A sufficient condition for the logarithm of the derivative of a Denjoy $C^1$ diffeomorphism to be a measurable coboundary

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ABSTRACT
We give a sufficient condition under which the logarithm of the derivative of a Denjoy $C^1$ diffeomorphism of the circle is a measurable coboundary on the unique Cantor minimal set. This condition also guarantees the existence of an automorphic measure which is equivalent to the unique invariant Borel probability measure.

1 Introduction
One of the most important examples of uniquely ergodic homeomorphisms are the orientation preserving homeomorphisms of the circle $S^1$ which have irrational rotation numbers and are not topologically conjugate to rotations. Among them special place occupy the Denjoy $C^1$ diffeomorphisms, named after A. Denjoy who gave explicit constructions of such $C^1$ examples and proved that they cannot be $C^2$ in [4] (see also [3]). For an exposition of the theory of Denjoy $C^1$ diffeomorphisms of $S^1$ we refer to [2] and [7].

Let $T : S^1 \to S^1$ be a Denjoy $C^1$ diffeomorphism with unique Cantor minimal set $K$ and unique invariant Borel probability measure $\mu$. Then $K$ is the nonwandering set of $T$ and $\mu$ is supported on $K$. The original motivation of this note was to examine whether $\mu$ is in some sense geometric with respect to $T$. This is closely related to a problem stated in [1]. To be more precise, we want to find conditions under which $\mu$ is equivalent to a Borel probability measure $\nu$ on $K$ such that

$$\int_K \phi d\nu = \int_K (\phi \circ T) T' d\nu$$

for every continuous function $\phi : K \to \mathbb{R}$. A measure $\nu$ with this property is called automorphic for $T$. The existence of an automorphic measure for $T$ which is equivalent to $\mu$ is closely related to the existence of measurable solutions of the cohomological equation $\log T' = u - u \circ T$. This relation is explained in section 2.

The automorphic measures for $T$ are defined as automorphic measures of exponent 1 in [6], but they have appeared in the literature much earlier in the more general setting of homeomorphisms on compact metric spaces. Let $X$ be a compact metric space, $T : X \to X$ be a homeomorphism and let $f : X \to \mathbb{R}$ be a continuous function. A Borel
probability measure \( \nu \) on \( X \) is called an \( e^f \)-automorphic measure for \( T \) if \( \nu \) is equivalent to \( T_* \nu \) and \( \frac{d\nu}{d(T_* \nu)} = e^f \). This kind of measure has been used without a particular name in [8].

In this note we study the existence of automorphic measures for a uniquely ergodic homeomorphism which is absolutely continuous to its invariant Borel probability measure. In section 3 we present a sufficient condition for that, using the Schauder-Tychonoff fixed point theorem applied to the dual Perron-Frobenius operator on an appropriate convex set. This approach was inspired by [6]. As a corollary we get the following, which is the main result of this paper.

**Theorem 1.1.** Let \( T : S^1 \to S^1 \) be a Denjoy \( C^1 \) diffeomorphism with unique minimal set \( K \) and unique \( T \)-invariant Borel probability measure \( \mu \). If there exists a positive \( g \in L^1(\mu) \) and an integer \( m \geq 0 \) such that

\[
(T^n)' \leq g \int_K (T^n)' d\mu
\]

on \( K \) for every \( n \geq m \), then there exists an automorphic measure for \( T \) which is equivalent to \( \mu \) and \( \log T' \) is a measurable coboundary on \( K \). \( \square \)

2 Automorphic measures

Let \( T : X \to X \) be a homeomorphism of a compact metric space \( X \) and let \( f : X \to \mathbb{R} \) be a continuous function. An \( e^f \)-automorphic measure for \( T \) is a Borel probability measure \( \nu \) on \( X \) such that

\[
\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu
\]

for every continuous function \( \phi : X \to \mathbb{R} \). Evidently, an \( e^f \)-automorphic measure for \( T \) is \( T \)-quasi-invariant and is an \( e^{-f \circ T^{-1}} \)-automorphic measure for \( T^{-1} \).

It is easy to see that if \( h : X \to X \) is a homeomorphism and \( S = h \circ T \circ h^{-1} \), then \( h_* \nu \) is an \( e^{f \circ h^{-1}} \)-automorphic measure for \( S \) for every \( e^f \)-automorphic measure \( \nu \) for \( T \).

The construction of an automorphic measure can be described as follows (see [5]).

Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers and let \( c = \limsup_{n \to +\infty} \frac{a_n}{n} \). The series \( \sum_{n=1}^{\infty} e^{a_n - ns} \) converges for \( s > c \), diverges for \( s < c \) and we cannot tell for \( s = c \), by the root test. There exists a sequence of positive real numbers \((b_n)_{n \in \mathbb{N}}\) such that \( \lim_{n \to +\infty} \frac{b_n}{b_{n+1}} = 1 \) and the series \( \sum_{n=1}^{\infty} b_n e^{a_n - ns} \) converges for \( s > c \) and diverges for \( s \leq c \).

Let now \( x \in X \) be such that the limit

\[
c = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x))
\]

exists in \( \mathbb{R} \). It is well known that the set consisting of these points has measure 1 with respect to every \( T \)-invariant Borel probability measure, and is therefore non-empty. Let
$a_n = nc - \sum_{k=1}^{n} f(T^{-k}(x))$ and $M_s = \sum_{n=1}^{\infty} b_n e^{a_n-n s}$, where $(b_n)_{n \in \mathbb{N}}$ is the corresponding sequence as above. Each accumulation point in the weak* topology as $s \downarrow 0$ of the directed family

$$\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n-n s} \delta_{T^n(x)}, \quad s > 0$$

is an $\epsilon^f$-automorphic measure for $T$.

There is a close relation between $\epsilon^f$-automorphic measures for a homeomorphism $T : X \to X$ of a compact metric space and solvability of the cohomological equation $f = u - u \circ T$, where $f : X \to \mathbb{R}$ is continuous.

Let $\mu$ be any $T$-invariant Borel probability measure. If there exists a measurable solution $u$ of the above cohomological equation defined $\mu$-almost everywhere such that $e^{-u} \in L^1(\mu)$, then there exists an $\epsilon^f$-automorphic measure $\nu$ for $T$ equivalent to $\mu$ with density

$$\frac{d\nu}{d\mu} = \frac{e^{-u}}{\int_X e^{-u} d\mu}.$$

Thus, if there exists a continuous solution $u$, then for every $T$-invariant Borel probability measure we get an equivalent $\epsilon^f$-automorphic measure for $T$. Moreover, in this case, every $\epsilon^f$-automorphic measure $\nu$ for $T$ is obtained in this way. Indeed, we have

$$\int_X \phi e^u d\nu = \int_X (\phi \circ T)e^u d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$, and so the equivalent measure $\mu$ to $\nu$ with density

$$\frac{d\mu}{d\nu} = \frac{e^u}{\int_X e^u d\nu}$$

is $T$-invariant. Consequently, if $f$ is a continuous coboundary, then the $\epsilon^f$-automorphic measures for $T$ are in one-to-one correspondence with the $T$-invariant Borel probability measures and each $\epsilon^f$-automorphic measure for $T$ is equivalent to its corresponding $T$-invariant measure.

Conversely, suppose that $\mu$ is an ergodic $T$-invariant Borel probability measure and $f : X \to \mathbb{R}$ is a continuous function such that $\int_X f d\mu = 0$. Suppose further that there exists an $\epsilon^f$-automorphic measure $\nu \in \mathcal{M}(X)$ for $T$ which is absolutely continuous with respect to $\mu$ and $g = \frac{d\nu}{d\mu}$. For every measurable set $A \subset X$ we have

$$\int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T)e^f d\nu = \int_X (\chi_A \circ T)e^f gd\mu$$

and therefore

$$\int_{T^{-1}(A)} [ge^f - (g \circ T)] d\mu = 0.$$

Since $\mu$ is $T$-invariant, it follows that $g \circ T = ge^f$ $\mu$-almost everywhere. The ergodicity of $\mu$ implies now that $g > 0$ $\mu$-almost everywhere. So, $u = -\log g$ is a measurable
solution of the cohomological equation \( f = u - u \circ T \). If \( \log g \in L^\infty(\mu) \) and \( T \) is a minimal homeomorphism, then there exists some continuous function \( u : X \to \mathbb{R} \) such that \( f = u - u \circ T \), by Proposition 4.2 on page 46 in [7].

Note that \( \nu \) is equivalent to \( \mu \), because \( g > 0 \). We remark that this is actually a more general fact which holds for every \( T \)-quasi-invariant Borel probability measure. To see this, let \( T : X \to X \) be a homeomorphism of a compact metric space \( X \) and \( \mu \) be an ergodic \( T \)-invariant Borel probability measure. Let \( \nu \) is a \( T \)-quasi-invariant Borel probability measure which is absolutely continuous with respect to \( \mu \). Let \( g = \frac{d\nu}{d\mu} \) and \( A = g^{-1}(0) \). If \( S = \bigcup_{n \in \mathbb{Z}} T^n(A) \), then \( S \) is \( T \)-invariant and \( \nu(S) = 0 \). On the other hand \( \mu(X \setminus S) > 0 \), and since \( \mu \) is ergodic we get \( \mu(S) = 0 \), that is \( g > 0 \) \( \mu \)-almost everywhere.

