d(t_j x_j, x_{j+1}) < \epsilon$ for every $j = 0, 1, ..., q$. If for every ϵ, $T > 0$ there is an (ϵ, T)-chain from x to y, we write xPy. The binary relation P is closed, transitive, flow invariant and depends only on the topology of X and not on the particular metric. The set $\Omega^+(x) = \{y \in X \colon xPy\}$ is called the *positive chain limit set* of x and the set $\Omega^-(x) = \{y \in X \colon yPx\}$ its *negative chain limit set*. Clearly, $L^\pm(x) \subset \Omega^\pm(x)$. A point $x \in X$ is called *chain recurrent* if xPx and the set $R(\phi)$ of chain recurrent points is closed and invariant. If $X = R(\phi)$, the flow is called chain recurrent. It is known that the connected components of $R(\phi)$ are the classes of the following equivalence relation in $R(\phi)$: $x \sim y$ if and only if xPy and yPx (see [**4**, theorem 3·6D]). Moreover, the restricted flow on each connected component of $R(\phi)$ is chain recurrent. The connected components of $R(\phi)$ are called chain components of the flow ϕ. It is also known that the restricted flow on each positive or negative limit set in X is chain recurrent (see [**6**, theorem 3·1]).

It follows from the representation of the complement of the chain recurrent set given in corollary 3·6B in [**4**], that chain recurrence is an invariant of topological equivalence. More precisely, let X and Y be compact metrizable spaces carrying topologically equivalent flows ϕ and ψ, respectively. If $h : X \to Y$ is a topological equivalence between ϕ and ψ, that is h is a homeomorphism sending orbits of ϕ onto orbits of ψ preserving the time orientation, then $h(R(\phi)) = R(\psi)$.

In [**2**] the second author developed the Poincaré–Bendixson theory of chain recurrence for flows on S^2. The proofs of lemma 3·3 and theorem 3·4 in [**2**] combined with the fact that a null homotopic simple closed curve on a 2-manifold bounds a disc (see [**5**, proposition 1·7]) work to give the following:

PROPOSITION 2·1. *Let A be a 1-dimensional, invariant, chain recurrent continuum in a flow on a closed 2-manifold X. If A contains a periodic orbit C, that is null homotopic in X, then $A = C$.*

3. *Covering spaces and chain recurrence*

In this section we shall study the behaviour of chain recurrence with respect to regular covering maps between compact manifolds. We shall need some elementary topological preliminaries. Let $p : \tilde{M} \to M$ be a regular covering map of connected compact manifolds with group of deck transformations Γ. Then Γ is finite and there exist compatible metrics $\tilde{d}$ on $\tilde{M}$ and d on M such that every element of Γ is a $\tilde{d}$-isometry and there exists $\delta > 0$ such that p maps the open ball $S(\tilde{x}, \delta)$ in $\tilde{M}$ isometrically onto the open ball $S(p(\tilde{x}), \delta)$ in M, which is evenly covered, for every $\tilde{x} \in \tilde{M}$.

LEMMA 3·1. *Let $p \colon \tilde{M} \to M$ be a k-fold regular covering map of connected compact manifolds. If $A \subset M$ is a connected compact set, then*

(i) *$p(C) = A$ for every connected component C of $p^{-1}(A)$ and*

(ii) *$p^{-1}(A)$ has at most k connected components.*

Proof. We shall use the metrics $\tilde{d}$ and d as above. Let C be a connected component of $p^{-1}(A)$ and $\tilde{x} \in C$. Then,

$$C = \{\tilde{y} \in p^{-1}(A) \colon \text{ for every } \epsilon > 0 \text{ there exists an } \epsilon\text{-chain in } p^{-1}(A) \text{ from } \tilde{x} \text{ to } \tilde{y}\}.$$

Let $\epsilon > 0$ and $z \in A$. Since A is connected, $p(\tilde{x})$ can be ϵ-chained in A to z. If $0 < \epsilon < \delta$, where δ is as in the above remarks, then $\tilde{x}$ is ϵ-chained in $p^{-1}(A)$ to some point of $p^{-1}(z)$. Since the covering is finite, it follows from this that there exists $\tilde{z} \in p^{-1}(z)$ which is ϵ-chained in $p^{-1}(A)$ for every $\epsilon > 0$. Hence $p^{-1}(z) \bigcap C \neq \emptyset$ for every $z \in A$, which means that $p(C) = A$. The second assertion follows directly from this, because if Γ is the group of deck transformations, then $p^{-1}(A) = \bigcup_{\gamma \in \Gamma} \gamma(C)$, each $\gamma(C)$ is connected and $k = |\Gamma|$.

