On the existence of absolutely continuous conformal measures

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ABSTRACT
Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. We present sufficient conditions which imply the existence of absolutely continuous conformal measures for $T$ with respect to a given ergodic $T$-invariant Borel probability measure. The same conditions give measurable or $L^\infty$ solutions of the corresponding cohomological equation. We illustrate our results in an example of a sofic system.

1 Introduction
Let $X$ be a compact metric space, $T : X \to X$ a continuous surjection and let $f : X \to \mathbb{R}$ be a continuous function. We call a Borel probability measure $\nu$ on $X$ an $e^f$-conformal measure for $T$ if $\nu$ is equivalent to $T_*\nu$ and \[
\frac{d\nu}{d(T_*\nu)} = e^f.
\] This kind of measure has been used without a particular name in [5].

In this note we study the existence of absolutely continuous conformal measures with respect to a given ergodic $T$-invariant Borel probability measure. We present a sufficient condition for the existence of an absolutely continuous conformal measure for a continuous surjection. The problem of the existence of an $e^f$-conformal measure $\nu$ for a homeomorphism $T$ which is absolutely continuous with respect to an ergodic $T$-invariant Borel probability measure $\mu$ is closely related to the existence and regularity properties of solutions of the cohomological equation $f = u - u \circ T$. This relation is explained with details in section 2. If there exists a continuous solution $u$, then $f$ is called a continuous coboundary. According to the classical Gottschalk-Hedlund theorem (see page 102 in [4]), if $T$ is minimal, then $f$ is a continuous coboundary if and only if there exists $x_0 \in X$ such that

$$\sup\{|\sum_{k=0}^{n-1} f(T^k(x_0))| : n \in \mathbb{N}\} < +\infty.$$ 

The main result is Theorem 3.5 which can be stated as follows.
Main Theorem. Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, where $E_n(f) = e^{S_n(f)}$ and $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$, then there exists an $e^f$-conformal measure $\nu$ for $T$ which is absolutely continuous with respect to $\mu$. Moreover, $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ and $-\log\left(\frac{d\nu}{d\mu}\right)$ is a measurable solution of the cohomological equation $f = u - u \circ T$. □

If $T$ is a homeomorphism, then in Theorem 3.7 we prove that if the stronger condition

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$) holds for some constant $c \geq 1$, then an $e^f$-conformal measure $\nu$ for $T$ exists which is equivalent to $\mu$ and $\log\left(\frac{d\nu}{d\mu}\right) \in L^\infty(\mu)$. Also, $f$ is a $L^\infty(\mu)$ coboundary with transfer function $-\log\left(\frac{d\nu}{d\mu}\right)$. This result holds without the assumption that $T$ is minimal.

In a final section we illustrate our results in an example of a known sofic system which is attributed to B. Markus in [3]. In this example $T$ is the two-sided left shift restricted on a suitable compact subset $X$ of $\{-1,1\}^\mathbb{Z}$ and is a continuous factor of a subshift of finite type on $N + 1$ symbols for some integer $N \geq 2$. The system is not minimal, it is chaotic and it has the strong specification property.

2 Conformal measures

Let $T : X \to X$ be a continuous surjection of a compact metric space $X$ and let $f : X \to \mathbb{R}$ be a continuous function. An $e^f$-conformal measure for $T$ is a Borel probability measure $\nu$ on $X$ such that

$$\int_X \phi d\nu = \int_X (\phi \circ T)e^f d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$. Evidently, an $e^f$-conformal measure for $T$ is $T$-quasi-invariant and is an $e^{-f \circ T^{-1}}$-conformal measure for $T^{-1}$, in case $T$ is a homeomorphism.

It is easy to see that if $h : X \to X$ is a homeomorphism and $S = h \circ T \circ h^{-1}$, then $h_* \nu$ is an $e^{hf^{-1}}$-conformal measure for $S$ for every $e^f$-conformal measure $\nu$ for $T$.

For the reader’s convenience we shall describe a construction of conformal measures for homeomorphisms due to M. Denker and M. Urbanski given in [1]. Note that there may be no $e^f$-conformal measure for $T$ for a given continuous function $f : X \to \mathbb{R}$. 

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This is the case, for example, if $f > 0$, since we necessarily have \( \int_X e^f \, d\nu = 1 \) for every \( e^f \)-conformal measure. We need some preliminary observations.

Let \( (a_n)_{n \in \mathbb{N}} \) be a sequence of real numbers and let \( c = \limsup_{n \to +\infty} \frac{a_n}{n} \). The series \( \sum_{n=1}^{\infty} e^{a_n - ns} \) converges for \( s > c \), diverges for \( s < c \) and we cannot tell for \( s = c \), by the root test.