In particular, if \( T \) is uniquely ergodic, then every \( T \)-quasi-invariant measure for \( T \) which is absolutely continuous with respect its unique invariant Borel probability measure is equivalent to it.

### 3 Absolutely continuous automorphic measures for uniquely ergodic homeomorphisms

Let \( X \) be a compact metric space and \( \mu \in \mathcal{M}(X) \). The set

\[
A_\mu = \{ \nu \in \mathcal{M}(X) : \nu \ll \mu \}
\]

is not empty, since it contains \( \mu \), and is convex. In general, \( A_\mu \) is not a closed subset of \( \mathcal{M}(X) \) with respect to the weak* topology. For example, if we let \( \mu \) be the Lebesgue measure on the unit interval \([0,1]\) and for \( 0 < \epsilon < 1 \) we let \( \mu_\epsilon \) denote the Borel probability measure on \([0,1]\) with density \( \frac{1}{\epsilon} \chi_{(0,\epsilon]} \), then \( \lim_{\epsilon \to 0} \mu_\epsilon \) is the Dirac point measure at 0.

**Lemma 3.1.** Let \( X \) be a compact metric space and \( \mu \in \mathcal{M}(X) \). Let \( (\nu_n)_{n \in \mathbb{N}} \) be a sequence in \( A_\mu \) converging weakly* to some \( \nu \in \mathcal{M}(X) \) and let \( f_n = \frac{d\nu_n}{d\mu} \), \( n \in \mathbb{N} \). If there exist non-negative \( h, g \in L^1(\mu) \) such that \( h \leq f_n \leq g \) for every \( n \in \mathbb{N} \), then \( \nu \in A_\mu \) and \( h \leq \frac{d\nu}{d\mu} \leq g \).

**Proof.** Since \( \nu \) is a finite measure, there exists a (countable) basis \( \mathcal{U} \) of the topology of \( X \) such that \( \nu(\partial U) = 0 \) for every \( U \in \mathcal{U} \). So \( \mathcal{U} \) is contained in the algebra

\[
\mathcal{C}(\nu) = \{ A | A \subset X \text{ Borel and } \nu(\partial A) = 0 \}
\]

and since it generates the Borel \( \sigma \)-algebra of \( X \), so does \( \mathcal{C}(\nu) \). Let now \( A \subset X \) be a Borel set with \( \mu(A) = 0 \) and \( \epsilon > 0 \). There exists \( 0 < \delta < \epsilon \) such that \( \int_B gd\mu < \epsilon \) for every Borel set \( B \subset X \) with \( \mu(B) < \delta \), because \( g \in L^1(\mu) \). There exists some \( A_0 \in \mathcal{C}(\nu) \) such that \( \mu(A \Delta A_0) < \delta \) and \( \nu(A \Delta A_0) < \delta \). Thus \( \mu(A_0) < \delta \) and \( |\nu(A) - \nu(A_0)| < \delta \).
By weak* convergence, $\nu(A_0) = \lim_{n \to +\infty} \nu_n(A_0)$ and so there exists some $n_0 \in \mathbb{N}$ such that $|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \geq n_0$. Therefore,

$$\nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq \int_{A_0} g d\mu + \epsilon < 2\epsilon.$$  

It follows that $0 \leq \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in A_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(\mathcal{P}_n)_{n \in \mathbb{N}}$ of $X$ such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, the Borel $\sigma$-algebra of $X$ is generated by $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ and $\mu(\partial B) = 0$ for every $B \in \mathcal{P}_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of $X$ such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively $\mathcal{P}_n$ to be the finite family consisting of Borel sets with positive $\mu$ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in \mathcal{P}_{n-1}$, taking $\mathcal{P}_0 = \{X\}$.

Let $\mathcal{P}_n(x)$ denote the element of $\mathcal{P}_n$ which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n \to +\infty} \nu(\mathcal{P}_n(x)),$$

$\mu$-almost everywhere on $X$ and in $L^1(\mu)$ (see page 8 in [9]). On the other hand, by the weak* convergence and since $\nu \in A_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k}\mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \leq \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} \leq \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \leq \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} g d\mu.$$

Since

$$\lim_{k \to +\infty} \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} g d\mu = g(x)$$

$\mu$-almost everywhere on $X$ and in $L^1(\mu)$, it follows that $0 \leq \frac{d\nu}{d\mu}(x) \leq g(x)$ $\mu$-almost everywhere on $X$.