Now let $p \colon \tilde{M} \to M$ be a regular covering map of connected manifolds carrying flows, such that p is equivariant. It is easy to see that the action of the group of the deck transformations Γ commutes with the flow on $\tilde{M}$. This means that $\gamma(t\tilde{x}) = t\gamma(\tilde{x})$ for every $t \in \mathbf{R}$, $\tilde{x} \in \tilde{M}$ and $\gamma \in \Gamma$.

PROPOSITION 3·2. *Let $p \colon \tilde{M} \to M$ be a k-fold regular covering map of connected compact manifolds carrying flows such that p is equivariant.*

(i) *If $x \in M$ is a chain recurrent point, then every point of $p^{-1}(x)$ is chain recurrent.*

(ii) *If $A \subset M$ is an invariant chain recurrent continuum, then $p^{-1}(A)$ is an invariant compact chain recurrent set with at most k connected components.*

Proof. We consider the metrics $\tilde{d}$ and d as in the beginning of this section and the corresponding number $\delta > 0$. Let $0 < \epsilon < \delta$ and $T > 0$. It suffices to prove that at least one point $\tilde{y} \in p^{-1}(x)$ can be joined to itself with an (ϵ, T)-chain, because Γ acts by $\tilde{d}$-isometries and commutes with flow on $\tilde{M}$. Let $(x_0, ..., x_{q+1} | t_0, ..., t_q)$ be an (ϵ, T)-chain from x to x. If $\tilde{x} \in p^{-1}(x)$, then $p(t_0\tilde{x}) = t_0 x$ and $x_1 \in S(t_0 x_0, \epsilon)$. We can choose $\tilde{x}_1 \in S(t_0\tilde{x}, \epsilon) \bigcap p^{-1}(x_1)$. Continuing in this way we obtain an (ϵ, T)-chain from $\tilde{x}$ to some point $\tilde{x}_{q+1} \in p^{-1}(x)$, with times $t_0, ..., t_q$ and intermediate points $\tilde{x}_j \in p^{-1}(x_j)$, $j = 0, 1, ..., q+1$. There exists $\gamma \in \Gamma$ such that $\tilde{x}_{q+1} = \gamma(\tilde{x})$ and $(\gamma(\tilde{x}_0), ..., \gamma(\tilde{x}_{q+1}) | t_0, ..., t_q)$ is an (ϵ, T)-chain. Thus,

$$(\tilde{x}_0, ..., \tilde{x}_{q+1}, \gamma(\tilde{x}_1), ..., \gamma(\tilde{x}_{q+1}) | t_0, ..., t_q, t_0, ..., t_q)$$

is an (ϵ, T)-chain from $\tilde{x}_0$ to $\gamma(\tilde{x}_{q+1})$. If $g \in \Gamma$ is such that $g(\tilde{x}) = \gamma(\tilde{x}_{q+1})$, then in the same way we can obtain an (ϵ, T)-chain from $\tilde{x}$ to $(g\gamma)(\tilde{x}_{q+1})$. Continuing this process, at some step we have an (ϵ, T)-chain from $\tilde{x}$ to some $\tilde{y} \in p^{-1}(x)$ passing through $\tilde{y}$, because the covering is finite. Therefore, $\tilde{y}$ is joined with itself with an (ϵ, T)-chain. The same proof also works to prove (ii).

COROLLARY 3·3. *Let $p \colon \tilde{M} \to M$ be a regular covering map of connected compact manifolds carrying flows such that p is equivariant.*

(i) *If R is the chain recurrent set of the flow on M, then $p^{-1}(R)$ is the chain recurrent set of the flow on $\tilde{M}$.*

(ii) *If A is a chain component in M, then the connected components of $p^{-1}(A)$ are chain components in $\tilde{M}$.*

Proof. (i) Let $\tilde{R}$ be the chain recurrent set in $\tilde{M}$. On the one hand we obviously have $p(\tilde{R}) \subset R$ and on the other hand $p^{-1}(R) \subset \tilde{R}$, by Proposition 3·2(i).

(ii) The connected components of $p^{-1}(R)$ are precisely the connected components of $p^{-1}(A)$, where A varies in the set of chain components in M. Since $\tilde{R} = p^{-1}(R)$, the connected components of $p^{-1}(A)$ are chain components in $\tilde{M}$ for every chain component A in M.