**Lemma 2.1.** There exists a sequence of positive real numbers \( (b_n)_{n \in \mathbb{N}} \) such that \( \lim \frac{b_n}{b_{n+1}} = 1 \) and the series \( \sum_{n=1}^{\infty} b_n e^{a_n - ns} \) converges for \( s > c \) and diverges for \( s \leq c \).

**Proof.** If the series \( \sum_{n=1}^{\infty} e^{a_n - nc} \) diverges, we may take \( b_n = 1 \) for every \( n \in \mathbb{N} \). Suppose that it converges. We choose a sequence of positive integers \( (n_k)_{k \in \mathbb{N}} \) such that
\[
\lim_{k \to +\infty} \frac{n_k}{n_{k+1}} = 0 \quad \text{and} \quad \lim_{k \to +\infty} \frac{a_{n_k}}{n_k} = c.
\]
It suffices now to put \( \epsilon_k = \frac{a_{n_k}}{n_k} - c \) and take
\[
b_n = \exp \left[ n \left( \frac{n - n_k}{n_k - n_{k-1}} \epsilon_{k-1} + \frac{n - n_{k-1}}{n_k - n_{k-1}} \epsilon_k \right) \right]
\]
for \( n_{k-1} \leq n < n_k \). \( \square \)

Let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f \, d\mu = 0 \) for some ergodic \( T \)-invariant Borel probability measure \( \mu \). It is well known that the set of points \( x \in X \) such that the limit
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x))
\]
exists in \( \mathbb{R} \) has measure 1 with respect to every \( T \)-invariant Borel probability measure, and is therefore non-empty. So there exists a point \( x \in X \) such that
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x)) = \int_X f \, d\mu = 0,
\]
since \( \mu \) is assumed to be ergodic.

If we take \( a_n = -\sum_{k=1}^{n} f(T^{-k}(x)) \), then \( \lim_{n \to +\infty} \frac{a_n}{n} = 0 \). Let \( M_s = \sum_{n=1}^{\infty} b_n e^{a_n - ns} \), \( s > 0 \), where \( (b_n)_{n \in \mathbb{N}} \) is the corresponding sequence given from Lemma 2.1, and
\[
\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \delta_{T^{-n}(x)}, \quad s > 0.
\]
Proposition 2.2. Every accumulation point with respect to the weak* topology of the directed family of Borel probability measures \((\mu_s)_{s>0}\), as \(s \downarrow 0\), is a \(e^f\)-conformal measure for \(T\).

Proof. For every continuous function \(\phi : X \rightarrow \mathbb{R}\) we have on the one hand

\[
\int_X \phi d\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \phi(T^{-n}(x))
\]

and on the other

\[
\int_X (\phi \circ T) e^f d\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \phi(T^{-n+1}(x)) e^{f(T^{-n}(x))}
\]

\[
= \frac{1}{M_s} \left[ b_1 e^{-s} \phi(x) + \sum_{n=1}^{\infty} b_{n+1} e^{-s} e^{a_n - ns} \phi(T^{-n}(x)) \right].
\]

Since \(\lim_{s \downarrow 0} \frac{b_1 e^{-s} \phi(x)}{M_s} = 0\), we need to estimate the difference

\[
\frac{1}{M_s} \left| \sum_{n=1}^{\infty} b_n e^{a_n - ns} \phi(T^{-n}(x)) - \sum_{n=1}^{\infty} b_{n+1} e^{-s} e^{a_n - ns} \phi(T^{-n}(x)) \right| 
\leq \frac{\|\phi\|}{M_s} \sum_{n=1}^{\infty} \frac{b_{n+1}}{b_n} e^{-s} - 1 \left| b_n e^{a_n - ns} \right|.
\]

Given \(\epsilon > 0\) there exists \(n_0 \in \mathbb{N}\) such that for \(n \geq n_0\) we have

\[
\left| \frac{b_{n+1}}{b_n} - 1 \right| < \epsilon
\]

and therefore

\[
\left| \frac{b_{n+1}}{b_n} e^{-s} - 1 \right| < \epsilon e^{-s} + |1 - e^{-s}|.
\]

It follows that

\[
\frac{1}{M_s} \sum_{n=1}^{\infty} \left| \frac{b_{n+1}}{b_n} e^{-s} - 1 \right| b_n e^{a_n - ns} < \frac{1}{M_s} \sum_{n=1}^{n_0} \left| \frac{b_{n+1}}{b_n} e^{-s} - 1 \right| b_n e^{a_n - ns} + \frac{\epsilon e^{-s} + |1 - e^{-s}|}{M_s} \sum_{n=n_0}^{\infty} b_n e^{a_n - ns}
\]

\[
\leq \frac{1}{M_s} \sum_{n=1}^{n_0} \left| \frac{b_{n+1}}{b_n} e^{-s} - 1 \right| b_n e^{a_n - ns} + \epsilon e^{-s} + |1 - e^{-s}|.
\]

