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Notes on vector bundles and characteristic classes

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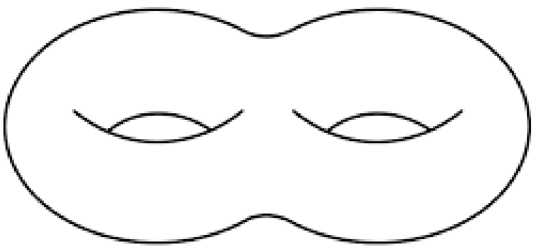

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Chapter 1

Vector bundles

1.1 Complex and real vector bundles

A complex, respectively real, vector bundle of rank n is a triple $\xi = (E, p, M)$, where E and M are topological spaces and $p : E \to M$ is a continuous map such that for every $x \in M$ the level set $p^{-1}(x)$ is a complex, respective;y real, vector space of dimension n and there exists an open cover $\mathcal{U}$ of M together with a family of homeomorphisms $h_U : p^{-1}(U) \to U \times \mathbb{C}^n$, respectively $h_U : p^{-1}(U) \to U \times \mathbb{R}^n$, $U \in \mathcal{U}$, so that h_U maps each level set $p^{-1}(x)$ linearly isomorphically onto $\{x\} \times \mathbb{C}^n$, respectively onto $\{x\} \times \mathbb{R}^n$, for $x \in U$. The homeomorphism h_U is called a local trivialization of the bundle over U. The space E is the total space and M is the base space of the bundle. The level sets $E_x = p^{-1}(x)$, $x \in M$, are called the fibres of the bundle.

The vector bundle $\xi = (E, p, M)$ is smooth, if E and M are smooth manifolds, the bundle map p is smooth and it has a family of local trivializations consisting of smooth diffeomorpisms.

Examples 1.1.1. (a) For every topological space M the projection onto the first factor $pr_1 : M \times \mathbb{C}^n \to M$ is a bundle map. The vector bundle $\epsilon^n_{\mathbb{C}} = (M \times \mathbb{C}^n, pr_1, M)$ is the complex trivial vector bundle of rank n.

(b) For every smooth n-manifold M its tangent bundle is a smooth real vector bundle of rank n with total space TM and base space M. In this case the bundle map $p : TM \to M$ is the canonical projection sending each tangent vector to its point of application.

(c) Let M be a regular m-dimensional submanifold of the euclidean space $\mathbb{R}^{m+n}$. Let

$$E = \bigcup_{x \in M} \{x\} \times (T_x M)^{\perp} \subset M \times \mathbb{R}^{m+n}$$

where the orthogonal complements are taken with respect to the euclidean inner product in $\mathbb{R}^{m+n}$. The map $p : E \to M$ with $p(x, v) = x$ is a bundle map defining a real smooth vector bundle over M called the normal bundle of M in $\mathbb{R}^{m+n}$. One way to construct local trivializations of p is the following. Let $x_0 \in M$. There exists an open neighbourhood U of x_0 on which there are smooth local coordinates. So, on U we have smooth basic tangent vector fields X_1,..., X_m to M. Let $\{v_1, ..., v_n\}$

be a basis of $(T_{x_0}M)^{\perp}$. There is now an open neighbourhood $W \subset U$ of x_0 such that

$$\det(X_1(x), ..., X_m(x), v_1, ..., v_n) \neq 0$$

for every $x \in W$. Applying Gram-Schmidt orthogonalization to the basis

$$\{X_1(x), ..., X_m(x), v_1, ..., v_n\}$$

we obtain an orthonormal basis

$$\{\tilde{X}_1(x), ..., \tilde{X}_m(x), Y_1(x), ..., Y_n(x)\}$$

such that $\{\tilde{X}_1(x), ..., \tilde{X}_m(x)\}$ is an orthonormal basis of T_xM and $\{Y_1(x), ..., Y_n(x)\}$ is an orthonormal basis of $(T_xM)^{\perp}$ for every $x \in W$. The map $g : W \times \mathbb{R}^n \to p^{-1}(W)$ defined by

$$g(x, t_1, ..., t_n) = \sum_{j=1}^{n} t_j Y_j(x)$$

is a diffeomorphism and $h = g^{-1}$ is a local trivialization of p over W. This shows that p is a vector bundle map.

(d) Let $n \in \mathbb{Z}^+$ and $E_n = S^{2n+1} \times \mathbb{C}/\sim$, where

$$(z_0, ..., z_n, u) \sim (\lambda z_0, ..., \lambda z_n, \lambda^{-1} u)$$

for $\lambda \in S^1$. The projection $pr_1 : S^{2n+1} \times \mathbb{C} \to S^{2n+1}$ onto the first factor induces a continuous map $q : E_n \to \mathbb{C}P^n$, which defines a smooth complex bundle of rank 1. A vector bundle of rank 1 is usually called line bundle.

There are local trivializations $h_j : q^{-1}(U_j) \to U_j \times \mathbb{C}$, $0 \leq j \leq n$, of q over the domains of the canonical atlas $\{(U_0, \phi_0), ..., (U_n, \phi_n)\}$ given by the formulas

$$h_j([z, u]) = ([z], u).$$

The inverse of h_j is given by

$$h_j^{-1}([z], u) = [\frac{z}{\|z\|}, u]$$

for $[z] \in U_j$. It is obvious that E_n becomes a smooth manifold and q a smooth vector bundle map. The complex line bundle $(E_n, q, \mathbb{C}P^n)$ is called the tautological (or canonical) line bundle over the complex projective space $\mathbb{C}P^n$.

Similarly, there is a tautological real line bundle over the real projective space $\mathbb{R}P^n$, where in this case the total space is $S^n \times \mathbb{R}/\sim$, and $(x, t) \sim (-x, -t)$. In particular, for $n = 1$ the total space is the Möbius strip and the base space is S^1.

Let $\xi_1 = (E_1, p_1, M_1)$ and $\xi_2 = (E_2, p_2, M_2)$ be two complex, respectively real, vector bundles. A vector bundle morphism from ξ_1 to ξ_2 is a pair $(\tilde{f}, f)$ of continuous maps $f : M_1 \to M_2$ and $\tilde{f} : E_1 \to E_2$ such that $p_2 \circ \tilde{f} = f \circ p_1$ and $\tilde{f}$ maps linearly $p_1^{-1}(x)$ into $p_2^{-1}(f(x))$ for every $x \in M_1$. In case the vector bundles are smooth we say that the morphism is smooth if both f and $\tilde{f}$ are smooth.

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{f}} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

If $\xi = (E, p, M)$ is a vector bundle and $A \subset M$, then the restriction of p on $p^{-1}(A)$ is a vector bundle map over A and the pair of the obvious inclusions is a vector bundle morphism from ξ to $\xi|_A = (p^{-1}(A), p|_{p^{-1}(A)}, A)$.

Two vector bundles ξ_1 and ξ_2 over the same base space $M = M_1 = M_2$ are called isomorphic if there are vector bundle morphisms $(\tilde{f}, id_M)$ from ξ_1 to ξ_2 and $(\tilde{g}, id_M)$ from ξ_2 to ξ_1 such that $\tilde{g} \circ \tilde{f} = id_{E_1}$ and $\tilde{f} \circ \tilde{g} = id_{E_2}$. In the sequel we shall simply write $\tilde{f}$ instead of $(\tilde{f}, id_M)$ and $\tilde{f} : \xi_1 \cong \xi_2$ to denote that $\tilde{f}$ is an isomorphism from ξ_1 to ξ_2. In the smooth case, ξ_1 and ξ_2 are called smoothly isomorphic if $\tilde{f}$ and $\tilde{g}$ are smooth diffeomorphisms.

Lemma 1.1.2. Let $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ be two complex, respectively real, vector bundles over the space M. If a vector bundle morphism $\tilde{f} : E_1 \to E_2$ maps each fiber $(p_1)^{-1}(x)$ isomorphically onto the fiber $(p_2)^{-1}(x)$, $x \in M$, then $\tilde{f} : \xi_1 \cong \xi_2$. If $\tilde{f}$ is smooth, then it is a smooth vector bundle isomorphism.

Proof. Our assumptions imply that $\tilde{f}$ is a bijection. Thus, we need only show that $\tilde{f}^{-1}$ is continuous and smooth in the smooth case. If $U \subset M$ is an open set and $h : (p_1)^{-1}(U) \to U \times \mathbb{C}^n$ and $g : (p_2)^{-1}(U) \to U \times \mathbb{C}^n$ are local trivializations, then

$$F = g \circ \tilde{f} \circ h^{-1} : U \times \mathbb{C}^n \to U \times \mathbb{C}^n$$

is an isomorphism of trivial vector bundles. Indeed, there is a continuous map $G : U \times \mathbb{C}^n \to \mathbb{C}^n$ such that $F(x, v) = (x, G(x, v))$ and $G(x, .) \in GL(n, \mathbb{C})$ for every $x \in U$. Also, taking the inverse in $GL(n, \mathbb{C})$ is a smooth map and $G(x, .)^{-1}$ depends continuously on x and smoothly in the smooth case. Since continuity and smoothness are local properties, the conclusion follows. □

Example 1.1.3. Let $\mathcal{H}_n = \{(\ell, u) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : u \in \ell\}$ and $p : \mathcal{H}_n \to \mathbb{C}P^n$ be the projection onto the first factor. The continuous map $f : S^{2n+1} \times \mathbb{C} \to \mathcal{H}_n$ defined by

$$f(z_0, ..., z_n, w) = ([z_0, ..., z_n], wz_0, ..., wz_n)$$

is onto and open. Moreover, $f(z_0, ..., z_n, w) = f(z'_0, ..., z'_n, w')$ if and only if there exists some $\lambda \in \mathbb{C}^*$ such that $z'_j = \lambda z_j$ for all $0 \leq j \leq n$ and $w' = \lambda^{-1} w$. This implies that f induces a homeomorphism $\tilde{f} : E_n \to \mathcal{H}_n$ such that $p \circ \tilde{f} = q$ and $\tilde{f}(q^{-1}(\ell)) = \ell \cup \{0\} \subset C^{n+1}$. Since $(E_n, q, \mathbb{C}P^n)$ is a smooth complex line bundle, the triple $(\mathcal{H}_n, p, \mathbb{C}P^n)$ becomes a smooth complex line bundle so that $\tilde{f}$ is a smooth vector bundle isomorphism. This is an alternative version of the tautological line bundle over the complex projective space.

1.2 Direct sums and inner products

Let $\xi_1 = (E_1, p_1, M_1)$ and $\xi_2 = (E_2, p_2, M_2)$ be two complex, respectively real, vector bundles. Then, the triple $(E_1 \times E_2, p_1 \times p_2, M_1 \times M_2)$ is a vector bundle with fibres $p_1^{-1}(x_1) \times p_2^{-1}(x_2)$, $(x_1, x_2) \in M_1 \times M_2$, because if $h_1 : p_1^{-1}(U_1) \to U \times \mathbb{C}^n$ and

$h_2 : p_2^{-1}(U_2) \to U \times \mathbb{C}^m$ are local trivializations, then $h_1 \times h_2$ is local trivialization of $p_1 \times p_2$ over $U_1 \times U_2$.

Suppose now that $M = M_1 = M_2$ and $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ are two vector bundles over the same space M. We put

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\}$$

and let $p : E_1 \oplus E_2 \to M$ be defined by $p(v_1, v_2) = p_1(v_1) = p_2(v_2)$. In other words, p is the restriction of $p_1 \times p_2$ over the diagonal in $M \times M$. The vector bundle $\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, p, M)$ is called the direct (or Whitney) sum of ξ_1 and ξ_2 and it has fibres the direct sums of the corresponding fibres of ξ_1 and ξ_2.

It is evident that the direct sum of two trivial vector bundles is a trivial vector bundle. However, the direct sum of two vector bundles neither of which is trivial may be trivial. For instance, if $M \subset R^{m+n}$ is a regular m-dimensional submanifold with normal bundle ν in $\mathbb{R}^{m+n}$, then $TM \oplus \nu \cong \epsilon_{\mathbb{R}}^{m+n}$, the trivial real vector bundle of rank $m + n$ over M.

An inner product on a complex (or real) vector bundle $\xi = (E, p, M)$ is a continuous function $g : E \oplus E \to \mathbb{C}$ (respectively $\mathbb{R}$ in the real case) such that its restriction g_x on each fibre E_x is a hermitian (respectively euclidean) inner product.

Lemma 1.2.1. *If M is a paracompact space, then every vector bundle $\xi = (E, p, M)$ of rank n over M admits an inner product.*

Proof. Let $\mathcal{U}$ be an open cover of M for which there is a family of local trivializations $h_U : pi^{-1}(U) \to U \times \mathbb{C}^n$, $U \in \mathcal{U}$. Since M is assumed to be paracompact, there exists a partition of unity $\{f_U : U \in \mathcal{U}\}$ subordinated to $\mathcal{U}$. For $x \in M$ and v, $w \in E_x$ the formula

$$g_x(v, w) = \sum_{U \in \mathcal{U}} f_U(x) \langle h_U(v), h_U(w) \rangle$$

defines an inner product on ξ, where $\langle , \rangle$ is the usual hermitian product on $\{x\} \times \mathbb{C}^n$ or the euclidean inner product on $\{x\} \times \mathbb{R}^n$ in the real case. □

As the proof of the preceding lemma shows, if the vector bundle $\xi = (E, p, M)$ over a smooth manifold M is smooth, then it admits a smooth inner product, by the existence of smooth partitions of unity. A smooth inner product on the tangent bundle of a smooth manifold M is a Riemannian metric on M.

As an application of the existence of inner products we shall prove that two isomorphic smooth vector bundles over a compact smooth manifold are smoothly isomorphic.

A section of a vector bundle $\xi = (E, p, M)$ is a continuous map $s : M \to E$ such that $p \circ s = id_M$, that is $s(x) \in E_x$ for every $x \in M$. The set $\Gamma(\xi)$ of all sections of ξ becomes a vector space in the obvious way. In the smooth case we shall denote by $\Omega^0(\xi)$ the vector subspace of $\Gamma(\xi)$ consisting of the smooth sections of ξ. If $h : p^{-1}(U) \to U \times \mathbb{C}^n$ is a local trivialization over the open set $U \subset M$ and $\{e_1, ..., e_n\}$ is the canonical (or any) basis of $\mathbb{C}^n$, then the formulas $s_j(x) = h^{-1}(x, e_j)$, $x \in U$, $1 \leq j \leq n$, define sections of $\xi|_U$ and $\{s_1(x), ..., s_n(x)\}$ is a basis of E_x for every

$x \in U$. The set $\{s_1, ..., s_n\}$ is called a frame of ξ over U. Conversely, each frame over an open subset U of M gives a trivialization of ξ over U. If we have an inner product on the bundle, then applying the Gram-Schmidt orthogonalization process we can construct orthonormal sections over U. In the smooth case, the above can be carried out smoothly.

Let now $\xi' = (E', p', M)$ be a second vector bundle of rank n over M and $f : E \to E'$ be a vector bundle morphisms of vector bundles over the same base space M. If $\{s_1, ..., s_n\}$ is a frame of ξ over U and $\{s'_1, ..., s'_n\}$ a frame of ξ' over U, then $f_x = f|_{E_x}$ is represented by a $n \times n$ matrix. In this way we get a continuous map $ad(f) : U \to \mathbb{C}^{n\times n}$, which depends on the choice of the local frames. If everything is smooth, then $ad(f)$ is also smooth.

Lemma 1.2.2. *Let M be a compact space, $\xi = (E, p, M)$ and $\xi' = (E', p', M)$ two vector bundles of rank n equipped with inner products. If $f : E \to E'$ is a vector bundle isomorphism, then there exists $\delta > 0$ any vector bundle morphism $\phi : E \to E'$ with $p' \circ \phi = p$ and such that* $\sup\{\|f_x - \phi_x\| : x \in M\} < \delta$ *is a vector bundle isomorphism.*

Proof. Since M is assumed a compact space, it can be covered by a finite number of compact subsets over each of which both bundles are trivial. Thus, it suffices to prove the conclusion only in the case where both bundles are trivial. Choosing frames, f is represented by a continuous map $ad(f) : U \to GL(n, \mathbb{C})$. Since $ad(f)(M)$ is a compact subset of the open subset $GL(n, \mathbb{C})$ of $\mathbb{C}^{n\times n}$, there exists $\delta > 0$ such that the ball of radius δ around $Ad(f)(M)$ is contained in $GL(n, \mathbb{C})$. This implies the assertion. □

Proposition 1.2.3. *Let $\xi = (E, p, M)$ and $\xi' = (E', p', M)$ be two smooth vector bundles of rank n over a compact smooth manifold M. If ξ is isomorphic to ξ', then it is smoothly isomorphic.*

Proof. Since M is assumed to be compact, there exists a finite open cover $\{U_1, ..., U_m\}$ of M and smooth orthonormal frames $\{s^j_1, ..., s^j_n\}$ and $\{t^j_1, ..., t^j_n\}$ of ξ and ξ', respectively, over U_j, $1 \le j \le m$. A vector bundle isomorphism $f : E \to E'$ gives rise to continuous maps $ad(f^j) : U_j \to GL(n, \mathbb{C})$, where $f^j = f|_{U_j}$, $1 \le j \le m$. There exists $\delta > 0$ as in Lemma 1.2.2. For every $1 \le j \le m$ there exists a smooth map $G^j : U_j \to GL(n, \mathbb{C})$ such that $\|G^j(x) - ad(f^j)(x)\| < \delta$ for every $x \in U_j$. Let $g^j : p^{-1}(U_j) \to (p')^{-1}(U_j)$ be defined by

$$g^j(\sum_{k=1}^{n} \lambda_k s^j(x)) = \sum_{k=1}^{n} \left(\sum_{l=1}^{n} G^j_{kl}(x)\lambda_l \right) t^j(x)$$

or in other words $ad(g^j) = G^j$. Obviously, $\|f^j(x) - g^j(x)\| < \delta$ for every $x \in U_j$. Let $\{\psi_1, ..., \psi_m\}$ be a smooth partition of unity subordinated to the open cover $\{U_1, ..., U_m\}$. Now we define $g : E \to E'$ by

$$g_x = g|_{E_x} = \sum_{j=1}^{m} \psi_j(x) g^j(x)$$

for every $x \in M$. Then,

$$\|f_x - g_x\| \leq \sum_{j=1}^{m} \psi_j(x)\|f_x^j - g_x^j\| < \delta$$

for every $x \in M$ and from Lemma 1.2.2 follows that g is a smooth isomorphism of vector bundles. □

1.3 The functors K and KO

As we have already mentioned in the preceding section, the direct sum of two non-trivial vector bundles can be trivial. Actually, the following general fact holds.

Theorem 1.3.1. *If M be a compact space, then for every vector bundle $\xi = (E, p, M)$ over M there exists another vector bundle $\tilde{\xi}$ such that $\xi \oplus \tilde{\xi}$ is trivial.*

Proof. Since M is compact, there exist a finite open cover $\{U_1, ..., U_m\}$ of M and local trivializations $h_j : p^{-1}(U_j) \to U_j \times \mathbb{C}^n$, $1 \leq j \leq m$. There is also a partition of unity $\{\psi_1, ..., \psi_m\}$ of M subordinated to this open cover. Let $f^j = pr_2 \circ h_j : p^{-1}(U_j) \to \mathbb{C}^n$, where pr_2 denotes the projection onto the second factor. Let $g : E \to M \times \mathbb{C}^{nm}$ be defined by

$$g(v) = (p(v), \psi_1(p(v))f^1(v), ..., \psi_m(p(v))f^m(v)).$$

It is obvious that g is a vector bundle morphism of vector bundles over M. Moreover, $g|_{E_x} : E_x \to \{x\} \times \mathbb{C}^{nm}$ is a monomorphism of vector spaces for every $x \in M$. We put

$$\tilde{E} = \{(x, v) \in M \times \mathbb{C}^{nm} : v \in g(E_x)^{\perp}\}$$

where the orthogonal complement is taken with respect to usual hermitian product on $\mathbb{C}^{nm}$. Then, $\tilde{\xi} = (\tilde{E}, pr_1, M)$ is a vector bundle (see Example 1.1.1(c)) and obviously $\xi \oplus \tilde{\xi} \cong \epsilon^{nm}$. □

In case M is a smooth manifold and the bundle ξ in Theorem 1.3.1 is smooth, then the vector bundle $\tilde{\xi}$ can be chosen to be also smooth, by the existence of smooth partitions of unity. In fact, Theorem 1.3.1 holds also under the assumption that the base space M is paracompact and has finite covering dimension. In particular, it holds if M is a topological manifold. We give a proof of this in the appendix to this chapter. For smooth real vector bundles there is an easier proof as consequence of Whitney's immersion theorem, which we present here.

Theorem 1.3.2. *Let M be a smooth manifold and let $\xi = (E, p, M)$ be a smooth real vector bundle of rank n. There exists a smooth real vector bundle $\tilde{\xi}$ over M of rank at most $n + 2\dim M$ such that $\xi \oplus \tilde{\xi}$ is trivial.*

Proof. Let M_0 be the copy of M in E, which is the image of the zero section of ξ. Then $TE|_{M_0} \cong TM \oplus \xi$, because each tangent space $T_{(x,0)}E$ is naturally isomorphic to $T_xM \oplus T_{(x,0)}E_x$.

$$\begin{array}{ccc} TM \oplus E \cong \; TE|_{M_0} & \longrightarrow & TE \\ \downarrow & & \downarrow \\ M & \longrightarrow & E \end{array}$$

By the Whitney immersion theorem, there exists an immersion $\phi : E \to \mathbb{R}^{2(n+\dim M)}$. The derivative of ϕ gives a monomorphism

$$\phi_* : TE|_{M_0} \to T\mathbb{R}^{2(n+\dim M)} = \mathbb{R}^{2(n+\dim M)} \times \mathbb{R}^{2(n+\dim M)}$$

of vector bundles. The orthogonal complement of the image of ϕ_* with respect to the euclidean inner product is the total space of a smooth vector bundle $\xi^\perp$ over M such that $TM \oplus \xi \oplus \xi^\perp$ is the trivial bundle over M of rank $2(n + \dim M)$. Thus, it suffices to take $\tilde{\xi} = TM \oplus \xi^\perp$. □

For any space M and non-negative integer n we let $\mathrm{Vect}_n^{\mathbb{C}}(M)$, respectively $\mathrm{Vect}_n^{\mathbb{R}}(M)$, denote the set of isomorphism classes of complex, respectively real, vector bundles over M. The direct sum of vector bundles makes

$$\mathrm{Vect}^{\mathbb{C}}(M) = \coprod_{n\geq 0} \mathrm{Vect}_n^{\mathbb{C}}(M)$$

an abelian semigroup whose neutral element is represented by the trivial bundle of rank 0 with total space $M \times \{0\}$. Similarly, for $\mathrm{Vect}^{\mathbb{R}}(M)$.

From any abelian semigroup one can construct an abelian group more or less in the same way the integers can be contructed from the set of natural numbers. It is worth to note however that in contrast to the case of the natural numbers the cancellation law may not hold in the semigroups $\mathrm{Vect}^{\mathbb{R}}(M)$ and $\mathrm{Vect}^{\mathbb{C}}(M)$. Indeed, consider for example the 2-sphere S^2. Its normal bundle ν in $\mathbb{R}^3$ is a trivial line bundle over S^2 and $TS^2 \oplus \nu$ is also trivial. So, $\nu \cong \epsilon^1$ and

$$TS^2 \oplus \nu \cong \epsilon^3 \cong \epsilon^2 \oplus \epsilon^1.$$

However, TS^2 is not trivial, by the Hairy Ball Theorem.

Lemma 1.3.3. (A. Grothendieck) *For every abelian semigroup $(V, \oplus)$ there exist a unique abelian group $(K(V), +)$ and a semigroup homomorphism $\gamma : V \to K(V)$ with the universal property that for every abelian group G and every semigroup homomorphism $f : V \to G$ there is a unique group homomorphism $\tilde{f} : K(V) \to G$ such that $\tilde{f} \circ \gamma = f$.*

Proof. Let $(F(V), +)$ denote the free abelian group with basis the set V and let R be its subgroup which is generated by the elements of V of the form $x \oplus y - x - y$, for x, $y \in V$. We put $K(V) = F(V)/R$ and let $\gamma : V \to K(V)$ be defined by $\gamma(x) = x + R$. Then, $\gamma(0) = R$ and

$$\gamma(x \oplus y) = (x \oplus y) + R = (x + y) + R = (x + R) + (y + R)$$

for every x, $y \in V$, from the choice of R.

Let now G be an abelian group and $f : V \to G$ be any semigroup hoomomorphism. There is unique linear extension of f to a group homomorphism $\hat{f} : F(V) \to G$. Obviously, R is contained in Ker$\hat{f}$ and so we get an induced group homomorphism $\tilde{f} : K(V) \to G$ such that $\tilde{f} \circ \gamma = f$. The uniquness of $\tilde{f}$ follows from the fact that if $\tilde{f} \circ \gamma = 0$, then $\tilde{f}(x + R) = 0$ for every $x \in V$ and since the set $\{x + R : x \in V\}$ generates $K(V)$ we must have $\tilde{f} = 0$. This universal property of $K(V)$ and γ implies their uniqueness. $\square$

The abelian group $K(V)$ is called the Grothendieck group of the semigroup V and can be realized as follows. On $V \times V$ we consider the equivalence relation with $(x_1, x_2) \sim (y_1, y_2)$ if and only if there exists some $z \in V$ such that

$$z \oplus x_1 \oplus y_2 = z \oplus y_1 \oplus x_2.$$

On the quotient $\tilde{V} = V \times V/\sim$ we have a well defined addition $+$ if we set

$$[x_1, x_2] + [a_1, a_2] = [x_1 \oplus a_1, x_2 \oplus a_2].$$

Note that $[x, y] = [x, 0] + [0, y]$ and $[0, b] + [b, 0] = [b, b] = [0, 0]$. Thus, $(\tilde{V}, +)$ is an abelian group with neutral element $[0, 0]$. Also, $-[x, y] = [y, x]$ and every $[x, y] \in \tilde{V}$ has the expression $[x, y] = [x, 0] - [y, 0]$. The map $\gamma : V \to \tilde{V}$ defined by $\gamma(x) = [x, 0]$ is obviously a semigroup homomorphism. We shall prove that it has the universal property. Let G be an abelian group and let $f : V \to G$ be a semigroup homomorphism. We define $\tilde{f} : \tilde{V} \to G$ by $\tilde{f}[x, y] = f(x) - f(y)$. The definition of $\tilde{f}$ is good, because if $[x, y] = [a, b]$, there exists some $z \in V$ such that $z \oplus x \oplus b = z \oplus a \oplus y$ and therefore $f(x) - f(y) = f(a) - f(b)$, since G is a group. Also, $\tilde{f}(\gamma(x)) = f(x) - f(0) = f(x) - 0 = g(x)$, because f is a semigroup homomorphism. Finally, $\tilde{f}$ is unique, because $\gamma(V)$ generates $\tilde{V}$. From the uniqueness of $K(V)$ follows now that $K(V) = \tilde{V}$.

Applying Grothendieck's Lemma, we get for every space M the abelian groups $K(M) = K(\mathrm{Vect}^{\mathbb{C}}(M))$ and $KO(M) = K(\mathrm{Vect}^{\mathbb{R}}(M))$. We shall make K and KO functors describing their effect on continuous and smooth maps.

Proposition 1.3.4. *Let $f : X \to M$ be a continuous map of topological spaces. To every vector bundle $\xi = (E, p, M)$ over M correspond a vector bundle $f^*\xi = (f^*E, q, X)$ over X and a continuous map $\tilde{f} : f^*E \to E$ which maps the fibres of $f^*\xi$ linearly isomorphically onto the fibres of ξ so that the pair $(\tilde{f}, f)$ is a vector bundle morphism.*

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ \downarrow q & & \downarrow p \\ X & \xrightarrow{f} & M \end{array}$$

Moreover, $f^*\xi$ is unique with these properties up to isomorphism of vector bundles over X.

Proof. Let $f^*E = \{(x, v) \in X \times E : f(x) = p(v)\}$ and define the continuous maps $q : f^*E \to X$ by $q(x, v) = x$ and $\tilde{f} : f^*E \to E$ by $\tilde{f}(x, v) = v$. Obviously,

$p \circ \tilde{f} = f \circ q$. Moreover, if $\Gamma(f) = \{(x, f(x)) : x \in X\} \subset X \times M$ is the graph of f, then q is precisely the composition

$$f^*E \overset{id \times p}{\longrightarrow} \Gamma(f) \overset{\approx}{\longrightarrow} X$$

and $id \times p|_{f^*E}$ is a vector bundle map, because $(X \times E, id \times p, X \times M)$ is a vector bundle. This means that the triple (f^*E, q, X) is a vector bundle. By its definition, $\tilde{f}$ maps the fibres of q linearly isomorphically onto the fibres of p.

In order to prove that the vector bundle $f^*\xi = (f^*E, q, X)$ is unique with these properties, suppose that $\zeta = (E', q', X)$ is another such bundle and continuous map $\tilde{f}'$. We consider the continuous map $F : E' \to f^*E$ defined by

$$F(u) = (q'(u), \tilde{f}'(u)).$$

From the definitions follows that $q \circ F = q'$ and

$$F((q')^{-1}(x)) = \{(x, \tilde{f}'(u)) \in f^*E : q'(u) = x\}$$

for every $x \in X$. Since $\tilde{f}'$ maps the fibres of q' linearly isomorphically onto the fibers of p, it follows from Lemma 1.1.2 that F is a vector bundle isomorphism of vector bundles over X. $\square$

The vector bundle $f^*\xi$ is called the induced (or pull-back) vector bundle of ξ by f. It is clear from the proof that if ξ is a smooth vector bundle and f is a smooth map, then $f^*\xi$ is smooth as well. Also, the induced bundle of ξ by the identity map is ξ itself and $(f \circ g)^*\xi \cong g^*(f^*\xi))$. If $X \subset M$ and $f : X \to M$ is the inclusion, then $f^*\xi \cong \xi|_X$. Finally, the pull-back preserves the direct sums. More precisely, let $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ be two vector bundles over the same base space M and let $f : X \to M$ be a continuous map. Then,

$$f^*E_1 \oplus f^*E_2 = \{(x, v_1, x, v_2) \in X \times E_1 \times X \times E_2 : p_1(v_1) = p_2(v_2) = f(x)\}.$$

If $q : f^*E_1 \oplus f^*E_2 \to X$ is the continuous map defined by $q(x, v_1, x, v_2) = x$ and $\tilde{f} : f^*E_1 \oplus f^*E_2 \to E_1 \oplus E_2$ is defined by $\tilde{f}(x, v_1, x, v_2) = (v_1, v_2)$, then $p \circ \tilde{f} = f \circ q$ and $\tilde{f}$ maps the fibres of q linearly isomorphically onto the fibres of p.

$$\begin{array}{ccc} f^*E_1 \oplus f^*E_2 & \overset{\tilde{f}}{\longrightarrow} & E_1 \oplus E_2 \\ \downarrow q & & \downarrow p \\ X & \overset{f}{\longrightarrow} & M \end{array}$$

The uniqueness now implies that $f^*\xi_1 \oplus f^*\xi_2 \cong f^*(\xi_1 \oplus \xi_2)$.