From the above, the main results of [**2**] and lemma 1 in [**10**], we obtain the following.

COROLLARY 3·4. *For every continuous flow on the projective plane $\mathbf{R}P^2$ the following are true.*

(i) *If $x \in \mathbf{R}P^2$ is a non-periodic chain recurrent point, then $L^+(x)$ and $L^-(x)$ consist of singular points.*

(ii) *If a 1-dimensional invariant chain recurrent continuum in $\mathbf{R}P^2$ contains a periodic orbit, then it is identical with the periodic orbit.*

(iii) *If a 1-dimensional invariant chain recurrent continuum in $\mathbf{R}P^2$ contains no singular point, then it is a periodic orbit.*

(iv) *If the flow on $\mathbf{R}P^2$ has finitely many singularities, then every 1-dimensional chain component consists of finitely many orbits and is homeomorphic to a finite graph.*

4. *One dimensional chain recurrent sets on the Klein bottle*

Recall that $\pi_1(K^2) =< \alpha, \beta : \alpha\beta = \beta\alpha^{-1} >$. A simple closed curve C in K^2, which is not null homotopic, represents one of the following elements of $\pi_1(K^2)$: α, α^{-1}, β^2, β^{-2} or $\beta^{-1}\alpha^n$, for some $n \in \mathbf{Z}$. If C represents $\alpha^{\pm 1}$, then $K^2 \setminus C$ is an open annulus. If C represents $\beta^{\pm 2}$, then $K^2 \setminus C$ consists of two open Möbius strips. Finally, if C represents $\beta^{-1}\alpha^n$, $n \in \mathbf{Z}$, then $K^2 \setminus C$ is an open Möbius strip (see [**7**, chapter IV, section 2]).

Let ϕ be a flow on K^2 and let $A \subset K^2$ be a 1-dimensional invariant chain recurrent continuum. If A contains no singular point, then every positive and negative limit set in A is a periodic orbit, by the Poincaré–Bendixson theorem for K^2 and proposition 7·11 of chapter II in [**7**]. If A is not a null homotopic periodic orbit, then no periodic orbit in A is null homotopic, by Proposition 2·1.

LEMMA 4·1. *If A contains a periodic orbit C such that $K^2 \setminus C$ consists of two disjoint open Möbius strips, then $A = C$.*

Proof. Let $A_i = A \cap \overline{K_i}$, $i = 1, 2$, where K_1 and K_2 are the connected components of $K^2 \setminus C$. Then A_1 is a 1-dimensional invariant compact set and it is easy to see that it is also connected. We shall prove that it is chain recurrent. Let $x \in A_1$ and $\epsilon, T > 0$. There is an (ϵ, T)-chain $(x_0, ..., x_{q+1} | t_0, ..., t_q)$ in A from x to x, since A is chain recurrent. Recall that since K^2 is a manifold, there exists a compatible metric d such that every open ball is connected. If not all the points of the chain are in A_1, there is some $1 \leqslant k \leqslant q$ such that $x_j \in A_1$ for $0 \leqslant j \leqslant k$ and $x_{k+1} \notin A_1$. Since $S(t_k x_k, \epsilon)$ is connected, we conclude that $C \cap S(t_k x_k, \epsilon) \neq \emptyset$. Similarly, there is some $k < l \leqslant q+1$ such that $x_j \in A_1$ for $l \leqslant j \leqslant q+1$ but $x_{l-1} \notin A_1$ and $C \cap S(x_l, \epsilon) \neq \emptyset$. Since C is a periodic orbit, if $z \in C \cap S(t_k x_k, \epsilon)$, there is some $s \geqslant T$ such that $sz \in C \cap S(x_l, \epsilon)$. It is now obvious that $(x_0, ..., x_k, z, x_l, ..., x_{q+1} | t_0, ..., t_k, s, t_l, ..., t_q)$ is an (ϵ, T)-chain in A_1 from x to x. So, we have a flow on the closed Möbius strip $\overline{K_1}$ and A_1 is a 1-dimensional invariant chain recurrent continuum. Gluing a closed disc carrying any flow with periodic boundary suitably oriented to $\overline{K_1}$ along their boundaries, we obtain a flow on $\mathbf{R}P^2$ in which A_1 is embedded (topologically and dynamically) and is a 1-dimensional invariant chain recurrent continuum which contains the periodic orbit C. Hence $A_1 = C$, by Corollary 3·4(ii). Similarly, $A_2 = C$ and therefore $A = C$.

