On the existence of absolutely continuous conformal measures

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ABSTRACT

Let X be a compact metric space and $T : X \to X$ a continuous surjection. We present sufficient conditions which imply the existence of absolutely continuous conformal measures for T with respect to a given ergodic T-invariant Borel probability measure. The same conditions give measurable or L^∞ solutions of the corresponding cohomological equation. We illustrate our results in an example of a sofic system.

1 Introduction

Let X be a compact metric space, $T : X \to X$ a continuous surjection and let $f : X \to \mathbb{R}$ be a continuous function. We call a Borel probability measure ν on X an e^f-conformal measure for T if ν is equivalent to $T_*\nu$ and $\dfrac{d\nu}{d(T_*\nu)} = e^f$. This kind of measure has been used without a particular name in [7] and in a more general probabilistic setting in [10].

In this note we study the existence of absolutely continuous conformal measures with respect to a given ergodic T-invariant Borel probability measure. We present a sufficient condition for the existence of an absolutely continuous conformal measure for a continuous surjection. The problem of the existence of an e^f-conformal measure ν for a homeomorphism T which is absolutely continuous with respect to an ergodic T-invariant Borel probability measure μ is closely related to the existence and regularity properties of solutions of the cohomological equation $f = u - u \circ T$. This relation is explained with details in section 2. If there exists a continuous solution u, then f is called a continuous coboundary. According to the classical Gottschalk-Hedlund theorem (see page 102 in [5]), if T is minimal, then f is a continuous coboundary if and only if there exists $x_0 \in X$ such that

$$\sup\{|\sum_{k=0}^{n-1} f(T^k(x_0))| : n \in \mathbb{N}\} < +\infty.$$

The main result is Theorem 3.5 which can be stated as follows.

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Main Theorem. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, where $E_n(f) = e^{S_n(f)}$ and $S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k$, then there exists an e^f-conformal measure ν for T which is absolutely continuous with respect to μ. Moreover, $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ and $-\log(\frac{d\nu}{d\mu})$ is a measurable solution of the cohomological equation $f = u - u \circ T$. $\square$

If T is a homeomorphism, then in Theorem 3.7 we prove that if the stronger condition

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$) holds for some constant $c \geq 1$, then a e^f-conformal measure ν for T exists which is equivalent to μ and $\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu)$. Also, f is a $L^\infty(\mu)$ coboundary with transfer function $-\log(\frac{d\nu}{d\mu})$. This result holds without the assumption that T is minimal.

In a final section we illustrate our results in an example of a known sofic system which is attributed to B. Markus in [4]. In this example T is the two-sided left shift restricted on a suitable compact subset X of $\{-1, 1\}^{\mathbb{Z}}$ and is a continuous factor of a subshift of finite type on $N + 1$ symbols for some integer $N \geq 2$. The system is not minimal, it is chaotic and it has the strong specification property.

2 Conformal measures

Let $T : X \to X$ be a continuous surjection of a compact metric space X and let $f : X \to \mathbb{R}$ be a continuous function. A e^f-conformal measure for T is a Borel probability measure ν on X such that

$$\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$. Evidently, a e^f-conformal measure for T is T-quasi-invariant and is an $e^{-f \circ T^{-1}}$-conformal measure for T^{-1}, in case T is a homeomorphism.

It is easy to see that if $h : X \to X$ is a homeomorphism and $S = h \circ T \circ h^{-1}$, then $h_* \nu$ is a $e^{f \circ h^{-1}}$-conformal measure for S for every e^f-conformal measure ν for T.

According to the main result (Theorem 6.2) of [1] if $T : X \to X$ is a homeomorphism of a compact metric space X and $f : X \to \mathbb{R}$ is a continuous function, then there exists a e^f-conformal measure for T if and only if there exists a point $x \in X$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) \leq 0$$

and

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}-f(T^{-k}(x))\leq 0.$$

For the reader's convenience we shall describe a construction of conformal measures for homeomorphisms due to M. Denker and M. Urbanski given in [2] (see also section 9.2 in [6]). Note that there may be no e^f-conformal measure for T for a given continuous function $f : X \to \mathbb{R}$. This is the case, for example, if $f > 0$, since we necessarily have $\int_X e^f\,d\nu = 1$ for every e^f-conformal measure. We need some preliminary observations.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $c = \limsup_{n\to+\infty}\frac{a_n}{n}$. The series $\sum_{n=1}^{\infty} e^{a_n - ns}$ converges for $s > c$, diverges for $s < c$ and we cannot tell for $s = c$, by the root test.

