A sufficient condition for the logarithm of the derivative of a Denjoy C^1 diffeomorphism to be a measurable coboundary

Konstantin Athanassopoulos
Department of Mathematics and Applied Mathematics, University of Crete, GR-70013 University Campus, Voutes-Iraklion, Greece
e-mail: athanako@uoc.gr

ABSTRACT

We give a sufficient condition under which the logarithm of the derivative of a Denjoy C^1 diffeomorphism of the circle is a measurable coboundary on the unique Cantor minimal set. This condition also guarantees the existence of an automorphic measure which is equivalent to the unique invariant Borel probability measure.

1 Introduction

One of the most important examples of uniquely ergodic homeomorphisms are the orientation preserving homeomorphisms of the circle S^1 which have irrational rotation numbers and are not topologically conjugate to rotations. Among them special place occupy the Denjoy C^1 diffeomorphisms, named after A. Denjoy who gave explicit constructions of such C^1 examples and proved that they cannot be C^2 in [4]. Prior to A. Denjoy, similar examples had been constructed by P. Bohl in [3]. For an exposition of the theory of Denjoy C^1 diffeomorphisms of S^1 we refer to [2], [7] and [10].

Let $T : S^1 \to S^1$ be a Denjoy C^1 diffeomorphism with unique Cantor minimal set K and unique invariant Borel probability measure μ. Then K is the nonwandering set of T and μ is supported on K. The original motivation of this note was to examine whether μ is in some sense geometric with respect to T. This is closely related to a problem stated in [1]. To be more precise, we want to find conditions under which μ is equivalent to a Borel probability measure ν on K such that

$$\int_K \phi d\nu = \int_K (\phi \circ T) T' d\nu$$

for every continuous function $\phi : K \to \mathbb{R}$. A measure ν with this property is called automorphic for T. It is clear from the change of variable formula that the (normalized) Lebesgue measure of S^1 is automorphic for T, but is not equivalent to μ.

The automorphic measures for T are defined as automorphic measures of exponent 1 in [6], but they have appeared in the literature much earlier in the more general setting of homeomorphisms on compact metric spaces. Let X be a compact metric space, $T : X \to X$ be a homeomorphism and let $f : X \to \mathbb{R}$ be a continuous function

[0]2010 *Mathematics Subject Classification*: 37A05, 37A40, 37E10.
Key words and phrases: uniquely ergodic homeomorphism, Denjoy C^1 diffeomorphism, automorphic measure.

such that $\int_X f d\mu = 0$ for some ergodic T-invariant Borel probability measure μ. A Borel probability measure ν on X is called a e^f-conformal measure for T if ν is equivalent to $T_*\nu$ and $\frac{d\nu}{d(T_*\nu)} = e^f$. This kind of measure has been used without a particular name in [8]. If μ is an ergodic T-invariant Borel probability measure on X, as it is explained in section 2, the existence of a e^f-conformal measure for T which is absolutely continuous with respect to μ (actually equivalent to μ) is equivalent to the existence of a measurable solution of the cohomological equation $f = u - u \circ T$. In particular, if T is a Denjoy C^1 diffeomorphism with unique minimal Cantor set K, then there exists an automorphic measure for T which is equivalent to its unique invariant Borel probability measure if and only if $\log T'$ is a measurable coboundary on K.

In this note we study the existence of conformal measures for a uniquely ergodic homeomorphism which is absolutely continuous to its invariant Borel probability measure. In section 3 we present a sufficient condition for that, using the Schauder-Tychonoff fixed point theorem applied to the dual Perron-Frobenius operator on an appropriate convex set. This approach was inspired by [6]. As a corollary we get the following, which is the main result of this paper.