Since \(\lim_{s \downarrow 0} M_s = +\infty\), there exists some \(0 < s_0 < 1\) such that \(\epsilon e^{-s} + 1 - e^{-s} < 2\epsilon\)

\[
\frac{1}{M_s} \sum_{n=1}^{n_0} \left| \frac{b_{n+1}}{b_n} e^{-s} - 1 \right| b_n e^{a_n - ns} < \epsilon
\]

and \(\frac{b_1 e^{-s}}{M_s} < \epsilon\) for all \(0 < s < s_0\).
Summarizing, for every $\epsilon > 0$ there exists $0 < s_0 < 1$ such that

$$\left| \int_X \phi d\mu_s - \int_X (\phi \circ T)e^f d\mu_s \right| < 4\epsilon \| \phi \|
$$

for all $0 < s < s_0$ and every continuous function $\phi : X \to \mathbb{R}$. This proves the assertion. □

There is a close relation between $e^f$-conformal measures for a homeomorphism $T : X \to X$ of a compact metric space and solvability of the cohomologic al equation $f = u - u \circ T$, where $f : X \to \mathbb{R}$ is continuous.

Let $\mu$ be any $T$-invariant Borel probability measure. If there exists a measurable solution $u$ of the above cohomological equation defined $\mu$-almost everywhere such that $e^{-u} \in L^1(\mu)$, then there exists a $e^f$-conformal measure $\nu$ for $T$ equivalent to $\mu$ with density

$$\frac{d\nu}{d\mu} = \frac{e^{-u}}{\int_X e^{-u} d\mu}.
$$

Thus, if there exists a continuous solution $u$, then for every $T$-invariant Borel probability measure we get an equivalent $e^f$-conformal measure for $T$. Moreover, in this case, every $e^f$-conformal measure $\nu$ for $T$ is obtained in this way. Indeed, we have

$$\int_X \phi e^u d\nu = \int_X (\phi \circ T)e^u d\nu
$$

for every continuous function $\phi : X \to \mathbb{R}$, and so the equivalent measure $\mu$ to $\nu$ with density

$$\frac{d\mu}{d\nu} = \frac{e^u}{\int_X e^u d\nu}
$$

is $T$-invariant. Consequently, if $f$ is a continuous coboundary, then the $e^f$-conformal measures for $T$ are in one-to-one correspondence with the $T$-invariant Borel probability measures and each $e^f$-conformal measure for $T$ is equivalent to its corresponding $T$-invariant measure.

Conversely, suppose that $\mu$ is an ergodic $T$-invariant Borel probability measure and $f : X \to \mathbb{R}$ is a continuous function such that $\int_X f d\mu = 0$. Suppose further that there exists a $e^f$-conformal measure $\nu \in \mathcal{M}(X)$ for $T$ which is absolutely continuous with respect to $\mu$ and let $g = \frac{d\nu}{d\mu}$. For every measurable set $A \subset X$ we have

$$\int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T)e^f d\nu = \int_X (\chi_A \circ T)e^f g d\mu
$$

and therefore

$$\int_{T^{-1}(A)} [ge^f - (g \circ T)] d\mu = 0.
$$

Since $\mu$ is $T$-invariant, it follows that $g \circ T = ge^f$ $\mu$-almost everywhere. The ergodicity of $\mu$ implies now that $g > 0$ $\mu$-almost everywhere. So, $u = -\log g$ is a measurable solution of the cohomological equation $f = u - u \circ T$. If $\log g \in L^\infty(\mu)$ and $T$ is a
minimal homeomorphism, then there exists some continuous function \( u : X \to \mathbb{R} \) such that \( f = u - u \circ T \), by Proposition 4.2 on page 46 in [2].

Note that \( \nu \) is equivalent to \( \mu \), because \( g > 0 \). We remark that this is actually a more general fact which holds for every \( T \)-quasi-invariant Borel probability measure. To see this, let \( T : X \to X \) be a homeomorphism of a compact metric space \( X \) and \( \mu \) be an ergodic \( T \)-invariant Borel probability measure. Let \( \nu \) be a \( T \)-quasi-invariant Borel probability measure which is absolutely continuous with respect to \( \mu \). Let \( g = \frac{d\nu}{d\mu} \) and \( A = g^{-1}(0) \). If \( S = \bigcup_{n \in \mathbb{Z}} T^n(A) \), then \( S \) is \( T \)-invariant and \( \nu(S) = 0 \). On the other hand \( \mu(X \setminus S) > 0 \), and since \( \mu \) is ergodic we get \( \mu(S) = 0 \), that is \( g > 0 \) \( \mu \)-almost everywhere. In particular, if \( T \) is uniquely ergodic, then every \( T \)-quasi-invariant measure for \( T \) which is absolutely continuous with respect its unique invariant Borel probability measure is equivalent to it.