Similarly, from

$$\frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} \geq -\frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \geq -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} h d\mu$$

follows that $h(x) \leq \frac{d\nu}{d\mu}(x)$ $\mu$-almost everywhere on $X$. □

Let now $T : X \to X$ be a uniquely ergodic homeomorphism $T : X \to X$. Automorphic measures for $T$ can be obtained as fixed points of the dual Perron-Frobenius operator. Let $\mu$ be the unique $T$-invariant Borel probability measure and $c = \int_X f d\mu$, where $f : X \to \mathbb{R}$ is a continuous function. Let $\mathcal{M}(X)$ denote the set of Borel probability
measures on $X$ equipped with the weak* topology. The dual Perron-Frobenius operator is the continuous map $W : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by

$$W(\nu)(\phi) = \frac{1}{\int_X e^f d\nu} \cdot \int_X (\phi \circ T)e^f d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$ (see page 185 in [8]). Since $T$ is a homeomorphism, $W$ is a homeomorphism and its inverse is given by the formula

$$W^{-1}(\nu)(\phi) = \frac{1}{\int_X e^{-f \circ T^{-1}} d\nu} \cdot \int_X (\phi \circ T^{-1})e^{-f \circ T^{-1}} d\nu.$$

It follows from the Schauder-Tychonoff theorem that $W$ has a fixed point in $\mathcal{M}(X)$. If $\nu$ is a fixed point of $W$, then $\int_X e^f d\nu = e^c$, and therefore $\nu$ is an $e^{f-c}$-automorphic measure for $T$. Indeed, if $\nu$ is a fixed point of $W$, then for every $n \in \mathbb{N}$ we have

$$\left( \int_X e^f d\nu \right)^n = \int_X \exp \left( \sum_{k=0}^{n-1} f \circ T^k \right) d\nu,$$

as one easily verifies by induction. It follows that

$$n \log \left( \int_X e^{f-c} d\nu \right) = \log \left( \int_X \exp \left( -nc + \sum_{k=0}^{n-1} f \circ T^k \right) d\nu \right) \leq \| -nc + \sum_{k=0}^{n-1} f \circ T^k \|,$$

and therefore

$$\left| \log \left( \int_X e^{f-c} d\nu \right) \right| \leq \| \lim_{n \to +\infty} \left( -c + \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right) \| = 0.$$

For any continuous function $f : X \to \mathbb{R}$ we put

$$E_n(f) = \begin{cases} \exp \sum_{k=1}^{n} f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp \left( - \sum_{k=0}^{\lfloor n \rfloor - 1} f \circ T^k \right), & \text{if } n < 0. \end{cases}$$

We can use the Schauder-Tychonoff theorem to get the following result for the existence of absolutely continuous automorphic measures in the case of uniquely ergodic homeomorphisms.

**Theorem 3.2.** Let $X$ be a compact metric space and $T : X \to X$ a uniquely ergodic homeomorphism with unique invariant Borel probability measure $\mu$. Let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a non-negative $g \in L^1(\mu)$ and an integer $m \geq 0$ such that

$$E_n(f) \leq g \int_X E_n(f) d\mu$$
for every \( n \geq m \), then there exists an \( \epsilon \)-automorphic measure for \( T \) equivalent to \( \mu \) and \( f \) is a measurable coboundary.

**Proof.** Let \( W : \mathcal{M}(X) \to \mathcal{M}(X) \) be the dual Perron-Frobenius operator. One can prove by induction that

\[
W^n(\nu)(\phi) = \frac{1}{\int_X (\exp \sum_{k=0}^{n-1} f(T^k))d\nu} \cdot \int_X (\phi \circ T^n)(\exp \sum_{k=0}^{n-1} f(T^k))d\nu
\]

and

\[
W^{-n}(\nu)(\phi) = \frac{1}{\int_X (\exp(-\sum_{k=1}^{n} f(T^{-k})))d\nu} \cdot \int_X (\phi \circ T^{-n})(\exp(-\sum_{k=1}^{n} f(T^{-k})))d\nu
\]

for every \( \nu \in \mathcal{M}(X) \) and \( n \in \mathbb{N} \). From the invariance of \( \mu \) we get

\[
W^n(\mu)(\phi) = \frac{1}{\int_X E_n(f)d\mu} \cdot \int_X \phi E_n(f)d\mu
\]

for every \( n \in \mathbb{Z} \). Note also that

\[
\int_X (\phi \circ T)e^f E_n(f)d\mu = \int_X \phi E_{n+1}(f)d\mu
\]

for every continuous \( \phi : X \to \mathbb{R} \).