Lemma 2.1. *There exists a sequence of positive real numbers $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\frac{b_n}{b_{n+1}} = 1$ and the series $\sum_{n=1}^{\infty} b_n e^{a_n-ns}$ converges for $s > c$ and diverges for $s \leq c$.*

Proof. If the series $\sum_{n=1}^{\infty} e^{a_n-nc}$ diverges, we may take $b_n = 1$ for every $n \in \mathbb{N}$. Suppose that it converges. We choose a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty}\frac{n_k}{n_{k+1}} = 0 \quad \text{and} \quad \lim_{k\to+\infty}\frac{a_{n_k}}{n_k} = c.$$

It suffices now to put $\epsilon_k = \frac{a_{n_k}}{n_k} - c$ and take

$$b_n = \exp\left[n\left(\frac{n_k - n}{n_k - n_{k-1}}\epsilon_{k-1} + \frac{n - n_{k-1}}{n_k - n_{k-1}}\epsilon_k\right)\right]$$

for $n_{k-1} \leq n < n_k$. $\square$

Let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$ for some ergodic T-invariant Borel probability measure μ. It is well known that the set of points $x \in X$ such that the limit

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} f(T^{-k}(x))$$

exists in $\mathbb{R}$ has measure 1 with respect to every T-invariant Borel probability measure, and is therefore non-empty. So there exists a point $x \in X$ such that

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} f(T^{-k}(x)) = \int_X f d\mu = 0,$$

since μ is assumed to be ergodic.

If we take $a_n = -\sum_{k=1}^{n} f(T^{-k}(x))$, then $\lim_{n\to+\infty} \frac{a_n}{n} = 0$. Let $M_s = \sum_{n=1}^{\infty} b_n e^{a_n - ns}$, $s > 0$, where $(b_n)_{n\in\mathbb{N}}$ is the corresponding sequence given from Lemma 2.1, and

$$\mu_s = \frac{1}{M_s}\sum_{n=1}^{\infty} b_n e^{a_n - ns}\delta_{T^{-n}(x)}, \quad s > 0.$$

Proposition 2.2. *Every accumulation point with respect to the weak* topology of the directed family of Borel probability measures $(\mu_s)_{s>0}$, as $s \downarrow 0$, is a e^f-conformal measure for T.*

Proof. For every continuous function $\phi : X \to \mathbb{R}$ we have on the one hand

$$\int_X \phi d\mu_s = \frac{1}{M_s}\sum_{n=1}^{\infty} b_n e^{a_n - ns}\phi(T^{-n}(x))$$

and on the other

$$\int_X (\phi\circ T)e^f d\mu_s = \frac{1}{M_s}\sum_{n=1}^{\infty} b_n e^{a_n - ns}\phi(T^{-n+1}(x))e^{f(T^{-n}(x))}$$

$$= \frac{1}{M_s}\left[b_1 e^{-s}\phi(x) + \sum_{n=1}^{\infty} b_{n+1}e^{-s}e^{a_n - ns}\phi(T^{-n}(x))\right].$$

Since $\lim_{s\downarrow 0} \frac{b_1 e^{-s}\phi(x)}{M_s} = 0$, we need to estimate the difference

$$\frac{1}{M_s}\left|\sum_{n=1}^{\infty} b_n e^{a_n - ns}\phi(T^{-n}(x)) - \sum_{n=1}^{\infty} b_{n+1}e^{-s}e^{a_n - ns}\phi(T^{-n}(x))\right| \le \frac{\|\phi\|}{M_s}\sum_{n=1}^{\infty}\left|\frac{b_{n+1}}{b_n}e^{-s} - 1\right| b_n e^{a_n - ns}.$$