### 3 Absolutely continuous conformal measures

Let \( X \) be a compact metric space and \( \mu \in M(X) \). The set

\[ A_\mu = \{ \nu \in M(X) : \nu \ll \mu \} \]

is not empty, since it contains \( \mu \), and is convex. In general, \( A_\mu \) is not a closed subset of \( M(X) \) with respect to the weak* topology. For example, if we let \( \mu \) be the Lebesgue measure on the unit interval \([0,1]\) and for \( 0 < \epsilon < 1 \) we let \( \mu_\epsilon \) denote the Borel probability measure on \([0,1]\) with density \( \frac{1}{\epsilon} \chi_{[0,\epsilon]} \), then \( \lim_{\epsilon \to 0} \mu_\epsilon \) is the Dirac point measure at 0.

**Lemma 3.1.** Let \( X \) be a compact metric space and \( \mu \in M(X) \). Let \((\nu_n)_{n \in \mathbb{N}}\) be a sequence in \( A_\mu \) converging weakly* to some \( \nu \in M(X) \) and let \( f_n = \frac{d\nu_n}{d\mu} \), \( n \in \mathbb{N} \). If there exist non-negative \( h \), \( g \in L^1(\mu) \) such that \( h \leq f_n \leq g \) for every \( n \in \mathbb{N} \), then \( \nu \in A_\mu \) and \( h \leq \frac{d\nu}{d\mu} \leq g \).

**Proof.** Since \( \nu \) is a finite measure, there exists a (countable) basis \( \mathcal{U} \) of the topology of \( X \) such that \( \nu(\partial U) = 0 \) for every \( U \in \mathcal{U} \). So \( \mathcal{U} \) is contained in the algebra \( \mathcal{C}(\nu) = \{ A | A \subset X \text{ Borel and } \nu(\partial A) = 0 \} \)

and since it generates the Borel \( \sigma \)-algebra of \( X \), so does \( \mathcal{C}(\nu) \). Let now \( A \subset X \) be a Borel set with \( \mu(A) = 0 \) and \( \epsilon > 0 \). There exists \( 0 < \delta < \epsilon \) such that \( \int_B gd\mu < \epsilon \) for every Borel set \( B \subset X \) with \( \mu(B) < \delta \), because \( g \in L^1(\mu) \). There exists some \( A_0 \in \mathcal{C}(\nu) \) such that \( \mu(A \Delta A_0) < \delta \) and \( \nu(A \Delta A_0) < \delta \). Thus \( \mu(A_0) < \delta \) and \( |\nu(A) - \nu(A_0)| < \delta \).

By weak* convergence, \( \nu(A_0) = \lim_{n \to +\infty} \nu_n(A_0) \) and so there exists some \( n_0 \in \mathbb{N} \) such that \( |\nu_n(A_0) - \nu(A_0)| < \epsilon \) for \( n \geq n_0 \). Therefore,

\[ \nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq \int_{A_0} gd\mu + \epsilon < 2\epsilon. \]
It follows that $0 \leq \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in A_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(P_n)_{n \in \mathbb{N}}$ of $X$ such that $P_{n+1}$ is a refinement of $P_n$, the Borel $\sigma$-algebra of $X$ is generated by $\bigcup_{n=1}^{\infty} P_n$ and $\mu(\partial B) = 0$ for every $B \in P_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of $X$ such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively $P_n$ to be the finite family consisting of Borel sets with positive $\mu$ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in P_{n-1}$, taking $P_0 = \{X\}$.

Let $P_n(x)$ denote the element of $P_n$ which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n \to +\infty} \frac{\nu(P_n(x))}{\mu(P_n(x))},$$

$\mu$-almost everywhere on $X$ and in $L^1(\mu)$ (see page 8 in [6]). On the other hand, by the weak* convergence and since $\nu \in A_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(P_k(x)) - \nu_{n_k}(P_k(x))| < \frac{1}{k}\mu(P_k(x)).$$

It follows that

$$0 \leq \frac{\nu(P_k(x))}{\mu(P_k(x))} < \frac{1}{k} + \frac{\nu_{n_k}(P_k(x))}{\mu(P_k(x))} = \frac{1}{k} + \frac{1}{\mu(P_k(x))} \int_{P_k(x)} f_{n_k} d\mu < \frac{1}{k} + \frac{1}{\mu(P_k(x))} \int_{P_k(x)} g d\mu.$$

Since

$$\lim_{k \to +\infty} \frac{1}{\mu(P_k(x))} \int_{P_k(x)} g d\mu = g(x)$$

$\mu$-almost everywhere on $X$ and in $L^1(\mu)$, it follows that $0 \leq \frac{d\nu}{d\mu}(x) \leq g(x)$ $\mu$-almost everywhere on $X$.