Theorem 1.1. *Let $T : S^1 \to S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K and unique T-invariant Borel probability measure μ. If there exists a positive $g \in L^1(\mu)$ and an integer $m \geq 0$ such that*

$$(T^n)' \leq g \int_K (T^n)' d\mu$$

on K for every $n \geq m$, then there exists an automorphic measure for T which is equivalent to μ and $\log T'$ is a measurable coboundary on K. $\square$

2 Conformal measures

Let $T : X \to X$ be a homeomorphism of a compact metric space X and let $f : X \to \mathbb{R}$ be a continuous function. A e^f-conformal measure for T is a Borel probability measure ν on X such that

$$\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$. Evidently, a e^f-conformal measure for T is T-quasi-invariant and is an $e^{-f\circ T^{-1}}$-conformal measure for T^{-1}.

It is easy to see that if $h : X \to X$ is a homeomorphism and $S = h \circ T \circ h^{-1}$, then $h_*\nu$ is a $e^{f\circ h^{-1}}$-conformal measure for S for every e^f-conformal measure ν for T.

The construction of a conformal measure can be described as follows (see [5]). Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $c = \limsup_{n\to+\infty} \frac{a_n}{n}$. The series $\sum_{n=1}^{\infty} e^{a_n - ns}$ converges for $s > c$, diverges for $s < c$ and we cannot tell for $s = c$, by the root test. There exists a sequence of positive real numbers $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \frac{b_n}{b_{n+1}} = 1$ and the series $\sum_{n=1}^{\infty} b_n e^{a_n - ns}$ converges for $s > c$ and diverges for $s \leq c$.

Let $f : X \to \mathbb{R}$ ba a continuous function such that $\int_X f d\mu = 0$ for some ergodic T-invariant Borel probability measure μ. It is well known that the set of points $x \in X$ such that the limit

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x))$$

exists in $\mathbb{R}$ has measure 1 with respect to every T-invariant Borel probability measure, and is therefore non-empty. So there exists a point $x \in X$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x)) = 0.$$

Let $a_n = -\sum_{k=1}^{n} f(T^{-k}(x))$ and $M_s = \sum_{n=1}^{\infty} b_n e^{a_n - ns}$, where $(b_n)_{n\in\mathbb{N}}$ is the corresponding sequence as above. Each accumulation point in the weak* topology as $s \downarrow 0$ of the directed family

$$\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \delta_{T^{-n}(x)}, \quad s > 0$$

is a e^f-conformal measure for T.

There is a close relation between e^f-conformal measures for a homeomorphism $T : X \to X$ of a compact metric space and solvability of the cohomological equation $f = u - u \circ T$, where $f : X \to \mathbb{R}$ is continuous.

Let μ be any T-invariant Borel probability measure. If there exists a measurable solution u of the above cohomological equation defined μ-almost everywhere such that $e^{-u} \in L^1(\mu)$, then there exists a e^f-conformal measure ν for T equivalent to μ with density

$$\frac{d\nu}{d\mu} = \frac{e^{-u}}{\displaystyle\int_X e^{-u} d\mu}.$$

Thus, if there exists a continuous solution u, then for every T-invariant Borel probability measure we get an equivalent e^f-conformal measure for T. Moreover, in this case, every e^f-conformal measure ν for T is obtained in this way. Indeed, we have

$$\int_X \phi e^u d\nu = \int_X (\phi \circ T) e^u d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$, and so the equivalent measure μ to ν with density

$$\frac{d\mu}{d\nu} = \frac{e^u}{\displaystyle\int_X e^u d\nu}$$

is T-invariant. Consequently, if f is a continuous coboundary, then the e^f-conformal measures for T are in one-to-one correspondence with the T-invariant Borel probability measures and each e^f-conformal measure for T is equivalent to its corresponding T-invariant measure.