Given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ we have

$$\left|\frac{b_{n+1}}{b_n} - 1\right| < \epsilon$$

and therefore

$$\left|\frac{b_{n+1}}{b_n}e^{-s} - 1\right| < \epsilon e^{-s} + |1 - e^{-s}|.$$

It follows that

$$\frac{1}{M_s}\sum_{n=1}^{\infty}\left|\frac{b_{n+1}}{b_n}e^{-s} - 1\right| b_n e^{a_n - ns}$$

$$< \frac{1}{M_s}\sum_{n=1}^{n_0 - 1}\left|\frac{b_{n+1}}{b_n}e^{-s} - 1\right| b_n e^{a_n - ns} + \frac{\epsilon e^{-s} + |1 - e^{-s}|}{M_s}\cdot\sum_{n=n_0}^{\infty} b_n e^{a_n - ns}$$

$$\le \frac{1}{M_s}\sum_{n=1}^{n_0 - 1}\left|\frac{b_{n+1}}{b_n}e^{-s} - 1\right| b_n e^{a_n - ns} + \epsilon e^{-s} + |1 - e^{-s}|.$$

Since $\lim_{s \downarrow 0} M_s = +\infty$, there exists some $0 < s_0 < 1$ such that $\epsilon e^{-s} + 1 - e^{-s} < 2\epsilon$,

$$\frac{1}{M_s} \sum_{n=1}^{n_0-1} \left| \frac{b_{n+1}}{b_n} e^{-s} - 1 \right| b_n e^{a_n - ns} < \epsilon$$

and $\frac{b_1 e^{-s}}{M_s} < \epsilon$ for all $0 < s < s_0$.

Summarizing, for every $\epsilon > 0$ there exists $0 < s_0 < 1$ such that

$$\left| \int_X \phi d\mu_s - \int_X (\phi \circ T) e^f d\mu_s \right| < 4\epsilon \|\phi\|$$

for all $0 < s < s_0$ and every continuous function $\phi : X \to \mathbb{R}$. This proves the assertion. $\square$

There is a close relation between e^f-conformal measures for a homeomorphism $T : X \to X$ of a compact metric space and solvability of the cohomological equation $f = u - u \circ T$, where $f : X \to \mathbb{R}$ is continuous (see also Proposition 4.4 in [1]).

Let μ be any T-invariant Borel probability measure. If there exists a measurable solution u of the above cohomological equation defined μ-almost everywhere such that $e^{-u} \in L^1(\mu)$, then there exists a e^f-conformal measure ν for T equivalent to μ with density

$$\frac{d\nu}{d\mu} = \frac{e^{-u}}{\int_X e^{-u} d\mu}.$$

Thus, if there exists a continuous solution u, then for every T-invariant Borel probability measure we get an equivalent e^f-conformal measure for T. Moreover, in this case, every e^f-conformal measure ν for T is obtained in this way. Indeed, we have

$$\int_X \phi e^u d\nu = \int_X (\phi \circ T) e^u d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$, and so the equivalent measure μ to ν with density

$$\frac{d\mu}{d\nu} = \frac{e^u}{\int_X e^u d\nu}$$

is T-invariant. Consequently, if f is a continuous coboundary, then the e^f-conformal measures for T are in one-to-one correspondence with the T-invariant Borel probability measures and each e^f-conformal measure for T is equivalent to its corresponding T-invariant measure.