Similarly, from

$$\frac{\nu(P_k(x))}{\mu(P_k(x))} > \frac{1}{k} + \frac{\nu_{n_k}(P_k(x))}{\mu(P_k(x))} = \frac{1}{k} + \frac{1}{\mu(P_k(x))} \int_{P_k(x)} f_{n_k} d\mu \geq \frac{1}{k} + \frac{1}{\mu(P_k(x))} \int_{P_k(x)} h d\mu$$

follows that $h(x) \leq \frac{d\nu}{d\mu}(x)$ $\mu$-almost everywhere on $X$. $\square$

Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. For any continuous function $f : X \to \mathbb{R}$ we put $S_0(f) = -\sum_{k=0}^{n-1} f \circ T^k$ and $E_n(f) = e^{S_n(f)}$.

Let $M_n = \sup\{S_n(f)(x) : x \in X\}$ and $L_n = \inf\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{N}$. Since $S_n(f) \circ T = S_{n+1}(f) + f$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T)e^{-f} - g_n = E_n(f) - e^{-f}.$$
Let now \( \mu \in \mathcal{M}(X) \) be \( T \)-invariant and suppose that \( \int_X f \, d\mu = 0 \). So, \( L_n \leq 0 \leq M_n \) for every \( n \in \mathbb{N} \). Putting \( h_n = \frac{g_n}{\int_X g_n \, d\mu} \), we get

\[
(h_n \circ T) - h_n e^f = \frac{e^f - e^{-S_n(f)}}{e^{-S_n(f)}} \int_X g_n \, d\mu,
\]

for every \( n \in \mathbb{N} \).

Suppose that there exists a positive \( h \in L^1(\mu) \) such that \( E_n(f) \leq h \int_X E_n(f) \, d\mu \) for every \( n \in \mathbb{N} \). Then also \( 0 \leq h_n \leq h \) for \( n \in \mathbb{N} \). If \( \nu_n \) denotes the element of \( E_\mu \) with \( h_n = \frac{d\nu_n}{d\mu} \), then \( \{\nu_n : n \in \mathbb{N}\} \subset E_\mu \), by Lemma 3.1.

**Proposition 3.2.** Let \( X \) be a compact metric space and \( T : X \to X \) a continuous surjection. Let \( \mu \in \mathcal{M}(X) \) be \( T \)-invariant and let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f \, d\mu = 0 \). Suppose that

(i) there exists a positive \( h \in L^1(\mu) \) such that \( E_n(f) \leq h \int_X E_n(f) \, d\mu \) for every \( n \in \mathbb{N} \), and

(ii) the sequence \( e^{-M_n} \sum_{k=0}^{n-1} e^{L_k} \), \( n \in \mathbb{N} \), is unbounded.

Then there exists an \( e^f \)-conformal measure for \( T \) which is absolutely continuous with respect to \( \mu \).

**Proof.** Using the above notations, it suffices to prove that there exists a sequence of positive integers \( n_j \to +\infty \) such that \( \lim_{j \to +\infty} \int_X (h_{n_j} \circ T) - h_{n_j} e^f \, d\nu = 0 \) \( \mu \)-almost everywhere on \( X \). Indeed, passing to a subsequence if necessary, there exists \( \nu \in E_\mu \) such that \( \nu = \lim_{j \to +\infty} \nu_{n_j} \), by Lemma 3.1. Since \( \mu \) is \( T \)-invariant, for every continuous function \( \phi : X \to \mathbb{R} \) we have

\[
\int_X (\phi - (\phi \circ T)e^f) \, d\nu = \lim_{j \to +\infty} \int_X (\phi \circ T)((h_{n_j} \circ T) - h_{n_j} e^f) \, d\mu = 0,
\]

by dominated convergence, because

\[
|(\phi \circ T)((h_n \circ T) - h_n e^f)| \leq \|\phi\|((h \circ T) + he^f) \in L^1(\mu).
\]

Since

\[
|(h_n \circ T) - h_n e^f| = e^f \frac{|E_n(f) - e^{-f}|}{\int_X g_n \, d\mu},
\]

we need only prove that there exist \( n_j \to +\infty \) such that

\[
\lim_{j \to +\infty} \mu\{x \in X : |E_{n_j}(f)(x) - e^{-f(x)}| \geq \delta \int_X g_{n_j}(x) \, d\mu\} = 0
\]