LEMMA 4·2. *If A contains a periodic orbit C such that $K^2 \setminus C$ is an open Möbius strip, then $A = C$.*

Proof. Suppose that $A \neq C$. If there is a point $x \in K^2 \setminus C$ such that $L^+(x) = C$, then C is positively asymptotically stable in K^2 and therefore $\boldsymbol{\Omega}^+(x) = C$ for x in a neighbourhood of C by lemma 2·1 of [**2**], a contradiction. Similarly, there is no point $x \in K^2 \setminus C$ such that $L^-(x) = C$. It follows that C is approximated by periodic orbits in $A \setminus C$. Let N be a tubular neighbourhood of C in K^2 which contains no singular point of the flow. Then N is a Möbius strip and there exists a periodic orbit $C' \subset N \cap A$. It is clear now that C' bounds a Möbius strip $M \subset N$ and $K^2 \setminus M$ is also a Möbius strip. Hence $K^2 \setminus C'$ consists of two Möbius strips. This contradicts Lemma 4·1.

If A contains a periodic orbit, which represents α or α^{-1}, then A may not be identical with this periodic orbit. For example, consider the flow on $[1,3] \times S^1$ defined by the differential equation (in polar coordinates)

$$r' = (r-1)(3-r) \quad \text{and} \quad \theta' = r - 2,$$

which has only two periodic orbits, the boundary components oppositely oriented. We can identify them suitably to get a smooth flow on K^2 with only one periodic orbit C and such that every other orbit spirals bilateraly towards C. If $x \notin C$ and $A = C(x) \cup C$, then A is a 1-dimensional invariant chain recurrent continuum without singularities. If we multiply the infinitesimal generator of the flow on K^2 with a smooth non-negative function which vanishes only at one point on C, then we get a smooth flow on K^2 with non-periodic chain recurrent points whose positive and negative limit sets do not consist of singularities. Thus, corollary 3·2 of [**2**] does not carry over to flows on K^2 also.

In general, if A is not a periodic orbit, but contains periodic orbits, it has a special position in the phase portrait of the flow, as the following shows.

LEMMA 4·3. *If A is a 1-dimensional invariant chain recurrent continuum, that is not a periodic orbit, but contains at least one periodic orbit, then it contains every non-null homotopic periodic orbit of the flow on K^2.*

Proof. Let C be a periodic orbit in K^2, which is not null homotopic. If $p: T^2 \to K^2$ is the canonical double covering map, then $p^{-1}(C)$ consists of at most two periodic orbits, which are not null homotopic in T^2 and the complement of each one of them in T^2 is an open annulus. If C is not contained in A, then $p^{-1}(A)$ does not intersect $p^{-1}(C)$ and therefore it is contained in an invariant open annulus Y. By Proposition 3·2 and our assumptions, $p^{-1}(A)$ has at most two connected components and each one of them is not a periodic orbit but is a 1-dimensional invariant chain recurrent continuum, which contains at least one periodic orbit. Compactifying the flow on Y, we get a flow on S^2 with repect to which the connected components of $p^{-1}(A)$ are 1-dimensional invariant chain recurrent continua, which contain periodic orbits, but are not periodic orbits. This contradicts theorem 3·4 of [**2**].

From the above and the Poincaré–Bendixson theorem for the Klein bottle we have the following.

THEOREM 4·4. *Let ϕ be a flow on the Klein bottle and $A \subset K^2$ be a 1-dimensional invariant chain recurrent continuum which contains no singular point. Then, one of the following holds.*

(i) *A is a periodic orbit, or*

(ii) *A contains every non-null homotopic periodic orbit of the flow on K^2 and the complement in K^2 of every such orbit is an open annulus.*

In the sequel, we shall be concerned with the topological structure of the 1-dimensional chain components of a flow on K^2 with finitely many singular points.