Conversely, suppose that μ is an ergodic T-invariant Borel probability measure and $f : X \to \mathbb{R}$ is a continuous function such that $\int_X f d\mu = 0$. Suppose further that there exists a e^f-conformal measure $\nu \in \mathcal{M}(X)$ for T which is absolutely continuous with respect to μ and let $g = \frac{d\nu}{d\mu}$. For every measurable set $A \subset X$ we have

$$\int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T) e^f d\nu = \int_X (\chi_A \circ T) e^f g d\mu$$

and therefore

$$\int_{T^{-1}(A)} [ge^f - (g \circ T)]d\mu = 0.$$

Since μ is T-invariant, it follows that $g \circ T = ge^f$ μ-almost everywhere. The ergodicity of μ implies now that $g > 0$ μ-almost everywhere. So, $u = -\log g$ is a measurable solution of the cohomological equation $f = u - u \circ T$. If $\log g \in L^\infty(\mu)$ and T is a minimal homeomorphism, then there exists some continuous function $u : X \to \mathbb{R}$ such that $f = u - u \circ T$, by Proposition 4.2 on page 46 in [3].

Note that ν is equivalent to μ, because $g > 0$. We remark that this is actually a more general fact which holds for every T-quasi-invariant Borel probability measure. To see this, let $T : X \to X$ be a homeomorphism of a compact metric space X and μ be an ergodic T-invariant Borel probability measure. Let ν is a T-quasi-invariant Borel probability measure which is absolutely continuous with respect to μ. Let $g = \dfrac{d\nu}{d\mu}$ and $A = g^{-1}(0)$. If $S = \bigcup\limits_{n \in \mathbb{Z}} T^n(A)$, then S is T-invariant and $\nu(S) = 0$. On the other hand $\mu(X \setminus S) > 0$, and since μ is ergodic we get $\mu(S) = 0$, that is $g > 0$ μ-almost everywhere. In particular, if T is uniquely ergodic, then every T-quasi-invariant measure for T which is absolutely continuous with respect to its unique invariant Borel probability measure is equivalent to it.

3 Absolutely continuous conformal measures

Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. The set

$$A_\mu = \{\nu \in \mathcal{M}(X) : \nu \ll \mu\}$$

is not empty, since it contains μ, and is convex. In general, A_μ is not a closed subset of $\mathcal{M}(X)$ with respect to the weak* topology. For example, if we let μ be the Lebesgue measure on the unit interval $[0, 1]$ and for $0 < \epsilon < 1$ we let μ_ϵ denote the Borel probability measure on $[0, 1]$ with density $\dfrac{1}{\epsilon}\chi_{[0,\epsilon]}$, then $\lim\limits_{\epsilon \to 0} \mu_\epsilon$ is the Dirac point measure at 0.

Lemma 3.1. *Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence in A_μ converging weakly* to some $\nu \in \mathcal{M}(X)$ and let $f_n = \dfrac{d\nu_n}{d\mu}$, $n \in \mathbb{N}$. If there exist non-negative h, $g \in L^1(\mu)$ such that $h \le f_n \le g$ for every $n \in \mathbb{N}$, then $\nu \in A_\mu$ and $h \le \dfrac{d\nu}{d\mu} \le g$.*

Proof. Since ν is a finite measure, there exists a (countable) basis $\mathcal{U}$ of the topology of X such that $\nu(\partial U) = 0$ for every $U \in \mathcal{U}$. So $\mathcal{U}$ is contained in the algebra

$$\mathcal{C}(\nu) = \{A | A \subset X \text{ Borel and } \nu(\partial A) = 0\}$$

and since it generates the Borel σ-algebra of X, so does $\mathcal{C}(\nu)$. Let now $A \subset X$ be a Borel set with $\mu(A) = 0$ and $\epsilon > 0$. There exists $0 < \delta < \epsilon$ such that $\int_B g d\mu < \epsilon$ for every Borel set $B \subset X$ with $\mu(B) < \delta$, because $g \in L^1(\mu)$. There exists some $A_0 \in \mathcal{C}(\nu)$ such that $\mu(A \triangle A_0) < \delta$ and $\nu(A \triangle A_0) < \delta$. Thus $\mu(A_0) < \delta$ and $|\nu(A) - \nu(A_0)| < \delta$. By weak* convergence, $\nu(A_0) = \lim_{n\to+\infty} \nu_n(A_0)$ and so there exists some $n_0 \in \mathbb{N}$ such that $|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \geq n_0$. Therefore,

$$\nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq \int_{A_0} g d\mu + \epsilon < 2\epsilon.$$

It follows that $0 \leq \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in A_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of X such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, the Borel σ-algebra of X is generated by $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ and $\mu(\partial B) = 0$ for every $B \in \mathcal{P}_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of X such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively $\mathcal{P}_n$ to be the finite family consisting of Borel sets with positive μ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in \mathcal{P}_{n-1}$, taking $\mathcal{P}_0 = \{X\}$.