Conversely, suppose that μ is an ergodic T-invariant Borel probability measure and $f : X \to \mathbb{R}$ is a continuous function such that $\int_X f d\mu = 0$. Suppose further that there exists a e^f-conformal measure $\nu \in \mathcal{M}(X)$ for T which is absolutely continuous with respect to μ and let $g = \frac{d\nu}{d\mu}$. For every measurable set $A \subset X$ we have

$$\int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T) e^f d\nu = \int_X (\chi_A \circ T) e^f g d\mu$$

and therefore

$$\int_{T^{-1}(A)} [ge^f - (g \circ T)] d\mu = 0.$$

Since μ is T-invariant, it follows that $g \circ T = ge^f$ μ-almost everywhere. The ergodicity of μ implies now that $g > 0$ μ-almost everywhere. So, $u = -\log g$ is a measurable solution of the cohomological equation $f = u - u \circ T$. If $\log g \in L^\infty(\mu)$ and T is a minimal homeomorphism, then there exists some continuous function $u : X \to \mathbb{R}$ such that $f = u - u \circ T$, by Proposition 4.2 on page 46 in [7].

Note that ν is equivalent to μ, because $g > 0$. We remark that this is actually a more general fact which holds for every T-quasi-invariant Borel probability measure. To see this, let $T : X \to X$ be a homeomorphism of a compact metric space X and μ be an ergodic T-invariant Borel probability measure. Let ν is a T-quasi-invariant Borel probability measure which is absolutely continuous with respect to μ. Let $g = \frac{d\nu}{d\mu}$ and $A = g^{-1}(0)$. If $S = \bigcup_{n \in \mathbb{Z}} T^n(A)$, then S is T-invariant and $\nu(S) = 0$. On the other hand $\mu(X \setminus S) > 0$, and since μ is ergodic we get $\mu(S) = 0$, that is $g > 0$ μ-almost everywhere. In particular, if T is uniquely ergodic, then every T-quasi-invariant measure for T which is absolutely continuous with respect its unique invariant Borel probability measure is equivalent to it.

3 Absolutely continuous conformal measures for uniquely ergodic homeomorphisms

Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. The set

$$A_\mu = \{\nu \in \mathcal{M}(X) : \nu \ll \mu\}$$

is not empty, since it contains μ, and is convex. In general, A_μ is not a closed subset of $\mathcal{M}(X)$ with respect to the weak* topology. For example, if we let μ be the Lebesgue measure on the unit interval $[0, 1]$ and for $0 < \epsilon < 1$ we let μ_ϵ denote the Borel probability measure on $[0, 1]$ with density $\frac{1}{\epsilon}\chi_{[0,\epsilon]}$, then $\lim_{\epsilon \to 0} \mu_\epsilon$ is the Dirac point measure at 0.

Lemma 3.1. *Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence in A_μ converging weakly* to some $\nu \in \mathcal{M}(X)$ and let $f_n = \frac{d\nu_n}{d\mu}$, $n \in \mathbb{N}$. If* *there exist non-negative $h, g \in L^1(\mu)$ such that $h \leq f_n \leq g$ for every $n \in \mathbb{N}$, then $\nu \in A_\mu$ and $h \leq \dfrac{d\nu}{d\mu} \leq g$.*

Proof. Since ν is a finite measure, there exists a (countable) basis $\mathcal{U}$ of the topology of X such that $\nu(\partial U) = 0$ for every $U \in \mathcal{U}$. So $\mathcal{U}$ is contained in the algebra

$$\mathcal{C}(\nu) = \{A | A \subset X \text{ Borel and } \nu(\partial A) = 0\}$$

and since it generates the Borel σ-algebra of X, so does $\mathcal{C}(\nu)$. Let now $A \subset X$ be a Borel set with $\mu(A) = 0$ and $\epsilon > 0$. There exists $0 < \delta < \epsilon$ such that $\int_B g d\mu < \epsilon$ for every Borel set $B \subset X$ with $\mu(B) < \delta$, because $g \in L^1(\mu)$. There exists some $A_0 \in \mathcal{C}(\nu)$ such that $\mu(A \triangle A_0) < \delta$ and $\nu(A \triangle A_0) < \delta$. Thus $\mu(A_0) < \delta$ and $|\nu(A) - \nu(A_0)| < \delta$. By weak* convergence, $\nu(A_0) = \lim_{n\to+\infty} \nu_n(A_0)$ and so there exists some $n_0 \in \mathbb{N}$ such that $|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \geq n_0$. Therefore,

$$\nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq \int_{A_0} g d\mu + \epsilon < 2\epsilon.$$