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for every $\delta > 0$. Let

$$A_{n,\delta} = \{ x \in X : E_n(f)(x) \geq e^{-f(x)} + \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x) \},$$

and

$$A'_{n,\delta} = \{ x \in X : E_n(f)(x) \leq e^{-f(x)} - \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x) \}.$$ 

Our assumption (i) implies that it suffices to prove the existence of a sequence of positive integers $n_j \to +\infty$ such that

$$\lim_{j \to +\infty} \mu(A_{n_j,\delta}) = \lim_{j \to +\infty} \mu(A'_{n_j,\delta}) = 0$$

for every $\delta > 0$.

For every $x \in A_{n,\delta}$ we have

$$\frac{h(x)}{\delta} > e^{-M_n} \sum_{k=0}^{n-1} E_k(f)(x)$$

and integrating over $A_{n,\delta}$ we obtain

$$\frac{1}{\delta} \int_X h \, d\mu \geq \mu(A_{n,\delta}) e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}.$$ 

Similarly, for every $x \in A'_{n,\delta}$ we have

$$\sum_{k=0}^{n-1} E_k(f)(x) < \frac{h(x)}{\delta} e^{-f(x)}$$

and integrating over $A'_{n,\delta}$ we get

$$\mu(A'_{n,\delta}) \sum_{k=0}^{n-1} e^{L_k} \leq \frac{1}{\delta} \int_X h e^{-f} \, d\mu.$$ 

Our assumption (ii) means that there exist $n_j \to +\infty$ such that

$$e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{L_k} \to +\infty,$$

and therefore we also have $\sum_{k=0}^{n_j-1} e^{L_k} \to +\infty$, because $L_n \leq 0 \leq M_n$. Consequently,

$$\lim_{j \to +\infty} \mu(A_{n_j,\delta}) = \lim_{j \to +\infty} \mu(A'_{n_j,\delta}) = 0.$$

□

In the next proposition we make a more restrictive assumption (i) and a weaker assumption (ii).

**Proposition 3.3.** Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be $T$-invariant and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f \, d\mu = 0$. Suppose that

1. there exists a constant $c \geq 1$ such that $E_n(f) \leq c \int_X E_n(f) \, d\mu$ for every $n \in \mathbb{N}$, and

...
(ii) the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}, \ n \in \mathbb{N}$, is unbounded.

Then there exists an $\mathcal{L}T$-conformal measure for $T$ which is absolutely continuous with respect to $\mu$.

Proof. Our assumption (ii) means that there exists a sequence of positive integers $n_j \to +\infty$ such that $e^{-M_n} \sum_{k=0}^{n_j-1} e^{M_k} \to +\infty$, as $j \to +\infty$. Using the same notations as above we have $\int_X g_{n_j} d\mu \to +\infty$ and

$$e^{-S_{n_j}} \int_X g_{n_j} d\mu \geq \frac{1}{c} \cdot e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{M_k} \to +\infty,$$

as $j \to +\infty$, by our assumptions. Therefore, $\lim_{j \to +\infty} (h_{n_j} \circ T - h_{n_j} e^f) = 0$ uniformly on $X$ and as in the proof of Proposition 3.2, every $\nu \in \{\nu_{n_j} : j \in \mathbb{N}\}$ is $\mathcal{L}T$-conformal measure for $T$ that is absolutely continuous with respect to $\mu$. $\square$

As the following Lemma shows, if in Proposition 3.3 the $T$-invariant measure $\mu \in \mathcal{M}(X)$ is ergodic, then condition (ii) is implied by condition (i).

**Lemma 3.4.** Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that there exists a constant $c \geq 1$ such that

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$.

(a) If $A_n = \{x \in X : S_n(x) > M_n - \log c - 1\}, \ n \in \mathbb{N}$, then $\mu(A_n) \geq \frac{e - 1}{ec - 1}$ for $n \in \mathbb{N}$.

(b) For every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$.

(c) The sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}, \ n \in \mathbb{N}$, is unbounded.

Proof. (a) From our assumption we have

$$e^{M_n - \log c} \leq \int_X E_n(f) d\mu \leq e^{M_n} \mu(A_n) + e^{M_n - \log c - 1} \mu(X \setminus A_n),$$

from which the required inequality follows.