LEMMA 4·5. *Let ϕ be a flow on K^2 and let A be a 1-dimensional invariant chain recurrent continuum, such that the positive and the negative limit set of every non-periodic orbit in A consist of singular points. If A contains a periodic orbit C, then $A = C$.*

Proof. Suppose that $A \neq C$. Then A contains every periodic orbit of the flow and the complement in K^2 of every such orbit is an open annulus, by Proposition 2·1 and Theorem 4·4. Our assumption, that the positive and the negative limit set of every non-periodic orbit in A consist of singular points, implies that every neighbourhood of C contains a periodic orbit different from C. Let N be a tubular neighbourhood of C, which contains no singular point of the flow. Since C is two-sided, N is an open annulus. Let $S \subset N$ be a local section to the flow, which passes through exactly one point of C and is a compact arc (see [**9**, chapter VII, 1·6]). By the above, there is a periodic orbit $C' \subset N$ sufficiently close to C, such that $C' \cap S$ is a singleton. Obviously, C and C' are the boundary curves of an invariant annulus $N' \subset N$. Note that $A \cap N'$ consists entirely of periodic orbits. The interval $J \subset S$ with endpoints $C \cap S$ and $C' \cap S$ is contained in N'. Since $J \setminus A$ is an open subset of J, there is a subinterval $I \subset J \setminus A$ with endpoints in A. Then, $\mathbf{R}I$ is an open annulus in $K^2 \setminus A$ with boundary two periodic orbits in A. Thus, $Y = K^2 \setminus \mathbf{R}I$ is an invariant closed annulus which contains A and its boundary curves are periodic orbits. We attach now to the two boundary components of Y discs, each one carrying a flow with periodic boundary suitably oriented. In this way we obtain a flow on S^2, which contains A as 1-dimensional invariant chain recurrent continuum, that is not a periodic orbit, but contains periodic orbits. This contradicts theorem 3·4 in [**2**].

PROPOSITION 4·6. *Let ϕ be a flow on K^2 with finitely many singular points and let $A \subset K^2$ be a 1-dimensional chain component. If the positive and the negative limit set of every non-periodic orbit in A consist of singular points, then A consists of finitely many orbits and is homeomorphic to a finite graph.*

Proof. Because of Lemma 4·5 we need to consider only the case where A contains no periodic orbit. Let $p\colon T^2 \to K^2$ be the canonical double covering map. The flow on K^2 can be lifted to a flow on T^2 with finitely many singular points (see [**10**, lemma 1]). By Corollary 3·3, $p^{-1}(A)$ has at most two connected components and each one of them is a 1-dimensional chain component of the flow on T^2. It is easy to see that the positive and the negative limit set of every non-periodic orbit in $p^{-1}(A)$ consist of singular points, by our assumptions. The proof of theorem 4·1 of [**2**] now works to show that $p^{-1}(A)$ is locally an arc at its non-singular points. Since p is an equivariant covering map, the same is true for A. In exactly the same way as in theorem 4·2 and corollary 4·3 of [**2**] it follows that every singular point in A is an isolated invariant set and is the positive and negative limit set of finitely many orbits in A. So A consists of finitely many orbits and is homeomorphic to a finite graph.

If a 1-dimensional chain component A of a flow on K^2 with finitely many singular points does not satisfy the hypothesis of Proposition 4·6, then it may not be locally an arc at its non-singular points. For example, consider the flow on the closed annulus illustrated in Fig. 1. Identifying the two boundary components suitably we get a flow on K^2 with two singular points y and z and no periodic orbit. There is exactly one non-singular orbit C_1 such that $L^+(C_1) = L^-(C_1) = \{z\}$ and exactly one orbit C_2 such that $L^-(C_2) = \{z\}$ and $L^+(C_2) = \overline{C_1}$. The set $A = \overline{C_1 \cup C_2}$ is a 1-dimensional chain component, which does not satisfy the hypothesis of Proposition 4·6, but is not locally an arc at the points of C_1.