It follows that $0 \leq \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in A_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of X such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, the Borel σ-algebra of X is generated by $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ and $\mu(\partial B) = 0$ for every $B \in \mathcal{P}_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of X such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively $\mathcal{P}_n$ to be the finite family consisting of Borel sets with positive μ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in \mathcal{P}_{n-1}$, taking $\mathcal{P}_0 = \{X\}$.

Let $\mathcal{P}_n(x)$ denote the element of $\mathcal{P}_n$ which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n\to+\infty} \frac{\nu(\mathcal{P}_n(x))}{\mu(\mathcal{P}_n(x))},$$

μ-almost everywhere on X and in $L^1(\mu)$ (see page 8 in [9]). On the other hand, by the weak* convergence and since $\nu \in A_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k}\mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \leq \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} < \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} f_{n_k} d\mu \leq \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} g d\mu.$$

Since

$$\lim_{k\to+\infty} \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} g d\mu = g(x)$$

μ-almost everywhere on X and in $L^1(\mu)$, it follows that $0 \leq \dfrac{d\nu}{d\mu}(x) \leq g(x)$ μ-almost everywhere on X.

Similarly, from

$$\frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} > -\frac{1}{k}+\frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k}+\frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} f_{n_k} d\mu \geq -\frac{1}{k}+\frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} h d\mu$$

follows that $h(x) \leq \frac{d\nu}{d\mu}(x)$ μ-almost everywhere on X. $\square$

Let now $T : X \to X$ be a uniquely ergodic homeomorphism $T : X \to X$. Conformal measures for T can be obtained as fixed points of the dual Perron-Frobenius operator. Let μ be the unique T-invariant Borel probability measure and $c = \int_X f d\mu$, where $f : X \to \mathbb{R}$ is a continuous function. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X equipped with the weak* topology. The dual Perron-Frobenius operator is the continuous map $W : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by

$$W(\nu)(\phi) = \frac{1}{\int_X e^f d\nu} \cdot \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$ (see page 185 in [8]). Since T is a homeomorphism, W is a homeomorphism and its inverse is given by the formula

$$W^{-1}(\nu)(\phi) = \frac{1}{\int_X e^{-f\circ T^{-1}} d\nu} \cdot \int_X (\phi \circ T^{-1}) e^{-f\circ T^{-1}} d\nu.$$

It follows from the Schauder-Tychonoff theorem that W has a fixed point in $\mathcal{M}(X)$. If ν is a fixed point of W, then $\int_X e^f d\nu = e^c$, and therefore ν is a e^{f-c}-conformal measure for T. Indeed, if ν is a fixed point of W, then for every $n \in \mathbb{N}$ we have

$$\Big(\int_X e^f d\nu\Big)^n = \int_X \exp(\sum_{k=0}^{n-1} f \circ T^k) d\nu,$$

as one easily verifies by induction. It follows that

$$n\big|\log\Big(\int_X e^{f-c} d\nu\Big)\big| = \big|\log\Big(\int_X \exp\big(-nc+\sum_{k=0}^{n-1} f\circ T^k\big) d\nu\Big)\big| \leq \| -nc + \sum_{k=0}^{n-1} f \circ T^k \|,$$

and therefore

$$\big|\log\Big(\int_X e^{f-c} d\nu\Big)\big| \leq \| \lim_{n\to+\infty} \Big(-c + \frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k\Big)\| = 0.$$

For any continuous function $f : X \to \mathbb{R}$ we put

$$E_n(f) = \begin{cases} \exp \sum_{k=1}^{n} f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp\,(-\sum_{k=0}^{|n|-1} f \circ T^k), & \text{if } n < 0. \end{cases}$$

We can use the Schauder-Tychonoff theorem to get the following result for the existence of absolutely continuous conformal measures in the case of uniquely ergodic homeomorphisms.