Thus, to every continuous map $f : X \to M$ corresponds a group homomorphism $f^* : K(M) \to K(X)$ such that $id_M^* = id_{K(M)}$ and $(f \circ g)^* = g^* \circ f^*$. These mean that K is a contravariant functor from the topological category to the category of abelian groups. In the rest of this section we shall show that K is actually a homotopy functor (for paracompact spaces) with values in the category of commutative rings with unity. Similar facts hold for the functor KO.

Lemma 1.3.5. *If X is a paracompact space, then for every open cover $\mathcal{U}$ of X there exists a countable open cover $\mathcal{V}$ of X consisting of open sets which are disjoint unions of open sets each of which is contained in some element of $\mathcal{U}$.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is paracompact, there exists a partition of unity $\{\phi_U : U \in \mathcal{U}\}$ subordinated to $\mathcal{U}$. For each finite set $S \subset \mathcal{U}$ we define

$$V_S = \{x \in X : \phi_U(x) > \phi_W(x) \quad \text{for all} \quad U \in S \quad \text{and} \quad W \in \mathcal{U} \setminus S\}.$$

Since for every $x \in X$ the set $\{U \in \mathcal{U} : \phi_U(x) > 0\}$ is finite, V_S is an open set. Also, $V_S \subset U$ for every $U \in S$, because $x \in V_S$ implies that $\phi_U(x) > 0$ for $U \in S$. Let now

$$V_n = \bigcup \{V_S : S \subset \mathcal{U} \quad \text{and} \quad |S| = n\}$$

for $n \in \mathbb{N}$. This is a disjoint union of open sets. Finally, $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is an open cover of X, because for every $x \in X$ the set $S = \{U \in \mathcal{U} : \phi_U(x) > 0\}$ is finite and $x \in V_S$. □

Theorem 1.3.6. *Let $\xi = (E, p, M)$ be a vector bundle and f, $g : X \to M$ be two continuous maps from a paracompact space X to M. If $f \simeq g$, then $f^*\xi \cong g^*\xi$.*

Proof. If $H : [0,1] \times X \to M$ is a homotopy with $H(0,.) = f$ and $H(1,.) = g$, then $H^*\xi|_{\{0\}\times X} \cong f^*\xi$ and $H^*\xi|_{\{1\}\times X} \cong g^*\xi$. Thus, it suffices to prove that if $\xi = (E, p, [0,1] \times X)$ is a vector bundle over $[0,1] \times X$ and X is a paracompact space, then $\xi|_{\{0\}\times X} \cong \xi|_{\{1\}\times X}$.

We observe that if for some $0 < c < 1$ the restrictions $\xi_{[0,c]\times X}$ and $\xi_{[c,1]\times X}$ are trivial, then ξ is trivial. Indeed, let $E_1 = p^{-1}([0,c] \times X)$ and $E_2 = p^{-1}([c,1] \times X)$, and suppose that $h_1 : E_1 \to [0,c] \times X \times \mathbb{C}^n$ and $h_2 : E_2 \to [c,1] \times X \times \mathbb{C}^n$ are vector bundles isomorphisms. Since $h_1 \circ h_2^{-1} : \{c\} \times X \times \mathbb{C}^n \to \{c\} \times X \times \mathbb{C}^n$ is an isomorphism of trivial vector bundles over $\{c\} \times X$, there exists a continuous map $\rho : X \to GL(n, \mathbb{C})$ such that

$$h_1 \circ h_2^{-1}(c, x, v) = (c, v, \rho(x)(v))$$

for every $x \in X$, $v \in \mathbb{C}^n$. The map $\sigma : [c,1] \times X \times \mathbb{C}^n \to [c,1] \times X \times \mathbb{C}^n$ defined by $\sigma(t, x.v) = (t, v, \rho(x)(v))$ is an isomorphism of trivial vector bundles over $[c,1] \times X$ and so is $\sigma \circ h_2 : E_2 \to [c,1] \times X \times \mathbb{C}^n$. Since h_1 and $\sigma \circ h_2$ coincide on $E_1 \cap E_2$, they fit together to a form an isomorphism from ξ to the trivial vector bundle over $[0,1] \times X$.

A second observation is that there exists an open cover of $\mathcal{U}$ of X such that $\xi|_{[0,1]\times U}$ is trivial for every $U \in \mathcal{U}$. This follows easily from our first observation and the compactness of $[0,1]$.

From Lemma 1.3.5 there exists a countable open cover $\mathcal{V} = \{V_k : k \in \mathbb{N}\}$ of X consisting of open sets which are disjoint unions of open sets each of which is contained in some element of $\mathcal{U}$. Thus, $\xi|_{[0,1]\times V_k}$ is trivial for every $k \in \mathbb{N}$. Let $\{\phi_k : k \in \mathbb{N}\}$ be a partition of unity subordinated to $\mathcal{V}$. We set $\psi_0 = 0$ and $\psi_k = \phi_1 + \cdots + \psi_k$, $k \in \mathbb{N}$. Let $X_k = \{(\psi_k(x), x) : x \in X\} \approx X$ and $\xi_k = \xi|_{X_k}$. The homeomorphism $\eta_k : X_k \to X_{k-1}$ defined by $\eta(\psi_k(x), x) = (\psi_{k-1}(x), x)$ can be lifted to a homeomorphism $\tilde{\eta}_k : p^{-1}(X_k) \to p^{-1}(X_{k-1})$ such that $\tilde{\eta}_k = id$ on $p^{-1}(X_k) \setminus p^{-1}([0,1] \times V_k)$ and

$$\tilde{\eta}_k = h_{k-1}^{-1} \circ (id \times (\eta_k|_{V_k})) \circ h_k$$

on $p^{-1}([0,1] \times V_k \cap X_k)$, where $h_k : p^{-1}(V_k) \to [0,1] \times V_k \times \mathbb{C}^n$ is a trivialization of ξ over $[0,1] \times V_k$. So, $\tilde{\eta}_k$ takes each fiber of ξ_k linearly isomorphically onto the corresponding fiber of ξ_{k-1}. Now the infinite composition $\tilde{\eta} = \tilde{\eta}_1 \circ \tilde{\eta}_2 \circ \cdots$ is well defined, because $\{\mathrm{supp}\phi_k : k \in \mathbb{N}\}$ is a locally finite closed cover of X, and is a vector bundle isomorphism from $\xi|_{\{1\}\times X}$ to $\xi|_{\{0\}\times X}$. □

Corollary 1.3.7. *Every homotopy equivalence $f : X \to Y$ of paracompact spaces induces an isomorphism $f^* : K(Y) \to K(X)$ and similarly for the KO groups. In particular, every vector bundle over a contractible paracompact space is trivial.* □

We shall now define a ring structure on $K(M)$ for any space M using the tensor product of vector bundles in the same way we used the direct sum to define the group structure. Let $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ be two complex (respectively real) vector bundles over the same base space M. We define

$$E_1 \otimes E_2 = \coprod_{x \in M} p_1^{-1}(x) \otimes p_2^{-1}(x)$$

where the tensor product is taken over $\mathbb{C}$ (respectively over $\mathbb{R}$ in the real case). On $E_1 \otimes E_2$ one can define a topology and make it the total space of a vector bundle over M. Indeed, let V, $W \subset M$ be two open sets such that $V \cap W \neq \varnothing$ for which there are trivializations $h_j : p^{-1}(V) \to V \times \mathbb{C}^{n_j}$ and $g_j : p^{-1}(W) \to W \times \mathbb{C}^{n_j}$, $j = 1, 2$, for ξ_1 and ξ_2, respectively. There exist continuous functions $G^j : V \cap W \to GL(n_j, \mathbb{C})$ such that

$$(g_j \circ h_j^{-1})(x, v) = (x, G^j(x)(v))$$

for $j = 1, 2$. Defining the map

$$h_1 \otimes h_2 : \coprod_{x \in V} p_1^{-1}(x) \otimes p_2^{-1}(x) \to V \times (\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$$

by the formula $(h_1 \otimes h_2)(v_1 \otimes v_2) = (x, h_1(v_1) \otimes h_2(v_2))$, for every $v_1 \in p_1^{-1}(x)$ and $v_2 \in p_2^{-1}(x)$, we see that

$$((g_1 \otimes g_2) \circ (h_1 \otimes h_2)^{-1})(x, u_1 \otimes u_2) = (x, (G^1(x) \otimes G^2(x))(u_1 \otimes u_2)).$$

Since $G^1(x) \otimes G^2(x)$ is a continuous function of $x \in V \cap W$, it is a standard fact that there exists a unique topology on $E_1 \otimes E_2$ such that each set of the form

$$\coprod_{x \in V} p_1^{-1}(x) \otimes p_2^{-1}(x)$$

as above is open and the maps like $h_1 \otimes h_2$ are homeomorphisms. It is obvious now that the triple $\xi_1 \otimes \xi_2 = (E_1 \otimes E_2, q, M)$ is a vector bundle over M of rank $n_1 n_2$, where q is the canonical projection, and each map $h_1 \otimes h_2$ as above is a local trivialization. The vector bundle $\xi_1 \otimes \xi_2$ is called the tensor product of the vector bundles ξ_1 and ξ_2.

The basic properties of the tensor product of vector spaces carry over immediately to the case of vector bundles over a space M. So,

(i) if $\xi_1 \cong \zeta_1$ and $\xi_2 \cong \zeta_2$, then $\xi_1 \otimes \xi_2 \cong \zeta_1 \otimes \zeta_2$.

(ii) $\xi_1 \otimes \xi_2 \cong \xi_2 \otimes \xi_1$.

(iii) $(\xi_1 \otimes \xi_2) \otimes \xi_3 \cong \xi_1 \otimes (\xi_2 \otimes \xi_3)$.

(iv) $\xi \otimes \epsilon^1 \cong \xi$.

(v) $\xi \otimes (\xi_1 \oplus \xi_2) \cong \xi \otimes \xi_1 \oplus \xi \otimes \xi_2$.

(vi) If $f : X \to M$ is a continuous map then $f^*(\xi_1 \otimes \xi_2) \cong f^*\xi_1 \otimes f^*\xi_2$. This follows from the uniqueness of the induced bundle.

The tensor product defines an associative commutative multiplication with unity on $\mathrm{Vect}^{\mathbb{C}}(M)$ and on $\mathrm{Vect}^{\mathbb{R}}(M)$ which is compatible with the direct sum. From this we get a commutative ring structure on $K(M)$ and $KO(M)$. More abstractly, let V be an abelian semigroup on which we have a commutative associative multiplication with unity which is compatible with the addition. A multiplication on $K(V)$ can be defined by putting

$$[a, b] \cdot [x, y] = [ax, ay] - [bx, by]$$

for every $[a, b]$, $[x, y] \in K(V)$. Indeed, if $[a_1, b_1] = [a_2, b_2]$ and $[x_1, y_1] = [x_1, y_2]$, there exist c, $d \in V$ such that $c + a_1 + b_2 = c + a_2 + b_1$ and $d + x_1 + y_2 = d + x_2 + y_1$. Then, $[a_1x_1, a_1y_1] = [a_1x_2, a_1y_2]$ and $[b_1x_1, b_1y_1] = [b_1x_2, b_1y_2]$. On the other hand,

$$(cx_2 + cy_2) + (a_1 + b_2)x_2 + (a_2 + b_1)y_2 = (cx_2 + cy_2) + (a_2 + b_1)x_2 + (a_1 + b_2)y_2$$

which means that $[(a_1 + b_2)x_2, (a_1 + b_2)y_2] = [(a_2 + b_1)x_2, (a_2 + b_1)y_2]$. This implies that

$$[a_1x_1, a_1y_1] - [b_1x_1, b_1y_1] = [a_1x_2, a_1y_2] - [b_1x_2, b_1y_2] = [a_2x_2, a_2y_2] - [b_2x_2, b_2y_2].$$

In this way $K(V)$ turns into a commutative ring with unity, called the Grothendieck ring of V. In particular for every space M we have the Grothendieck ring $K(M)$ of complex vector bundles over M and the Grothendieck ring $KO(M)$ of real vector bundles. The unity is represented by ϵ^1 in both cases.

1.4 The classification of vector bundles

In this section we shall show that the functor $\mathrm{Vect}^{\mathbb{C}}(M)$ is representable for paracompact spaces by constructing an explicit classifying space. Although we present the case of complex vector bundles, everything holds verbatim for the functor $\mathrm{Vect}^{\mathbb{R}}(M)$ also, replacing the unitary groups involved by orthogonal groups and the complex Grassmannians by the real ones.

Let $1 \leq k \leq n$ be positive integers and let

$$V_k(\mathbb{C}^n) = \{(v_1, ..., v_k) \in (S^{2n+1})^k : \langle v_l, v_j \rangle = \delta_{lj}, \quad 1 \leq l, j \leq k\}$$

be the space of all orthonormal k-frames in $\mathbb{C}^n$, where $\langle , \rangle$ denotes the usual hermitian product on $\mathbb{C}^n$. Obviously, $V_k(\mathbb{C}^n)$ is a compact space and there is a continuous surjection $\eta_k^n : U(n) \to V_k(\mathbb{C}^n)$ defined by $\eta_k^n(A) = (Ae_1, ..., Ae_k)$. We observe that if A, $B \in U(n)$, then $\eta_k^n(A) = \eta_k^n(B)$ if and only if $B^{-1}A \in U(n-k)$, where we consider the inclusion $U(n-k) \subset U(n)$ so that each element of $U(n-k)$ fixes e_1,..., e_k in $\mathbb{C}^n$. This implies that η_k^n induces a homeomorphism

$$\tilde{\eta}_k^n : \frac{U(n)}{U(n-k)} \approx V_k(\mathbb{C}^n).$$

The inclusion $SU(n) \hookrightarrow U(n)$ induces a continuous injection of the homogeneous space $SU(n)/SU(n-k)$ into $U(n)/U(n-k)$ which is moreover a surjection, because for every $A \in U(n)$ there exists $B \in SU(n)$ such that $B^{-1}A \in U(n-k)$. Thus,

$$\frac{SU(n)}{SU(n-k)} \approx \frac{U(n)}{U(n-k)} \approx V_k(\mathbb{C}^n).$$

The homogeneous space $V_k(\mathbb{C}^n)$ is called the Stiefel manifold of orthonormal k-frames in $\mathbb{C}^n$.

Each element of $V_k(\mathbb{C}^n)$ generates a k-dimensional vector subspace of $\mathbb{C}^n$. Let $G_k(\mathbb{C}^n)$ be the space of all k-dimensional vector subspaces of $\mathbb{C}^n$ endowed with the quotient topology with respect to the natural surjection $q : V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$. The group $U(k)$ acts smoothly on $V_k(\mathbb{C}^n)$ from the right and $G_k(\mathbb{C}^n)$ is the orbit space of the action. Here we consider $U(k)$ embedded in $U(n)$ so that each element of $U(k)$ fixes e_{k+1},...,e_n in $\mathbb{C}^n$. The right action of $U(k)$ on $V_k(\mathbb{C}^n)$ is defined by

$$(v_1, ..., v_k)A = \left(\sum_{l=1}^{k} a_{l1}v_l, ..., \sum_{l=1}^{k} a_{lk}v_l\right),$$

for $A = (a_{lj})_{1\leq l,j\leq n} \in U(k) \subset U(n)$, where $a_{lj} = \delta_{lj}$, $1 \leq l \leq n$, $k+1 \leq j \leq n$.

If A, $B \in U(n)$, then the orthonormal k-frames $(Ae_1, ..., Ae_k)$ and $(Be_1, ..., Be_k)$ generate the same vector subspace of $\mathbb{C}^n$ if and only if there exists $C \in U(k) \subset U(n)$ such that $Ae_j = BCe_j$ for $1 \leq j \leq k$. Thus, $(B^{-1}A)(\{0\} \times \mathbb{C}^{n-k}) = \{0\} \times \mathbb{C}^{n-k}$, because $\{0\} \times \mathbb{C}^{n-k} = (\mathbb{C}^k \times \{0\})^{\perp}$. If $D \in U(n-k)$ is defined by $De_j = e_j$ for $1 \leq j \leq k$ and $De_j = (B^{-1}A)e_j$ for $k+1 \leq j \leq n$, then $B^{-1}A = CD \in U(k)$. This implies that the $q \circ \tilde{\eta}_k^n$ induces a homeomorphism

$$\frac{U(n)}{U(k) \times U(n-k)} \approx G_k(\mathbb{C}^n).$$

The homogeneous space $G_k(\mathbb{C}^n)$ is called the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. Note that $G_k(\mathbb{C}^n) \approx G_{n-k}(\mathbb{C}^n)$ and $G_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$.

Now we consider the standard inclusions $\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \cdots$ and the union $\mathbb{C}^\infty = \bigcup_{n=0}^{\infty} \mathbb{C}^n = \lim\limits_{\to} \mathbb{C}^n$, which is the vector space of all sequences of complex numbers with only a finite number of non-zero terms. The hermitian product extends to $\mathbb{C}^\infty$. Also, $\mathbb{C}^\infty$ becomes a topological space equipped with the weak topology. Correspondingly, we get a sequence of inclusions

$$V_k(\mathbb{C}^k) \subset V_k(\mathbb{C}^{k+1}) \subset \cdots \subset V_k(\mathbb{C}^n) \cdots$$

and the space $V_k(\mathbb{C}^\infty) = \bigcup_{n=k}^{\infty} V_k(\mathbb{C}^n)$ equipped with the weak topology.

Similarly, we construct the infinite Grassmannian $G_k(\mathbb{C}^\infty) = \bigcup_{n=k}^{\infty} G_k(\mathbb{C}^n)$ endowed with the weak topology. In particular we have an infinite complex projective space $\mathbb{C}P^\infty = G_1(\mathbb{C}^\infty) = \bigcup_{n=1}^{\infty} \mathbb{C}P^n$.

There is a canonical smooth vector bundle γ_n^k of rank k over $G_k(\mathbb{C}^n)$ with total space

$$E(\gamma_n^k) = \{(V, z) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n : z \in V\}.$$

The bundle map $p_{n,k} : E(\gamma_n^k) \to G_k(\mathbb{C}^n)$ is the restriction to $E(\gamma_n^k)$ of the projection onto the first factor. Since $p_{n,k}^{-1}(V) = \{V\} \times V$ for every $V \in G_k(\mathbb{C}^n)$, the vector bundle $\gamma_n^k = (E(\gamma_n^k), p_{n,k}, G_k(\mathbb{C}^n))$ is called the tautological bundle over $G_k(\mathbb{C}^n)$. It is a generalization of Example 1.1.3. In the sequel we shall prove that γ_n^k is indeed a smooth vector bundle.

Lemma 1.4.1. *Suppose that $(v_1, v_2, ..., v_k)$, $(v_1', v_2', ..., v_k') \in V_k(\mathbb{C}^n)$ are such that $q(v_1, v_2, ..., v_k) = q(v_1', v_2', ..., v_k')$. Then*

$$\sum_{j=1}^{k} \langle z, v_j \rangle v_j = \sum_{j=1}^{k} \langle z, v_j' \rangle v_j'$$

for every $z \in \mathbb{C}^n$.

Proof. There exists some $A = (a_{lj})_{1 \le l,j \le k} \in U(k)$ such that $(v_1, v_2, ..., v_k)A = (v_1', v_2', ..., v_k')$. This means that

$$v_j' = \sum_{l=1}^{k} a_{lj} v_l$$

for every $1 \le j \le k$. Therefore,

$$\sum_{j=1}^{k} \langle z, v_j' \rangle v_j' = \sum_{j,l,r=1}^{k} \bar{a}_{lj} a_{rj} \langle z, v_l \rangle v_r = \sum_{r,l=1}^{k} \left(\sum_{j=1}^{k} \bar{a}_{lj} a_{rj} \right) \langle z, v_l \rangle v_r = \sum_{l=1}^{k} \langle z, v_l \rangle v_l$$

because $\overline{A}^T = A^{-1}$. □

The preceding Lemma 1.4.1 implies that there is a well-defined smooth map $h : G_k(\mathbb{C}^n) \times \mathbb{C}^n \to \mathbb{C}^n$ with

$$h(q(v_1, v_2, ..., v_k), z) = \sum_{j=1}^{k} \langle z, v_j \rangle v_j$$

which is the projection of the vector $z \in \mathbb{C}^n$ on the vector subspace of $\mathbb{C}^n$ spanned by the orthonormal k-frame $(v_1, v_2, ..., v_k)$.

Also the smooth symmetric function $\sigma : G_k(\mathbb{C}^n) \times G_k(\mathbb{C}^n) \to \mathbb{R}$ with

$$\sigma((q(v_1, v_2, ..., v_k), (q(v'_1, v'_2, ..., v'_k)) = |\det(\langle v_l, v'_j\rangle)_{1\le l,j\le k}|$$

is well-defined, because if A, $B \in U(k)$ and $(v_1, v_2, ..., v_k)A = (u_1, u_2, ..., u_k)$ and $(v'_1, v'_2, ..., v'_k)B = (u'_1, u'_2, ..., u'_k)$, then

$$|\det(\langle u_l, u'_j\rangle)_{1\le l,j\le k}| = |\det(A^T \cdot (\langle v_l, v'_j\rangle)_{1\le l,j\le k} \cdot \overline{B})| = |\det(\langle v_l, v'_j\rangle)_{1\le l,j\le k}|.$$

It is obvious that $\sigma((q(v_1, v_2, ..., v_k), (q(v'_1, v'_2, ..., v'_k)) > 0$ if and only if $h(q(v'_1, v'_2, ..., v'_k), v_j)$, $1 \le j \le k$, are linearly independent and form a basis of $q(v'_1, v'_2, ..., v'_k)$, because the entries of the l row of the matrix $(\langle v_l, v'_j\rangle)_{1\le l,j\le k}$ are the coordinates of the orthogonal projection of v_l on $q(v'_1, v'_2, ..., v'_k)$ with respect to its ordered basis $(v'_1, v'_2, ..., v'_k)$. In this case, $h(q(v'_1, v'_2, ..., v'_k), .)$ maps $q(v_1, v_2, ..., v_k)$ linearly isomorphically onto $q(v'_1, v'_2, ..., v'_k)$.

For every $q(v_1, v_2, ..., v_k) \in G_k(\mathbb{C}^n)$ the set

$$U_{q(v_1,v_2,...,v_k)} = \{q(v'_1, v'_2, ..., v'_k) \in G_k(\mathbb{C}^n) : \sigma((q(v_1, v_2, ..., v_k), (q(v'_1, v'_2, ..., v'_k)) > 0\}$$

is an open neighbourhood of $q(v_1, v_2, ..., v_k)$ and

$$G_k(\mathbb{C}^n) = \bigcup\{U_{\mathbb{C}^\Gamma} : \Gamma \subset \{1, 2, ..., n\} \quad \text{with} \quad |\Gamma| = k\},$$

where $\mathbb{C}^\Gamma = \oplus_{j\in\Gamma}\mathbb{C}e_j$.

For each $\Gamma \subset \{1, 2, ..., n\}$ with $|\Gamma| = k$ let $j_\Gamma : \mathbb{C}^k \to \mathbb{C}^\Gamma$ be the linear isomorphism which sends $e_1 \in \mathbb{C}^k$ to $e_{j(1)} \in \mathbb{C}^n$, where $j(1) = \min\Gamma$ and so on taking into account the ordering of Γ. The map $\phi_\Gamma : U_{\mathbb{C}^\Gamma} \times \mathbb{C}^k \to p^{-1}(U_{\mathbb{C}^\Gamma})$ defined by

$$\phi_\Gamma(V, z) = (V, h(V, j_\Gamma(z)))$$

is a diffeomorphism which maps $\{V\} \times \mathbb{C}^k$ linearly isomorphically onto the fibre $p^{-1}_{n,k}(V)$ from the above remarks concerning h. This shows that the triple $\gamma^k_n = (E(\gamma^k_n), p_{n,k}, G_k(\mathbb{C}^n))$ is a smooth complex vector bundle of rank k.

In the same way we have a tautological complex vector bundle of rank k $\gamma^k_\infty = (E(\gamma^k_\infty), p_{n,k}, G_k(\mathbb{C}^\infty))$ over $G_k(\mathbb{C}^\infty)$, whose restriction to each $G_k(\mathbb{C}^n))$ is γ^k_n.

Definition 1.4.2. Let $\xi = (E, p, M)$ be a complex vector bundle of rank k. A Gauss map of ξ is a continuous map $g : E \to \mathbb{C}^n$ for some $k \le n \le \infty$ such that $g|_{p^{-1}(x)} : p^{-1}(x) \to \mathbb{C}^n$ is a linear monomorphism for every $x \in M$.

For example, the restriction of the projection onto the second factor to $E(\gamma^k_n)$, that is the map $g : E(\gamma^k_n) \to \mathbb{C}^n$ with $g(V, z) = z$, is a Gauss map of the tautological bundle γ^k_n.

If a complex vector bundle $\xi = (E, p, M)$ of rank k admits a continuous Gauss map $g : E \to \mathbb{C}^n$, then there are two continuous maps $f : M \to G_k(\mathbb{C}^n)$ with $f(x) = g(E_x)$ and $\tilde{f} : E \to E(\gamma^n_k)$ with $\tilde{f}(v) = (f(p(v), g(v))$ such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E(\gamma^k_n) \\ \Big\downarrow p & & \Big\downarrow p_{n,k} \\ M & \xrightarrow{f} & G_k(\mathbb{C}^n) \end{array}$$

Thus, the pair $(f, \tilde{f})$ is a vector bundle morphism, whose restriction on each fibre is a linear isomorphism. It follows from Proposition 1.3.4 that $\xi \cong f^*\gamma_n^k$. Conversely, if we start from a vector bundle morphism $(f, \tilde{f})$ which is a linear isomorphism on fibres so that the above diagram commutes, then $pr \circ \tilde{f} : E \to \mathbb{C}^n$ is a Gauss map of ξ. This shows that a complex vector bundle $\xi = (E, p, M)$ of rank k admits a Gauss map $g : E \to \mathbb{C}^n$ for some $k \leq n \leq \infty$ if and only if there exists a continuous map $f : M \to G_k(\mathbb{C}^n)$ such that $\xi \cong f^*\gamma_n^k$.

Theorem 1.4.3. *Every complex vector bundle $\xi = (E, p, M)$ of rank k over a paracompact space M admits a continuous Gauss map $g : E \to \mathbb{C}^\infty$. Moreover, if there exists a finite open cover $\{U_1, ..., U_n\}$ of M such that $\xi|_{U_j}$ is trivial for all $1 \leq j \leq n$, then there exists a continuous Gauss map $g : E \to \mathbb{C}^{kn}$ of ξ.*

Proof. Since M is assumed to be paracompact, there exists a countable open cover $\{U_j : j \in \mathbb{N}\}$ of M such that $\xi|_{U_j}$ is trivial for every $j \in \mathbb{N}$, by Lemma 1.3.5. Let $\phi_j : p^{-1}(U_j) \to U_j \times \mathbb{C}^k$ be a trivialization of $\xi|_{U_j}$. Then $pr \circ \phi_j : p^{-1}(U_j) \to \mathbb{C}^k$ is a Gauss map for $\xi|_{U_j}$, where $pr : U_j \times \mathbb{C}^k \to \mathbb{C}^k$ is the projection onto the second factor. Let $\{f_j : j \in \mathbb{N}\}$ be a partition of unity subordinated to the open cover $\{U_j : j \in \mathbb{N}\}$ and for each $j \in \mathbb{N}$ let $g_j : E \to \mathbb{C}^k$ be the continuous map defined by

$$g_j(v) = \begin{cases} 0, & \text{if } v \in E \setminus p^{-1}(U_j), \\ f_j(p(v)) \cdot pr(\phi_j(v)), & \text{if } x \in p^{-1}(U_j). \end{cases}$$

The map

$$g = \sum_{j \in \mathbb{N}} g_j : E \to \bigoplus_{j \in \mathbb{N}} \mathbb{C}^k = \mathbb{C}^\infty$$

is now continuous. Since each g_j maps E_x linearly isomorphically onto $\mathbb{C}^k$ for $f_j(x) > 0$ and the images of different g_j's belong to different factors of the direct sum, it follows that $g|_{E_x}$ is a linear monomorphism for every $x \in M$. Hence g is a continuous Gauss map of ξ. The second assertion is now obvious, because in this case we begin with the finite open cover $\{U_1, ..., U_n\}$ and the direct sum is finite. □

Corollary 1.4.4. *For every complex vector bundle $\xi = (E, p, M)$ of rank k over a paracompact space M there exists a continuous map $f : M \to G_k(\mathbb{C}^\infty)$ such that $\xi \cong f^*\gamma_\infty^k$. If M is compact, there exists a continuous map $f : M \to G_k(\mathbb{C}^n)$ for some large enough $n \in \mathbb{N}$ such that $\xi \cong f^*\gamma_n^k$.* □

Actually, the second part of Corollary 1.4.4 holds under the more general assumption that the base space M is paracompact and has finite covering dimension. We refer for this to Corollary A.4 in the appendix to this chapter. In particular this holds for vector bundles over topological manifolds.

The continuous map f in Corollary 1.4.4 is not unique, but its homotopy class is, as we shall prove shortly. We set

$$\mathbb{C}^{ev} = \{(z_n)_{n \geq 0} \in \mathbb{C}^\infty : z_{2m+1} = 0 \quad \text{for all} \quad m \in \mathbb{Z}^+\},$$

$$\mathbb{C}^{odd} = \{(z_n)_{n \geq 0} \in \mathbb{C}^\infty : z_{2m} = 0 \quad \text{for all} \quad m \in \mathbb{Z}^+\}$$

and consider the homotopies g^{ev}, $g^{odd} : [0,1] \times \mathbb{C}^\infty \to \mathbb{C}^\infty$ defined by

$$g_t^{ev}(z_0, z_1, z_2, ...) = (1-t) \cdot (z_0, z_1, z_2, ...) + t(z_0, 0, z_1, 0, z_2, ...),$$

$$g_t^{odd}(z_0, z_1, z_2, ...) = (1-t) \cdot (z_0, z_1, z_2, ...) + t(0, z_0, 0, z_1, 0, ...).$$

The continuous map $g_1^{ev} \circ pr|_{E(\gamma_n^k)} : E(\gamma_n^k) \to \mathbb{C}^{2n}$ is a Gauss map of γ_n^k from which we get a vector bundle morphism $(f^{ev}, \tilde{f}^{ev})$ from γ_n^k to γ_{2n}^k. Similarly, we get a vector bundle morphism $(f^{odd}, \tilde{f}^{odd})$ from γ_n^k to γ_{2n}^k for every $1 \leq n \leq \infty$. Since f^{ev} and f^{odd} are induced by g_1^{ev} and g_1^{odd}, the homotopies g^{ev} and g^{odd} induce homotopies of f^{ev} and f^{odd} with the canonical inclusion $j : G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^{2n})$, because $g_t^{ev}(\mathbb{C}^n) \subset \mathbb{C}^{2n}$, $g_t^{odd}(\mathbb{C}^n) \subset \mathbb{C}^{2n}$ and in particular $g_1^{ev}(\mathbb{C}^n) = \mathbb{C}^{2n} \cap \mathbb{C}^{ev}$ and $g_1^{odd}(\mathbb{C}^n) = \mathbb{C}^{2n} \cap \mathbb{C}^{odd}$.