Let $\mathcal{P}_n(x)$ denote the element of $\mathcal{P}_n$ which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n\to+\infty} \frac{\nu(\mathcal{P}_n(x))}{\mu(\mathcal{P}_n(x))},$$

μ-almost everywhere on X and in $L^1(\mu)$ (see page 8 in [8]). On the other hand, by the weak* convergence and since $\nu \in A_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k}\mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \leq \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} < \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} f_{n_k} d\mu \leq \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} g d\mu.$$

Since

$$\lim_{k\to+\infty} \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} g d\mu = g(x)$$

μ-almost everywhere on X and in $L^1(\mu)$, it follows that $0 \leq \frac{d\nu}{d\mu}(x) \leq g(x)$ μ-almost everywhere on X.

Similarly, from

$$\frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} > -\frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} f_{n_k} d\mu \geq -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} h d\mu$$

follows that $h(x) \leq \frac{d\nu}{d\mu}(x)$ μ-almost everywhere on X. $\square$

Let X be a compact metric space and $T : X \to X$ a continuous surjection. For any continuous function $f : X \to \mathbb{R}$ we put $S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k$ and $E_n(f) = e^{S_n(f)}$.

Let $M_n = \sup\{S_n(f)(x) : x \in X\}$ and $L_n = \inf\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{N}$. Since $S_n(f) \circ T = S_{n+1}(f) + f$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T)e^{-f} - g_n = E_n(f) - e^{-f}.$$

Let now $\mu \in \mathcal{M}(X)$ be T-invariant and suppose that $\int_X f d\mu = 0$. So, $L_n \leq 0 \leq M_n$ for every $n \in \mathbb{N}$. Putting $h_n = \dfrac{g_n}{\int_X g_n d\mu}$, we get

$$(h_n \circ T) - h_n e^f = \frac{e^f - e^{-S_n(f)}}{e^{-S_n(f)} \int_X g_n d\mu},$$

for every $n \in \mathbb{N}$.

Suppose that there exists a positive $h \in L^1(\mu)$ such that $E_n(f) \leq h \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$. Then also $0 \leq h_n \leq h$ for $n \in \mathbb{N}$. If ν_n denotes the element of E_μ with $h_n = \dfrac{d\nu_n}{d\mu}$, then $\overline{\{\nu_n : n \in \mathbb{N}\}} \subset E_\mu$, by Lemma 3.1.

Proposition 3.2. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be T-invariant and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that*

(i) there exists a positive $h \in L^1(\mu)$ such that $E_n(f) \leq h \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and

(ii) the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}$, $n \in \mathbb{N}$, is unbounded.

Then there exists an e^f-conformal measure for T which is absolutely continuous with respect to μ.

Proof. Using the above notations, it suffices to prove that there exists a sequence of positive integers $n_j \to +\infty$ such that $\lim_{j \to +\infty} \big((h_{n_j} \circ T) - h_{n_j} e^f\big) = 0$ μ-almost everywhere on X. Indeed, passing to a subsequence if necessary, there exists $\nu \in E_\mu$ such that $\nu = \lim_{j \to +\infty} \nu_{n_j}$, by Lemma 3.1. Since μ is T-invariant, for every continuous function $\phi : X \to \mathbb{R}$ we have

$$\int_X \big(\phi - (\phi \circ T)e^f\big) d\nu = \lim_{j \to +\infty} \int_X (\phi \circ T)\big((h_{n_j} \circ T) - h_{n_j} e^f\big) d\mu = 0,$$

by dominated convergence, because

$$|(\phi \circ T)((h_n \circ T) - h_n e^f)| \leq \|\phi\|((h \circ T) + h e^f) \in L^1(\mu).$$