Theorem 3.2. *Let X be a compact metric space and $T : X \to X$ a uniquely ergodic homeomorphism with unique invariant Borel probability measure μ. Let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a non-negative $g \in L^1(\mu)$ and an integer $m \geq 0$ such that*

$$E_n(f) \leq g \int_X E_n(f) d\mu$$

for every $n \geq m$, then there exists a e^f-conformal measure for T equivalent to μ and f is a measurable coboundary.

Proof. Let $W : \mathcal{M}(X) \to \mathcal{M}(X)$ be the dual Perron-Frobenius operator. One can prove by induction that

$$W^n(\nu)(\phi) = \frac{1}{\displaystyle\int_X (\exp \sum_{k=0}^{n-1} f \circ T^k) d\nu} \cdot \int_X (\phi \circ T^n)(\exp \sum_{k=0}^{n-1} f \circ T^k) d\nu$$

and

$$W^{-n}(\nu)(\phi) = \frac{1}{\displaystyle\int_X (\exp(-\sum_{k=1}^{n} f \circ T^{-k})) d\nu} \cdot \int_X (\phi \circ T^{-n})(\exp(-\sum_{k=1}^{n} f \circ T^{-k})) d\nu$$

for every $\nu \in \mathcal{M}(X)$ and $n \in \mathbb{N}$. From the invariance of μ we get

$$W^n(\mu)(\phi) = \frac{1}{\displaystyle\int_X E_n(f) d\mu} \cdot \int_X \phi E_n(f) d\mu$$

for every $n \in \mathbb{Z}$. Note also that

$$\int_X (\phi \circ T) e^f E_n(f) d\mu = \int_X \phi E_{n+1}(f) d\mu$$

for every continuous $\phi : X \to \mathbb{R}$.

If now $A \subset X$ is a measurable set, it follows from regularity that

$$W^n(\mu)(A) = \frac{1}{\displaystyle\int_X E_n(f) d\mu} \cdot \int_A E_n(f) d\mu$$

which implies that $W^n(\mu) \in A_\mu$ and

$$\frac{dW^n(\mu)}{d\mu} = \frac{E_n(f)}{\displaystyle\int_X E_n(f) d\mu}$$

for every $n \in \mathbb{Z}$. If C_m is the convex hull of $\{W^n(\mu) : n \geq m\}$, then $W(C_m) \subset C_m$. Indeed, let

$$t_n = \frac{\int_X E_{n+1}(f)d\mu}{\int_X E_n(f)d\mu}$$

for all $n \in \mathbb{Z}$. If a_1,...,$a_n \geq 0$ are such that $a_1 + \cdots + a_n = 1$ and j_1,...,$j_n \in \mathbb{Z}$, then

$$W\Big(\sum_{k=1}^{n} a_k W^{j_k}(\mu)\Big) = \sum_{k=1}^{n} \frac{a_k t_{j_k}}{a_1 t_{j_1} + \cdots a_n t_{j_n}} \cdot W^{j_k+1}(\mu).$$

This shows that $W(C_m) \subset C_m$ and by continuity $W(\overline{C_m}) \subset \overline{C_m}$. Since $\overline{C_m}$ is a compact convex subset of $\mathcal{M}(X)$, it follows from the Schauder-Tychonoff theorem that W has a fixed point in $\overline{C_m}$ or in other words there is a e^f-conformal measure for T in $\overline{C_m}$. Moreover, our assumption and Lemma 3.1 imply that $\overline{C_m} \subset A_\mu$. This proves the conclusion. □

The conclusion of Theorem 3.2 remains true under the assumption that there exists an integer $m \leq 0$ such that

$$E_n(f) \leq g \int_X E_n(f)d\mu$$

for every $n \leq m$, by considering W^{-1}.