Proposition 1.4.5. *Let $1 \leq n \leq \infty$, $k \in \mathbb{N}$ and M be a topological space. Let f_0, $f_1 : M \to G_k(\mathbb{C}^n)$ be two continuous maps such that $f_0^*\gamma_n^k \cong f_1^*\gamma_n^k$ as vector bundles over M. Then, $j \circ f_0 \simeq j \circ f_1$, where $j : G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^{2n})$ is the canonical inclusion.*

Proof. The hypothesis says that there exists a complex vector bundle $\xi = (E, p, M)$ and two vector bundle morphisms $(f_0, \tilde{f}_0)$ and $(f_1, \tilde{f}_1)$ from ξ to γ_n^k, which are linear isomorphisms of fibres. As before we get two continuous Gauss maps g_0, $g_1 : E \to \mathbb{C}^n$ of ξ as well as two vector bundle morphisms $(f^{ev} \circ f_0, \tilde{f}^{ev} \circ \tilde{f}_0)$, $(f^{odd} \circ f_1, \tilde{f}^{odd} \circ \tilde{f}_1)$ to γ_{2n}^k and corresponding Gauss maps $g^{ev} \circ g_0 : E \to \mathbb{C}^{2n}$, $g^{odd} \circ g_1 : E \to \mathbb{C}^{2n}$. The continuous map $h : [0,1] \times E \to \mathbb{C}^{2n}$ defined by

$$h(t, v) = (1-t) \cdot g_1^{ev}(g_0(v)) + t g_1^{odd}(g_1(v))$$

is now a Gauss map of the vector bundle $1 \times \xi = ([0,1] \times E, id \times p, [0,1] \times M)$ from which we get a vector bundle morphism $(H, \tilde{H})$ from $1 \times \xi$ to γ_{2n}^k. The map $H : [0,1] \times M \to G_k(\mathbb{C}^{2n})$ is a homotopy from $f^{ev} \circ f_0$ to $f^{odd} \circ f_1$. Since $f^{ev} \circ f_0 \simeq j \circ f_0$ and $f^{odd} \circ f_1 \simeq j \circ f_1$, it follows that $j \circ f_0 \simeq j \circ f_1$. □

Combining the above with Theorem 1.3.6 we get a natural one-to-one correspondence of the set of isomorphism classes of complex vector bundles of rank k over a paracompact space M onto the set of homotopy classes of maps $[M, G_k(\mathbb{C}^\infty)]$. To every homotopy class $[f] \in [M, G_k(\mathbb{C}^\infty)]$ corresponds (the isomorphism class of) $f^*\gamma_\infty^k$. Thus, the problem of the classification of complex vector bundles of rank k over a paracompact space M is equivalent to the calculation of the set $[M, G_k(\mathbb{C}^\infty)]$.

Let H be a contravariant functor on a category of spaces and continuous maps with values in the category of commutative semigroups. A characteristic class of complex vector bundles with values in H is a natural transformation Φ from the functor $\mathrm{Vect}^{\mathbb{C}}$ to H. If for each space M in the category of spaces we consider the image of $\Phi_M : \mathrm{Vect}^{\mathbb{C}}(M) \to H(M)$ is contained in a subgroup of $H(M)$, then Φ factors through the functor K. In this case we say that the characteristic class is stable. Let R be a commutative ring with unity. If Φ is a natural transformation from the functor K to the (singular) cohomology functor $H^*(-; R)$ with coefficients in R, then to every continuous map of paracmpact spaces $f : M \to N$ corresponds the commutative diagram

$$\begin{array}{ccc} K(N) & \xrightarrow{\Phi_N} & H^*(N;R) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ K(M) & \xrightarrow{\Phi_M} & H^*(M;R). \end{array}$$

If $c = \Phi_{G_k(\mathbb{C}^\infty)}(\gamma_\infty^k) \in H^*(G_k(\mathbb{C}^\infty);R)$, then for every complex vector bundle ξ of rank k over the paracompact space M there is a continuous map $f : M \to G_k(\mathbb{C}^\infty)$ such that $\xi \cong f^*\gamma_\infty^k$ and $\Phi_M(\xi) = f^*(c)$.

1.5 Operations with vector bundles and their sections

In this section we shall describe some useful constructions using vector bundles and their sections, which are analogous to the ones in the category of finite dimensional vector spaces.

As for vector spaces, to every vector bundle $\xi = (E, p, M)$ over a space M corresponds its dual vector bundle $\xi^* = (E^*, p^*, M)$ over M which is defined in an analogous way as the cotangent bundle of a smooth manifold. Its total space is the disjoint union

$$E^* = \coprod_{x \in M} (p^{-1}(x))^*$$

with the obvious topology.

Recall that if V is a finite dimensional vector space then choosing a basis of V we have a linear isomorphism $V \cong V^*$, but the isomorphism is not natural as it depends on the initial choice of the basis. If V is real and carries an inner product $\langle , \rangle$, then the map which sends $v \in V$ to $\langle ., v\rangle$ is a natural linear isomorphism of V to its dual V^*. Since every vector bundle over a paracompact space admits an inner product, it follows that if ξ is a real vector bundle over a paracompact space, then $\xi \cong \xi^*$.

To every finite dimensional complex vector space V corresponds its conjugate $\overline{V}$ with the same additive structure and exterior multiplication sending $\lambda \in \mathbb{C}$ and $v \in V$ to $\overline{\lambda}v$. If $\langle , \rangle$ is a hermitian inner product on V, then the map which sends $v \in \overline{V}$ to $\langle ., v\rangle \in V^*$ is a linear isomorphism $\overline{V} \cong V^*$. To every complex vector bundle $\xi = (E, p, M)$ corresponds its conjugate vector bundle $\overline{\xi}$ in the obvious way and if the base space M is paracompact, then $\overline{\xi} \cong \xi^*$.

In any case V is naturally isomorphic to V^{**} and therefore $\xi \cong \xi^{**}$ for any vector bundle ξ.

Let now V and W be two finite dimensional vector spaces (both complex or real). The linear map $\mu : V^* \otimes W \to \mathrm{Hom}(V, W)$ defined by

$$\mu(a \otimes w)(v) = a(v)w$$

for every $a \in V^*$, $w \in W$ and $v \in V$, is an isomorphism. This carries over to vector bundles. If $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ are two vector bundles over the same base space M, there is a vector bundle $\mathrm{Hom}(\xi_1, \xi_2)$ and

$$\xi_1^* \otimes \xi_2 \cong \mathrm{Hom}(\xi_1, \xi_2).$$

If $\xi = (E, p, M)$ is a real vector bundle, the complex vector bundle $\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}}$ is called the complexification of ξ, where $\epsilon^1_{\mathbb{C}}$ is the trivial complex line bundle over M. On the other hand, every complex vector bundle ζ of rank n can be considered as a real vector bundle of rank $2n$ denoted by $\zeta_{\mathbb{R}}$. Now we have

$$(\xi_{\mathbb{C}})_{\mathbb{R}} \cong \xi \otimes_{\mathbb{R}} (\epsilon^1_{\mathbb{R}} \oplus \epsilon^1_{\mathbb{R}}) \cong \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{R}} \oplus \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{R}} \cong \xi \oplus \xi.$$

For the converse we have the following.

Lemma 1.5.1. *(i) If V is a complex vector space then $V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$ as complex vector spaces.*

(ii) If $\xi = (E, p, M)$ is a complex vector bundle over a paracompact space M, then $(\xi_{\mathbb{R}})_{\mathbb{C}} \cong \xi \oplus \xi^*$.

Proof. Since the exterior multiplication on $V \otimes_{\mathbb{R}} \mathbb{C}$ is defined by $\lambda(v \otimes_{\mathbb{R}} z) = v \otimes_{\mathbb{R}} (\lambda z)$ for $v \in V$ and λ, $z \in \mathbb{C}$, the formula

$$\phi(v \otimes_{\mathbb{R}} z) = (zv, \overline{z}v)$$

defines a $\mathbb{C}$-linear isomorphism $V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$. This proves (i) and (ii) follows from this choosing a hermitian inner product on ξ. □

In the rest of this section we shall describe the spaces of smooth sections of the vector bundles defined above corresponding to a given smooth vector bundle $\xi = (E, p, M)$ of rank n over a smooth manifold M. The vector space $\Omega^0(\xi)$ of the smooth sections of ξ is a $C^\infty(M)$-module. From Theorem 1.3.2 there exists a smooth vector bundle $\tilde{\xi}$ over M of some rank m such that $\xi \oplus \tilde{\xi} \cong \epsilon^{n+m}$ and therefore

$$\Omega^0(\xi) \oplus \Omega^0(\tilde{\xi}) \cong \Omega^0(\xi \oplus \tilde{\xi}) \cong \Omega^0(\epsilon^{n+m}).$$

Since $\Omega^0(\epsilon^{n+m})$ is a finitely generated free $C^\infty(M)$-module, we conclude that $\Omega^0(\xi)$ is a finitely generated projective $C^\infty(M)$-module.

We shall need the following algebraic lemma.

Lemma 1.5.2. *Let R be a commutative ring with unity, A a projective R-module and B a finitely generated R-module. Then,*

$$\mathrm{Hom}_R(A, R) \otimes_R B \cong \mathrm{Hom}_R(A, B).$$

Proof. Let $\mu : \mathrm{Hom}_R(A, R) \otimes_R B \to \mathrm{Hom}_R(A, B)$ be the natural homomorphism defined by $\mu(f \otimes b)(a) = f(a)b$. If $B = R$ or a finitely generated free R-module, then μ is an isomorphism. If B is a finitely generated R-module, there is a short exact sequence of R-modules

$$0 \to K \to F \to B \to 0$$

where K and F are free and finitely generated. Since μ is natural, we get the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(A, R) \otimes_R K & \longrightarrow & \mathrm{Hom}_R(A, R) \otimes_R F & \longrightarrow & \mathrm{Hom}_R(A, R) \otimes_R B & \longrightarrow & 0 \\
\downarrow \mu & & \downarrow \mu & & \downarrow & & \\
\mathrm{Hom}_R(A, K) & \longrightarrow & \mathrm{Hom}_R(A, F) & \longrightarrow & \mathrm{Hom}_R(A, B) & \longrightarrow & 0
\end{array}$$

in which the rows are exact, because A is assumed to be projective and therefore $\mathrm{Hom}_R(A, .)$ is an exact functor. The assertion follows now from the five lemma. □

The previous Lemma 1.5.2 is a special case of the more general statement

$$\mathrm{Hom}_R(A, G) \otimes_R B \cong \mathrm{Hom}_R(A, G \otimes_R B)$$

which holds under the same assumptions on A and B for every R-module G. The isomorphism now is given by $\mu(f \otimes b)(a) = f(a) \otimes b$ and the proof is essentially the same.

Theorem 1.5.3. *If $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ are two smooth vector bundles over the same smooth manifold M then the following hold.*

(i) $\Omega^0(\mathrm{Hom}(\xi_1, \xi_2)) \cong \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi_1), \Omega^0(\xi_2))$.

(ii) $\Omega^0(\xi_1 \otimes \xi_2) \cong \Omega^0(\xi_1) \otimes_{C^\infty(M)} \Omega^0(\xi_2)$.

(iii) $\Omega^0(\xi_1^*) \cong \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi_1), C^\infty(M))$.

Proof. Let

$$F : \Omega^0(\mathrm{Hom}(\xi_1, \xi_2)) \to \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi_1), \Omega^0(\xi_2))$$

be the $C^\infty(M)$-linear map defined by $F(\hat{\phi})(s)(x) = \hat{\phi}(x)(s(x))$, for every $x \in M$ and $\hat{\phi} \in \Omega^0(\mathrm{Hom}(\xi_1, \xi_2))$, $s \in \Omega^0(\xi_1)$.

First, we observe that F is injective, because if $F(\hat{\phi}) = 0$, then for every $x \in M$ and $v \in p_1^{-1}(x)$ there exists $s_v \in \Omega^0(\xi_1)$ with $s_v(x) = v$ and therefore $\hat{\phi}(x)(v) = F(\hat{\phi})(s)(x) = 0$.

In order to prove that F is onto let $\phi \in \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi_1), \Omega^0(\xi_2))$. In the beginning we shall show that if $s \in \Omega^0(\xi_1)$ and $x \in M$ are such that $s(x) = 0$, then $\phi(s)(x) = 0$. Let s_1, s_2,..., $s_{n_1} \in \Omega^0(\xi_1)$ be a local frame of ξ_1 on some open neighbourhood U of x. Then

$$s|_U = \sum_{j=1}^{n_1} f_j s_j$$

for some $f_j \in C^\infty(U)$, $1 \leq j \leq n_1$. Let $g \in C^\infty(M)$ be such that $g(x) = 1$ and $\mathrm{supp} g \subset U$. Then,

$$\phi(s) = \phi((1-g)s + sg) = (1-g)\phi(s) + \phi(gs)$$

and

$$g(s|_U) = \sum_{j=1}^{n_1} (g f_j) s_j.$$

Now each $g f_j$ can be extended to a smooth function $\tilde{f}_j \in C^\infty(M)$ by setting it zero outside U. Thus,

$$\phi(gs) = \sum_{j=1}^{n_1} \tilde{f}_j \phi(s_j) \in \Omega^0(\xi_2)$$

and $\phi(s)(x) = \phi(gs)(x) = 0$.

We define now $\hat{\phi}$ setting $\hat{\phi}(x)(v) = \phi(s_v)(x)$, for every $x \in M$, where $s_v \in \Omega^0(\xi_1)$ is any with $s_v(x) = v$. From the above, ϕ is well defined and obviously $F(\hat{\phi}) = \phi$. This concludes the proof of (i), while (iii) follows as a special case by taking $\xi_2 = \epsilon^1$.

The proof of (ii) is the following chain of isomorphisms

$$\begin{aligned}\Omega^0(\xi_1 \otimes \xi_2) &\cong \Omega^0(\mathrm{Hom}(\xi_1^*, \xi_2)) \\ &\cong \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi_1^*), \Omega^0(\xi_2)) \\ &\cong \mathrm{Hom}_{C^\infty(M)}(\mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi_1)C^\infty(M)), \Omega^0(\xi_2)) \\ &\cong \Omega^0(\xi_1) \otimes_{C^\infty(M)} \Omega^0(\xi_2)\end{aligned}$$

where the last isomorphism is given by Lemma 1.5.2. □

Appendix

A vector bundle $\xi = (E, p, M)$ is said to be of finite type if M is a normal space and may be covered by a finite number of open sets over each of which ξ is trivial. If M is a compact space, then every vector bundle over M is of finite type. The main purpose of this section is to prove that every vector bundle over a finite dimensional paracompact space is of finite type.

Recall that a Hausdorff space X is said to have covering dimension not greater than m if every open cover of X has an open refinement such that no point of X is contained in more than $m + 1$ elements of the refinement. In this case we write $\dim X \leq m$. If $\dim X \leq m$ and $\dim X \nleq m - 1$, we say that the covering dimension of X is m and write $\dim X = m$. If $\dim X \nleq m$ for every $m \in \mathbb{Z}^+$, we say that X is infinite dimensional and set $\dim X = \infty$. If M is a topological m-manifold, then $\dim M \leq m$.

Proposition A.1. *If M is a paracompact space of finite covering dimension, then every vector bundle $\xi = (E, p, M)$ over M is of finite type.*

Proof. Let $\mathcal{U}$ be an open cover of M such that $\xi|_U$ is trivial for every $U \in \mathcal{U}$. Suppose that $\dim M \leq m$ and let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ such that no point of M is contained in more than $m + 1$ elements of $\mathcal{V}$. Since M is assumed to be paracompact, we may take $\mathcal{V}$ to be locally finite and there exists a partition of unity $\{\phi_V : V \in \mathcal{V}\}$ subordinated to $\mathcal{V}$. Let

$$\mathcal{A}_i = \{a \subset \mathcal{V} : |a| = i + 1\}$$

for each $i \in \mathbb{Z}^+$. For each $a \in \mathcal{A}_i$ with $a = \{V_0, ..., V_i\}$ the set

$$W_{i,a} = \{x \in M : \phi_V(x) < \min\{\phi_{V_0}(x), ..., \phi_{V_i}(x)\} \quad \text{for} \quad V \neq V_0, ..., V_i\}$$

is open and contained in $V_0 \cap \cdots \cap V_i$. So, $\xi|_{W_{i,a}}$ is trivial. Moreover, if a, $b \in \mathcal{A}_i$, then $W_{i,a}$ and $W_{i,b}$ are disjoint. Thus, if we put

$$X_i = \bigcup_{a \in \mathcal{A}_i} W_{i,a}$$

then $\xi|_{X_i}$ is trivial as well and it suffices to show that $\{X_0, ..., X_m\}$ is an open cover of M. Indeed, if a point $x \in M$ is contained in at most $m+1$ of $\mathcal{V}$ and so at most $m+1$ of the functions ϕ_V, $V \in \mathcal{V}$ are positive at x. In other words, there exist some $0 \leq i \leq m$ and V_0,..., $V_i \in \mathcal{V}$ such that $\phi_{V_0}(x) > 0$,..., $\phi_{V_i}(x) > 0$ and $\phi_V(x) = 0$ for $V \neq V_0, ..., V_i$. This implies that $x \in W_{i,a}$, where $a = \{V_0, ..., V_i\}$. This concludes the proof. □

Corollary A.2. *Every (complex or real) vector bundle over a topological manifold is of finite type.* □

The proof of Theorem 1.3.1 together with Corollary A.2 show that the following topological version of Theorem 1.3.2 holds.

Corollary A.3. *If M is a paracompact space of finite covering dimension and ξ is a (complex or real) vector bundle over M, then there exists a vector bundle $\tilde{\xi}$ over M such that $\xi \oplus \tilde{\xi}$ is trivial. In particular, this holds if M is a topological manifold. Moreover, if ξ is a smooth vector bundle over a smooth manifold M, then there exists a smooth vector bundle $\tilde{\xi}$ over M such that $\xi \oplus \tilde{\xi}$ is trivial.* □

Corollary A.4. *If M is a paracompact space of finite covering dimension and ξ is a complex vector bundle over M, then there exists some $n \in \mathbb{N}$ and a continuous map $f : M \to G_k(\mathbb{C}^n)$ such that $\xi \cong f^*\gamma_n^k$. In particular this holds in case M is a topological manifold. The same is true for real vector bundles if we replace $G_k(\mathbb{C}^n)$ with the real Grassmann manifold $G_k(\mathbb{R}^n)$.* □

Chapter 2

Characteristic classes

2.1 Connections

Let $\xi = (E, p, M)$ be a smooth vector bundle of rank n over a smooth manifold M. A (linear) connection on ξ is a linear map

$$\nabla : \Omega^0(\xi) \to A^1(M) \otimes_{C^\infty(M)} \Omega^0(\xi)$$

with the additional property (Leibniz formula)

$$\nabla(fs) = df \otimes s + f\nabla s$$

for every $f \in C^\infty(M)$ and $s \in \Omega^0(\xi)$, where $A^1(M)$ denotes the space of smooth 1-forms of M. If ξ is real then linear means $\mathbb{R}$-linear. If ξ is a smooth complex vector bundle, a connection on ξ is a $\mathbb{C}$-linear map

$$\nabla : \Omega^0(\xi) \to A^1(M; \mathbb{C}) \otimes_{C^\infty(M;\mathbb{C})} \Omega^0(\xi)$$

satisfying the Leibniz formula for all $f \in C^\infty(M; \mathbb{C})$. We will write $A^k(M)$ and $C^\infty(M)$ in both cases, as the meaning will usually be clear from the context.

Since $A^1(M) = \Omega^0(T^*M)$ and $A^1(M; \mathbb{C}) = \Omega^0((T^*M)_{\mathbb{C}})$, from Theorem 1.5.3 we have

$$A^1(M) \otimes_{C^\infty(M)} \Omega^0(\xi) \cong \Omega^0(T^*M \otimes \xi)$$
$$\cong \Omega^0(\mathrm{Hom}(TM, \xi)) \cong \mathrm{Hom}_{C^\infty(M)}(\Omega^0(TM), \Omega^0(\xi)).$$

So a connection on ξ is a map $\nabla : \Omega^0(\xi) \times \Omega^0(TM) \to \Omega^0(\xi)$ which is linear with respect to the factor $\Omega^0(\xi)$, is $C^\infty(M)$-linear with respect to the factor $\Omega^0(TM)$ and if we write $\nabla_X = \nabla(., X)$, then

$$\nabla_X(fs) = f\nabla_X s + (Xf)s$$

for every $X \in \Omega^0(TM)$, $s \in \Omega^0(\xi)$ and $f \in C^\infty(M)$. In other words a connection is a way to differentiate smooth sections of ξ in the directions of smooth vector fields of M. From the above isomorphisms a connection can be thought of as a linear map $\nabla : \Omega^0(\xi) \to \Omega^0(\mathrm{Hom}(TM, \xi))$, and so the value $(\nabla_X s)(x) \in E_x = p^{-1}(x)$ depends only of the vector $X(x) \in T_xM$ and the values of s on an open neighbourhood of

$x \in M$, because if $s|_U = 0$ and $U \subset M$ is an open neighbourhood of x, there exists some $f \in C^\infty(M)$ such that $f(x) = 1$ and supp$f \subset U$, and therefore $f \cdot s = 0$ on M, which gives

$$0 = \nabla_X(fs)(x) = f(x)(\nabla_X s)(x) + (Xf)(x)s(x) = (\nabla_X s)(x).$$

Thus, a connection can be localized to $\xi|_U$ for every open set $U \subset M$.

Let $U \subset M$ be an open set over which ξ is trivial and let $\{e_1, ..., e_n\}$ be a smooth local frame of ξ on U. Every element of $A^1(U) \otimes_{C^\infty(U)} \Omega^0(\xi|_U)$ can be written in a unique way as

$$\sum_{j=1}^{n} a_j \otimes e_j$$

for some $a_j \in C^\infty(U)$, $1 \leq j \leq n$. Therefore,

$$\nabla e_k = \sum_{j=1}^{n} A_{jk} \otimes e_j$$

where $A = (A_{jk})$ is a $n \times n$ matrix of smooth 1-forms on U, called the connection form with respect to the frame $\{e_1, ..., e_n\}$. Conversely, for any $n \times n$ matrix of smooth 1-forms on U and any smooth frame $\{e_1, ..., e_n\}$ of $\xi|_U$ one can define a connection on $\xi|_U$ by setting

$$\nabla(\sum_{k=1}^{n} f_k e_k) = \sum_{k=1}^{n} df_k \otimes e_k + \sum_{k,j=1}^{n} f_k A_{jk} \otimes e_j$$

for every f_1,..., $f_n \in C^\infty(M)$.

Example 2.1.1. If $\xi = (E, p, M)$ is a smooth vector bundle of rank n on a smooth manifold M, there exists a smooth vector bundle $\tilde{\xi}$ of some rank k such that $\xi \oplus \tilde{\xi} \cong \epsilon^{n+k}$. Let $f : E \to M \times \mathbb{C}^{n+k}$ be the inclusion and $g : M \times \mathbb{C}^{n+k} \to E$ the projection. Let ∇_0 be the connection on ϵ^{n+k} with zero connection form. Equivalently, $\nabla_0 = d \oplus \cdots \oplus d$, since $\Omega^0(\epsilon^{n+k}) \cong C^\infty(M) \oplus \cdots C^\infty(M)$ $n + k$ times and therefore

$$A^1(M) \otimes_{C^\infty(M)} \Omega^0(\epsilon^{n+k}) \cong A^1(M) \oplus \cdots \oplus A^1(M)$$

We have $C^\infty(M)$-linear maps $f_* : \Omega^0(\xi) \to \Omega^0(\epsilon^{n+k})$ and $g_* : \Omega^0(\epsilon^{n+k}) \to \Omega^0(\xi)$ and the composition $\nabla = (id \otimes g_*) \circ \nabla_0 \circ f_*$

$$\Omega^0(\xi) \xrightarrow{f_*} \Omega^0(\epsilon^{n+k}) \xrightarrow{\nabla_0} A^1(M) \otimes_{C^\infty(M)} \Omega^0(\epsilon^{n+k}) \xrightarrow{id \otimes g_*} A^1(M) \otimes_{C^\infty(M)} \Omega^0(\xi)$$

is a connection on ξ. Thus, every (complex or real) smooth vector bundle over a smooth manifold admits at least one connection.

In the sequel we denote $\Omega^k(\xi) = A^k(M) \otimes_{C^\infty(M)} \Omega^0(\xi)$ for every $k \in \mathbb{Z}^+$ and every smooth vector bundle $\xi = (E, p, M)$.

If $\xi_1 = (E_1, p_1, M)$ and $\xi_1 = (E_2, p_2, M)$ be two smooth vector bundles over the same smooth manifold M. We define the $C^\infty(M)$-bilinear form

$$\Omega^k(\xi_1) \otimes_{C^\infty(M)} \Omega^l(\xi_2) \xrightarrow{\wedge} \Omega^{k+l}(\xi_1 \otimes \xi_2) \cong A^{k+l}(M)) \otimes_{C^\infty(M)} (\Omega^0(\xi_1) \otimes_{C^\infty(M)} \Omega^0(\xi_2))$$

which sends $(\omega \otimes s) \otimes (\theta \otimes t)$ to $(\omega \wedge \theta) \otimes (s \otimes t)$, where $\omega \wedge \theta$ is the usual wedge product of smooth forms on M.

Since $\Omega^0(\epsilon^1_{\mathbb{R}}) \cong C^\infty(M)$ and $\Omega^k(\epsilon^1_{\mathbb{R}}) \cong A^k(M)$, taking $\xi_1 = \epsilon^1_{\mathbb{R}}$ and $k = 0$ the above bilinear form gives just the $C^\infty(M)$-module structure of $\Omega^l(\xi_2)$ for a real vector bundle ξ_2. Similarly, $\Omega^0(\epsilon^1_{\mathbb{C}}) \cong C^\infty(M;\mathbb{C})$ and $\Omega^k(\epsilon^1_{\mathbb{C}}) \cong A^k(M;\mathbb{C})$, the $\mathbb{C}$-valued smooth k-forms on M. Moreover, if ξ_2 is a complex vector bundle, for $\omega \in A^k(M;\mathbb{C})$ and $s \in \Omega^0(\xi_2)$ we have $\omega \wedge s = \omega \otimes s$, which means that

$$A^k(M;\mathbb{C}) \otimes_{C^\infty(M;\mathbb{C})} \Omega^0(\xi_2) \quad \stackrel{\wedge}{\longrightarrow} \quad \Omega^k(\xi_2) \cong A^k(M;\mathbb{C}) \otimes_{C^\infty(M;\mathbb{C})} \Omega^0(\xi_2)$$

is the identity map. analogously, in case ξ_2 is real.

Obviously, $1 \wedge s = s$ and $(\omega \wedge \theta) \wedge s = \omega \wedge (\theta \wedge s)$ for every $\omega \in A^k(M)$, $\theta \in A^l(M)$ and $s \in \Omega^j(\xi_2)$.

Lemma 2.1.2. *If ∇ is a connection on the smooth vector bundle $\xi = (E, p, M)$, then there exists a linear map $d^\nabla : \Omega^k(\xi) \to \Omega^{k+1}(\xi)$ for $k \in \mathbb{Z}^+$ such that*

(i) $d^\nabla = \nabla : \Omega^0(\xi) \to \Omega^1(\xi)$ for $k = 0$ and

(ii) $d^\nabla(\omega \wedge s) = d\omega \wedge s + (-1)^k \omega \wedge d^\nabla s$ for every $\omega \in A^k(M)$ and $s \in \Omega^l(\xi)$ and k, $l \in \mathbb{Z}^+$.

Proof. For every $\omega \in A^k(M)$ and $s \in \Omega^0(\xi)$ we put

$$d^\nabla(\omega \otimes s) = d\omega \wedge s + (-1)^k \omega \wedge (\nabla s)$$

and observe that d^∇ is well defined on $\Omega^k(\xi)$, because

$$d^\nabla(\omega \otimes (fs)) = fd\omega \wedge s + (-1)^k \omega \wedge (df \otimes s + f\nabla s)$$

$$= fd\omega \wedge s + (-1)^k \omega \wedge f\omega \wedge \nabla s + (df \wedge \omega) \wedge s = d^\nabla((f\omega) \otimes s)$$

for every $f \in C^\infty(M)$. Since $d\omega \wedge s = d\omega \otimes s$, we have (i).

To prove (ii) suppose that $s = \theta \otimes t$, where $\theta \in A^l(M)$ and $t \in \Omega^0(\xi)$. Then,

$$d^\nabla(\omega \wedge s) = d^\nabla(\omega \wedge (\theta \otimes t)) = d^\nabla((\omega \wedge \theta) \otimes t)$$

$$= d(\omega \wedge \theta) \otimes t + (-1)^{k+l}(\omega \wedge \theta) \wedge (\nabla t)$$

$$= (d\omega \wedge \theta + (-1)^k \omega \wedge \theta) \otimes t + (-1)^{k+l}(\omega \wedge \theta) \wedge (\nabla t)$$

$$= d\omega \wedge (\theta \otimes t) + (-1)^k \omega \wedge [d\theta \otimes t + (-1)^l \theta \wedge (\nabla t)]$$

$$= d\omega \wedge (\theta \otimes t) + (-1)^k \omega \wedge d^\nabla(\theta \otimes t)$$

$$= d\omega \wedge s + (-1)^k \omega \wedge d^\nabla s. \quad \square$$

Thus, for every connection on a smooth vector bundle $\xi = (E, p, M)$ we get the sequence of linear maps

$$0 \longrightarrow \Omega^0(\xi) \stackrel{\nabla}{\longrightarrow} \Omega^1(\xi) \stackrel{d^\nabla}{\longrightarrow} \Omega^2(\xi) \stackrel{d^\nabla}{\longrightarrow} \cdots$$

In the particular space $\xi = \epsilon^1$, it coincides with the deRham complex of M. However, as we shall see, this is not a cochain complex in general. In any case, the map $F^\nabla = d^\nabla \circ \nabla : \Omega^0(\xi) \to \Omega^2(\xi)$ is $C^\infty(M)$-linear. Indeed, for every $f \in C^\infty(M)$ and $s \in \Omega^0(\xi)$ we have

$$d^\nabla(\nabla(fs)) = d^\nabla(df \otimes s + f\nabla s) = d^\nabla(df \wedge s + f \wedge \nabla s)$$

$$= d(df) \wedge s - df \wedge (\nabla s) + df \wedge (\nabla s) + f d^\nabla(\nabla s) = f d^\nabla(\nabla s)).$$

On the other hand, from Theorem 1.5.3 we have

$$\mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi), \Omega^2(\xi)) \cong \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi), \Omega^0(\xi)) \otimes_{C^\infty(M)} A^2(M)$$

$$\cong \Omega^0(\mathrm{Hom}(\xi,\xi)) \otimes_{C^\infty(M)} A^2(M) = \Omega^2(\mathrm{Hom}(\xi,\xi)).$$

Thus, F^∇ is a smooth 2-form with values in $\mathrm{Hom}(\xi,\xi)$ which is called the curvature form of ∇. For every X, $Y \in \Omega^0(TM)$ the evaluation at (X,Y) induces a $C^\infty(M)$-linear map from $\Omega^2(\mathrm{Hom}(\xi,\xi))$ to $\Omega^0(\mathrm{Hom}(\xi,\xi))$ which sends F^∇ to an element $F^\nabla_{X,Y}$. Because of the $C^\infty(M)$-linearity, for every $x \in M$ the value $F^\nabla_{X,Y}(x)$ depends only on the values $X(x)$ and $Y(x)$. For every $\omega \in A^1(M)$ and $s \in \Omega^0(\xi)$ we have

$$d^\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s$$

and therefore

$$d^\nabla(\omega \otimes s)(X,Y) = [X\omega(Y) - Y\omega(X) - \omega([X,Y])] \cdot s - [\omega(X)\nabla_Y s - \omega(Y)\nabla_X s]$$

$$= \nabla_X(\omega(Y)s) - \nabla_Y(\omega(X)s) - \omega([X,Y])s$$

from which follows the traditional formula of the curvature tensor

$$F^\nabla_{X,Y}(s) = d^\nabla(\nabla s)(X,Y) = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s.$$

In order to carry out explicit calculations it is useful to have a local formula for the curvature 2-form. Let $A = (A_{jk})$ be the connection form with respect to some local smooth frame $\{e_1, ..., e_n\}$. Then,

$$d^\nabla(\nabla e_k) = \sum_{j=1}^n dA_{jk} \otimes e_j - \sum_{j=1}^n A_{jk} \wedge \nabla e_j$$

$$= \sum_{j=1}^n dA_{jk} \otimes e_j - \sum_{j=1}^n A_{jk} \wedge \big(\sum_{l=1}^n A_{lj} \otimes e_l\big)$$

$$= \sum_{l=1}^n \Big(dA_{lk} \otimes e_l + \big(\sum_{j=1}^n A_{lj} \wedge A_{jk}\big) \otimes e_l\Big).$$

Thus, in matrix form we have

$$F^\nabla|_{\text{locally}} = dA + A \wedge A$$

and for every X, $Y \in \Omega^0(TM)$ the matrix of the linear map $F^{\nabla}_{X,Y}(x) : E_x \to E_x$ with respect to the basis $\{e_1(x), ..., e_n(x)\}$ is $(dA + A \wedge A)(X, Y)$.