Since

$$|(h_n \circ T) - h_n e^f| = e^f \frac{|E_n(f) - e^{-f}|}{\int_X g_n d\mu},$$

we need only prove that there exist $n_j \to +\infty$ such that

$$\lim_{j\to+\infty} \mu(\{x \in X : |E_{n_j}(f)(x) - e^{-f(x)}| \geq \delta \int_X g_{n_j} d\mu\}) = 0$$

for every $\delta > 0$. Let

$$A_{n,\delta} = \{x \in X : E_n(f)(x) \geq e^{-f(x)} + \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x)\}, \quad \text{and}$$

$$A'_{n,\delta} = \{x \in X : E_n(f)(x) \leq e^{-f(x)} - \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x)\}.$$

Our assumption (i) implies that it suffices to prove the existence of a sequence of positive integers $n_j \to +\infty$ such that $\lim\limits_{j\to+\infty} \mu(A_{n_j,\delta}) = \lim\limits_{j\to+\infty} \mu(A'_{n_j,\delta}) = 0$ for every $\delta > 0$.

For every $x \in A_{n,\delta}$ we have

$$\frac{h(x)}{\delta} > e^{-M_n} \sum_{k=0}^{n-1} E_k(f)(x)$$

and integrating over $A_{n,\delta}$ we obtain

$$\frac{1}{\delta} \int_X h d\mu \geq \mu(A_{n,\delta}) e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}.$$

Similarly, for every $x \in A'_{n,\delta}$ we have

$$\sum_{k=0}^{n-1} E_k(f)(x) < \frac{h(x)}{\delta} e^{-f(x)}$$

and integrating over $A'_{n,\delta}$ we get

$$\mu(A'_{n,\delta}) \sum_{k=0}^{n-1} e^{L_k} \leq \frac{1}{\delta} \int_X h e^{-f} d\mu.$$

Our assumption (ii) means that there exist $n_j \to +\infty$ such that $e^{-M_{n_j}} \sum\limits_{k=0}^{n_j-1} e^{L_k} \to +\infty$, and therefore we also have $\sum\limits_{k=0}^{n_j-1} e^{L_k} \to +\infty$, because $L_n \leq 0 \leq M_n$. Consequently, $\lim\limits_{j\to+\infty} \mu(A_{n_j,\delta}) = \lim\limits_{j\to+\infty} \mu(A'_{n_j,\delta}) = 0$. □

In the next proposition we make a more restrictive assumption (i) and a weaker assumption (ii).

Proposition 3.3. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be T-invariant and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that*

(i) there exists a constant $c \geq 1$ such that $E_n(f) \leq c \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and

(ii) the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}$, $n \in \mathbb{N}$, is unbounded,

Then there exists an e^f-conformal measure for T which is absolutely continuous with respect to μ.

Proof. Our assumption (ii) means that there exists a sequence of positive integers $n_j \to +\infty$ such that $e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{M_k} \to +\infty$, as $j \to +\infty$. Using the same notations as above we have $\int_X g_{n_j} d\mu \to +\infty$ and

$$e^{-S_{n_j}} \int_X g_{n_j} d\mu \geq \frac{1}{c} \cdot e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{M_k} \to +\infty,$$

as $j \to +\infty$, by our assumptions. Therefore, $\lim_{j \to +\infty} \left((h_{n_j} \circ T) - h_{n_j} e^f \right) = 0$ uniformly on X and as in the proof of Proposition 3.2, every $\nu \in \overline{\{\nu_{n_j} : j \in \mathbb{N}\}}$ is e^f-conformal measure for T that is absolutely continuous with respect to μ. $\square$

As the following Lemma shows, if in Proposition 3.3 the T-invariant measure $\mu \in \mathcal{M}(X)$ is ergodic, then condition (ii) is implied by condition (i).

Lemma 3.4. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$.

(a) If $A_n = \{x \in X : S_n(x) > M_n - \log c - 1\}$, $n \in \mathbb{N}$, then $\mu(A_n) \geq \dfrac{e-1}{ec-1}$ for $n \in \mathbb{N}$.