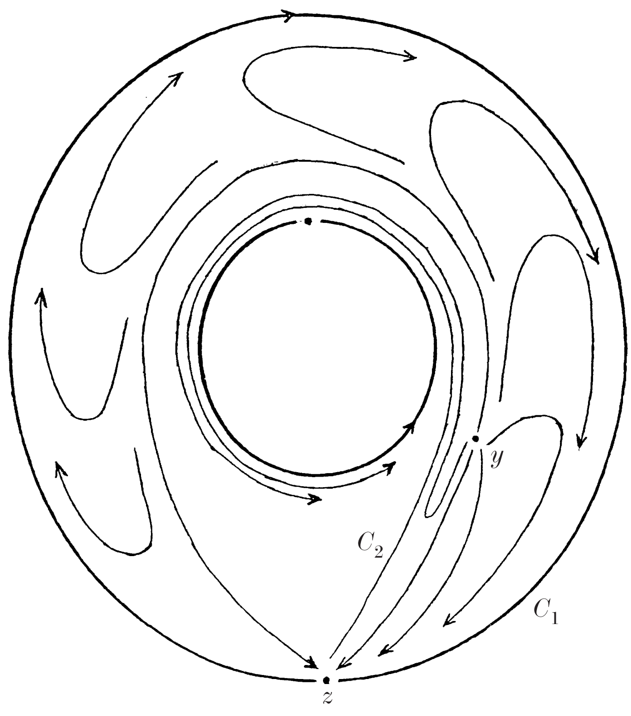

Fig. 1

If a 1-dimensional chain component A of a flow on K^2 with finitely many singularities is not locally an arc at its non-singular points, then it has a special position in the phase portrait of the flow. The situation is analogous to Lemma 4·3, but now we have to consider also orbit cycles. Recall that an *orbit chain* is a finite sequence $C_1, \ldots, C_n$ of orbits, such that $L^-(C_i)$ is a singular point z_{i-1} and $L^+(C_i)$ is a singular point z_i (see [**3**]). The singular points $z_0, \ldots, z_n$ are the *nodes* of the orbit chain. If $z_0 = z_n$, then we have an *orbit cycle*. If moreover the support $C = \bigcup_{i=1}^n \overline{C_i}$ of the orbit cycle is a simple closed curve, then we have a *simple orbit cycle*. Note that every orbit cycle is a chain recurrent set and contains a simple orbit cycle.

Lemma 4·7. *If some non-null homotopic simple orbit cycle C is not contained in A, then A consists of finitely many orbits and is homeomorphic to a finite graph.*

Proof. Since A is a chain component, C must be disjoint from A. Thus, A is contained in an invariant open set Y with boundary C and Y is either an open Möbius strip or an open annulus. Compactifying the flow on Y, we get a flow on $\mathbf{R}P^2$ or S^2, respectively, with finitely many singularities, with respect to which A is still a 1-dimensional chain component. This is so, because otherwise the points of A would be chained to the points of C and vice versa, which would mean that A is not a chain component of the flow in K^2. Corollary 4·4 in [**2**] and Corollary 3·4 imply now that A consists of finitely many orbits and is homeomorphic to a finite graph.

Lemma 4·8. *If A contains a simple orbit cycle C such that $K^2 \setminus C$ consists of one or two disjoint open Möbius strips, then A consists of finitely many orbits and is homeomorphic to a finite graph.*

Proof. The proof in the case of the two Möbius strips is a simple modification of the proof of Lemma 4·1 and is therefore omitted. Thus, we assume that $K^2 \setminus C$ is an open Möbius strip, which is invariant. Compactifying the flow on $K^2 \setminus C$ we get a flow on $\mathbf{R}P^2$ with finitely many singular points, the point at infinity being one. The set $\overline{A \setminus C}$ (closure taken in $\mathbf{R}P^2$) is a 1-dimensional chain component of the flow on $\mathbf{R}P^2$, if it is not empty. Hence, $\overline{A \setminus C}$ consists of finitely many orbits and is homeomorphic to a finite graph, by Corollary 3·4(iv). Thus, there are finitely many orbits in $A \setminus C$ whose positive or negative limit set is contained in C. Note also that if the positive (or negative) limit set of an orbit in $A \setminus C$ has non-empty intersection with C, then it is contained in C. Let $x \in A \setminus C$ be such that $L^+(x) \subset C$. Then, either $L^+(x)$ is a singular point or $L^+(x) = C$, because C is a simple orbit cycle and $L^+(x)$ is a chain recurrent set. If $L^+(x) = C$, then C is positively asymptotically stable in K^2 and therefore $\mathbf{\Omega}^+(x) = C$, which contradicts the fact that A is a chain component. This shows that $L^+(x)$ is a singular point for every non-periodic point $x \in A$ and similarly for $L^-(x)$. By Proposition 4·6, A is homeomorphic to a finite graph.

Summarizing, the structure of the 1-dimensional chain components can be described as follows.

Theorem 4·9. *Let ϕ be a flow on K^2 with finitely many singular points and let $A \subset K^2$ be a 1-dimensional chain component. Then, one of the following holds.*

(i) *A consists of finitely many orbits and is homeomorphic to a finite graph, or*

(ii) *A contains every non-null homotopic periodic orbit and every non-null homotopic simple orbit cycle and the complement in K^2 of every such periodic orbit and simple orbit cycle is an open annulus.*

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