Example 2.1.3. Let $\gamma_1 = (\mathcal{H}_1, p, \mathbb{C}P^1)$ be the tautological complex line bundle over $\mathbb{C}P^1 \approx S^2$. Recall that $\mathcal{H}_1 = \{(\ell, u) \in \mathbb{C}P^1 \times \mathbb{C}^2 : u \in \ell\}$ and let

$$\mathcal{H}_1^{\perp} = \{(\ell, u) \in \mathbb{C}P^1 \times \mathbb{C}^2 : u \in \ell^{\perp}\}$$

with respect to the usual hermitian product on $\mathbb{C}^2$. Then, $\mathcal{H}_1^{\perp}$ is the total space of an obvious smooth complex vector bundle $\gamma_1^{\perp}$ over $\mathbb{C}P^1$ such that $\gamma_1 \oplus \gamma_1^{\perp} \cong \epsilon^2_{\mathbb{C}}$. We shall compute the connection form and the curvature form of the connection ∇ defined as in Example 2.1.1. using the same notations. Thus, $\nabla = (id \otimes g_*) \circ (d \oplus d) \circ f_*$, where $f : \mathcal{H}_1 \to \mathbb{C}P^1 \times \mathbb{C}^2$ is the inclusion and $g : \mathbb{C}P^1 \times \mathbb{C}^2 \to \mathcal{H}_1$ is the projection. If $\ell = [z_0, z_1]$, then

$$g([z_0, z_1], (u_0, u_1)) = (\overline{z}_0 u_0 + \overline{z}_1 u_1) \cdot (z_0, z_1) = (|z_0|^2 u_0 + \overline{z}_1 z_0 u_1, \overline{z}_1 z_0 u_1 + |z_1|^2 u_1).$$

Let $\{(U_0, \phi_0), (U_1, \phi_1)\}$ be the canonical atlas of $\mathbb{C}P^1$. Over U_0 we have the smooth section s defined by $s([1, z]) = (1, z)$ and $(d \oplus d)s([1, z]) = ([1, z], (0, dz))$. Therefore,

$$(\nabla s)([1, z]) = \left([1, z], \frac{1}{1 + |z|^2} \cdot 0 + \frac{\overline{z}}{1 + |z|^2} dz, \frac{z}{1 + |z|^2} \cdot 0 + \frac{|z|^2}{1 + |z|^2} dz\right)$$
$$= \left([1, z], (\frac{\overline{z}}{1 + |z|^2} dz) \cdot (1, z)\right) = (\frac{\overline{z}}{1 + |z|^2} dz) \otimes s.$$

So, the connection form on U_0 with respect to the frame $\{s\}$ is

$$A = \frac{\overline{z}}{1 + |z|^2} dz.$$

Since $A \wedge A = 0$, we have $F^{\nabla}|_{U_0} = dA$ and so

$$F^{\nabla}|_{U_0} = d(\frac{\overline{z}}{1 + |z|^2}) \wedge dz = \left[d(\frac{1}{1 + |z|^2})\overline{z} + \frac{1}{1 + |z|^2} d\overline{z}\right] \wedge dz$$
$$= \left[-\frac{d(1 + \overline{z}z)}{(1 + |z|^2)^2}\overline{z} + \frac{1}{1 + |z|^2} d\overline{z}\right] \wedge dz = \frac{1}{(1 + |z|^2)^2} d\overline{z} \wedge dz.$$

Note that $\mathrm{Hom}(\gamma_1, \gamma_1) \cong \epsilon^1_{\mathbb{C}}$, because it is a complex line bundle and admits the global smooth section whose value at ℓ is the identity map of the corresponding fiber of γ_1. Thus,

$$F^{\nabla} \in \Omega^2(\mathrm{Hom}(\gamma_1, \gamma_1)) \cong A^2(\mathbb{C}P^1) \otimes_{C^{\infty}(\mathbb{C}P^1)} C^{\infty}(\mathbb{C}P^1; \mathbb{C}) = A^2(\mathbb{C}P^1; \mathbb{C})$$

is indeed a $\mathbb{C}$-valued smooth 2-form on $\mathbb{C}P^1$.

So far we have dealt with $F^{\nabla} = d^{\nabla} \circ \nabla$. It turns out that in higher degrees the composition $d^{\nabla} \circ d^{\nabla} : \Omega^k(\xi) \to \Omega^{k+2}(\xi)$ for $k \geq 2$ is completely determined by F^{∇}. To see this, we consider the $C^{\infty}(M)$-bilinear map

$$\Omega^k(\xi) \times \mathrm{Hom}_{C^{\infty}(M)}(\Omega^0(\xi), \Omega^2(\xi)) \quad \stackrel{\wedge}{\longrightarrow} \quad \Omega^{k+2}(\xi)$$

defined by $(\omega \otimes s) \wedge G = \omega \wedge G(s)$, for every $\omega \in A^k(M)$, $s \in \Omega^0(\xi)$ and $G \in \mathrm{Hom}_{C^\infty(M)}(\Omega^0(\xi), \Omega^2(\xi))$, where the wedge in the right hand side is the one previously defined.

Proposition 2.1.4. *$(d^\nabla \circ d^\nabla)(t) = t \wedge F^\nabla$ for every $t \in \Omega^k(\xi)$.*

Proof. Indeed, if $t = \omega \otimes s \in \Omega^k(\xi)$, we have

$$(d^\nabla \circ d^\nabla)(\omega \otimes s) = d^\nabla(d\omega \otimes s + (-1)^k \omega \wedge \nabla s)$$

$$= d(d\omega) \otimes s + (-1)^{k+1} d\omega \wedge \nabla s + (-1)^k d\omega \wedge \nabla s + \omega \wedge (d^\nabla(\nabla s)) = \omega \wedge F^\nabla(s). \quad \square$$

2.2 Induced connections

Let $f : N \to M$ be a smooth map between smooth manifolds and let $\xi = (E, p, M)$ be a (complex or real) smooth vector bundle of rank n over M. Since the induced map $f^* : C^\infty(M) \to C^\infty(N)$ is a ring homomorphism, every $C^\infty(N)$-module is also a $C^\infty(M)$-module. In particular, $\Omega^0(f^*\xi)$ has a $C^\infty(M)$-module structure and the map $f^* : \Omega(\xi) \to \Omega^0(f^*\xi)$ defined by

$$(f^*(s))(x) = (x, s(f(x)))$$

for every $x \in N$, is $C^\infty(M)$-linear.

Lemma 2.2.1. *The well defined $C^\infty(N)$-linear map*

$$f^* : C^\infty(N) \otimes_{C^\infty(M)} \Omega^0(\xi) \to \Omega^0(f^*\xi)$$

which sends $\phi \otimes s$ to $\phi \cdot f^*(s)$ is an isomorphism.

Proof. If ξ is trivial, then $f^*\xi$ is the trivial vector bundle of rank n over N and $\Omega^0(\xi) \cong C^\infty(M) \oplus \cdots \oplus C^\infty(M)$ and $\Omega^0(f^*\xi) \cong C^\infty(N) \oplus \cdots \oplus C^\infty(N)$, n-times. It is immediate from the definitions that in this case f^* is an isomorphism, essentially the identity map.

In the general case, there exists a smooth vector bundle $\tilde{\xi} = (\tilde{E}, \tilde{p}, M)$ over M of some rank m such that $\xi \oplus \tilde{\xi} \cong \epsilon^{n+m}$. Then, $f^*\xi \oplus f^*\tilde{\xi} \cong \epsilon^{n+m}$ over N and from the trivial case

$$f^* : (C^\infty(N) \otimes_{C^\infty(M)} \Omega^0(\xi)) \oplus (C^\infty(N) \otimes_{C^\infty(M)} \Omega^0(\xi)) \cong \Omega^0(f^*\xi) \oplus \Omega^0(f^*\tilde{\xi})$$

where the first factor on the left hand side is send to the first factor on the right hand side. $\square$

It is evident that the $C^\infty(M)$-linear map $f^* : \Omega^0(\xi) \to \Omega^0(f^*\xi)$ induces a $C^\infty(M)$-linear map $f^* : A^1(M) \otimes_{C^\infty(M)} \Omega^0(\xi) \to A^1(N) \otimes_{C^\infty(N)} \Omega^0(f^*\xi)$.

Lemma 2.2.2. *For every connection ∇ on ξ and every smooth map $f : N \to M$ there exists a unique connection $f^*\nabla$ on $f^*\xi$, which makes the following diagram commutative.*

$$\begin{array}{ccc} \Omega^0(\xi) & \xrightarrow{\nabla} & \Omega^1(\xi) \\ \downarrow f^* & & \downarrow f^* \\ \Omega^0(f^*\xi) & \xrightarrow{f^*\nabla} & \Omega^1(f^*\xi) \end{array}$$

Proof. From the preceding Lemma 2.2.1 it follows that we have an $C^\infty(N)$-isomorphism $\Omega^k(f^*\xi) \cong A^k(N) \otimes_{C^\infty(M)} \Omega^0(\xi)$ for every $k \in \mathbb{Z}^+$. On the other hand, the pull-back map $f^* : A^k(M) \to A^k(N)$ induces a $C^\infty(M)$-linear map from $C^\infty(N) \otimes_{C^\infty(M)} A^k(M)$ to $A^k(N)$ which sends $\phi \otimes \omega$ to $\phi \cdot f^*(\omega)$. Taking tensor products (over $C^\infty(M)$) with $\Omega^0(\xi)$ we obtain a $C^\infty(M)$-linear map

$$\rho : C^\infty(N) \otimes_{C^\infty(M)} \Omega^k(\xi) \to A^k(N) \otimes_{C^\infty(M)} \Omega^0(\xi).$$

It suffices now to take

$$f^*\nabla = (d \otimes id) + \rho(id \otimes \nabla) : \Omega^0(f^*\xi) \to A^1(N) \otimes_{C^\infty(N)} \Omega^0(f^*\xi),$$

since from Lemma 2.2.1 we have a $C^\infty(N)$-isomorphism

$$f^* : C^\infty(N) \otimes_{C^\infty(M)} \Omega^0(\xi) \cong \Omega^0(f^*\xi). \quad \square$$

Let $U \subset M$ be an open set over which ξ is trivial and let $\{e_1, ..., e_n\}$ be a local frame of ξ on U. Let A be the connection form of a connection ∇ on U with respect to this frame. Then, $\{f^*(e_1), ..., f^*(e_n)\}$ is a frame of $f^*\xi$ on $f^{-1}(U)$ and the corresponding connection form of $f^*\nabla$ on $f^{-1}(U)$ is f^*A. The commutative diagram of Lemma 2.2.2 extends to the commutative diagram

$$\begin{array}{ccc} \Omega^1(\xi) & \xrightarrow{d^\nabla} & \Omega^2(\xi) \\ \downarrow f^* & & \downarrow f^* \\ \Omega^1(f^*\xi) & \xrightarrow{d^{f^*\nabla}} & \Omega^2(f^*\xi) \end{array}$$

from which we get a commutative diagram

$$\begin{array}{ccc} \Omega^0(\xi) & \xrightarrow{F^\nabla} & \Omega^2(\xi) \\ \downarrow f^* & & \downarrow f^* \\ \Omega^0(f^*\xi) & \xrightarrow{F^{f^*\nabla}} & \Omega^2(f^*\xi) \end{array}$$

Since $f^*(\mathrm{Hom}(\xi, \xi)) \cong \mathrm{Hom}(f^*\xi, f^*\xi)$, we arrive at $f^*(F^\nabla) = F^{f^*\nabla}$. This can also be seen by computing locally

$$f^*(F^\nabla) = f^*(dA + A \wedge A) = f^*(dA) + f^*(A \wedge A) = d(f^*(A)) + f^*(A) \wedge f^*(A) = F^{f^*\nabla}.$$

A connection ∇ on a smooth vector bundle $\xi = (E, p, M)$ induces a connection on the dual vector bundle ξ^* as follows. We consider the composition

$$(.,.) \quad : \quad \Omega^k(\xi) \otimes_{C^\infty(M)} \Omega^l(\xi^*) \xrightarrow{\wedge} \Omega^{k+l}(\xi \otimes \xi^*) \longrightarrow A^{k+l}(M)$$

where the second map is induced by the vector bundle morphism $\xi\otimes\xi^* \to \epsilon^1$ defined by evaluation on the fibres. So,

$$(\omega \otimes s, \theta \otimes s^*) = s^*(s) \cdot \omega \wedge \theta$$

for every $\omega \in A^k(M)$, $\theta \in A^l(M)$ and $s \in \Omega^0(\xi)$, $s^* \in \Omega(\xi^*)$. Since $(.,.)$ is non-degenerate for $(k,l) = (0,0)$ and for $(k,l) = (0,1)$, the equation

$$d(s, s^*) = (\nabla s, s^*) + (s, \nabla^* s^*)$$

defines a connection ∇^* on ξ^*.

If $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ are two smooth vector bundles over the same smooth manifold M with connections ∇^1 and ∇^2, respectively, then the wedge

$$\Omega^0(\xi_1) \otimes_{C^\infty(M)} \Omega^0(\xi_2) \xrightarrow{\wedge} \Omega^0(\xi_1 \otimes \xi_2)$$

coincides with the isomorphism $\Omega^0(\xi_1) \otimes_{C^\infty(M)} \Omega^0(\xi_2) \cong \Omega^0(\xi_1 \otimes \xi_2)$ of Theorem 1.5.3(ii), and we can define a connection ∇ on the tensor product $\xi_1 \otimes \xi_2$ by the formula

$$\nabla(s \otimes t) = (\nabla^1 s) \wedge t + s \wedge (\nabla^2 t).$$

In particular, this gives a way to define a connection ∇ on $\mathrm{Hom}(\xi_1, \xi_2) \cong \xi_1^* \otimes \xi_2$ putting

$$\nabla(s^* \otimes t) = (\nabla^{1*} s^*) \wedge t + s^* \wedge (\nabla^2 t).$$

There is another, perhaps more direct, way to define this connection on $\mathrm{Hom}(\xi_1, \xi_2)$, as follows. The evaluation map

$$\Omega^0(\xi_1) \times \Omega^0(\mathrm{Hom}(\xi_1, \xi_2)) \to \Omega^0(\xi_2)$$

induces a $C^\infty(M)$-bilinear map

$$(.,.) : \Omega^k(\xi_1) \times \Omega^l(\mathrm{Hom}(\xi_1, \xi_2)) \to \Omega^{k+l}(\xi_2)$$

which for $(k,l) = (0,1)$ is given by the formula $(s, \omega \otimes \phi) = \omega \otimes \phi(s)$. Thus, it is non-degenerate and the equation

$$\nabla^2(s, \phi) = (\nabla^1 s, \phi) + (s, \nabla' \phi)$$

defines a connection ∇' on $\mathrm{Hom}(\xi_1, \xi_2)$.

We shall prove that the connections ∇ and ∇' on $\mathrm{Hom}(\xi_1, \xi_2)$ coincide through the isomorphism $a : \xi_1^* \otimes \xi_2 \cong \mathrm{Hom}(\xi_1, \xi_2)$. It suffices to show that

$$(s, \nabla' a(s^* \otimes t)) = (s, \nabla(s^* \otimes t))$$

for every $s \in \Omega^0(\xi_1)$, $t \in \Omega^0(\xi_2)$ and $s^* \in \Omega^0(\xi_1^*)$. Indeed, there is a commutative diagram of vector bundle morphisms

$$\begin{array}{ccc} \xi_1 \otimes \xi_1^* \otimes \xi_2 & \xrightarrow{id\otimes a} & \xi_1 \otimes \mathrm{Hom}(\xi_1, \xi_2) \\ \downarrow{\scriptstyle (.,.)\otimes id} & & \downarrow{\scriptstyle (.,.)} \\ \epsilon^1 \otimes \xi_2 & \longrightarrow & \xi_2 \end{array}$$

where the bottom map is scalar multiplication, because

$$(s, a(s^* \otimes t)) = (s, s^* \cdot t) = s^*(s)t = (s, s^*)t.$$

Thus,

$$(s, \nabla(s^* \otimes t)) = (s, \nabla^1 s^*) \wedge t + (s, s^*)\nabla^2 t.$$

From the definitions now we have

$$(s, \nabla' a(s^* \otimes t)) = \nabla^2(s, a(s^* \otimes t)) - (\nabla^1 s, a(s^* \otimes t)) = \nabla^2((s, s^*)t) - (\nabla^1 s, s^*) \wedge t$$

$$= d(s, s^*) \wedge t + (s, s^*)\nabla^2 t - (\nabla^1 s, s^*) \wedge t = (s, \nabla^{1*} s^*) \wedge t + (s, s^*)\nabla^2 t = (s\nabla(s^* \otimes t)).$$

Finally, it is easy to check following.

(i) $d(s, s^*) = (d^\nabla s, s^*) + (-1)^k (s, d^\nabla s^*)$ for every $s \in \Omega^k(\xi)$ and $s^* \in \Omega^k(\xi^*)$, and

(ii) $d^\nabla(s \otimes t) = (d^\nabla s) \otimes t + (-1)^k s \otimes (d^\nabla t)$,

(iii) $d(s, \phi) = (d^\nabla s, \phi) + (-1)^k (s, d^\nabla \phi)$ for every $s \in \Omega^k(\xi_1)$, $t \in \Omega^l(\xi_2)$ and $\phi \in \Omega^l(\mathrm{Hom}(\xi_1, \xi_2))$.

2.3 Invariant complex polynomials

A complex polynomial P in n^2 variables of degree k is homogeneous if it is the sum of monomials of the same degree k. Such a polynomial can be considered as a function $P : \mathbb{C}^{n\times n} \to \mathbb{C}$, by arranging the n^2 variables in a $n \times n$ matrix. So $P(A)$ is determined as a polynomial function of the entries of the matrix $A \in \mathbb{C}^{n\times n}$ with the property $P(\lambda A) = \lambda^k P(A)$ for every $\lambda \in \mathbb{C}$.

A homogeneous polynomial $P : \mathbb{C}^{n\times n} \to \mathbb{C}$ is called invariant if it is an invariant function under the action of $GL(n, \mathbb{C})$ on $\mathbb{C}^{n\times n}$ by conjugation, that is

$$P(gAg^{-1}) = P(A)$$

for every $g \in GL(n, \mathbb{C})$ and $A \in \mathbb{C}^{n\times n}$. In this case, P induces a well defined function $P : \mathrm{Hom}(V, V) \to \mathbb{C}$ for every complex vector space of dimension n, since the value $P(A)$ does not depend on the choice of basis.

Examples 2.3.1. (a) For every $A \in \mathbb{C}^{n\times n}$ the "characteristic polynomial" of $-A$ is

$$\sigma(t) = \det(I_n + tA) = \sum_{k=0}^{n} \sigma_k(A)t^k$$

and $\sigma_0(A) = 1$. Each coefficient $\sigma_k(A)$ is obviously an invariant homogeneous polynomial of degree k. Note that $\sigma_n(A) = \det A$.

(b) For every $A \in \mathbb{C}^{n\times n}$ the trace $\mathrm{Tr}(A^k)$ is an invariant homogeneous polynomial of A of degree k. There is an alternative description which relates this example with the previous one. Let

$$s(t) = -t\frac{d}{dt}\log\det(I_n - tA) = \sum_{k=0}^{\infty} s_k(A)t^k$$

where log is considered as the formal power series

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

and $\frac{d}{dt}$ denotes the formal derivative

$$\frac{d}{dt}\left(\sum_{k=0}^{\infty} a_k t^k\right) = \sum_{k=0}^{\infty} k a_k t^{k-1}.$$

We shall show that $s_k(A) = \mathrm{Tr}(A^k)$ for every $k \in \mathbb{N}$. In the special case of a diagonal matrix $A = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ we have

$$s(t) = -t\frac{d}{dt}\log\prod_{k=1}^{n}(1-t\lambda_k) = -t\frac{d}{dt}\sum_{k=1}^{n}\log(1-t\lambda_k) = \sum_{k=1}^{n}\frac{t\lambda_k}{1-t\lambda_k}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{\infty}\lambda_k^j t^j = \sum_{j=0}^{\infty}\left(\sum_{k=1}^{n}\lambda_k^j\right)t^j.$$

This implies that $s_k(A) = \mathrm{Tr}(A^k)$ for every diagonal matrix $A \in \mathbb{C}^{n\times n}$. The general case is a consequence of continuity and the following.

Lemma 2.3.2. *The set of diagonalisable complex $n \times n$ matrices is dense in $\mathbb{C}^{n\times n}$.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ have eigenvalues λ_1,..., $\lambda_j \in \mathbb{C}$ with multiplicities n_1,..., n_j, respectively. There exists $R \in GL(n, \mathbb{C})$ such that $R^{-1}AR$ is upper triangular. Let $\epsilon > 0$. We choose any

$$0 < \rho < \frac{1}{2}\min\{\epsilon, |\lambda_k - \lambda_l| : 1 \le k \ne l \le j\}.$$

We also choose distinct points z_1^k,..., $z_{n_k}^k \in \mathbb{C}$ of distance at most ρ from λ_k. Let T_ϵ be the matrix which results in from $R^{-1}AR$ by replacing the diagonal entries with the complex numbers

$$z_1^1, ..., z_{n_1}^1, ..., z_{n_j}^j.$$

Then, $A_\epsilon = RT_\epsilon R^{-1}$ is diagonalisable, because it has distinct eigenvalues, and

$$\|A - A_\epsilon\| \le n\|R\| \cdot \|R^{-1}\| \cdot \|R^{-1}AR - T_\epsilon\| \le n\|R\| \cdot \|R^{-1}\| \cdot \rho$$

where $\|.\|$ denotes the maximum norm. □

Note that the preceding Lemma 2.3.2 is not true over the field of real numbers. For instance the matrix of the rotation $R_{\pi/2}$ by the angle $\pi/2$ has characteristic polynomial $t^2 + 1$ which has negative discriminant. Since the discriminant of the characteristic polynomial is a continuous function of the matrix and the characteristic polynomial of a diagonalisable real 2×2 matrix must have non-negative discriminant, it follows that $R_{\pi/2}$ cannot be approximated by diagonalisable elements of $\mathbb{R}^{2\times 2}$.

The invariant homogeneous polynomials $\sigma_k(A)$ and $s_k(A)$, $0 \le k \le n$ are related through the Newton identities

$$s_k(A) - s_{k-1}(A)\sigma_1(A) + s_{k-2}(A)\sigma_2(A) + \cdot + (-1)^k k\sigma_k(A) = 0.$$

To see this, we apply again Lemma 2.3.2, so that it suffices to prove the identities for diagonal $A = \text{diag}(\lambda_1, ..., \lambda_n)$. In this case, on the one hand we have

$$\left(\sum_{k=0}^{n}(-1)^k\sigma_k(A)t^k\right)\cdot\left(\sum_{k=1}^{\infty}s_k(A)t^k\right) = \left(\sum_{j=1}^{n}\frac{t\lambda_j}{1-t\lambda_j}\right)\cdot\prod_{j=1}^{n}(1-t\lambda_j)$$

$$= \sum_{j=1}^{n} t\lambda_j(1-t\lambda_1)\cdots(1-t\lambda_{j-1})(1-t\lambda_{j+1})\cdots(1-t\lambda_n)$$

$$= -t\frac{d}{dt}\prod_{j=1}^{n}(1-t\lambda_j) = -t\frac{d}{dt}\sum_{k=0}^{n}(-1)^k\sigma_k(A)t^k = \sum_{k=1}^{n}(-1)^{k-1}k\sigma_k(A)t^k$$

and on the other hand

$$\left(\sum_{k=0}^{n}(-1)^k\sigma_k(A)t^k\right)\cdot\left(\sum_{k=1}^{\infty}s_k(A)t^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^k\sigma_j(A)s_{k-j}(A)\right)t^k,$$

where we have set $\sigma_k(A) = 0$ for $k > n$ and $s_0(A) = 0$. Comparing the coefficients we obtain the Newton identities.

It follows from the Newton identities that $s_k(A)$ can be determined inductively as a polynomial function with integer coefficients of $\sigma_1(A)$,..., $\sigma_k(A)$. Conversely, $\sigma_k(A)$ is a polynomial function with rational coefficients of $s_1(A)$,..., $s_k(A)$. For instance, for $k = 1$ we have $s_1(A) = \sigma_1(A)$ and for $k = 2$ we have

$$s_2(A) = s_1(A)\sigma_1(A) - 2\sigma_2(A) = (\sigma_1(A))^2 - 2\sigma_2(A).$$

For $k = 3$ we have

$$s_3(A) = s_2(A)\sigma_1(A) - s_1(A)\sigma_2(A) + 3\sigma_3(A) = (\sigma_1(A))^3 - 3\sigma_1(A)\sigma_2(A) + 3\sigma_3(A)$$

and so on.

It is immediate from the definitions that $s_k(\text{diag}(A_1, A_2)) = s_k(A_1) + s_k(A_2)$ and

$$\sigma_k(\text{diag}(A_1, A_2)) = \sum_{j=0}^{k}\sigma_j(A_1)\sigma_{k-j}(A_2).$$

Also, $s_k(A_1 \otimes A_2) = s_k(A_1)\cdot s_k(A_2)$, since $\text{Tr}(A_1 \otimes A_2) = \text{Tr}(A_1)\cdot\text{Tr}(A_2)$, where $A_1 \otimes A_2$ denotes the matrix of the tensor product of the linear maps with matrices A_1 and A_2.

The invariant homogeneous polynomials can be described as polynomial functions of the elementary symmetric polynomials. Recall that the elementary symmetric polynomials $\sigma_j(X_1,...,X_n)$, $1 \le j \le n$ in n variables are determined from the identity

$$\prod_{j=1}^{n}(1+tX_j) = \sum_{j=0}^{n}\sigma_j(X_1,...,X_n)t^j.$$

Obviously, $\sigma_1(X_1,...,X_n) = X_1 + \cdots + X_n$ and $\sigma_n(X_1,...,X_n) = X_1X_2\cdots X_n$. Every symmetric complex polynomial of n variables is a polynomial function of σ_1,..., σ_n.

Theorem 2.3.3. *For every invariant homogeneous polynomial $P : \mathbb{C}^{n\times n} \to \mathbb{C}$ there exists a polynomial p of n variables such that $P(A) = p(\sigma_1(A),...,\sigma_n(A))$ for every $A \in \mathbb{C}^{n\times n}$.*

Proof. Let $D_n \subset \mathbb{C}^{n\times n}$ be the set of all diagonal matrices. By Lemma 2.3.2, the set

$$\bigcup_{g\in GL(n,\mathbb{C})} gD_ng^{-1}$$

is dense in $\mathbb{C}^{n\times n}$ and so P is completely determined by its values on D_n. Every permutation s in n symbols determines an element $g \in GL(n,\mathbb{C})$ such that

$$g\mathrm{diag}(\lambda_1,...,\lambda_n)g^{-1} = \mathrm{diag}(\lambda_{s(1)},...,\lambda_{s(n)})$$

for every λ_1,..., $\lambda_n \in \mathbb{C}$. Since P is invariant, it follows that $P(\mathrm{diag}(X_1,...,X_n))$ is a symmetric polynomial and so there exists a polynomial p of n variables such that

$$P(\mathrm{diag}(X_1,...,X_n)) = p(\sigma_1(X_1,...,X_n),...,\sigma_n(X_1,...,X_n)).$$

The conclusion follows now by continuity. □

The set $I_n^*(\mathbb{C})$ of invariant homogeneous polynomials of n^2 complex variables equipped with the usual operations is a commutative algebra. Similarly, the set $S_n^*(\mathbb{C})$ of all symmetric homogeneous polynomials of n variables is a commutative algebra and $S_n^*(\mathbb{C}) = \mathbb{C}[\sigma_1,...,\sigma_n]$. The preceding Theorem 2.3.3 says that the map $\rho : I_n^*(\mathbb{C}) \to S_n^*(\mathbb{C})$ defined by

$$\rho(\sigma)(X_1,...,X_n) = \sigma(\mathrm{diag}(X_1,...,X_n))$$

is an isomorphism.

2.4 Chern classes

Let $\xi = (E,p,M)$ be a smooth complex vector bundle of rank n over a smooth manifold M. Let $U \subset M$ be an open set over which ξ is trivial and let $\{e_1,...,e_n\}$ be a frame of ξ on U. There is a corresponding isomorphism of the restriction

$\mathrm{Hom}(\xi,\xi)|_U$ with the trivial bundle of rank $n \times n$ over U. From this we get an isomorphism

$$\Omega^2(\mathrm{Hom}(\xi,\xi)|_U) \cong A^2(U;\mathbb{C}^{n\times n}) \cong A^2(U;\mathbb{C})^{n\times n}.$$

Thus, every 2-form R on $\mathrm{Hom}(\xi,\xi)$ gives a matrix $(R_{kl}) \in A^2(U;\mathbb{C})^{n\times n}$, which depends on the initial choice of the frame $\{e_1,...,e_n\}$. For every invariant homogeneous complex polynomial P of n^2 variables and degree k we have a corresponding element $P((R_{kl})) \in A^{2k}(U;\mathbb{C})$, because the wedge product of differential forms of even degree is commutative.