(b) For every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$.

(c) The sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}$, $n \in \mathbb{N}$, is unbounded.

Proof. (a) From our assumption we have

$$e^{M_n - \log c} \leq \int_X E_n(f) d\mu \leq e^{M_n} \mu(A_n) + e^{M_n - \log c - 1} \mu(X \setminus A_n),$$

from which the required inequality follows.

(b) We proceed to prove the assertion by contradiction assuming that there exists some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $1 \leq j_n \leq N$ such that $M_{n+j_n} > M_n+1$. Inductively, if we put $n_k = 1+j_1+\cdots+j_k$, then $M_{n_k} > M_1+k$ and $1+k \leq n_k \leq 1+kN$ for every $k \in \mathbb{N}$. Therefore,

$$\frac{M_{n_k}}{n_k} > \frac{1}{N+1}$$

for every $k \in \mathbb{N}$. If now $k_0 \in \mathbb{N}$ is such that $\left|\dfrac{\log c - 1}{n_k}\right| < \dfrac{1}{2(N+1)}$ for $k \geq k_0$, then for $x \in A_n$ we have

$$\frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N+1)}$$

and by (a) we get

$$\mu(\{x \in X : \frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N+1)}\}) \geq \frac{e-1}{ec-1} > 0$$

for every $k \geq k_0$. Hence the sequence $(\frac{1}{n} S_n)_{n\in\mathbb{N}}$ does not converge in measure to zero. This contradicts the Ergodic Theorem of Birkhoff, since we assume that μ is an ergodic T-invariant Borel probability measure.

(c) Suppose on the contrary that there exists a real number $a > 0$ such that $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k} \leq a$, for every $n \in \mathbb{N}$. By (b), for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$, and so

$$\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} \leq a \sum_{j=0}^{N} e^{M_{n+j}} \leq eae^{M_n} + a\left(\sum_{k=0}^{n+N-1} e^{M_k} - \sum_{k=0}^{n-1} e^{M_k}\right)$$

$$\leq ea(1+a)e^{M_n} - a\sum_{k=0}^{n-1} e^{M_k}.$$

Substituting

$$\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} = (N+1)\sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N-i)e^{M_{n+i}},$$

we arrive at

$$(N+1+a)\sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N-i)e^{M_{n+i}} \leq ea(1+a)e^{M_n}$$

and therefore $N \leq ea(1+a)$ for every $N \in \mathbb{N}$, contradiction. □

The above immediately imply the following theorem which is the main result of this note.

Theorem 3.5. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, then there exists an e^f-conformal measure ν for T which is absolutely continuous with respect to μ. Moreover, $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ and $-\log(\frac{d\nu}{d\mu})$ is a measurable solution of the cohomological equation $f = u - u \circ T$. □

The preceding Theorem 3.5 combined with the main result of [9] gives the following.

Corollary 3.6. *Let X be a compact metric space and $T : X \to X$ a continuous surjection which is a locally eventually onto local homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, then there exists an e^f-conformal measure ν for T which is absolutely continuous with respect to μ. Moreover, $-\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu)$ and in case μ has full support the cohomological equation $f = u - u \circ T$ has a continuous solution. □

If X is a compact metric space and $T : X \to X$ is a homeomorphism, for any continuous function $f : X \to \mathbb{R}$ we put

$$E_n(f) = \begin{cases} \exp \sum_{k=1}^{n} f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp\left(-\sum_{k=0}^{|n|-1} f \circ T^k\right), & \text{if } n < 0. \end{cases}$$

As before we also put $S_n(f) = \log E_n(f)$ and $M_n = \sup\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{Z}$.