(b) We proceed to prove the assertion by contradiction assuming that there exists some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $1 \leq j_n \leq N$ such that $M_{n+j} > M_n + 1$. Inductively, if we put $n_k = 1 + j_1 + \cdots + j_k$, then $M_{n_k} > M_1 + k$ and $1 + k \leq n_k \leq 1 + kN$ for every $k \in \mathbb{N}$. Therefore,

$$\frac{M_{n_k}}{n_k} > \frac{1}{N + 1}$$
for every \( k \in \mathbb{N} \). If now \( k_0 \in \mathbb{N} \) is such that 
\[
\left| \frac{\log c - 1}{n_k} \right| < \frac{1}{2(N + 1)}
\]
for \( k \geq k_0 \), then for \( x \in A_n \) we have
\[
\frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N + 1)}
\]
and by (a) we get
\[
\mu(\{x \in X : \frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N + 1)}\}) \geq \frac{e - 1}{ec - 1} > 0
\]
for every \( k \geq k_0 \). Hence the sequence \( \frac{1}{n} S_n \) does not converge in measure to zero. This contradicts the Ergodic Theorem of Birkhoff, since we assume that \( \mu \) is an ergodic \( T \)-invariant Borel probability measure.

(c) Suppose on the contrary that there exists a real number \( a > 0 \) such that 
\[
e^{-M_n} \sum_{k=0}^{n-1} e^{M_k} \leq a,
\]
for every \( n \in \mathbb{N} \). By (b), for every \( N \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that \( M_{n+j} \leq M_n + 1 \) for all \( 0 \leq j \leq N \), and so
\[
\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} \leq a \sum_{j=0}^{N} e^{M_{n+j}} \leq ea e^{M_n} + a \left( \sum_{k=0}^{n-N-1} e^{M_k} - \sum_{k=0}^{n-1} e^{M_k} \right)
\]
\[
\leq ea(1 + a)e^{M_n} - a \sum_{k=0}^{n-1} e^{M_k}.
\]
Substituting
\[
\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} = (N + 1) \sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N - i)e^{M_{n+i}},
\]
we arrive at
\[
(N + 1 + a) \sum_{k=0}^{n-1} e^{M_k} + Ne^M_n + \sum_{i=1}^{N-1} (N - i)e^{M_{n+i}} \leq ea(1 + a)e^{M_n}
\]
and therefore \( N \leq ea(1 + a) \) for every \( N \in \mathbb{N} \), contradiction. \( \square \)

The above immediately imply the following theorem which is the main result of this note.

**Theorem 3.5.** Let \( X \) be a compact metric space and \( T : X \to X \) a continuous surjection. Let \( \mu \in \mathcal{M}(X) \) be an ergodic \( T \)-invariant measure and let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f d\mu = 0 \). If there exists a constant \( c \geq 1 \) such that
\[
E_n(f) \leq c \int_X E_n(f) d\mu
\]
for every \( n \in \mathbb{N} \), then there exists an \( \mathcal{U} \)-conformal measure \( \nu \) for \( T \) which is absolutely continuous with respect to \( \mu \). Moreover, \( \frac{d\nu}{d\mu} \in L^\infty(\mu) \) and \(-\log(\frac{d\nu}{d\mu})\) is a measurable solution of the cohomological equation \( f = u - u \circ T \). □

The preceding Theorem 3.5 combined with the main result of [7] gives the following.

**Corollary 3.6.** Let \( X \) be a compact metric space and \( T : X \to X \) a continuous surjection which is a locally eventually onto local homeomorphism. Let \( \mu \in \mathcal{M}(X) \) be an ergodic \( T \)-invariant measure and let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f d\mu = 0 \).

If there exists a constant \( c \geq 1 \) such that
\[
\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu
\]
for every \( n \in \mathbb{N} \), then there exists an \( \mathcal{U} \)-conformal measure \( \nu \) for \( T \) which is absolutely continuous with respect to \( \mu \). Moreover, \( -\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu) \) and in case \( \mu \) has full support the cohomological equation \( f = u - u \circ T \) has a continuous solution. □

If \( X \) is a compact metric space and \( T : X \to X \) is a homeomorphism, for any continuous function \( f : X \to \mathbb{R} \) we put
\[
E_n(f) = \begin{cases} \exp \sum_{k=1}^{n} f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp \left( - \sum_{k=0}^{\lfloor |n| \rfloor} f \circ T^k \right), & \text{if } n < 0. \end{cases}
\]

As before we also put \( S_n(f) = \log E_n(f) \) and \( M_n = \sup \{ S_n(f)(x) : x \in X \} \), \( n \in \mathbb{Z} \).