If $\{e'_1,...,e'_n\}$ is another frame on U from which we have a corresponding matrix $(R'_{kl}) \in A^2(U;\mathbb{C})^{n\times n}$, there exists a smooth function $g : U \to GL(n,\mathbb{C})$ such that $(R_{kl}) = g(R'_{kl})g^{-1}$. Since P is invariant, we have $P((R_{kl})) = P((R'_{kl}))$. This shows that there is a global well defined complex smooth $2k$-form $P(R) \in A^{2k}(M;\mathbb{C})$.

In particular, if ∇ is a connection on ξ with curvature form $F^\nabla \in \Omega^2(\mathrm{Hom}(\xi,\xi))$, then for every invariant homogeneous polynomial $P : \mathbb{C}^{n\times n} \to \mathbb{C}$ we have a well defined $\mathbb{C}$-valued smooth $2k$-form $P(F^\nabla) \in A^{2k}(M;\mathbb{C})$.

Lemma 2.4.1. *Let $P : \mathbb{C}^{n\times n} \to \mathbb{C}$ be an invariant homogeneous polynomial. If $P' = \left(\frac{\partial P}{\partial x^{kl}}\right)^T$, where T means transpose, then $P'(X)\cdot X = X \cdot P'(X)$ for every $X \in \mathbb{C}^{n\times n}$.*

Proof. Since P is invariant, we have

$$P((I_n + tE_{kl})X) = P(X(I_n + tE_{kl}))$$

for every $|t| < 1$, where E_{kl} is the basic $n \times n$ matrix whose (k,l)-entry is equal to 1 and has zeros everywhere else. Differentiating at $t = 0$ for $X = (a_{kl})$ the left hand side gives

$$DP(X)XE_{kl} = DP(X)\left(\sum_{j=1}^n a_{lj}E_{kj}\right) = \sum_{j=1}^n a_{lj}\frac{\partial P}{\partial x^{kj}}(X)$$

which is the (l,k)-entry of $P'(X)X$. Similarly, the right hand side gives

$$DP(X)E_{kl}X = DP(X)\left(\sum_{j=1}^n a_{jk}E_{jl}\right) = \sum_{j=1}^n a_{jk}\frac{\partial P}{\partial x^{jl}}(X)$$

which is the (l,k)-entry of $XP'(X)$. □

Proposition 2.4.2. *If ∇ is a connection on ξ with curvature form $F^\nabla \in \Omega^2(\mathrm{Hom}(\xi,\xi))$, then for every invariant homogeneous polynomial $P : \mathbb{C}^{n\times n} \to \mathbb{C}$ the complex smooth $2k$-form $P(F^\nabla) \in A^{2k}(M;\mathbb{C})$ is closed.*

Proof. (J. Milnor and J. Stasheff) It suffices to prove the assertion locally. Let $U \subset M$ be an open set over which ξ is trivial and let A be the connection form of ∇ on U with respect to some frame. Then $F^\nabla|_U = dA + A\wedge A$ and differentiating

$$dF^\nabla|_U = F^\nabla \wedge A - A \wedge F^\nabla.$$

This is called the (second) Bianchi identity. If $F^\nabla|_U = (F_{kl})$, then

$$dP(F^\nabla)|_U = \sum_{k,l=1}^{n} \frac{\partial P}{\partial x^{kl}}(F^\nabla) \wedge dF_{kl} = \mathrm{Tr}(P'(F^\nabla) \wedge dF^\nabla),$$

where P' is defined as in the preceding Lemma 2.4.1, by the use of which we get

$$\begin{aligned} dP(F^\nabla)|_U &= \mathrm{Tr}(P'(F^\nabla) \wedge F^\nabla \wedge A - P'(F^\nabla) \wedge A \wedge F^\nabla) \\ &= \mathrm{Tr}(F^\nabla \wedge P'(F^\nabla) \wedge A - P'(F^\nabla) \wedge A \wedge F^\nabla) = 0, \end{aligned}$$

because if $Y = P'(F^\nabla) \wedge A = (Y_{kl})$, then

$$dP(F^\nabla)|_U = \mathrm{Tr}(F^\nabla \wedge Y - Y \wedge F^\nabla) = \sum_{k,l=1}^{n} F_{lk} \wedge Y_{kl} - Y_{kl} \wedge F_{lk} = 0,$$

since F_{lk} is a 2-form. □

Proposition 2.4.3. *If P is an invariant homogeneous complex polynomial of n^2 variables of degree k, then the cohomology class $[P(F^\nabla)] \in H^{2k}(M;\mathbb{C})$ does not depend on the choice of the connection ∇ on ξ.*

Proof. Let ∇^0 and ∇^1 be two connections on ξ and let $pr : \mathbb{R} \times M \to M$ denote the projection. Let $\tilde{\nabla}^0 = pr^*\nabla^0$ and $\tilde{\nabla}^1 = pr^*\nabla^1$ be the induced connections on $pr^*\xi$. On $pr^*\xi$ we consider the connection $\tilde{\nabla}$ defined by

$$(\tilde{\nabla}s)(t,x) = (1-t)(\tilde{\nabla}^0 s)(t,x) + t(\tilde{\nabla}^1 s)(t,x)$$

for $(t,x) \in \mathbb{R} \times M$. From Lemma 2.2.2 we have $j_0^*\tilde{\nabla} = \nabla^0$ and $j_1^*\tilde{\nabla} = \nabla^1$, where j_0, $j_1 : M \to \mathbb{R} \times M$ are the inclusions $j_0(x) = (0,x)$ and $j_1(x) = (1,x)$. Moreover, $F^{\nabla^0} = j_0^*(F^{\tilde{\nabla}})$ and $F^{\nabla^1} = j_1^*(F^{\tilde{\nabla}})$. Therefore,

$$[P(F^{\nabla^0})] = [j_0^*(P(F^{\tilde{\nabla}}))] = j_0^*[P(F^{\tilde{\nabla}})] = j_1^*[P(F^{\tilde{\nabla}})] = [j_1^*(P(F^{\tilde{\nabla}}))] = [P(F^{\nabla^1})]$$

by homotopy invariance. □

It follows from Propositions 2.4.2 and 2.4.3 that if $\xi = (E,p,M)$ is a complex smooth vector bundle of rank n over a smooth manifold M, then for every invariant homogeneous complex polynomial P on n^2 variables of degree k there is a well defined cohomology class in $H^{2k}(M;\mathbb{C})$. If $\xi' = (E',p',M)$ is another complex smooth vector bundle isomorphic to ξ and $f : E' \to E$ is a smooth vector bundle isomorphism, then for every connection ∇ on ξ we can choose a connection ∇' on ξ' such that the following diagram commutes.

$$\begin{array}{ccc} \Omega^0(\xi') & \xrightarrow{\nabla'} & \Omega^1(\xi') \\ \downarrow f^* & & \downarrow f^* \\ \Omega^0(\xi) & \xrightarrow{\nabla} & \Omega^1(\xi) \end{array}$$

Then, the local matrices of F^{∇} and $F^{\nabla'}$ with respect to suitable local frames coincide and thus $P(F^{\nabla}) = P(F^{\nabla'})$, since P is invariant. More generally, if $f : N \to M$ is a smooth map and P is an invariant homogeneous polynomial, then for every connection ∇ on ξ we have $f^*(P(F^{\nabla})) = P(F^{f^*\nabla})$. This means that the correspondence which sends each isomorphism class of complex vector bundles over M to the cohomology class in $H^*(M;\mathbb{C})$ defined by P is a natural transformation from the K-functor to the cohomology functor $H^*(.;\mathbb{C})$.

For every $k \in \mathbb{Z}^+$ we define by

$$c_k(\xi) = \left[\sigma_k\left(\frac{-1}{2\pi i}F^{\nabla}\right)\right] \in H^{2k}(M;\mathbb{C})$$

the k-Chern class of ξ and by

$$ch_k(\xi) = \left[\frac{1}{k!}s_k\left(\frac{-1}{2\pi i}F^{\nabla}\right)\right] \in H^{2k}(M;\mathbb{C})$$

the k-Chern character of ξ. From the above, the definitions are independent of the choice of the connection ∇ on ξ. Obviously, $c_0(\xi) = 1$ and $ch_0(\xi) = n$. The Newton identities imply that $ch_k(\xi)$ is a polynomial function of $c_0(\xi)$,..., $c_k(\xi)$.

Examples 2.4.4 (a) Let M be a smooth manifold and let $\xi = (L, p, M)$ be a smooth complex line bundle over M. Then, $\Omega^2(\mathrm{Hom}(\xi,\xi) \cong A^2(M;\mathbb{C})$. Thus, if ∇ is a connection on ξ, then $F^{\nabla} \in A^2(M;\mathbb{C})$ and

$$s_k(F^{\nabla}) = F^{\nabla} \wedge \cdots \wedge F^{\nabla} \quad k\text{-times}.$$

Since $\sigma_1(F^{\nabla}) = F^{\nabla}$, it follows that

$$ch_k(\xi) = \frac{1}{k!}c_1(\xi)^k.$$

(b) We shall compute the first Chern class $c_1(\gamma_1)$ of the tautological complex line bundle $\gamma_1 = (\mathcal{H}_1, p, \mathbb{C}P^1)$ over $\mathbb{C}P^1 \approx S^2$. Since the integration

$$\int_{\mathbb{C}P^1} : H^2(\mathbb{C}P^1;\mathbb{C}) \to \mathbb{C}$$

is an isomorphism, by Poincaré duality, it suffices to calculate the integral

$$\int_{\mathbb{C}P^1} c_1(\gamma_1).$$

We use the connection ∇ of Example 2.1.3 and the calculations therein according to which if $\{(U_0,\phi_0),(U_1,\phi_1)\}$ is the canonical atlas of $\mathbb{C}P^1$, then

$$F^{\nabla}|_{U_0} = \frac{1}{(1+|z|^2)^2}d\overline{z} \wedge dz = \frac{2i}{(1+x^2+y^2)^2}dx \wedge dy$$

where $z = x + iy$. Since $\mathbb{C}P^1 \setminus U_0$ is a singleton, we have

$$\int_{\mathbb{C}P^1} F^{\nabla} = 2i\int_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^2}dxdy = 2i\int_0^{2\pi}\int_0^{+\infty} \frac{r}{(1+r^2)^2}drd\theta = 2\pi i.$$

Since $\sigma_1(F^\nabla) = F^\nabla$, it follows that

$$\int_{\mathbb{C}P^1} c_1(\gamma_1) = \int_{\mathbb{C}P^1} \left(\frac{-1}{2\pi i}\right) F^\nabla = -1.$$

In particular γ_1 is not trivial.

(c) In the Newton identities we see that the coefficient of σ_n in s_n is $(-1)^{n-1}n$. Let now ξ be a smooth complex vector bundle of rank n such that $c_k(\xi) = 0$ for $1 \leq k \leq n-1$. In this case the Newton identities imply that the n-Chern character of ξ is

$$ch_n(\xi) = \frac{1}{n!}(-1)^{n-1} n c_n(\xi) = \frac{(-1)^{n-1}}{(n-1)!} c_n(\xi).$$

In particular this holds for every smooth complex vector bundle ξ of rank n over the $2n$-dimensional sphere S^{2n}.

The following proposition is useful in calculations.

Proposition 2.4.5. *If ξ_1 and ξ_2 are two smooth complex vector bundles over a smooth manifold M, then*

(a) $ch_k(\xi_1 \oplus \xi_2) = ch_k(\xi_1) + ch_k(\xi_2)$ and

(b) $c_k(\xi_1 \oplus \xi_2) = \sum_{j=0}^{k} c_j(\xi_1) \wedge c_{k-j}(\xi_2)$.

Proof. We take connections ∇^1 and ∇^2 on ξ_1 and ξ_2, respectively. Then,

$$\nabla^1 \oplus \nabla^2 : \Omega^0(\xi_1) \oplus \Omega^0(\xi_2) \cong \Omega^0(\xi_1 \oplus \xi_2) \to \Omega^1(\xi_1 \oplus \xi_2) \cong \Omega^1(\xi_1) \oplus \Omega^0(\xi_2)$$

is a connection on $\xi_1 \oplus \xi_2$ with curvature form

$$F^{\nabla^1} \oplus F^{\nabla^2} \in \Omega^2(\mathrm{Hom}(\xi_1 \oplus \xi_2, \xi_1 \oplus \xi_2)).$$

So,

$$ch_k(\xi_1 \oplus \xi_2) = \left[\frac{1}{k!} s_k\left(\frac{-1}{2\pi i}\mathrm{diag}(F^{\nabla^1}, F^{\nabla^2})\right)\right] = ch_k(\xi_1) + ch_k(\xi_2).$$

This proves (a) and (b) follows in the same way. □

Let $\xi = (E, p, M)$ be a complex smooth vector bundle of rank n over a smooth manifold M. Let $I_n^*(\mathbb{C})$ be the commutative graded algebra of invariant homogeneous complex polynomials. More precisely, we set $I_n^{2k+1}(\mathbb{C}) = 0$ and let $I_n^{2k}(\mathbb{C})$ be the space of invariant homogeneous polynomials of degree k. For each $P \in I_n^*(\mathbb{C})$ let $\phi_\xi(P) \in H^*(M; \mathbb{C})$ denote the cohomology class defined by P as above choosing any connection on ξ. In this way we have a well defined homomorphism of graded algebras $\phi_\xi : I_n^*(\mathbb{C}) \to H^*(M; \mathbb{C})$, which is called the Chern-Weil homomorphism for the complex vector bundle ξ. The subalgebra $\phi_\xi(I_n^*(\mathbb{C}))$ of $H^*(M; \mathbb{C})$ is called the Chern algebra of ξ and is generated (as an algebra) by the set of the Chern classes

$$c_k(\xi) = \left(\frac{-1}{2\pi i}\right) \phi_\xi(\sigma_k), \quad k \in \mathbb{Z}^+$$

of ξ, by Theorem 2.3.3.

2.5 The Pfaffian polynomial

Let $n \in \mathbb{N}$ and let $\mathfrak{so}(2n, \mathbb{R})$ denote the Lie algebra of the special orthogonal group $SO(2n, \mathbb{R})$, which consists of the skew-symmetric $2n \times 2n$ real matrices. If $A = (A_{kl}) \in \mathfrak{so}(2n, \mathbb{R})$, we let

$$\omega(A) = \sum_{k<l} A_{kl} e_k^* \wedge e_l^*$$

where $\{e_1^*, ..., e_{2n}^*\}$ is the dual of the canonical basis $\{e_1, ..., e_{2n}\}$ of $\mathbb{R}^{2n}$, and define $\mathrm{Pf}(A)$ by the equality

$$\omega(A) \wedge \cdots \wedge \omega(A) = n!\mathrm{Pf}(A) \cdot e_1^* \wedge \cdots \wedge e_{2n}^*.$$

It is obvious that $\mathrm{Pf}(A)$ is a homogeneous polynomial of degree n of the $2n^2 - n$ real variables A_{kl}, $1 \leq k < l \leq n$ and is called the Pfaffian polynomial. Explicitly,

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} (\mathrm{sgn}\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2n-1)\sigma(2n)}.$$

Example 2.5.1. Let a_1,..., $a_n \in \mathbb{R}$ and $A \in \mathfrak{so}(2n, \mathbb{R})$ be the matrix with the 2×2 blocks

$$\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_n \\ -a_n & 0 \end{pmatrix}$$

along the diagonal and zeros elsewhere. Then,

$$\omega(A) = a_1 e_1^* \wedge e_2^* + \cdots + a_n e_{2n-1}^* \wedge e_{2n}^*$$

and thus

$$\omega(A) \wedge \cdots \wedge \omega(A) = n! a_1 \cdots a_n e_1^* \wedge \cdots \wedge e_{2n}^*.$$

So in this case $\mathrm{Pf}(A) = a_1 \cdots a_n$. Note that $(\mathrm{Pf}(A))^2 = \det A$. We shall generalize this property of the Pfaffian for every element of $\mathfrak{so}(2n, \mathbb{R})$. We shall need the following.

Lemma 2.5.2. *If $A = (A_{kl}) \in \mathfrak{so}(2n, \mathbb{R})$ and $B \in \mathbb{R}^{2n \times 2n}$, then*

$$\mathrm{Pf}(BAB^T) = \mathrm{Pf}(A) \cdot \det B.$$

Proof. Let $B = (B_{kl})$ and let $u_l = Be_l$. From the equalities

$$\sum_{k<l} A_{kl} u_k^* \wedge u_l^* = \sum_{k<l} \sum_{\mu,\nu} B_{\nu k} A_{kl} B_{\mu l} e_\nu^* \wedge e_\mu^* = \sum_{k<l} \sum_{\nu<\mu} (BAB^T)_{\nu\mu} e_\nu^* \wedge e_\mu^* = \omega(BAB^T)$$

follows that

$$\omega(BAB^T) \wedge \cdots \wedge \omega(BAB^T) = \left(\sum_{k<l} A_{kl} u_k^* \wedge u_l^* \right) \wedge \cdots \wedge \left(\sum_{k<l} A_{kl} u_k^* \wedge u_l^* \right)$$

$$= n!\mathrm{Pf}(A) \cdot u_1^* \wedge \cdots \wedge u_n^* = n!\mathrm{Pf}(A) \cdot (\det B) \cdot e_1^* \wedge \cdots \wedge e_n^*. \quad \square$$

Corollary 2.5.3. *The Pffafian polynomial is invariant under the action of* $SO(2n,\mathbb{R})$ *by conjugation.*

If $A \in \mathfrak{so}(2n,\mathbb{R})$, then A is normal as a complex matrix and by the Spectral Theorem there exists an orthonormal basis $\{e_1, e_2, ..., e_{2n}\}$ of $\mathbb{C}^{2n}$ with respect to the usual hermitian product consisting of eigenvectors of A. Let λ_1, λ_2,..., $\lambda_{2n} \in \mathbb{C}$ be the corresponding eigenvalues. Since A is real, $\bar{\lambda}_1$, $\bar{\lambda}_2$,..., $\bar{\lambda}_{2n}$ are also eigenvalues with corresponding eigenvectors $\bar{e}_1$, $\bar{e}_2$,..., $\bar{e}_{2n}$ and since A is skew-symmetric, λ_1, λ_2,..., $\lambda_{2n} \in i\mathbb{R}$. It is possible to arrange this orthonormal basis so that $e_{2k} = \bar{e}_{2k-1}$ for all $1 \leq k \leq n$. This is trivial, if $A = 0$. If $A \neq 0$ and $\lambda_1 \neq 0$, we have $A\bar{e}_1 = \bar{\lambda}_1\bar{e}_1 = -\lambda_1\bar{e}_1$ and e_1, $\bar{e}_1$ are orthogonal. So, we may take $\lambda_2 = -\lambda_1$ and $e_2 = \bar{e}_1$. Inductively now, if H is the linear subspace of $\mathbb{C}^{2n}$ with basis $\{e_1, \bar{e}_1\}$, then H, $H^\perp$ and $\overline{H}$ are A-invariant and we can repeat this for the restriction of A on $H^\perp$.

Theorem 2.5.4. $(\mathrm{Pf}(A))^2 = \det A$ *for every* $A \in \mathfrak{so}(2n,\mathbb{R})$.

Proof. Since A is skew-symmetric, it has eigenvalues

$$\lambda_1, \lambda_2 = -\lambda_1, ..., \lambda_{2n-1}, \lambda_{2n} = -\lambda_{2n-1} \in i\mathbb{R}$$

and corresponding eigenvectors

$$e_1, e_2 = \bar{e}_1, ..., e_{2n-1}, e_{2n} = \bar{e}_{2n-1} \in \mathbb{C}^{2n}$$

which comprise an orthonormal basis of $\mathbb{C}^{2n}$. Putting

$$v_k = \frac{1}{\sqrt{2}}(e_{2k-1} + e_{2k}) \quad \text{and} \quad w_k = \frac{1}{i\sqrt{2}}(e_{2k-1} - e_{2k}), \quad 1 \leq k \leq n$$

we get an orthonormal basis of $\mathbb{R}^{2n}$. If $a_k = -i\lambda_{2k-1}$, then $Av_k = -a_k w_k$ and $Aw_k = a_k v_k$. This means that there exists $g \in O(2n,\mathbb{R})$ such that gAg^{-1} is the matrix with the 2×2 blocks

$$\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_n \\ -a_n & 0 \end{pmatrix}$$

along the diagonal and zeros everywhere else. From Example 2.5.1 and Lemma 2.5.2, we have on the one hand

$$(\mathrm{Pf}(gAg^{-1}))^2 = (a_1 \cdots a_n)^2 = \det A$$

and on other other hand

$$(\mathrm{Pf}(gAg^{-1}))^2 = (\mathrm{Pf}(gAg^T))^2 = (\mathrm{Pf}(A))^2(\det A)^2 = (\mathrm{Pf}(A))^2. \quad \square$$

If $A \in \mathfrak{su}(n,\mathbb{C})$, then $A = -\overline{A}^T$ and from it we get an element $A_\mathbb{R} \in \mathfrak{so}(2n,\mathbb{R})$.

Corollary 2.5.5. *If* $A \in \mathfrak{su}(n,\mathbb{C})$, *then* $\mathrm{Pf}(A_\mathbb{R}) = i^n \det A$.

Proof. Since A is normal, there exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A. Thus, we may assume that $A = \mathrm{diag}(ia_1, ..., ia_n)$, for some a_1,..., $a_n \in \mathbb{R}$. Since ia_k corresponds to the 2×2 block

$$\begin{pmatrix} 0 & -a_k \\ a_k & 0 \end{pmatrix}$$

from Example 2.5.1 we have $\mathrm{Pf}(A_{\mathbb{R}}) = (-1)^n a_1 \cdots a_n$ and on the other hand $\det A = i^n a_1 \cdots a_n$. The conclusion follows now from Lemma 2.5.2. □

2.6 The Euler class

Let $\xi = (E, p, M)$ be a smooth real vector bundle of rank n over a smooth manifold M. A smooth inner product $\langle , \rangle$ on ξ induces a bilinear map

$$\langle , \rangle : \Omega^k(\xi) \times \Omega^l(\xi) \to A^{k+l}(M)$$

defined by $\langle \omega_1 \otimes s_1, \omega_2 \otimes s_2 \rangle = \langle s_1, s_2 \rangle \omega_1 \wedge \omega_2$.

A connection ∇ on ξ is said to be compatible with the inner product (or a metric connection with respect to $\langle , \rangle$) if

$$d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle$$

for every s_1, $s_2 \in \Omega^0(\xi)$.

Let $U \subset M$ be an open set over which ξ is trivial and let $\{e_1, ..., e_n\}$ be an orthonormal frame on U. Let $A = (A_{kl})$ be the connection form with respect to this frame. Then,

$$0 = d\langle e_k, e_l \rangle = \langle \sum_{j=1}^{n} A_{jk} \otimes e_j, e_l \rangle + \langle e_k, \sum_{j=1}^{n} A_{jl} \otimes e_j \rangle$$

$$= \sum_{j=1}^{n} A_{jk} \langle e_j, e_l \rangle + \sum_{j=1}^{n} A_{jl} \langle e_k, e_j \rangle = A_{lk} + A_{kl}.$$

Thus, the connection form A is skew-symmetric and an easy calculation shows that the converse is also true. More precisely, if the connection form A of ∇ on U with respect to an orthonormal frame is skew-symmetric, then the restriction of ∇ on U is a metric connection. The curvature form F^{∇} is also skew-symmetric, since on U it is given by the formula $F^{\nabla}|_U = dA + A \wedge A$.

We note that if $\{f_j : j \in J\}$ is a smooth partition of unity on the base space M and $\{\nabla^j : j \in J\}$ is a family of connections on ξ, then

$$\nabla = \sum_{j \in J} f_j \nabla^j$$

is a connection on ξ. Moreover, if each ∇^j is a metric connection with respect to the same inner product on ξ for every $j \in J$, then ∇ is also a metric connection.

Using smooth partitions of unity one can construct connections which are compatible with a given inner product on ξ. Indeed, let $\mathcal{U}$ be an open cover of M

consisting of open sets over which ξ is trivial. For $U \in \mathcal{U}$ we choose an orthonormal frame $\{e_1, ..., e_n\}$ on U. On U we consider the connection ∇^U defined by the formula

$$\nabla^U_X \left(\sum_{k=1}^{n} \phi_k e_k \right) = \sum_{k=1}^{n} d\phi_k(X) e_k$$

for every smooth vector field X on U. Then, ∇^U is compatible with the inner product. If $\{f_U : U \in \mathcal{U}\}$ is a smooth partition of unity subordinated to $\mathcal{U}$, then

$$\nabla = \sum_{U \in \mathcal{U}} f_U \nabla^U$$

is a connection on ξ compatible with the inner product.

The real vector bundle ξ of rank n is called orientable if there exists an open cover $\mathcal{U}$ of its base space M such that ξ is trivial over each element of $\mathcal{U}$ and for any U, $V \in \mathcal{U}$ such that $U \cap V \neq \varnothing$ and there are trivializations h_U, h_V of ξ over U and V, respectively, such that

$$(h_U \circ h_V^{-1})(x, v) = (x, g_{UV}(x)v)$$

for every $x \in U \cap V$ and $v \in \mathbb{R}^n$, where $g_{UV} : U \cap V \to SO(n, \mathbb{R})$ is a smooth map. Applying the Gram-Schmidt orthogonalization method, it is always possible to find such an open cover with the corresponding maps g_{UV} taking values in $O(n, \mathbb{R})$. The bundle is orientable if g_{UV} take values in the connected component of the identity of $O(n, \mathbb{R})$.

We shall assume now that the rank of ξ is even and equal to $2n$. Then, $\mathrm{Pf}(F^\nabla|_U)$ is a smooth $2n$-form on U, which depends on the choice of the initial orthonormal frame on U. If we choose another orthonormal frame on U, then the curvature form with respect to the new frame is $B \cdot (F^\nabla|_U) \cdot B^{-1}$, where $B : U \to O(2n, \mathbb{R})$ is some smooth map. It follows from Lemma 2.5.2 that the Pfaffian of the curvature form with respect to the new frame is $\pm\mathrm{Pf}(F^\nabla|_U)$, assuming that U is connected. Thus, in case ξ is orientable, we have a well defined global smooth $2n$-form $\mathrm{Pf}(F^\nabla)$ on M, for which the proof of Proposition 2.4.2 works and shows that it is closed. We shall prove in the sequel that its cohomology class does not depend on the choices of the metric connection and the initial inner product.

Lemma 2.6.1. *Let j_0, $j_1 : M \to \mathbb{R} \times M$ be the inclusions with $j(x) = (0, x)$ and $j(x) = (1, x)$ and $pr : \mathbb{R} \times M \to M$ the projection. If g_0, g_1 are two inner products on ξ and ∇^0 a connection compatible with g_0 and ∇^1 a connection compatible with g_1, then there exists an inner product g on $pr^*\xi$ and a connection ∇ compatible with g such that $j_0^* g = g_0$, $j_1^* g = g_1$ and $j_0^* \nabla = \nabla^0$, $j_1^* \nabla = \nabla^1$.*

Proof. Let $\{f_0, f_1\}$ be smooth partition of unity subordinated to the open cover

$$\{(-\infty, \frac{3}{4}) \times M, (\frac{1}{4}, +\infty) \times M\}$$

of $\mathbb{R} \times M$. Then,

$$g = f_0 pr^* g_0 + f_1 pr^* g_1$$

is an inner product on $pr^*\xi$ such that $j_0^*g = g_0$ and $j_1^*g = g_1$. Now $pr^*\nabla^0$ is a connection which is compatible with g only on $(-\infty, \frac{1}{4}) \times M$ and $pr^*\nabla^1$ is compatible with g on $(\frac{3}{4}, +\infty) \times M$. Taking any connection $\tilde{\nabla}$ on M which is compatible with g, we can glue these three connections using a smooth partition of unity subordinated to the open cover

$$\{(-\infty, \frac{1}{4}) \times M, (\frac{1}{8}, \frac{7}{8}) \times M, (\frac{3}{4}, +\infty) \times M\}$$

of $\mathbb{R} \times M$ with the required properties. □

Corollary 2.6.2. *The cohomology class of* $\mathrm{Pf}(F^\nabla)$ *in* $H^{2n}(M;\mathbb{R})$ *does not depend on the choices of the inner product and the compatible connection* ∇ *on* ξ.

Proof. Let g_0, ∇^0 and g_1, ∇^1 be two choices of inner products and compatible connections on ξ. Applying the preceding Lemma 2.6.1 and using the same notations, there exists an inner product g on $pr^*\xi$ and a compatible connection such that $j_0^*(F^\nabla) = F^{\nabla^0}$ and $j_1^*(F^\nabla) = F^{\nabla^1}$. Hence $j_0^*(\mathrm{Pf}(F^\nabla)) = \mathrm{Pf}(F^{\nabla^0})$ and $j_1^*(\mathrm{Pf}(F^\nabla)) = \mathrm{Pf}(F^{\nabla^1})$. By homotopy invariance, the cohomology classes of these two closed $2n$-forms coincide. □

If $\xi = (E, p, M)$ ia a smooth orientable real vector bundle of rank $2n$ over a smooth manifold M, then the cohomology class

$$e(\xi) = \left[\mathrm{Pf}\left(\frac{F^\nabla}{2\pi}\right)\right] \in H^{2n}(M;\mathbb{R})$$

is called the Euler class of ξ.

Example 2.6.3. A connection ∇ on the cotangent bundle T^*M of a smooth manifold M of any dimension n is said to be symmetric if the composition

$$\Omega^0(T^*M) = A^1(M) \xrightarrow{\nabla} \Omega^1(T^*M) = A^1(M) \otimes_{C^\infty(M)} A^1(M) \xrightarrow{\wedge} A^2(M)$$

coincides with the exterior derivation d.

On a local chart $(U; x^1, ..., x^n)$ of M there are smooth functions $\Gamma^j_{kl} : U \to \mathbb{R}$ such that

$$\nabla(dx^j) = \sum_{k,l=1}^{n} \Gamma^j_{kl} dx^k \otimes dx^l, \quad 1 \le j \le n,$$

which are traditionally called the Christoffel symbols. If ∇ is symmetric, we have

$$\sum_{k,l=1}^{n} \Gamma^j_{kl} dx^k \wedge dx^l = d(dx^j) = 0$$

and therefore $\Gamma^j_{kl} = \Gamma^j_{lk}$ for all $1 \le j, k, l \le n$.

More generally, for every $f \in C^\infty(M)$ we can compute on U that

$$\nabla(df) = \sum_{k,l=1}^{n} \left(\frac{\partial^2 f}{\partial x^k \partial x^l} + \sum_{j=1}^{n} \Gamma^j_{kl} \frac{\partial f}{\partial x^j}\right) dx^k \otimes dx^l.$$

If ∇ is symmetric, then the coefficient of $dx^k \otimes dx^l$ is symmetric with respect to the indices k, l. The converse is also true.