Let now $\mu \in \mathcal{M}(X)$ be T-invariant and suppose that $\int_X f d\mu = 0$. Then, $M_n \geq 0$ for every $n \in \mathbb{Z}$. Since $S_n(f) \circ T^{-1} = S_{n+1}(f) - f \circ T^{-1}$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T^{-1}) e^{f \circ T^{-1}} - g_n = E_n(f) - 1.$$

Putting $h_n = \dfrac{g_n}{\displaystyle\int_X g_n d\mu}$, we get

$$(h_n \circ T^{-1}) e^{f \circ T^{-1}} - h_n = \frac{1 - e^{-S_n(f)}}{e^{-S_n(f)} \displaystyle\int_X g_n d\mu},$$

for every $n \in \mathbb{N}$. So the same reasoning as above and Lemma 3.1 give the following.

Theorem 3.7. *Let X be a compact metric space and $T : X \to X$ a homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_X f d\mu = 0$.*

(a) If there exists a constant $c \geq 1$ such that

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then there exists an e^f-conformal measure ν for T which is equivalent to μ such that $\dfrac{d\nu}{d\mu} \in L^\infty(\mu)$.

(b) Moreover, if

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then $\log(\dfrac{d\nu}{d\mu}) \in L^\infty(\mu)$. □

Combining Theorem 3.7 with section 2 we get the following.

Corollary 3.8. *Let X be a compact metric space and $T : X \to X$ a minimal homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_X f d\mu = 0$. Then the following assertions are equivalent.*

(i) f is a continuous coboundary.

(ii) There exists a constant $c \geq 1$ such that

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$). □

4 An example

We shall illustrate the results of the preceding section by applying them to a specific homeomorphism and continuous function. Let $N \geq 2$ be an integer and X_N be the compact subset of $\{-1, 1\}^{\mathbb{Z}}$ consisting of all sequences $(x_n)_{n \in \mathbb{Z}}$ such that

$$\left| \sum_{k=m}^{n} x_k \right| \leq N$$

for every m, $n \in \mathbb{Z}$ with $m < n$. Obviously, X_N is invariant under the shift. The restriction T of the shift on X_N defines a symbolic dynamical system which is sofic, that is a continuous factor of a subshift of finite type. To see this, we consider the shift $S : \{0, 1, ..., N\}^{\mathbb{Z}} \to \{0, 1, ..., N\}^{\mathbb{Z}}$ on $N+1$ symbols and the transition matrix $A = (a_{ij})_{0 \le i,j \le N}$ where $a_{ij} = 1$, if $|i - j| = 1$, and $a_{ij} = 0$ otherwise. The corresponding subshift of finite type is defined on

$$\Omega_A = \{(y_n)_{n\in\mathbb{Z}} \in \{0, 1, ..., N\}^{\mathbb{Z}} : |y_{n+1} - y_n| = 1 \quad \text{for all} \quad n \in \mathbb{Z}\}.$$

The continuous surjection $h : \Omega_A \to X_N$ defined by

$$h((y_n)_{n\in\mathbb{Z}}) = (y_{n+1} - y_n)_{n\in\mathbb{Z}}$$

satisfies $h \circ S = T \circ h$. Since A is an irreducible 0-1 matrix, the subshift (Ω_A, S) is topologically transitive and has a dense subset of periodic points. Since the symbolic system (X_N, T) is a continuous factor of (Ω_A, S), it has the same properties and so it is chaotic.

Let $f : X \to \{-1, 1\}$ be the restriction to X_N of the projection to the 0-th coordinate. It is proved in Proposition 11.16 in [4] that f is a Borel measurable coboundary with a bounded measurable transfer function but it is not a continuous coboundary for T.

A Markov measure on Ω_A defined by a stochastic matrix which is compatible with A and a corresponding probability vector is ergodic for S (see page 161 in [8]) and is projected by h to an ergodic T-invariant Borel probability measure μ on X_N. Since f is an $L^\infty(\mu)$-coboundary, we have

$$\int_{X_N} f d\mu = 0.$$

In this case we have $E_n(f)((x_n)_{n\in\mathbb{Z}}) = e^{-(x_0+x_1+\cdots+x_{n-1})}$ and therefore

$$e^{-N} \le E_n(f) \le e^N$$

for every $n \in \mathbb{N}$. It follows from Theorem 3.7 that there exists an e^f-conformal measure ν for T on X_N which is equivalent to μ such that $\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu)$.

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