4 The derivative of Denjoy C^1 diffeomorphisms

Let $T : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number $\rho(T)$. It is well known (see [2], [7], [10]) that T is uniquely ergodic and there exists a unique minimal set $K \subset S^1$ which is the support of the unique T-invariant Borel probability measure μ, and either $K = S^1$, in which case T is topologically conjugate to the rotation by the angle $2\pi\rho(T)$ or K is a Cantor set and T is only topologically semi-conjugate to the rotation by the angle $2\pi\rho(T)$. In the latter case T is a Denjoy C^1 diffeomorphism and the semi-conjugation is never C^1. In both cases, K is the non-wandering set of T and

$$\int_{S^1} \log(T^n)' d\mu = 0$$

for every $n \in \mathbb{Z}$.

A T'-conformal measure ν for T on K will be called automorphic for T and is a Borel probability measure on K such that

$$\int_K \phi d\nu = \int_K (\phi \circ T)T' d\nu$$

for every continuous function $\phi : K \to \mathbb{R}$. By the change of variable formula, the (normalized) Lebesgue measure of S^1 is automorphic for T. It is also T-quasi-invariant from the mean value theorem.

If $h : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism and $S = h \circ T \circ h^{-1}$, then S is a Denjoy C^1 diffeomorphism with unique minimal set $h(K)$ and unique S-invariant Borel probability measure $h_*\mu$. If ν is an automorphic measure for T, then

$$\nu' = \frac{h'}{\displaystyle\int_K h' d\nu} \cdot h_*\nu$$

is automorphic for S. It follows that if $\nu \ll \mu$, then $\nu' \ll h_*\mu$.

The proof of Theorem 1.1 is now an immediate consequence of Theorem 3.2 and the chain rule.

If $\log T'$ is a continuous coboundary on K, then there exists a unique automorphic measure for T which is absolutely continuous with respect to μ, since T is uniquely ergodic. We note however that one can construct examples of Denjoy C^1 diffeomorphisms where the logarithm of the derivative is not a continuous coboundary on the unique minimal set and others where it is. In any case, $\log T'$ is never a continuous coboundary on S^1, by an argument due to M. Herman [7]. See also section 6 in [2].

References

[1] K. Athanassopoulos, *Divergence of C^1 vector fields and nontrivial minimal sets on 2-manifolds*, J. Differential Equations 243 (2007), 24-35.

[2] K. Athanassopoulos, *Denjoy C^1 diffeomorphisms of the circle and McDuff's question*, Expositiones Math. 33 (2015), 48-66.

[3] P. Bohl, *Über die hinsichtlich der unabhängigen variabeln periodische Differentialgleichung erster Ordnung*, Acta Math. 40 (1916) 321-336.

[4] A. Denjoy, *Sur les courbes définies par les équations différentielles a la surface du tore*, J. Math. Pures et Appl. 11 (1932) 333-375.

[5] M. Denker and M.Urbanski, *On the existence of conformal measures*, Trans. Amer. Math. Soc. 328 (1991), 563-587.

[6] R. Douady and J.-C. Yoccoz, *Nombre de rotation des difféomorphisms du cercles et mesures automorphes*, Regular Chaotic Dyn. 4 (1999), 3-24.

[7] M. Herman, *Sur la conjugaision différentiable des difféomorphismes du cercle á des rotations*, Publications Mathématiques I.H.E.S 49 (1979), 5-233.

[8] W. Krieger, *On quasi-invariant measures in uniquely ergodic systems*, Invent. Math. 14 (1971), 184-196.

[9] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, New York, 1983.

[10] A. Navas, *Groups of circle diffeomorphisms*, Chicago Lectures in Mathematics Series, The University of Chicago Press, Chicago and London, 2011.