A Riemannian metric on M is a (smooth) inner product on TM and gives rise to a natural smooth vector bundle isomorphism $T^*M \cong TM$ by the use of which we can transfer the inner product to T^*M. For every Riemannian metric on M there exists a unique symmetric connection on T^*M which is compatible with the inner product and is called the Levi-Civita connection of the Riemannian metric. This can be proved in our context as follows. It suffices to prove that for every local chart $(U; x^1, ..., x^n)$ of M and every orthonormal frame $\{\theta_1, ..., \theta_n\}$ of T^*M on U there exists a unique skew-symmetric matrix (A_{kl}) of smooth 1-forms on U such that

$$d\theta_l = \sum_{k=1}^{n} A_{kl} \wedge \theta_k, \quad 1 \leq l \leq n,$$

because the local formulas

$$\nabla\theta_l = \sum_{k=1}^{n} A_{kl} \otimes \theta_k, \quad 1 \leq l \leq n,$$

define a symmetric metric connection on U which is actually defined globally on M by uniqueness. Indeed, there are smooth functions $A_{klj} : U \to \mathbb{R}$ such that

$$d\theta_j = \sum_{k,l=1}^{n} A_{klj}\theta_k \wedge \theta_l.$$

If we take

$$B_{klj} = \frac{1}{2}[A_{klj} + A_{lkj} - A_{jkl} - A_{jlk} + A_{ljk} + A_{kjl}]$$

and

$$C_{klj} = \frac{1}{2}[A_{klj} - A_{lkj} + A_{jkl} - A_{jlk} - A_{ljk} - A_{kjl}]$$

then B_{klj} is symmetric with respect to k, l and C_{klj} is skew-symmetric with respect to l, j. Moreover, $A_{klj} = B_{klj} + C_{klj}$ and this decomposition is unique, because if $A_{klj} = B'_{klj} + C'_{klj}$ and B'_{klj}, C'_{klj} have the same symmetry properties as B_{klj} and C_{klj}, then $D_{klj} = B_{klj} - B'_{klj} = C_{klj} - C'_{klj}$ is at the same time symmetric with respect to k, l and skew-symmetric with respect to l, j, which implies that

$$D_{klj} = D_{lkj} = -D_{ljk} = -D_{jlk} = D_{jkl} = D_{kjl} = -D_{klj}$$

and therefore $D_{klj} = 0$. It follows now that

$$d\theta_j = \sum_{k,l=1}^{n} C_{klj}\theta_k \wedge \theta_l$$

and it suffices to take

$$A_{kl} = \sum_{k,l=1}^{n} C_{jkl}\theta_j, \quad 1 \leq k, l \leq n.$$

Specializing to the case where M is an oriented compact Riemannian 2-manifold, let again $\{\theta_1, \theta_2\}$ be an orthonormal frame of T^*M on U. Then $\theta_1 \wedge \theta_2$ is the restriction to U of the Riemannian volume $\mathrm{vol}(M)$. The corresponding connection form of the Levi-Civita connection is

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

where $\omega \in A^1(U)$. Also, we have the structure equations

$$d\theta_1 = -\omega \wedge \theta_2, \quad d\theta_2 = \omega \wedge \theta_1$$

and the curvature form is

$$F^\nabla|_U = dA + A \wedge A = \begin{pmatrix} 0 & d\omega \\ -d\omega & 0 \end{pmatrix}.$$

Hence, $\mathrm{Pf}(F^\nabla)|_U = d\omega$, which is called the Gauss-Bonnet 2-form of M, and there exists a unique smooth function $K : M \to \mathbb{R}$ such that $\mathrm{Pf}(F^\nabla) = K \cdot \mathrm{vol}(M)$ which is called the Gauss curvature of M. Then,

$$\int_M K\mathrm{vol}(M) = 2\pi \int_M e(T^*M).$$

The Euler class is natural in the sense that if $f : N \to M$ is a smooth map of smooth manifolds and $\xi = (E, p, M)$ is an smooth, orientable real vector bundle of rank $2n$ over M, then

$$e(f^*\xi) = f^*(e(\xi)).$$

Also, if $\xi_1 = (E_1, p_1, M)$ and $\xi_2 = (E_2, p_2, M)$ are two smooth, orientable real vector bundles of even ranks over M, then

$$e(\xi_1 \oplus \xi_2) = e(\xi_1) \wedge e(\xi_2).$$

Both assertions are proved in the same way as the corresponding assertions for Chern classes.

So far in this section we have considered real vector bundles. It is obvious however that the notion of metric connection or hermitian connection can be defined on a smooth complex vector bundle equipped with a hermitian inner product. In the same way as in the real case, it is easy to show that the connection form A of a hermitian connection with respect to an orthonormal local frame is skew-hermitian, that is $A = -\overline{A}^T$.

Let $\xi = (E, p, M)$ be a smooth complex vector bundle of rank n over a smooth manifold M. As a real vector bundle ξ has rank $2n$ and is orientable, because $U(n) \subset SO(2n, \mathbb{R})$, expanding the entries of $U(n)$ to 2×2 real blocks in the usual way. Let h be a smooth hermitian inner product on ξ and let ∇ be a compatible connection. The underlying real vector bundle $\xi_\mathbb{R}$ inherits the real inner product $\mathrm{Re}h$ and a corresponding compatible connection $\nabla^\mathbb{R}$. The connection form A of ∇ with respect to some orthonormal local frame of ξ on an open set $U \subset M$ corresponds to a connection form $A_{\mathbb{R}}$ of $\xi_{\mathbb{R}}$. For instance, if ξ is a complex line bundle, that is $n = 1$, then $A = (i\omega) \in A^1(U;\mathbb{C})^{1\times 1}$ for some smooth 1-form ω on U and

$$A_{\mathbb{R}} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

In case $n = 2$, there are smooth 1-forms ω_1, ω_2 and θ on U such that

$$A = \begin{pmatrix} i\omega_1 & \theta \\ -\overline{\theta} & i\omega_2 \end{pmatrix}.$$

and

$$A_{\mathbb{R}} = \begin{pmatrix} 0 & -\omega_1 & \mathrm{Re}\theta & \mathrm{Im}\theta \\ \omega_1 & 0 & -\mathrm{Im}\theta & \mathrm{Re}\theta \\ -\mathrm{Re}\theta & \mathrm{Im}\theta & 0 & -\omega_2 \\ -\mathrm{Im}\theta & -\mathrm{Re}\theta & \omega_2 & 0 \end{pmatrix}.$$

From Corollary 2.5.5 we have $\mathrm{Pf}(F^{\nabla^{\mathbb{R}}}) = i^n \det(F^{\nabla})$.

Theorem 2.6.4. *If $\xi = (E, p, M)$ is a smooth complex vector bundle of rank n over a smooth manifold M, then $e(\xi_{\mathbb{R}}) = c_n(\xi)$. In particular $c_n(\xi) \in H^{2n}(M;\mathbb{R})$.*

Proof. We compute

$$\mathrm{Pf}\left(\frac{1}{2\pi}F^{\nabla^{\mathbb{R}}}\right) = \left(\frac{i}{2\pi}\right)^n \det(F^{\nabla}) = \left(\frac{i}{2\pi}\right)^n \sigma_n(F^{\nabla}) = \sigma_n\left(\frac{-1}{2\pi i}F^{\nabla}\right). \quad \square$$

Theorem 2.6.5. *Let $\xi = (E, p, M)$ is a smooth orientable real vector bundle of rank $2n$ over a smooth manifold M. If there exists a nowhere vanishing smooth section of ξ, then $e(\xi) = 0$.*

*Proof.*We choose any smooth inner product on ξ. Normalising we may assume that there exists a nowhere vanishing smooth section s of ξ of unit length. There is an open cover $\mathcal{U}$ of M consisting of open sets over which ξ is trivial. Applying the Gram-Schmidt process on each $U \in \mathcal{U}$ we can construct a smooth local orthonormal frame $\{e_1, ..., e_{2n}\}$ such that $e_1 = s|_U$. Using a smooth partition of unity subordinated to $\mathcal{U}$ as in the beginning of this section, we can construct a metric connection ∇ on ξ such that $\nabla s = 0$. The connection form A of ∇ with respect to the orthonormal frame $\{e_1, ..., e_{2n}\}$ on U has zeros in the first column. The same is true for the curvature form $F^{\nabla}|_U = dA + A \wedge A$. This implies that $\mathrm{Pf}(F^{\nabla}) = 0$ and therefore $e(\xi) = 0$. $\square$

Example 2.6.6. As an illustration we shall compute the Euler class of the tangent bundle TS^{2n} of the $2n$-dimensional sphere using the Levi-Civita connection of the standard euclidean round Riemannian metric $\langle , \rangle$ of constant sectional curvature 1. The curvature in then given by the formula

$$F^{\nabla}_{X,Y}(Z) = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

for every X, Y, $Z \in \Omega^0(TS^{2n})$.

Let $\{v_1, v_2, ..., v_{2n}\}$ be a positively oriented smooth local orthonormal frame of TS^{2n} on $U = S^{2n} \setminus \{e_{n+1}\}$ and $\{v_1^*, v_2^*, ..., v_{2n}^*\}$ be its dual. For every $1 \leq j \leq 2n$ we have

$$F^{\nabla}_{X,Y}(v_j) = \langle Y, v_j \rangle X - \langle X, v_j \rangle Y = \sum_{k=1}^{2n} \Big(\langle X, v_k \rangle \cdot \langle Y, v_j \rangle - \langle X, v_j \rangle \cdot \langle Y, v_k \rangle \Big) v_k$$

$$= \sum_{k=1}^{2n} \big(v_k^* \wedge v_j^* \big)(X, Y) \cdot v_k.$$

Therefore

$$F^{\nabla}|_U = \big(v_k^* \wedge v_j^* \big)_{1 \leq k,j \leq 2n}$$

and on U the Euler class is represented by the smooth closed $2n$-form

$$\mathrm{Pf}\bigg(\frac{F^{\nabla}}{2\pi} \bigg) = \frac{1}{2^n n! (2\pi)^n} \sum_{\sigma \in S_{2n}} (\mathrm{sgn}\sigma) v^*_{\sigma(1)} \wedge v^*_{\sigma(2)} \wedge \cdots \wedge v^*_{\sigma(2n-1)} \wedge v^*_{\sigma(2n)}$$

$$= \frac{(2n)!}{2^n n! (2\pi)^n} \cdot v_1^* \wedge v_2^* \wedge \cdots \wedge v_{2n-1}^* \wedge v_{2n}^*.$$

It follows that

$$\int_{S^{2n}} \mathrm{Pf}\bigg(\frac{F^{\nabla}}{2\pi} \bigg) = \frac{(2n)!}{2^n n! (2\pi)^n} \cdot \mathrm{Vol}(S^{2n}) = \frac{(2n)!}{2^n n! (2\pi)^n} \cdot \frac{2\pi^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} = \frac{(2n)!}{2^{2n} n!} \cdot \frac{2\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}.$$

Since $\Gamma(t) = \dfrac{2^{t-1}}{\sqrt{\pi}} \Gamma(\frac{t}{2}) \Gamma(\frac{t+1}{2})$ for every $t > 0$, taking $t = 2n$ we get

$$\frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} = \frac{2^{2n}\Gamma(n)}{\Gamma(2n)} = \frac{2^{2n} n!}{(2n)!}.$$

Substituting we arrive at

$$\int_{S^{2n}} \mathrm{Pf}\bigg(\frac{F^{\nabla}}{2\pi} \bigg) = 2$$

which means that $e(TS^{2n})$ is twice the standard generator of $H^{2n}(S^{2n}; \mathbb{R})$.

For $n = 1$ it follows from the above that for every Riemannian metric on S^2 with Gauss curvature K of the corresponding Levi-Civita connection we have

$$\int_{S^2} K \mathrm{vol}(S^2) = 2\pi \int_{S^2} e(T^* S^2) = 4\pi.$$

This is the Gauss-Bonnet Theorem for the 2-sphere. The Gauss-Bonnet Theorem for the 2-torus $T^2 = S^1 \times S^1$ takes the form

$$\int_{T^2} K \mathrm{vol}(T^2) = 2\pi \int_{T^2} e(T^* T^2) = 0,$$

by Theorem 2.6.5, because T^2 is parallelizable.

We shall conclude this section with the statement and proof of the Gauss-Bonnet Theorem for oriented compact 2-manifolds. Let M be an oriented compact Riemannian 2-manifold with Levi-Civita connection ∇. We shall use the notations of the end of Example 2.6.3. The total space T^1M of the unit tangent bundle of M can be identified with the set L of triples (x, v_1, v_2), where $x \in M$ and (v_1, v_2) is an ordered positively oriented orthonormal basis of T_xM, through the bijection $f : L \to T^1M$ with $f(x, v_1, v_2) = (x, v_1)$. In other words, the unit tangent bundle of M can be identified with the frame bundle of positively oriented orthonormal frames. There is a natural smooth action of S^1 on T^1M defined by the diffeomorphisms $R_\phi : T^1M \to T^1M$ with

$$R_\phi(x, v_1, v_2) = (x, \cos\phi \cdot v_1 + \sin\phi \cdot v_2, -\sin\phi \cdot v_1 + \cos\phi \cdot v_2)$$

for all $e^{i\phi} \in S^1$.

Let $U \subset M$ be an open set which is diffeomorphic to $\mathbb{R}^2$ and let (e_1, e_2) be an ordered positively oriented orthonormal frame on U. Let (θ_1, θ_2) be its dual frame with respect to the Riemannian metric. If $(\hat{e}_1, \hat{e}_2)$ is a second ordered positively oriented orthonormal frame on U with dual frame $(\hat{\theta}_1, \hat{\theta}_2)$, there exists a smooth function $\tau : U \to \mathbb{R}$ such that

$$\hat{e}_1(x) = \cos\tau(x) \cdot e_1(x) + \sin\tau(x) \cdot e_2(x)$$

$$\hat{e}_2(x) = -\sin\tau(x) \cdot e_1(x) + \cos\tau(x) \cdot e_2(x)$$

and correspondingly

$$\hat{\theta}_1(x) = \cos\tau(x) \cdot \theta_1(x) + \sin\tau(x) \cdot \theta_2(x)$$

$$\hat{\theta}_2(x) = -\sin\tau(x) \cdot \theta_1(x) + \cos\tau(x) \cdot \theta_2(x)$$

for every $x \in U$. Of course $\mathrm{vol}(M)|_U = \theta_1 \wedge \theta_2 = \hat{\theta}_1 \wedge \hat{\theta}_2$.

If A and $\hat{A}$ are the corresponding connection forms on U and

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & \hat{\omega} \\ -\hat{\omega} & 0 \end{pmatrix},$$

then $\hat{\omega} = \omega - d\tau$, by uniqueness, because

$$d\hat{\theta}_1 = -(\omega - d\tau) \wedge \hat{\theta}_2, \quad d\hat{\theta}_2 = (\omega - d\tau) \wedge \hat{\theta}_1.$$

On T^1M we consider the smooth 1-forms ω_1, ω_2 defined by

$$(\omega_j)_{(x,v_1,v_2)}(w) = \langle v_j, p_{*(x,v_1,v_2)}(w)\rangle$$

for $w \in T_{(x,v_1,v_2)}T^1M$, $(x, v_1, v_2) \in T^1M$, $j = 1, 2$, where $\langle , \rangle$ is the Riemannian metric on M and $p : T^1M \to M$ is the unit tangent bundle projection. It is useful to find local expressions of ω_1, ω_2 on $p^{-1}(U)$. The map $h_U : U \times S^1 \to p^{-1}(U)$ defined by

$$h_U(x.e^{i\phi}) = (x, \cos\phi \cdot e_1(x) + \sin\phi \cdot e_2(x), -\sin\phi \cdot e_1(x) + \cos\phi \cdot e_2(x))$$

is a diffeomorphism and $pr = p \circ h_U : U \times S^1 \to U$ is the projection. It follows from the definitions that

$$(h_U)^*\omega_1 = \cos\phi \cdot pr^*\theta_1 + \sin\phi \cdot pr^*\theta_2$$

$$(h_U)^*\omega_2 = -\sin\phi \cdot pr^*\theta_1 + \cos\phi \cdot pr^*\theta_2$$

and therefore

$$(h_U)^*(\omega_1 \wedge \omega_2) = pr^*(\theta_1 \wedge \theta_2)$$

or equivalently

$$\omega_1 \wedge \omega_2|_{p^{-1}(U)} = p^*(\theta_1 \wedge \theta_2) = p^*(\mathrm{vol}(M)|_U).$$

Since U is an arbitrary open subset of M diffeomorphic to $\mathbb{R}^2$, it follows that

$$\omega_1 \wedge \omega_2 = p^*(\mathrm{vol}(M))$$

on T^1M.

Lemma 2.6.7. *There exists a smooth* 1*-form* α *on* T^1M *such that*
(i) $d\omega_1 = -\alpha \wedge \omega_2$ *and* $d\omega_2 = \alpha \wedge \omega_1$*,*
(ii) $d\alpha = p^*(\mathrm{Pf}(F^\nabla))$ *on* T^1M *and*
(iii) α *is invariant under the smooth action of* S^1 *on* T^1M*.*

Proof. Using the above notations, let again $U \subset M$ be an open set which is diffeomorphic to $\mathbb{R}^2$. Differentiating we see that

$$(h_U)^*(d\omega_1) = -(pr^*\omega - d\phi) \wedge (h_U)^*\omega_2, \quad (h_U)^*(d\omega_2) = (pr^*\omega - d\phi) \wedge (h_U)^*\omega_1.$$

If $\hat{h}_U$ is taken from another frame $(\hat{e}_1, \hat{e}_2)$ on U, then

$$(h_U^{-1} \circ \hat{h}_U)(x, e^{i\hat{\phi}}) = (x, \hat{\phi} + \tau(x))$$

and so $d\phi = d\hat{\phi} + d\tau$, from which follows that

$$(h_U^{-1} \circ \hat{h}_U)^*(pr^*\omega - d\phi) = pr^*\hat{\omega} - d\hat{\phi}$$

since $\hat{\omega} = \omega - d\tau$. This means hat there exists a globally defined smooth 1-form α on T^1M such that

$$\alpha|_{p^{-1}(U)} = (h_U^{-1})^*(pr^*\omega - d\phi) = p^*\omega - (h_U^{-1})^*(d\phi)$$

for every open set $U \subset M$ diffeomorphic to $\mathbb{R}^2$. Differentiating

$$d\alpha|_{p^{-1}(U)} = p^*(d\omega) = p^*(\mathrm{Pf}(F^\nabla)|_U).$$

Finally, it is evident from the definitions that

$$(h_U^{-1} \circ R_\beta \circ h_U)(x, e^{i\phi}) = (x, e^{i(\phi+\beta)})$$

from which follows immediately that α is invariant unser the action of S^1. □

The tangent bundle of M is actually a smooth complex line bundle over M, because $U(1) = SO(2,\mathbb{R})$. In section 3.2 we shall generalize the above construction of α to any smooth complex line bundle over a smooth manifold.

Let now $I \subset \mathbb{R}$ be an open interval and $\sigma : I \to M$ be a smooth curve parametrized by arclength. For the lifted smooth curve $\gamma : I \to T^1M$ defined by $\gamma(s) = (\sigma(s), \dot\sigma(s))$ we have $\gamma^*\omega_1 = ds$ and $\gamma^*\omega_2 = 0$. There exists a unique smooth function $\kappa : I \to \mathbb{R}$ such that

$$\gamma^*\alpha = -\kappa(s)ds$$

which is called the geodesic curvature of σ. Locally, on an open set $U \subset M$ diffeomorphic to $\mathbb{R}^2$ with respect to an ordered positively oriented orthonormal frame (e_1, e_2), if $\sigma(I) \subset U$, there exists a smooth function $\phi : I \to \mathbb{R}$ such that $h_U^{-1}(\gamma(s)) = (c(s), e^{i\phi(s)})$ for every $s \in I$. The smooth map $e^{i\phi} : I \to S^1$ is the angle between e_1 and $\dot c$ and

$$-\kappa(s)ds = \gamma^*\alpha = (h_U^{-1} \circ \gamma)^*(pr^*\omega - d\phi) = c^*\omega - d\phi$$

as the proof of Lemma 2.6.7 shows.

Theorem 2.6.8. (C.F. Gauss - P.O. Bonnet) *If M is an oriented compact Riemannian 2-manifold with Riemannian volume form* $\mathrm{vol}(M)$ *and Gauss curvature $K : M \to \mathbb{R}$, then*

$$\int_M K \cdot \mathrm{vol}(M) = 2\pi\chi(M).$$

Proof. The assertion has been proved in case M is the 2-torus T^2, by Theorem 2.6.5 (and in the case of the 2-sphere, by Example 2.6.6). Let $V = T^2 \setminus D_1 \cup D_2$, where D_1, $D_2 \subset T^2$ are two disjoint closed discs with smooth boundary. Since T^2 is parallelizable, there exists a global ordered positively oriented orthonormal frame (e_1, e_2) on T^2. If ϕ_j is the angle between e_1 and ∂D_j and κ_j is the geodesic curvature of ∂D_j, $j = 1, 2$, we have

$$\int_V K \cdot \mathrm{vol}(M) = -\int_{T^2\setminus V} K \cdot \mathrm{vol}(M) = -\int_{D_1} K \cdot \mathrm{vol}(M) - \int_{D_2} K \cdot \mathrm{vol}(M)$$

$$= -\int_{D_1} d\omega - \int_{D_2} d\omega = -\int_{\partial D_1} \omega - \int_{\partial D_1} \omega$$

$$= -\int_{\partial D_1} (d\phi - \kappa_1(s))ds - \int_{\partial D_2} (d\phi - \kappa_2(s))ds$$

$$= -2\pi + \int_{\partial D_1} \kappa_1(s)ds - 2\pi + \int_{\partial D_2} \kappa_2(s)ds.$$

Suppose now that the genus of M is $g > 1$. Then,

$$M = V_0 \cup V_1 \cup \cdots \cup V_g \cup V_{g+1}$$

where V_0, V_{g+1} are closed discs with smooth boundaries $\partial V_0 = C_0$, $\partial V_{g+1} = C_{g+1}$, and each V_j is diffeomorphic to V for $1 \leq j \leq g$ with $\partial V_j = C_j \cup C'_j$ so that $C'_j = -C_{j+1}$ homologically, $0 \leq j \leq g$. We have

$$\int_M K \cdot \mathrm{vol}(M) = \sum_{j=0}^{g+1} \int_{V_j} K \cdot \mathrm{vol}(M).$$

If κ_j denotes the geodesic curvature of C_j and κ'_j the geodesic curvature of C'_j, we have

$$\int_{V_0} K \cdot \mathrm{vol}(M) + \int_{V_1} K \cdot \mathrm{vol}(M)$$

$$= 2\pi - \int_{C_0} \kappa_0(s)ds - 4\pi + \int_{C_1} \kappa_1(s)ds + \int_{C'_1} \kappa'_1(s)ds$$

$$= 2\pi - 4\pi - \int_{C_2} \kappa_2(s)ds.$$

Similarly,

$$\int_{V_g} K \cdot \mathrm{vol}(M) + \int_{V_{g+1}} K \cdot \mathrm{vol}(M) = 2\pi - 4\pi + \int_{C_g} \kappa_g(s)ds.$$

For $2 \leq j \leq g-2$ we have

$$\int_{V_j} K \cdot \mathrm{vol}(M) + \int_{V_{j+1}} K \cdot \mathrm{vol}(M)$$

$$= -4\pi + \int_{C_j} \kappa_j(s)ds + \int_{C'_j} \kappa_j(s)ds - 4\pi + \int_{C_{j+1}} \kappa_j(s)ds + \int_{C'_{j+1}} \kappa_j(s)ds$$

$$= -4\pi + \int_{C_j} \kappa_j(s)ds - 4\pi + \int_{C'_{j+1}} \kappa'_j(s)ds.$$

Consequently,

$$\int_M K \cdot \mathrm{vol}(M) = 4\pi - 4\pi g = 2\pi\chi(M). \quad \square$$

In purely topological terms the Gauss-Bonnet Theorem can be stated as follows.

Corollary 2.6.9. *If M is an oriented compact 2-manifold, then*

$$\int_M e(T^*M) = \chi(M). \quad \square$$

2.7 The splitting principle for complex vector bundles

The notion of vector bundle is a special case of the more general notion of fibre bundle. A fibre bundle is a quadruple (E, p, M, F) where E, M and F are topological spaces and $p : E \to M$ is a continuous onto map such that there exists an open cover $\mathcal{U}$ of M consisting of open sets $U \subset M$ for each of which there exists a homeomorphism $h_U : p^{-1}(U) \to U \times F$ such that $pr \circ h_U = p$, where $pr : U \times F \to U$ is the projection. The space E is called the total space, the space M is the base space and F is the fibre. Each homeomorphism like h_U is a local trivialization of the bundle on U. The fibre bundle is said to be smooth if E, B and F are smooth manifolds and $p : E \to M$ is a smooth map. It is obvious from the definitions that a vector bundle is fibre bundle with fibre a vector space and local trivializations which are linear on fibres. The fibre bundle $(M \times F, pr, M, F)$ is the trivial fibre bundle over M with fibre F.

Examples 2.7.1 (a) If $\xi = (E, p, M)$ is a real vector bundle of rank n equipped with an inner product $\langle , \rangle$ and we put $S(\xi) = \{v \in E : \langle v, v\rangle = 1\}$, then $(S, p|_S, M, S^{n-1})$ is a fibre bundle, which is called the corresponding sphere bundle of ξ. Indeed, if $U \subset M$ is an open set over which ξ is trivial, then applying the Gram-Schmidt orthogonalization process to any local frame of ξ on U we obtain a local trivialization of $p|_S$ on U.

(b) Let $\xi = (E, p, M)$ be a (real or complex) vector bundle of rank n and

$$P(\xi) = \{(x, \ell) : x \in M \quad \text{and} \quad \ell \in P(p^{-1}(x))\}$$

where $P(p^{-1}(x))$ denotes the projective space corresponding to the vector space $p^{-1}(x)$. The projection $q : P(\xi) \to M$ with $q(x, \ell) = x$ is a fibre bundle map. The total space is $P(\xi)$, base space M and fibre $\mathbb{R}P^{n-1}$, in case ξ is real or $\mathbb{C}P^{n-1}$, if ξ is a complex vector bundle. This is the projective vector bundle which corresponds to ξ. If the initial vector bundle ξ is smooth, then its corresponding projective fibre bundle is also smooth.

In the case of a vector bundle the total space and the base space have the same homotopy type and actually (a copy of) the base space is a strong deformation retract of the total space. This is not the case in general for fibre bundles. If (E, p, M, F) is a smooth fibre bundle, then on $H^*(E; \mathbb{R})$ one can define an exterior multiplication

$$\cdot : H^*(M; \mathbb{R}) \otimes H^*(E; \mathbb{R}) \to H^*(E; \mathbb{R})$$

by setting $a \cdot e = p^*(a) \wedge e$, for $a \in H^*(M; \mathbb{R})$, $e \in H^*(E; \mathbb{R})$. In this way the cohomology algebra $H^*(E; \mathbb{R})$ of the total space becomes a graded module over the graded cohomology algebra $H^*(M; \mathbb{R})$ of the base space.

Theorem 2.7.2. (J. Leray and G. Hirsch) *Let (E, p, M, F) be a smooth fibre bundle. We assume that $H^*(F; \mathbb{R})$ is a finite dimensional vector space and that there exist n_1,..., $n_k \in \mathbb{N}$ and cohomology classes $e_j \in H^{n_j}(E; \mathbb{R})$, $1 \le j \le k$, such that*

$$\{e_j|_{p^{-1}(x)} : j = 1, 2, ..., k\}$$

is a basis of $H^*(p^{-1}(x);\mathbb{R}) \cong H^*(F;\mathbb{R})$ *for every* $x \in M$. *Then,* $H^*(E;\mathbb{R})$ *is the free* $H^*(M;\mathbb{R})$*-module with basis* $\{e_1, ..., e_k\}$.

Proof. Let $\mathcal{V}$ be an open cover of M consisting of open subsets of M over each of which the fibre bundle is trivial. Let also $\mathcal{U}$ denote the family of all open sets $U \subset M$ such that the assertion is true for $\xi|_U$. By Proposition B.1 in the appendix to this chapter, it suffices to prove the following:

(i) $\varnothing \in \mathcal{U}$.

(ii) If $V \in \mathcal{V}$ and $U \subset V$ is an open subset of M diffeomorphic to $\mathbb{R}^m$, where $m = \dim M$, then $U \in \mathcal{U}$.

(iii) If U_1, $U_2 \in \mathcal{U}$ are such that $U_1 \cap U_2 \in \mathcal{U}$, then $U_1 \cup U_2 \in \mathcal{U}$.

(iv) If $\{U_n : n \in \mathbb{N}\}$ is a countable family of mutually disjoint elements of $\mathcal{U}$, then $\bigcup_{n=1}^{\infty} U_n \in \mathcal{U}$.

The first point is trivially true as well as the second, because $H^*(\mathbb{R}^m \times F;\mathbb{R}) \cong H^*(F;\mathbb{R})$ is a real vector space, hence a free $H^*(\mathbb{R}^m;\mathbb{R}) \cong \mathbb{R}$-module. The fourth point is also clear from the facts

$$H^*(\bigcup_{n=1}^{\infty} U_n;\mathbb{R}) \cong \prod_{n=1}^{\infty} H^*(U_n;\mathbb{R}) \quad \text{and} \quad H^*(p^{-1}(\bigcup_{n=1}^{\infty} U_n);\mathbb{R}) \cong \prod_{n=1}^{\infty} H^*(p^{-1}(U_n);\mathbb{R})$$

and our assumption. The non-trivial point of the proof is (iii) which can be proved using Mayer-Vietoris sequences. For simplicity of notation we denote $E_1 = p^{-1}(U_1)$, $E_2 = p^{-1}(U_2)$ and $E_{12} = p^{-1}(U_1 \cap U_2)$. Let also $U = U_1 \cup U_2$ and $E_U = p^{-1}(U)$. We have the two Mayer-Vietoris long exact sequences

$$\cdots \longrightarrow H^{q-1}(E_{12};\mathbb{R}) \xrightarrow{\delta^*} H^q(E_U;\mathbb{R}) \xrightarrow{I} H^q(E_1;\mathbb{R}) \oplus H^q(E_2;\mathbb{R}) \xrightarrow{\rho} \cdots$$

$$\cdots \longrightarrow H^{q-1}(U_1 \cap U_2;\mathbb{R}) \xrightarrow{\delta^*} H^q(U;\mathbb{R}) \xrightarrow{I} H^q(U_1;\mathbb{R}) \oplus H^q(U_2;\mathbb{R}) \xrightarrow{\rho} \cdots$$

If $\sum_{j=1}^{k} a_j \cdot e_j = 0$ in $H^*(E_U;\mathbb{R})$, where $a_j \in H^*(U;\mathbb{R})$, $1 \leq j \leq k$, then $a_j = 0$, $1 \leq j \leq k$, because this holds in $H^*(E_1;\mathbb{R})$ and $H^*(E_2;\mathbb{R})$.