If now \( A \subset X \) is a measurable set, it follows from regularity that

\[
W^n(\mu)(A) = \frac{1}{\int_X E_n(f)d\mu} \cdot \int_A E_n(f)d\mu
\]

which implies that \( W^n(\mu) \in A_\mu \) and

\[
\frac{dW^n(\mu)}{d\mu} = \frac{E_n(f)}{\int_X E_n(f)d\mu}
\]

for every \( n \in \mathbb{Z} \). If \( C_m \) is the convex hull of \( \{W^n(\mu) : n \geq m\} \), then \( W(C_m) \subset C_m \).

Indeed, let

\[
t_n = \frac{\int_X E_{n+1}(f)d\mu}{\int_X E_n(f)d\mu}
\]

for all \( n \in \mathbb{Z} \). If \( a_1, \ldots, a_n \geq 0 \) are such that \( a_1 + \cdots + a_n = 1 \) and \( j_1, \ldots, j_n \in \mathbb{Z} \), then

\[
W\left(\sum_{k=1}^{n} a_k W^{j_k}(\mu)\right) = \sum_{k=1}^{n} \frac{a_k t_{j_k}}{a_1 t_{j_1} + \cdots + a_n t_{j_n}} \cdot W^{j_k+1}(\mu).
\]
This shows that $W(C_m) \subseteq C_m$ and by continuity $W(\overline{C_m}) \subseteq \overline{C_m}$. Since $\overline{C_m}$ is a compact convex subset of $\mathcal{M}(X)$, it follows from the Schauder-Tychonoff theorem that $W$ has a fixed point in $\overline{C_m}$ or in other words there is an $e^t$-automorphic measure for $T$ in $\overline{C_m}$. Moreover, our assumption and Lemma 3.1 imply that $\overline{C_m} \subseteq A_\mu$. This proves the conclusion. □

The conclusion of Theorem 3.2 remains true under the assumption that there exists an integer $m \leq 0$ such that
\[
E_n(f) \leq g \int_X E_n(f) d\mu
\]
for every $n \leq m$, by considering $W^{-1}$.

4 The derivative of Denjoy $C^1$ diffeomorphisms

Let $T : S^1 \to S^1$ be an orientation preserving $C^1$ diffeomorphism with irrational rotation number $\rho(T)$. It is well known (see [2], [7]) that $T$ is uniquely ergodic and there exists a unique minimal set $K \subset S^1$ which is the support of the unique $T$-invariant Borel probability measure $\mu$, and either $K = S^1$, in which case $T$ is topologically conjugate to the rotation by the angle $2\pi \rho(T)$ or $K$ is a Cantor set and $T$ is only topologically semi-conjugate to the rotation by the angle $2\pi \rho(T)$. In the latter case $T$ is a Denjoy $C^1$ diffeomorphism and the semi-conjugation is never $C^1$. In both cases, $K$ is the non-wandering set of $T$ and
\[
\int_{S^1} \log(T^n)' d\mu = 0
\]
for every $n \in \mathbb{Z}$.

A $T'$-automorphic measure $\nu$ for $T$ on $K$ will be called automorphic for $T$ and is a Borel probability measure on $K$ such that
\[
\int_K \phi d\nu = \int_K (\phi \circ T)T'd\nu
\]
for every continuous function $\phi : K \to \mathbb{R}$. If $h : S^1 \to S^1$ be an orientation preserving $C^1$ diffeomorphism and $S = h \circ T \circ h^{-1}$, then $S$ is a Denjoy $C^1$ diffeomorphism with unique minimal set $h(K)$ and unique $S$-invariant Borel probability measure $h_\ast \mu$. If $\nu$ is an automorphic measure for $T$, then
\[
\nu' = \frac{h'}{h'd\nu} \cdot h_\ast \nu
\]
is automorphic for $S$. It follows that if $\nu \ll \mu$, then $\nu' \ll h_\ast \mu$.

The proof of Theorem 1.1 is now an immediate consequence of Theorem 3.2 and the chain rule.

If $\log T'$ is a continuous coboundary on $K$, then there exists a unique automorphic measure for $T$ which is absolutely continuous with respect to $\mu$, since $T$ is uniquely ergodic. We note however that one can construct examples of Denjoy $C^1$ diffeomorphisms where the logarithm of the derivative is not a continuous coboundary on the unique minimal set and others where it is. In any case, $\log T'$ is never a continuous coboundary on $S^1$, by an argument due to M. Herman [7]. See also section 6 in [2].
References