Let now \( \mu \in \mathcal{M}(X) \) be \( T \)-invariant and suppose that \( \int_X f d\mu = 0 \). Then, \( M_n \geq 0 \) for every \( n \in \mathbb{Z} \). Since \( S_n(f) \circ T^{-1} = S_{n+1}(f) - f \circ T^{-1} \) for \( n \in \mathbb{N} \), if \( g_n = \sum_{k=0}^{n-1} E_k(f) \), then we have
\[
(g_n \circ T^{-1}) e^{f \circ T^{-1}} - g_n = E_n(f) - 1.
\]

Putting \( h_n = \frac{g_n}{\int_X g_n d\mu} \), we get
\[
(h_n \circ T^{-1}) e^{f \circ T^{-1}} - h_n = \frac{1 - e^{-S_n(f)}}{e^{-S_n(f)} \int_X g_n d\mu},
\]
for every \( n \in \mathbb{N} \). So the same reasoning as above and Lemma 3.1 give the following.
Theorem 3.7. Let $X$ be a compact metric space and $T : X \to X$ a homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$.

(a) If there exists a constant $c \geq 1$ such that
$$E_n(f) \leq c \int_X E_n(f) d\mu$$
for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then there exists an $e^f$-conformal measure $\nu$ for $T$ which is equivalent to $\mu$ such that $d\nu/d\mu \in L^\infty(\mu)$.

(b) Moreover, if
$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$
for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then $\log\left(d\nu/d\mu\right) \in L^\infty(\mu)$. □

Combining Theorem 3.7 with section 2 we get the following.

Corollary 3.8. Let $X$ be a compact metric space and $T : X \to X$ a minimal homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Then the following assertions are equivalent.

(i) $f$ is a continuous coboundary.

(ii) There exists a constant $c \geq 1$ such that
$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$
for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$). □

4 An example

We shall illustrate the results of the preceding section by applying them to a specific homeomorphism and continuous function. Let $N \geq 2$ be an integer and $X_N$ be the compact subset of $\{-1, 1\}^\mathbb{Z}$ consisting of all sequences $(x_n)_{n \in \mathbb{Z}}$ such that
$$\sum_{k=m}^{n} x_k \leq N$$
for every $m, n \in \mathbb{Z}$ with $m < n$. Obviously, $X_N$ is invariant under the shift. The restriction $T$ of the shift on $X_N$ defines a symbolic dynamical system which is sofic, that is a continuous factor of a subshift of finite type. To see this, we consider the shift $S : \{0, 1, ..., N\}^\mathbb{Z} \to \{0, 1, ..., N\}^\mathbb{Z}$ on $N + 1$ symbols and the transition matrix $A = (a_{ij})_{0 \leq i, j \leq N}$ where $a_{ij} = 1$, if $|i - j| = 1$, and $a_{ij} = 0$ otherwise. The corresponding subshift of finite type is defined on
$$\Omega_A = \{(y_n)_{n \in \mathbb{Z}} \in \{0, 1, ..., N\}^\mathbb{Z} : |y_{n+1} - y_n| = 1 \text{ for all } n \in \mathbb{Z}\}.$$
The continuous surjection \( h : \Omega_A \to X_N \) defined by
\[
h((y_n)_{n \in \mathbb{Z}}) = (y_{n+1} - y_n)_{n \in \mathbb{Z}}
\]
satisfies \( h \circ S = T \circ h \). Since \( A \) is an irreducible 0-1 matrix, the subshift \((\Omega_A, S)\) is topologically transitive and has a dense subset of periodic points. Since the symbolic system \((X_N, T)\) is a continuous factor of \((\Omega_A, S)\), it has the same properties and so it is chaotic.

Let \( f : X \to \{-1, 1\} \) be the restriction to \( X_N \) of the projection to the 0-th coordinate. It is proved in Proposition 11.16 in [3] that \( f \) is a Borel measurable coboundary with a bounded measurable transfer function but it is not a continuous coboundary for \( T \).

A Markov measure on \( \Omega_A \) defined by a stochastic matrix which is compatible with \( A \) and a corresponding probability vector is ergodic for \( S \) (see page 161 in [6]) and is projected by \( h \) to an ergodic \( T \)-invariant Borel probability measure \( \mu \) on \( X_N \). Since \( f \) is an \( L^\infty(\mu) \)-coboundary, we have
\[
\int_{X_N} f \, d\mu = 0.
\]
In this case we have \( E_n(f)((x_n)_{n \in \mathbb{Z}}) = e^{-(x_0 + x_1 + \cdots + x_{n-1})} \) and therefore
\[
e^{-N} \leq E_n(f) \leq e^N
\]
for every \( n \in \mathbb{N} \). It follows from Theorem 3.7 that there exists an \( e^f \)-conformal measure \( \nu \) for \( T \) on \( X_N \) which is equivalent to \( \mu \) such that \( \log\left(\frac{d\nu}{d\mu}\right) \in L^\infty(\mu) \).

References