It remains to prove that for every $e \in H^*(E_U;\mathbb{R})$ there exist $a_j \in H^*(U;\mathbb{R})$, $1 \leq j \leq k$, such that $e = a_1 \cdot e_1 + \cdots + a_k \cdot e_l$ in $H^*(E_U;\mathbb{R})$. If $i_1 : E_1 \to E_U$ and $i_2 : E_2 \to E_U$ are the inclusions, then our assumption implies that $i_1^*(e)$ and $i_2^*(e)$ can be written as

$$i_1^*(e) = \sum_{j=1}^{k} a_j^1 \cdot e_j \quad \text{and} \quad i_2^*(e) = \sum_{j=1}^{k} a_j^2 \cdot e_j.$$

If $g_1 : E_{12} \to E_1$ and $g_2 : E_{12} \to E_2$, it follows by exactness of the first Mayer-Vietoris sequence that

$$\sum_{j=1}^{k} g_1^*(a_j^1) \cdot e_j = \sum_{j=1}^{k} g_2^*(a_j^2) \cdot e_j$$

and therefore $g_1^*(a_j) = g_2^*(a_j)$, $1 \leq j \leq k$. By exactness of the second Mayer-Vietoris sequence, there are $a_j \in H^*(U;\mathbb{R})$, $1 \leq j \leq k$, such that $I(a_j) = (a_j^1, a_j^2)$ for every $1 \leq j \leq k$. Hence

$$I(e - \sum_{j=1}^{k} a_j \cdot e_j) = 0$$

and $e - \sum_{j=1}^{k} a_j \cdot e_j \in \mathrm{Im}\delta^*$, by exactness. Thus, it suffices to prove the assertion in $\mathrm{Im}\delta^*$. This follows from the assumption that it holds on E_{12} and the formula

$$\delta^*(a \cdot i_{12}^*(e)) = \delta^*(a) \cdot e$$

for every $a \in H^*(U;\mathbb{R})$ and $e \in H^*(E_U;\mathbb{R})$, where $i_{12} : E_{12} \to E_U$ is the inclusion. This formula follows immediately from the formula giving the connecting homomorphism δ^* using a smooth partition of unity $\{f_1, f_2\}$ subordinated to the open cover $\{U_1, U_2\}$ of U and the induced partition of unity $\{f_1 \circ p, f_2 \circ p\}$ subordinated to the open cover $\{E_1, E_2\}$ of E_U. □

Of course in the preceding Theorem 2.7.2 we could have used cohomology with complex coefficients. We recall now that for every $n \in \mathbb{N}$ the canonical inclusion $j : \mathbb{C}P^1 \to \mathbb{C}P^n$ with $j[z_0, z_1] = [z_0, z_1, 0, ..., 0]$ induces and isomorphism $j^* : H^2(\mathbb{C}P^n;\mathbb{C}) \to H^2(\mathbb{C}P^1;\mathbb{C})$ (also in cohomology with real or integer coefficients). Actually, if X generates $H^2(\mathbb{C}P^1;\mathbb{C}) \cong \mathbb{C}$, then $(j^*)^{-1}(X)$ generates the cohomology algebra of $\mathbb{C}P^n$. If $\gamma_n = (\mathcal{H}_n, p, \mathbb{C}P^n)$ is the tautological complex line bundle, then $j^*\gamma_n = \gamma_1$. Since the Chern classes are natural, from Example 2.4.4 we conclude that

$$j^*(c_1(\gamma_n)) = c_1(j^*\gamma_n)) = c_1(\gamma_1) = -X \neq 0$$

and hence $c_1(\gamma_n) = -(j^*)^{-1}(X) \neq 0$.

Let $\xi = (E, p, M)$ be a smooth complex vector bundle of rank $n + 1$ and let $(P(\xi), q, M, \mathbb{C}P^n)$ be the corresponding projective fibre bundle of Example 2.7.1(b). There exists a smooth complex line bundle $\zeta = (\mathcal{H}, \tau, P(\xi))$, where

$$\mathcal{H} = \{(x, \ell, v) : (x, \ell) \in P(\xi), v \in \ell\}$$

and $\tau(x, \ell, v) = (x, \ell)$. In case M is a singleton this is just the tautological complex line bundle γ_n over $\mathbb{C}P^n$. We consider any smooth hermitian inner product on ξ. This induces a smooth hermitian inner product on $q^*\xi$ and we have a splitting $q^*\xi \cong \zeta \oplus \zeta^\perp$, where the total space of $\zeta^\perp$ is $\mathcal{H}^\perp = \{(x, \ell, v) : (x.\ell) \in P(\xi), v \in \ell^\perp\}$.

$$\begin{array}{ccc} q^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ P(\xi) & \xrightarrow{q} & M \end{array}$$

Let $e = c_1(\zeta) \in H^2(P(\xi);\mathbb{C})$. Since the restriction of ζ on a fibre $q^{-1}(x)$ is isomorphic to the tautological complex line bundle γ_n, we conclude that $e|_{q^{-1}(x)}$ is (minus) the generator of $H^2(q^{-1}(x);\mathbb{C})$. This implies that the set of cohomology classes

$$\{1, e, ..., e^n\}$$

in $H^*(P(\xi);\mathbb{C})$, where powers are taken with respect to the wedge product, satisfies the assumptions of Theorem 2.7.2. Thus, $H^*(P(\xi);\mathbb{C})$ is the free $H^*(M;\mathbb{C})$-module with basis $\{1, e, ..., e^n\}$. In particular, for every $a \in H^*(M;\mathbb{C})$ we have

$$q^*(a) = q^*(a) \wedge 1 = a \cdot 1 \in H^*(P(\xi);\mathbb{C})$$

and so $q^* : H^*(M;\mathbb{C}) \to H^*(P(\xi);\mathbb{C})$ is a injective.

Theorem 2.7.3. (Splitting Principle) *If $\xi = (E, p, M)$ is a smooth complex vector bundle of rank n, then there exist a smooth manifold N, a proper smooth map $f : N \to M$ and smooth complex line bundles $\xi_j = (E_j, p_j, N)$, $1 \le j \le n$ such that*
(i) $f^* : H^*(M;\mathbb{C}) \to H^*(N;\mathbb{C})$ is injective and
(ii) $f^*\xi \cong \xi_1 \oplus \cdots \oplus \xi_n$.

Proof. Let $(P(\xi), q, M, \mathbb{C}P^{n-1})$ be the corresponding projective fibre bundle and let $\zeta = (\mathcal{H}, \tau, P(\xi))$ be the smooth complex line bundle which was defined above. We have the commutative diagrams

$$\begin{array}{ccc} q^*E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ P(\xi) & \xrightarrow{q} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} q_1^*(\mathcal{H}^\perp) & \longrightarrow & \mathcal{H}^\perp \\ \downarrow & & \downarrow p_1 \\ P(\mathcal{H}^\perp) & \xrightarrow{q_1} & P(\xi) \end{array}$$

and $q_1^*\zeta^\perp$ is isomorphic to the direct sum of a complex line bundle and another complex vector bundle (like $\zeta^\perp$). This implies a splitting

$$(q \circ q_1)^*\xi \cong \xi_1 \oplus \xi_2 \oplus \xi'$$

where $\xi_1 = \zeta$ and ξ_2 are complex line bundles. Moreover, the homomorphisms $q^* : H^*(M;\mathbb{C}) \to H^*(P(\xi);\mathbb{C})$ and $q_1^* : H^*(P(\xi);\mathbb{C}) \to H^*(P(\zeta^\perp);\mathbb{C})$ are injective and hence so is $(q \circ q_1)^*$.

Repeating this construction we get a finite sequence of smooth proper maps

$$P_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} P_1 \xrightarrow{q_1} P_0 = P(\xi) \xrightarrow{q} M$$

such that each q_j induces an injective homomorphism in cohomology and

$$(q \circ q_1 \circ \cdots q_j)^*\xi \cong \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_{j+1} \oplus \xi'$$

for $1 \le j \le n-1$, where ξ_1, ξ_2,..., ξ_{j+1} are smooth complex line bundles. Setting $f = q \circ q_1 \circ \cdots q_{n-1}$ and $N = P_{n-1}$ the assertion follows. □

The combination of the preceding Theorem 2.7.3 with Theorem 2.6.4 yields that the Chern classes of a smooth complex vector bundle are actually real.

Corollary 2.7.4. *If $\xi = (E, p, M)$ is a smooth complex vector bundle over a smooth manifold M, then $c_k(\xi) \in H^{2k}(M;\mathbb{R})$ for every $k \in \mathbb{Z}^+$.* □

Corollary 2.7.5. *If $\xi = (E, p, M)$ is a smooth complex vector bundle of rank n, then $c_k(\xi) = 0$ for $k > n$.* □

In particular, for the tautological complex line bundle γ_n over $\mathbb{C}P^n$ we have $c_k(\gamma_n) = 0$ for $k > 1$. From the Splitting Principle we obtain the following characterization of the Chern classes.

Theorem 2.7.6. *For every smooth manifold M there exists exactly one set consisting of cohomology classes $c_k(\xi) \in H^{2k}(M;\mathbb{R})$, $k \in \mathbb{Z}^+$, for each isomorphic class of smooth complex vector bundles ξ over M with the following properties:*

(i) $\int_{\mathbb{C}P^1} c_1(\gamma_1) = -1$ *and* $c_0(\gamma_n) = 1$, $c_k(\gamma_n) = 0$ *for* $k > 1$ *and for every* $n \in \mathbb{N}$.

(ii) $f^*(c_k(\xi)) = c_k(f^*(\xi))$ *for every smooth map* $f : N \to M$.

(iii) $c_k(\xi_1 \oplus \xi_2) = \sum_{j=0}^{k} c_j(\xi_1) \wedge c_{k-j}(\xi_2)$.

Proof. From what we have proved so far in this and the previous sections only the uniqueness needs proof. Suppose that we have a set of cohomology classes c_k, $k \in \mathbb{Z}^+$, with the properties (i), (ii) and (iii). From (i) we have immediately that $c_1(\gamma_1)$ is the first Chern class of γ_1.

Let now $\xi = (L, p, M)$ be a smooth complex line bundle over M. There exists a smooth complex vector bundle $\tilde{\xi}$ over M such that $\xi \oplus \tilde{\xi} \cong \epsilon_{\mathbb{C}}^{n+1}$. We consider the smooth map $f : M \to \mathbb{C}P^n$ with $f(x) = pr(L_x)$, where $pr : M \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the projection. In the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\hat{f}} & \mathcal{H}_n \\ \downarrow p & & \downarrow \\ M & \xrightarrow{f} & \mathbb{C}P^n \end{array}$$

each $\hat{f}|_{L_x}$ is a linear isomorphism for every $x \in M$, which implies that $f^*(\gamma_n) \cong \xi$ and from property (ii) we have $c_1(\xi) = f^*(c_1(\gamma_n))$ and $c_k(\xi) = 0$ for $k > 1$. These show that properties (i) and (ii) determine uniquely the Chern classes for smooth complex line bundles. Using inductively property (iii), it follows that $c_k(\xi_1 \oplus \cdots \oplus \xi_n)$ is uniquely determined from $c_1(\xi_j)$, $1 \le j \le k$, for every finite family $\xi_1,\dots, \xi_n$ of smooth complex line bundles. From Theorem 2.7.3 it follows immediately that $c_k(\xi)$, $k \in \mathbb{Z}^+$ is uniquely determined for every smooth complex line bundle ξ. $\square$

The total Chern class of a smooth complex vector bundle $\xi = (E, p, M)$ is by definition

$$c(\xi) = \sum_{k=0}^{\infty} c_k(\xi) \in H^*(M;\mathbb{R}).$$

In case ξ is a line bundle, then $c(\xi) = 1 + c_1(\xi)$. If $\xi \cong \xi_1 \oplus \cdots \oplus \xi_n$, where $\xi_1,\dots, \xi_n$ are line bundles, then

$$c(\xi) = \prod_{k=1}^{n} (1 + c_1(\xi_k)) = \sum_{k=0}^{n} \sigma_k(c_1(\xi_1), \dots, c_1(\xi_n))$$

and therefore $c_k(\xi) = \sigma_k(c_1(\xi_1), \dots, c_1(\xi_n))$ for every $k \in \mathbb{Z}^+$.

Analogously, the total Chern character of ξ is defined to be

$$ch(\xi) = \sum_{k=0}^{\infty} ch_k(\xi) \in H^*(M;\mathbb{R})$$

and

$$ch_k(\xi) = \sum_{j=0}^{n} ch_k(\xi_j) = \sum_{j=0}^{n} \frac{1}{k!} c_1(\xi_j)^k,$$

by Proposition 2.4.5(a) and Example 2.4.4(a). Therefore,

$$ch(\xi) = \sum_{j=1}^{n} e^{c_1(\xi_j)}$$

where for $a \in H^2(M;\mathbb{R})$ we have put

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k \in H^*(M;\mathbb{R}).$$

2.8 Pontryagin classes

Let $\xi = (E,p,M)$ be a smooth complex vector bundle of rank n. Recall that from it we derive its conjugate bundle $\overline{\xi}$ and its dual bundle ξ^* which are isomorphic. The Chern classes of ξ and ξ^* are related as follows.

Proposition 2.8.1. *If $\xi = (E,p,M)$ is a smooth complex vector bundle of rank n, then $c_k(\xi^*) = (-1)^k c_k(\xi)$ for every $k \in \mathbb{Z}^+$.*

Proof. There exists a hermitian inner product on ξ and a compatible connection ∇, which is also a connection on $\overline{\xi}$. The connection form A of ∇ with respect to an orthonormal local frame of ξ is skew-hermitian, that is $\overline{A}^T = -A$. The curvature $F^\nabla = dA + A \wedge A$ is also skew-hermitian. An orthonormal local frame of ξ is also orthonormal for $\overline{\xi}$ and the corresponding connection form of ∇ is $\overline{A}$. Thus, the connection form of F^∇ on $\overline{\xi}$ is $\overline{F^\nabla} = -(F^\nabla)^T$. Thus,

$$c_k(\overline{\xi}) = \left[\sigma_k\left(\frac{-1}{2\pi i}\overline{F^\nabla}\right)\right] = \left[\sigma_k\left(\frac{1}{2\pi i}(F^\nabla)^T\right)\right].$$

On the other hand, for every $B \in \mathbb{C}^{n\times n}$ we have

$$\det(I_n - tB^T) = \det(I_n - tB) = \sum_{k=1}^{n} \sigma_k(B)(-t)^k$$

which means that $\sigma_k(-B^T) = (-1)^k \sigma_k(B)$, $1 \leq k \leq n$. Therefore,

$$c_k(\overline{\xi}) = \left[\sigma_k\left(\frac{1}{2\pi i}(F^\nabla)^T\right)\right] = (-1)^k\left[\sigma_k\left(\frac{-1}{2\pi i}F^\nabla\right)\right] = (-1)^k c_k(\xi). \quad \square$$

Let now $\xi = (E, p, M)$ be a smooth real vector bundle of rank n and let $\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}}$ be its complexification. Then,

$$\overline{\xi_{\mathbb{C}}} \cong \xi \otimes_{\mathbb{R}} \overline{\epsilon^1_{\mathbb{C}}} \cong \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}} \cong \xi_{\mathbb{C}},$$

because the map $f : \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}} \to \xi \otimes_{\mathbb{R}} \overline{\epsilon^1_{\mathbb{C}}}$ defined by $f(v \otimes_{\mathbb{R}} z) = v \otimes_{\mathbb{R}} \overline{z}$ is an isomorphism of complex vector bundles since

$$f(i(v \otimes_{\mathbb{R}} z)) = f(v \otimes_{\mathbb{R}} (iz)) = v \otimes_{\mathbb{R}} (-i\overline{z}) = if(v \otimes_{\mathbb{R}} z).$$

Consequently, $(\xi_{\mathbb{C}})^* \cong \overline{\xi_{\mathbb{C}}} \cong \xi \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}} \cong \xi^* \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}} = (\xi^*)_{\mathbb{C}}$ and it follows from Proposition 2.8.1 that

$$c_k(\xi_{\mathbb{C}}) = c_k((\xi^*)_{\mathbb{C}}) = c_k((\xi_{\mathbb{C}})^*) = (-1)^k c_k(\xi_{\mathbb{C}}).$$

Hence $c_k(\xi_{\mathbb{C}}) = 0$, if k is odd.

The cohomology classes

$$p_k(\xi) = (-1)^k c_{2k}(\xi_{\mathbb{C}}) \in H^{4k}(M; \mathbb{R}), \quad k \in \mathbb{Z}^+,$$

are called the Pontryagin classes of the real vector bundle ξ. The total Pontryagin class of ξ is by definition

$$p(\xi) = \sum_{k=0}^{\infty} p_k(\xi) \in H^*(M; \mathbb{R}).$$

If now ξ is a smooth complex vector bundle, then the Pontryagin classes of the underlying real vector bundle and its Chern classes satisfy certain quadratic polynomial equations. To see this, let $p_k = p_k(\xi_{\mathbb{R}})$ and $c_k = c_k(\xi)$. Then, $(\xi_{\mathbb{R}})_{\mathbb{C}} \cong \xi \oplus \xi^*$, by Lemma 1.5.1, and so

$$p_k = (-1)^k c_{2k}(\xi \oplus \xi^*) = (-1)^k \sum_{j=0}^{2k} (-1)^j c_j(\xi) \wedge c_{2k-j}(\xi).$$

If we consider the total classes, we have

$$1 - p_1 + p_2 - \cdots + (-1)^n p_n = (1 + c_1 + c_2 + \cdots + c_n) \wedge (1 - c_1 + c_2 - \cdots + (-1)^n c_n).$$

Specifically, $p_1 = c_1^2 - 2c_2$, $p_2 = c_2^2 - 2c_1c_3 + 2c_4$, etc, where the powers are taken with respect to the wedge product. These polynomial equations can serve as obstructions for a smooth real vector bundle of even rank to admit a complex structure.

Example 2.8.2. We shall calculate the Chern classes of the tangent bundle of the n-dimensional complex projective space $\mathbb{C}P^n$, which is a complex manifold and so its tangent bundle $T\mathbb{C}P^n$ (when $\mathbb{C}P^n$ is considered as a real smooth $2n$-manifold) is a smooth complex vector bundle of rank n. We shall need a generalization of the canonical atlas of $\mathbb{C}P^n$. With the term line we mean a 1-dimensional (complex) linear subspace of $\mathbb{C}^{n+1}$. For each line ℓ let $g_\ell : \mathrm{Hom}(\ell, \ell^\perp) \to \mathbb{C}P^n$ be the map which sends $\phi \in \mathrm{Hom}(\ell, \ell^\perp)$ to its graph. The orthogonal complement $\ell^\perp$ is considered with respect to the usual hermitian inner product and $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{C}}$. Obviously, $g_\ell(0) = \ell$. For instance, if ℓ is the line which is generated by $(1, 0, ..., 0)$, then $\ell^\perp = \{(0, z_1, ..., z_n) : z_1, ..., z_n \in \mathbb{C}\}$ and the map which sends $\phi \in \mathrm{Hom}(\ell, \ell^\perp)$ to $\phi(1, 0, ..., 0)$ establishes an isomorphism $\mathrm{Hom}(\ell, \ell^\perp) \cong \mathbb{C}^n$. Using this identification, we have $g_\ell(\phi) = [1, u_1, ..., u_n]$, where $\phi(1, 0, ..., 0) = (0, u_1, ..., u_n)$. Similarly, if ℓ is generated by $(0, ..., 0, 1, 0, ..., 0)$, using an analogous identification we have

$$g_\ell(\phi) = [u_1, ..., 1, ..., u_n]$$

where $\phi((0, ..., 0, 1, 0, ..., 0) = (u_1, ..., 0, ..., u_n)$. The image of g_ℓ is the set U_ℓ of points in $\mathbb{C}P^n$, which as lines in $\mathbb{C}^{n+1}$ are not orthogonal to ℓ. The pair (U_ℓ, g_l) is a holomorphic chart of $\mathbb{C}P^n$.

Let $\gamma_n^\perp = (\mathcal{H}_n^\perp, p^\perp, \mathbb{C}P^n)$ be the smooth complex vector bundle with total space

$$\mathcal{H}_n^\perp = \{(\ell, u) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : u \in \ell^\perp\}$$

and $p^\perp$ the obvious projection. Then, $\gamma_n \oplus \gamma_n^\perp \cong \epsilon_{\mathbb{C}}^{n+1} \cong \epsilon_{\mathbb{C}}^1 \oplus \cdots \oplus \epsilon_{\mathbb{C}}^1$. Moreover, $\mathrm{Hom}(\gamma_n, \gamma_n^\perp) \cong T\mathbb{C}P^n$. Such a vector bundle isomorphism is for instance the map which restricted on the fibre over $\ell \in \mathbb{C}P^n$ is the complex derivative of g_ℓ at 0. We recall also that $\mathrm{Hom}(\gamma_n, \gamma_n) \cong \epsilon_{\mathbb{C}}^1$, since γ_n is a line bundle. Now we have

$$T\mathbb{C}P^n \oplus \epsilon_{\mathbb{C}}^1 \cong \mathrm{Hom}(\gamma_n, \gamma_n^\perp) \oplus \mathrm{Hom}(\gamma_n, \gamma_n) \cong \mathrm{Hom}(\gamma_n, \gamma_n^\perp \oplus \gamma_n) \cong \mathrm{Hom}(\gamma_n, \epsilon_{\mathbb{C}}^{n+1})$$

$$\cong \mathrm{Hom}(\gamma_n, \epsilon_{\mathbb{C}}^1 \oplus \cdots \oplus \epsilon_{\mathbb{C}}^1) \cong \mathrm{Hom}(\gamma_n, \epsilon_{\mathbb{C}}^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma_n, \epsilon_{\mathbb{C}}^1) \cong \gamma_n^* \oplus \cdots \oplus \gamma_n^*.$$

According to Proposition 2.8.1, the total Chern class of $T\mathbb{C}P^n$ is

$$c(T\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \epsilon_{\mathbb{C}}^1) = c(\gamma_n^*)^{n+1} = (1 - c_1(\gamma_n))^{n+1} = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} c_1(\gamma_n)^k$$

where powers are considered with respect to the wedge product. Hence

$$c_k(T\mathbb{C}P^n) = (-1)^k \binom{n+1}{k} c_1(\gamma_n)^k \neq 0, \quad 0 \leq k \leq n.$$

Example 2.8.3. We can use the calculation of the preceding Example 2.8.2 in order to prove that $\mathbb{C}P^{2n}$ is not the boundary of any compact smooth $(4n+1)$-manifold with boundary for all $n \in \mathbb{N}$. Suppose that there exists a compact smooth $(4n+1)$-manifold M with boundary $\partial M = \mathbb{C}P^{2n}$ and let $j : \mathbb{C}P^{2n} \to M$ be the inclusion. From the existence of collar along ∂M we conclude that

$$T\partial M \oplus \epsilon_{\mathbb{R}}^1 \cong j^*(TM).$$

Complexifying, it follows that

$$((T\mathbb{C}P^{2n})_{\mathbb{R}})_{\mathbb{C}} \oplus \epsilon_{\mathbb{C}}^1 \cong j^*((TM)_{\mathbb{C}}).$$

From Lemma 1.5.1 and the calculations of Example 2.8.2 we have

$$((T\mathbb{C}P^{2n})_{\mathbb{R}})_{\mathbb{C}} \oplus \epsilon^2_{\mathbb{C}} \cong (T\mathbb{C}P^{2n} \oplus \epsilon^1_{\mathbb{C}}) \oplus ((T\mathbb{C}P^{2n})^* \oplus \epsilon^1_{\mathbb{C}})$$
$$\cong \gamma^*_{2n} \oplus \cdots \oplus \gamma^*_{2n} \oplus \gamma_{2n} \oplus \cdots \oplus \gamma_{2n}.$$

The total Chern class is

$$c(((T\mathbb{C}P^{2n})_{\mathbb{R}})_{\mathbb{C}}) = (1 - c_1(\gamma_{2n}))^{2n+1} \wedge (1 + c_1(\gamma_{2n}))^{2n+1} = (1 - (c_1(\gamma_{2n})^2)^{2n+1}$$
$$= \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} (c_1(\gamma_{2n}))^{2k}.$$

If $\omega \in A^{4n}(M)$ represents $c_{2n}((TM)_{\mathbb{C}})$, then

$$[j^*\omega] = j^*(c_{2n}((TM)_{\mathbb{C}})) = c_{2n}(j^*(TM)_{\mathbb{C}}) = (-1)^n \binom{2n+1}{n} (c_1(\gamma_{2n}))^{2n} \neq 0.$$

It follows now from Stokes' formula that

$$0 = \int_M d\omega = \int_{\partial M} j^*\omega \neq 0.$$

This contradiction proves the assertion.

Example 2.8.4. The non-triviality of the Chern or the Pontryagin classes can be used as obstruction to embedding smooth manifolds into euclidean spaces. As an illustration, we consider $\mathbb{C}P^4$. Let X denote the standard generator of $H^2(\mathbb{C}P^4;\mathbb{R})$. The calculation of the preceding Example 2.8.3 gives

$$c(((T\mathbb{C}P^4)_{\mathbb{R}})_{\mathbb{C}}) = (1 - X^2)^5 = 1 - 5X^2 + 10X^4$$

in the cohomology ring $H^*(\mathbb{C}P^4;\mathbb{R})$.

Suppose that $\mathbb{C}P^4$ can be smoothly embedded in $\mathbb{R}^n$, where $n \geq 9$ is a positive integer. There is a normal bundle ξ over $\mathbb{C}P^4$ such that

$$(T\mathbb{C}P^4)_{\mathbb{R}} \oplus \xi \cong T\mathbb{R}^n|_{\mathbb{C}P^4} \cong \epsilon^n_{\mathbb{R}}.$$

From Proposition 2.4.5(b) we obtain $c(((T\mathbb{C}P^4)_{\mathbb{R}})_{\mathbb{C}}) \wedge c(\xi_{\mathbb{C}}) = 1$ and therefore

$$c(\xi_{\mathbb{C}}) = \frac{1}{(1 - X^2)^5} = 1 + 5X^2 + 15X^4$$

in $H^*(\mathbb{C}P^4;\mathbb{R})$. Since $5X^2$ and $15X^4$ are non-zero in $H^4(\mathbb{C}P^4;\mathbb{R})$ and $H^8(\mathbb{C}P^4;\mathbb{R})$, respectively, this implies that ξ must be of rank at least 4. In other words, $\mathbb{C}P^4$ cannot be embedded in $\mathbb{R}^{11}$.

Example 2.8.5. If $\xi = (E, p, M)$ is an orientable real smooth vector bundle of rank $2n$, then from the definitions and Theorem 2.6.4 we have

$$p_n(\xi) = c_{2n}(\xi_{\mathbb{C}}) = e((\xi_{\mathbb{C}})_{\mathbb{R}}) = e(\xi \oplus \xi) = e(\xi)^2.$$

Example 2.8.6. A (complex or real) vector bundle ξ of rank n is said to be stably trivial, if there exists $k \in \mathbb{N}$ such that $\xi \oplus \epsilon^k \cong \epsilon^{n+k}$. For instance the tangent bundle TS^n of the n-sphere is stably trivial for every $n \in \mathbb{N}$, because the normal bundle of S^n in $\mathbb{R}^{n+1}$ is trivial and so $TS^n \oplus \epsilon^1 \cong \epsilon^{n+1}$. From Proposition 2.4.5(b) follows that the Chern classes of a stably trivial smooth complex vector bundle are trivial. Similarly, the Pontryagin classes of a stably trivial real vector bundle are trivial. In particular, the Pontryagin classes of TS^n are trivial.

Example 2.8.7. Using characteristic classes we can prove that the $4k$-dimensional sphere S^{4k}, $k \in \mathbb{N}$, does not admit any almost complex structure. We recall that an almost complex structure on a smooth manifold M is a smooth vector bundle endomorphism $J : TM \to TM$ such that $J^2 = -id$. If M admits an almost complex structure J, then each tangent space T_xM, $x \in M$, becomes a complex vector space and M must be even dimensional. Also, J extends to a smooth vector bundle endomorphism of $(TM)_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \epsilon^1_{\mathbb{C}}$ and there exists a smooth complex vector bundle ξ over M such that $(TM)_{\mathbb{C}} = \xi \oplus \xi^*$. Actually, ξ is the i-eigenspace of J and ξ^* is the $(-i)$-eigenspace of J. Note that $\xi_{\mathbb{R}} \cong TM$.

In case $M = S^{4k}$ the rank of ξ is $2k$ and from the previous Example 2.8.6 we have

$$0 = (-1)^k p_k(TS^{4k}) = c_{2k}(\xi \oplus \xi^*) = \sum_{j=0}^{2k} c_j(\xi) \wedge c_{2k-j}(\xi^*)$$

$$= c_{2k}(\xi^*) + c_{2k}(\xi) = (-1)^{2k} c_{2k}(\xi) + c_{2k}(\xi) = 2c_{2k}(\xi) = 2e(TS^{4k}),$$

by Theorem 2.6.4. Thus, $e(TS^{4k}) = 0$, which contradicts the fact that $e(TS^{4k})$ is twice the standard generator of $H^{4k}(S^{4k}; \mathbb{R})$, as we have calculated in Example 2.6.6.

Appendix

In the proof of Theorem 2.7.2 we have used the following.

Proposition B.1 *Let M be a smooth m-manifold and let $\mathcal{U}$ be a set of open subsets of M with the following properties:*

(i) $\varnothing \in \mathcal{U}$.

(ii) If U is an open subset of M diffeomorphic to $\mathbb{R}^m$, then $U \in \mathcal{U}$.

(iii) If U_1, $U_2 \in \mathcal{U}$ are such that $U_1 \cap U_2 \in \mathcal{U}$, then $U_1 \cup U_2 \in \mathcal{U}$.

(iv) If $\{U_n : n \in \mathbb{N}\}$ is a countable family of mutually disjoint elements of $\mathcal{U}$, then $\bigcup_{n=1}^{\infty} U_n \in \mathcal{U}$.

Then, $M \in \mathcal{U}$.

The proof of Proposition B.1 relies on the following lemma.

Lemma B.2 *With the assumptions of Proposition B.1, let $\{U_n : n \in \mathbb{N}\}$ be a locally finite countable family of open and relatively compact subsets of M such that*

$\bigcap_{j\in J} U_j \in \mathcal{U}$ *for every finite set* $J \subset \mathbb{N}$. *Then,* $\bigcup_{n=1}^{\infty} U_n \in \mathcal{U}$.

Proof. First we show that finite unions of elements of the countable family belong to $\mathcal{U}$. Let $n \in \mathbb{N}$ and $i_1,\dots, i_n \in \mathbb{N}$. We shall show by induction on n that $U_{i_1}\cup\cdots\cup U_{i_n} \in \mathcal{U}$. For $n = 1, 2$ this is true by property (iii) and our assumption (in case J is a singleton). Let $n \geq 3$ and suppose that the assertion holds for finite subfamilies with $n-1$ elements. If $V = U_{1_2} \cup \cdots \cup U_{i_n}$, then

$$U_{i_1} \cap V = \bigcup_{k=2}^{n} U_{i_1} \cap U_{i_k} \in \mathcal{U}$$

from the inductive hypothesis. Moreover, from our assumption (iii) we have

$$U_{i_1} \cup \cdots \cup U_{i_n} = U_{i_1} \cup V \in \mathcal{U}.$$

Since finite unions of elements of the countable family belong to $\mathcal{U}$, for every $n \in \mathbb{N}$ and indices $i_1, j_1,\dots, i_n, j_n \in \mathbb{N}$ we have

$$\bigcup_{k=1}^{n} U_{i_k} \cap U_{j_k} \in \mathcal{U}.$$

Now we define inductively $I_1 = \{1\}$, $W_1 = U_1$ and

$$I_n = \{n\} \cup \{i \in \mathbb{N} : i > n \quad \text{and} \quad U_i \cap W_{n-1} \neq \varnothing\} \setminus \bigcup_{k=1}^{n-1} I_k, \quad W_n = \bigcup_{i\in I_n} U_i,$$

for $n \geq 2$. If I_{n-1} is finite, then W_{n-1} is relatively compact and intersects at most finitely many of the elements of the countable family, since the latter is assumed to be locally finite. Thus, inductively I_n is finite and W_n is relatively compact and belongs to $\mathcal{U}$ for every $n \in \mathbb{N}$. Moreover, $W_n \cap W_{n+1} \in \mathcal{U}$ and $W_n \cap W_k = \varnothing$, if $k > n + 1$, because otherwise there exists some $i \in I_k$ such that $W_n \cap U_i \neq \varnothing$ and thus $i \in I_j$ for some $j \leq n + 1$, contradiction. From property (iv) of $\mathcal{U}$ we have

$$\left(\bigcup_{k=1}^{\infty} W_{2k}\right) \cap \left(\bigcup_{k=1}^{\infty} W_{2k-1}\right) = \bigcup_{n=1}^{\infty} W_n \cap W_{n+1} \in \mathcal{U}$$

and from property (iii) the proof is concluded. □

Proof of Proposition B.1. In the beginning we consider the case where M is an open subset of $\mathbb{R}^m$. Then there exists a locally finite countable open cover of M which consists of open cubes (with edges parallel to the axis) and refines $\mathcal{U}$. Any finite intersection of open cubes is an open cube and thus again diffeomorphic to $\mathbb{R}^m$. From property (ii) and Lemma B.2 follows that $M \in \mathcal{U}$.

In the general case, for every chart (U, ϕ) of M the family

$$\mathcal{U}^{\phi} = \{B \subset \phi(U) : \quad B \text{ is open and } \quad \phi^{-1}(B) \in \mathcal{U}\}$$

has the properties (i), (ii), (iii) and (iv). Hence $\phi(U) \in \mathcal{U}^{\phi}$ and therefore $U \in \mathcal{U}$. Now we take any locally finite countable open cover of M consisting of relatively compact open sets which are domains of charts. Lemma B.2 gives immediately $M \in \mathcal{U}$. □

Chapter 3

Prequantization

3.1 Classification of complex line bundles

In this section we shall describe the smooth complex line bundles over a smooth manifold M in terms of the cohomology of M. Let $\xi = (L, p, M)$ be a smooth complex line bundle and let $\mathcal{U}$ be an open cover of M consisting of open sets U over each of which there is a trivialization $h_U : p^{-1}(U) \to U \times \mathbb{C}$ of ξ. If U, $V \in \mathcal{U}$ are such that $U \cap V \neq \varnothing$, there exists a smooth map $g_{UV} : U \cap V \to \mathbb{C}^\times$, called transition function, such that

$$(h_U \circ h_V^{-1})(x, z) = (x, g_{UV}(x)z)$$

for every $x \in U \cap V$ and $z \in \mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. It is obvious that $g_{VU} = g_{UV}^{-1}$ and $g_{UW} = g_{UV} g_{VW}$, if $U \cap V \cap W \neq \varnothing$.

We can change the local trivializations h_U, $U \in \mathcal{U}$ to new ones $\tilde{h}_U$ on each U so that the new corresponding transition functions take values in S^1 and are

$$\tilde{g}_{UV} = \frac{g_{UV}}{|g_{UV}|}.$$

Indeed, $s_U : U \to L$ defined by $s_U(x) = h_U^{-1}(x, 1)$ is a smooth local section and $g_{UV} s_U(x) = s_V(x)$ for every $x \in U \cap V$. Choosing any hermitiam inner product on ξ and defining $h_U : p^{-1}(U) \to U \times \mathbb{C}$ by

$$\tilde{h}_U\left(z \frac{s_U(x)}{\|s_U(x)\|}\right) = (x, z)$$

for every $z \in \mathbb{C}$, we have

$$(\tilde{h}_U \circ \tilde{h}_V^{-1})(x, z) = \tilde{h}_U\left(z \frac{s_V}{\|s_V(x)\|}\right) = \tilde{h}_U\left(z \frac{g_{UV}(x)}{|g_{UV}(x)|} \cdot \frac{s_U}{\|s_U(x)\|}\right) = \left(x, \frac{g_{UV}(x)}{|g_{UV}(x)|} z\right).$$

On the set of isomorphism classes of complex line bundles over a given smooth manifold M, one can define a group structure induced by the tensor product of complex line bundles. The inverse of the isomorphism class of the complex line bundle $\xi = (L, p, M)$ is represented by its dual bundle $\xi^* \cong \overline{\xi}$. Indeed, there exists an open cover $\mathcal{U}$ of M over the elements of which ξ is trivial such that the corresponding transitions functions g_{UV} for U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$ take values in S^1. Then, $\xi^*|_U$ is also trivial for every $U \in \mathcal{U}$ and the corresponding transition functions are $\overline{g_{UV}}$. Since the transition functions for the tensor product $\xi \otimes \xi^*$ are $g_{UV}\overline{g_{UV}} = 1$, it follows that $\xi \otimes \xi^* \cong \epsilon^1_{\mathbb{C}}$. We shall denote by $\mathrm{Pic}^\infty(M)$ the group of smooth complex line bundles over a smooth manifold M.

We shall call an open cover $\mathcal{U}$ of the smooth manifold M admissible if for any $n \in \mathbb{N}$ and U_1,..., $U_n \in \mathcal{U}$ the set $U_1 \cap \cdots \cap U_n$ is contractible, if non-empty. From the existence of geodesically convex neighbourhoods with respect to a Riemannian metric on M it follows that the set of admissible open covers of M is non-empty and cofinal in the set of all open covers of M. Thus, every open cover of M has an admissible refinement.

If now $\xi = (L, p, M)$ is a smooth complex line bundle and $\mathcal{U}$ is an admissible open cover of M, then $\xi|_U$ is trivial for every $U \in \mathcal{U}$. If U, $V \in \mathbb{U}$ are such that $U \cap V \neq \varnothing$ with transition function $g_{UV} : U \cap V \to S^1$, there exists a smooth function f_{UV} : $U \cap V \to \mathbb{R}$ such that $g_{UV} = e^{2\pi i f_{UV}}$, because $U \cap V$ is contractible. If $U \cap V \cap W \neq \varnothing$, then the relation $g_{UW} = g_{UV} g_{VW}$ implies that $a_{UVW} = f_{VW} - f_{UW} + f_{UV} \in \mathbb{Z}$, since $U \cap V \cap W$ is contractible, hence arcwise connected. Moreover, if U, V, W, $Y \in \mathcal{U}$ are such that $U \cap V \cap W \cap Y \neq \varnothing$, then

$$a_{VWY} - a_{UWY} + a_{UVY} - a_{UVW} = 0.$$

This means that $a = (a_{UVW})$ is a Čech 2-cocycle with respect to the open cover $\mathcal{U}$ with integer coefficients and so defines a Čech cohomology class

$$[a] \in \check{H}^2(\mathcal{U}; \mathbb{Z}) \cong \check{H}^2(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z}),$$

since $\mathcal{U}$ is an admissible open cover of M.

If $f'_{UV} : U \cap V \to \mathbb{R}$ is another set of smooth functions such that

$$g_{UV} = e^{2\pi i f_{UV}} = e^{2\pi i f'_{UV}},$$

then $n_{UV} = f_{UV} - f'_{UV} \in \mathbb{Z}$. If $a' = (a_{UVW})$ is the corresponding Čech 2-cocycle, we see that

$$a_{UVW} = a'_{UVW} + n_{UV} - n_{UW} + n_{VW}.$$

Thus, $a = a' + \delta n$, where $n = (n_{UV})$ and δ is the coboundary operator in Čech cohomology. Hence, the Čech class $[a]$ does not depend on the choice of the logarithms of the transition functions.

In the sequel we shall show that actually $[a] \in H^2(M; \mathbb{Z})$ depends only on the isomorphy class of the line bundle. Suppose that $\xi' = (L', q, M)$ is a smooth complex line bundle and $h : L \to L'$ be a smooth isomorphism of complex vector bundles over M. If $\mathcal{U}$ is an admissible open cover of M and h_U are local trivializations for of $\xi|_U$ and $U \in \mathcal{U}$ with transition functions g_{UV}, then $h_U \circ h^{-1}$ are local trivializations of $\xi'|_U$ with the same transition functions. Thus, it suffices to prove that if h_U, and h'_U, $U \in \mathcal{U}$, are two sets of local trivializations with corresponding transition functions g_{UV} and g'_{UV}, then they define the same element of $\check{H}^2(\mathcal{U}; \mathbb{Z})$. The smooth map $h'_U \circ h_U : U \times \mathbb{C} \to U \times \mathbb{C}$ is of the form

$$(h'_U \circ h_U)(x, z) = (x, \beta_U(x) z)$$

for some smooth function $\beta_U : U \to \mathbb{C}^\times$ and for every $x \in U \cap V$ we have

$$(x, g_{UV}(x)\beta_V(x)z) = (h'_U \circ h_V^{-1})(x, z) = (h'_U \circ h_U^{-1})(x, g_{UV}(x)z) = (x, \beta_U(x)g_{UV}(x)z).$$

Thus, $\beta_U g_{UV} = g'_{UV}\beta_V$ on $U \cap V$. Since U is contractible, there exists a smooth function $\mu_U : U \to \mathbb{R}$ such that $\beta_U = e^{2\pi i \mu_U}$. There exist $m_{UV} \in \mathbb{Z}$ such that $f_{UV}\mu_U = f'_{UV} + \mu_V + m_{UV}$, where $g'_{UV} = e^{2\pi i f'_{UV}}$. If now $a'_{UVW} = f'_{VW} - f'_{UW} + f'_{UV}$, then

$$a_{UVW} - a'_{UVW} = m_{VW} - m_{UW} + m_{UV}$$

which means that $a = a' + \delta m$, if $m = (m_{UV})$. Hence $[a] = [a'] \in H^2(M; \mathbb{Z})$.

Since the transition functions of the tensor product of two complex line bundles over M are the products of the transition functions of the line bundles, we obtain a well defined group homomorphism

$$c : \mathrm{Pic}^\infty(M) \to H^2(M; \mathbb{Z}).$$

Theorem 3.1.1. *c is an isomorphism of abelian groups.*

Proof. Let $\mathcal{U}$ be an admissible open cover of M and let $\{\psi_U : U \in \mathcal{U}\}$ be a smooth partition of unity subordinated to $\mathcal{U}$. In order to prove that c is injective, we need to show that if $\xi = (L, p, M)$ is a smooth complex line bundle and $c(\xi) = [a] = 0$, then ξ is trivial. For this it suffices to construct a nowhere vanishing smooth global section of ξ. For each $U \in \mathcal{U}$ let h_U be a trivialization of $\xi|_U$ and let g_{UV} be the corresponding transition functions. Since $[a] = 0$, there exists $\sigma \in \check{C}^1(\mathcal{U}; \mathbb{Z})$ such that $a = \delta\sigma$, that is

$$f_{VW} - f_{UW} + f_{UV} = a_{UVW} = \sigma_{VW} - \sigma_{UW} + \sigma_{UV}$$

on $U \cap V \cap W$ and using the same notation as above. Since σ_{UV}, σ_{UW}, $\sigma_{UV} \in \mathbb{Z}$ and $(f_{VW} - \sigma_{VW}) - (f_{UW} - \sigma_{UW}) + (f_{UV} - \sigma_{UV}) = 0$, we may assume from the very beginning that $a_{UVW} = 0$ for every U, V, $W \in \mathcal{U}$ such that $U \cap V \cap W \neq \varnothing$.

Let

$$\phi_U = \sum_{V \in \mathcal{U}} \psi_V \cdot f_{UV}$$

for $U \in \mathcal{U}$. Then, $\phi_U - \phi_V = f_{UV}$ for every U, $V \in \mathcal{U}$ such that $U \cap V \neq \varnothing$, because $a_{UVW} = 0$ for every U, V, $W \in \mathcal{U}$ such that $U \cap V \cap W \neq \varnothing$. Further, if we set $\beta_U = e^{2\pi i \phi_U}$, then $\beta_U = g_{UV}\beta_V$ on $U \cap V$. This implies that the formula

$$s(x) = h_U^{-1}(x, \beta_U(x)), \quad \text{for } x \in U,$$

defines a nowhere vanishing smooth global section $s : M \to L$, because

$$(h_U \circ h_V^{-1})(x, \beta_V(x)) = (x, \beta_U(x))$$

for $x \in U \cap V$. This shows that $\xi \cong \epsilon^1_{\mathbb{C}}$.

In order to show that c is surjective, let $a \in \check{C}^2(\mathcal{U}; \mathbb{Z})$ be a 2-cocycle. For each pair U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$ we define the smooth function

$$f_{UV} = \sum_{W \in \mathcal{U}} a_{UVW}\psi_W : U \cap V \to \mathbb{R}.$$

Then,

$$f_{VW}-f_{UW}+f_{UV} = \sum_{Y\in\mathcal{U}} \psi_Y(a_{VWY} - a_{UWY} + a_{UVY}) = \Big(\sum_{Y\in\mathcal{U}} \psi_Y\Big) a_{UVW} = a_{UVW} \in \mathbb{Z}$$

on $U \cap V \cap W$. If we define

$$g_{UV} = e^{2\pi i f_{UV}} : U \cap V \to S^1,$$

then $g_{UV}g_{VW} = g_{UW}$ on $U \cap V \cap W$. Since a is a 2-cocycle, taking $U = V$ we have $a_{UWY} - a_{UWY} + a_{UUY} - a_{UUW} = 0$ for all U, W, $Y \in \mathcal{U}$ such that $U \cap W \cap Y \neq \varnothing$, which implies that

$$f_{UU} = \sum_{Y\in\mathcal{U}} a_{UUY}\psi_Y = a_{UUU} \in \mathbb{Z}$$

and therefore $g_{UU} = 1$ for every $U \in \mathcal{U}$. There exists now a complex line bundle over M having transition functions g_{UV}, for U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$. For this it suffices to take

$$L = \coprod_{U\in\mathcal{U}} U \times \mathbb{C} \Big/ \sim$$

where $(x, z) \sim (x, g_{UV}(x)z)$, if $(x, z) \in (U \cap V) \times \mathbb{C}$, and take as vector bundle map $p : L \to M$ the obvious projection. This concludes the proof. □

3.2 Connections on complex line bundles

Let $\xi = (L, p, M)$ be a smooth complex line bundle over a smooth manifold M and

$$\nabla : \Omega^0(\xi) \to A^1(M;\mathbb{C}) \otimes_{C^\infty(M;\mathbb{C})} \Omega^0(\xi)$$

be a connection. Let $\mathcal{U}$ be an open cover of M consisting of open sets over each of which ξ is trivial. On each $U \in \mathcal{U}$ there exists a nowhere vanishing smooth section $e_U : U \to L$ and if $g_{UV} : U \cap V \to \mathbb{C}^\times$ are the corresponding transition functions, then $g_{UV}e_U = e_V$ on $U \cap V$.

For each $U \in \mathcal{U}$ we have a connection form $\omega_U \in A^1(U;\mathbb{C})$ which by definition satisfies $\nabla e_U = \omega_U \otimes e_U$. Thus,

$$g_{UV}\omega_V \otimes e_U = \omega_V \otimes e_V = \nabla e_V = \nabla(g_{UV}e_U) = dg_{UV} \otimes e_U + g_{UV}\omega_U \otimes e_U$$

and therefore on $U \cap V$ we have

$$\omega_V - \omega_U = \frac{dg_{UV}}{g_{UV}}.$$

Conversely, given a set of smooth 1-forms $\omega_U \in A^1(U;\mathbb{C})$, $U \in \mathcal{U}$, which satisfies the above condition for every U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$, we can define a connection on ξ by setting

$$\nabla s = df_U \otimes e_U + f_U\omega_U \otimes e_U$$

on U, where $s \in \Omega^(\xi)$ and $f_U \in C^\infty(U;\mathbb{C})$ is the unique function such that $s|_U = f_Ue_U$. Indeed, on $U \cap V$ we have $g_{UV}f_V = f_U$, because

$$f_Ue_U = s|_{U\cap V} = f_Ve_V = f_Vg_{UV}e_U,$$

and therefore

$$\begin{aligned}\nabla(f_V e_V) &= df_V \otimes e_V + f_V \omega_V \otimes e_V \\ &= g_{UV} df_V \otimes e_U + f_V g_{UV} \omega_U \otimes e_U + f_V \cdot \frac{dg_{UV}}{g_{UV}} \otimes (g_{UV} e_U) \\ &= g_{UV} df_V \otimes e_U + f_U \omega_U \otimes e_U + f_V dg_{UV} \otimes e_U \\ &= d(f_V g_{UV}) \otimes e_U + f_U \omega_U \otimes e_U \\ &= df_U \otimes e_U + f_U \omega_U \otimes e_U = \nabla(f_U e_U).\end{aligned}$$

A connection on a smooth complex line bundle $\xi = (L, p, M)$ can be described though a connection form on its associated principal $\mathbb{C}^\times$-bundle (or circle bundle). Let $L_0 = \{v \in L : v \neq 0\}$. The multiplicative group $\mathbb{C}^\times$ acts freely on L_0 by scalar multiplication and the orbit space of this action is M. Thus, $\mathcal{F}(\xi) = (L_0, p, M, \mathbb{C}^\times)$ is a fibre bundle from which ξ can be recovered as follows. The multiplicative group $\mathbb{C}^\times$ acts on $L_0 \times \mathbb{C}$ by

$$\lambda \cdot (v, z) = (\lambda^{-1} v, \lambda z)$$

and the map $f : L_0 \times \mathbb{C} \to L$ with $f(v, z) = zv$ is constant on orbits. So we get a smooth diffeomorphism $\tilde{f} : L_0 \times_{\mathbb{C}^\times} \mathbb{C} \to L$, where $L_0 \times_{\mathbb{C}^\times} \mathbb{C}$ denotes the orbit space. If $q[v, z] = p(v)$, then $(L_0 \times_{\mathbb{C}^\times} \mathbb{C}, q, M)$ is a smooth complex line bundle and $\tilde{f}$ is a vector bundle isomorphism.

The correspondence of $\mathcal{F}(\xi) = (L_0, p, M, \mathbb{C}^\times)$ to ξ is a functor $\mathcal{F}$ from the category $\mathcal{L}_M$ of complex line bundles over M to the category of principle $\mathbb{C}^\times$-bundles $\mathcal{P}_M$ over M. In both categories the morphisms are the bundle isomorphisms over M. Trivially, if f is a vector bundle isomorphism from ξ to some complex line bundle ξ', then $\mathcal{F}(f) = f|_{L_0}$ is a fibre bundle isomorphism.

Proposition 3.2.1. *The functor $\mathcal{F}$ is an equivalence of categories.*

Proof. We need to show that every object of $\mathcal{P}_M$ comes from $\mathcal{L}_M$ and if ξ, ξ' are two objects of $\mathcal{L}_M$, then the corresponding map

$$\mathrm{Hom}_{\mathcal{L}_M}(\xi, \xi') \to \mathrm{Hom}_{\mathcal{P}_M}(\mathcal{F}(\xi), \mathcal{F}(\xi'))$$

is bijective. The first assertion has already been shown above. For the second assertion, it is easy to see that if two principle $\mathbb{C}^\times$-bundles over M with total spaces L_0 and L_0' are isomorphic and $f : L_0 \to L_0$ is such an isomorphism, then the map $\tilde{f} : L_0 \times_{\mathbb{C}^\times} \mathbb{C} \to L_0' \times_{\mathbb{C}^\times} \mathbb{C}$ with $\tilde{f}[v, z] = [f(v), z]$ is a vector bundle isomorphism. □

According to Proposition 3.2.1, no piece of information is lost if instead of the smooth complex line bundle ξ we consider its associated principle $\mathbb{C}^\times$-bundle $\mathcal{F}(\xi)$. In order to describe a connection on ξ in terms of $\mathcal{F}(\xi)$, we note first that the $\mathbb{C}$-valued smooth 1-form

$$\frac{dz}{z} = \frac{1}{2r^2} d(r^2) + i d\theta = d(\log r) + i d\theta, \quad \text{(in polar coodinates } (r, \theta))$$

remains invariant under scalar multiplication with non-zero complex numbers. This implies that there exists a unique invariant $\mathbb{C}$-valued smooth 1-form β_x on each fibre

$p^{-1}(x) \cap L_0$ for $x \in M$, such that if $\tau : \mathbb{C}^\times \to p^{-1}(x) \cap L_0$ is any $\mathbb{C}^\times$-equivariant smooth map, we have

$$\tau^*(\beta_x) = \frac{dz}{z}$$

where the action of $\mathbb{C}^\times$ on itself is the scalar multiplication, because if we have two such $\mathbb{C}^\times$-equivariant smooth maps τ_1, $\tau_2 : \mathbb{C}^\times \to p^{-1}(x) \cap L_0$ and $\lambda = \tau_1^{-1}(\tau_2(1))$, then $\tau_2(z) = \tau_1(\lambda z)$ for every $z \in \mathbb{C}^\times$. Thus, $\tau_1^*(\beta_x) = \dfrac{dz}{z}$ implies that $\tau_2^*(\beta_x) = \dfrac{dz}{z}$.

A connection form on $\mathcal{F}(\xi)$ is a $\mathbb{C}$-valued smooth 1-form a on L_0 which is invariant under the action of $\mathbb{C}^\times$ and $a|_{p^{-1}(x)\cap L_0} = \beta_x$ for every $x \in M$.

Let now $U \subset M$ be an open set for which there exists a nowhere vanishing smooth section $s : U \to L_0$ of ξ. Let $\sigma : U \times \mathbb{C} \to p^{-1}(U)$ be the corresponding parametrization $\sigma(x, z) = z \cdot s(x)$, so that $h = \sigma^{-1}$ is a trivialization of $\xi|_U$. Suppose that a is a connection form on $\mathcal{F}(\xi)$. For every $x \in U$ we have

$$\sigma^* a|_{\{x\}\times\mathbb{C}^\times} = \frac{dz}{z}$$

because $\sigma|_{\{x\}\times\mathbb{C}^\times}$ is $\mathbb{C}^\times$-equivariant. On the other hand, for every $z \in \mathbb{C}^\times$ we have $\sigma^* a|_{U\times\{z\}} = s^* a$, because a is $\mathbb{C}^\times$-invariant. Consequently,

$$\sigma^* a = s^* a + \frac{dz}{z}.$$

Let $t : U \to L_0$ be another nowhere vanishing section of ξ on U and $\tau(x, z) = z \cdot t(x)$ be the corresponding parametrization of $p^{-1}(U)$. There exists a unique smooth function $g : U \to \mathbb{C}^\times$ such that

$$(\sigma^{-1} \circ \tau)(x, z) = (x, g(x)z)$$

for every $x \in U$ and $z \in \mathbb{C}^\times$. In other words, $\tau = \sigma \circ \rho$, where $\rho(x, z) = (x, g(x)z)$, and

$$\tau^* a = \rho^*(\sigma^* a) = \rho^*(s^* a, 0) + \rho^*(0, \frac{dz}{z}) = \sigma^* a + \frac{dg}{g}.$$

These remarks imply that if we choose an open cover $\mathcal{U}$ of M consisting of open sets U over which there exist a trivializations h_U of $\xi|_U$ with transition functions g_{UV}, then

$$(h_V^{-1})^* a = (h_U^{-1})^* a + \frac{dg_{UV}}{g_{UV}}$$

and therefore there exists a unique connection on ξ such that $\nabla e_U = (h_U^{-1})^* a \otimes e_U$, for every $U \in \mathcal{U}$, where $e_U = h_U^{-1}(.,1)$.

Conversely, if we start with a connection ∇ on ξ, using the same notation, we put

$$a_U = h_U^* \left(\omega_U + \frac{dz}{z} \right)$$

on every $p^{-1}(U) \cap L_0$. A similar computation as above gives

$$(h_V^{-1})^* a_U = \omega_U + \frac{dg_{UV}}{g_{UV}} + \frac{dz}{z} = \omega_V + \frac{dz}{z}$$

and thus $a_U = a_V$ on $p^{-1}(U \cap V) \cap L_0$. This means that we have a well defined connection form a on $\mathcal{F}(\xi)$ such that

$$(h_U^{-1})^* a = \omega_U + \frac{dz}{z}$$

which is unique with the property $\omega_U = e_U^* a$ for every $U \in \mathcal{U}$.

The curvature form F^∇ of a connection ∇ on the smooth complex line bundle $\xi = (L, p, M)$ is a $\mathbb{C}$-valued smooth 2-form on M, because $\mathrm{Hom}(\xi, \xi)$ is trivial. Taking an open cover $\mathcal{U}$ of M as above we have

$$F^\nabla|_U = d\omega_U - \omega_U \wedge \omega_U = d\omega_U.$$

If a is the corresponding connection form on $\mathcal{F}(\xi)$, it follows immediately that

$$da = (p|_{L_0})^*(F^\nabla)$$

and F^∇ is unique with this property, since $p|_{L_0} : L_0 \to M$ is a submersion.

3.3 Hermitian connections

Let $\xi = (L, p, M)$ bw a smooth complex line bundle over a smooth manifold M. Since M is paracompact, there exists a smooth hermitian inner product h on ξ. Given such a hermitian inner product, we recall that a connection ∇ on ξ is called hermitian (or the other way round h is called invariant under ∇) if it is compatible with h, that is

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t)$$

for every s, $t \in \Omega^0(\xi)$, where $h(\theta \otimes s, t) = \theta \cdot h(s,t)$ and $h(s, \theta \otimes t) = \overline{\theta} \cdot h(s,t)$ for $\theta \in A^1(M; \mathbb{C})$.

The curvature form F^∇ is then skew-hermitian and actually if $\mathcal{U}$ is an open cover of M over each element U of which there exists a nowhere vanishing smooth section $e_U : U \to L$ and $\nabla e_U = \omega_U \otimes e_U$, we have

$$dh(e_U, e_U) = h(\omega_U \otimes e_U, e_U) + h(e_U, \omega_U \otimes e_U) = (\omega_U + \overline{\omega_U}) h(e_U, e_U)$$

and so $\omega_U + \overline{\omega_U} = d(\log h(e_U, e_U))$. Therefore,

$$F^\nabla + \overline{F^\nabla} = d\omega_U + d\overline{\omega_U} = 0$$

on U. In other words $\dfrac{1}{2\pi i} F^\nabla$ is a real closed smooth 2-form on M, which represents $-c_1(\xi)$.

Let $h_U : p^{-1}(U) \to U \times \mathbb{C}$ be the trivialization of $\xi|_U$ such that $e_U = h_U^{-1}(.,1)$. If a is the connection 1-form on the associated principal $\mathbb{C}^\times$-bundle $\mathcal{F}(\xi) = (L_0, p, M)$ defined by ∇, then

$$a|_U = h_U^* \left(\omega_U + \frac{dz}{z} \right),$$

as we saw in the previous section and so

$$a|_U + \overline{a|_U} = h_U^*(d(\log(h(e_U, e_U^2))) + d(\log |z|^2)) = d(\log |H|^2)$$

where $|H|^2 : p^{-1}(U) \cap L_0 \to [0, +\infty)$ is the smooth function defined by

$$|H|^2(h_U^{-1}(x,z)) = h(ze_U(x), ze_U(x)) = h(h_U^{-1}(x,z), (h_U^{-1}(x,z)).$$

In other words, $|H|^2$ is the quadratic form defined by the hermitian inner product h, which is defined everywhere on L_0. Hence

$$a + \overline{a} = d(\log |H|^2), \quad \text{on } L_0$$

and since L_0 is connected, $|H|^2$ is unique with this property, up to a constant.

Proposition 3.3.1. *Given a connection ∇ on ξ with corresponding connection 1-form a on the associated principle $\mathbb{C}^\times$-bundle $\mathcal{F}(\xi)$, there exists an invariant hermitian inner product h on ξ if and only if $a + \overline{a}$ is exact. In this case, the invariant hermitian inner product is unique, up to a constant.*

Proof. The above considerations show that only the converse needs proof. Thus, suppose that there exists some smooth function $\psi : L_0 \to \mathbb{R}$ such that $a + \overline{a} = d\psi$. Putting $\phi = e^\psi$ we have

$$a + \overline{a} = \frac{d\phi}{\phi}, \quad \text{on } L_0$$

and

$$\frac{d\phi}{\phi} = a + \overline{a} = h_U^*\left(\omega_U + \overline{\omega_U} + \frac{1}{|z|^2} d(|z|^2)\right)$$

on $p^{-1}(U) \cap L_0$. If we fix a point $x \in U$ and let $\chi : \mathbb{C}^\times \to (0, +\infty)$ be the smooth function defined by $\chi(z) = \phi(h_U^{-1}(x,z))$, it follows that

$$\frac{d\chi}{\chi} = ((h_U|_{p^{-1}(x)})^{-1})^*(\frac{d\phi}{\phi}) = \frac{d(|z|^2)}{|z|^2}$$

or equivalently $d(\log \chi) = d(\log(|z|^2))$ on $\mathbb{C}^\times$. Integrating, we conclude

$$\log \chi(\lambda z) - \log \chi(z) = \log |\lambda z|^2 - \log |z|^2$$

or equivalently $\chi(\lambda z) = |\lambda|^2 \chi(z)$ for every $\lambda \in \mathbb{C}^\times$ and $z \in \mathbb{C}^\times$. Thus,

$$\phi(\lambda v) = |\lambda|^2 \phi(v)$$

for every $\lambda \in \mathbb{C}^\times$ and $v \in L_0$.

For every $u, v \in p^{-1}(x) \cap L_0$ there exists a unique $\lambda \in \mathbb{C}^\times$ such that $u = \lambda v$. We set then $h(u,v) = \lambda\phi(v)$. If either $u = 0$ or $v = 0$, we set $h(u,v) = 0$. It is easy to see now that h is a smooth hermitian inner product on ξ.

On $U \in \mathcal{U}$ we have

$$d(\log h(e_U, e_U)) = e_U^*\left(\frac{d\phi}{\phi}\right) = (e_U^* \circ h_U^*)\left(\omega_U + \overline{\omega_U} + \frac{d(|z|^2)}{|z|^2}\right)$$

$$= pr^*\left(\omega_U + \overline{\omega_U} + \frac{d(|z|^2)}{|z|^2}\right) = \omega_U + \overline{\omega_U}$$

and thus

$$dh(e_U, e_U) = \omega_U h(e_U, e_U) + \overline{\omega_U} h(e_U, e_U) = h(\nabla e_U, e_U) + h(e_U, \nabla e_U).$$

Finally, if f_1, $f_2 : U \to \mathbb{C}$ are two smooth functions we compute

$$\begin{aligned} & h(\nabla(f_1 e_U), f_2 e_U) + h(f_1 e_U, \nabla(f_2 e_U)) \\ &= h(df_1 \otimes e_U, f_2 e_U) + h(f_1 \nabla e_U, f_2 e_U) + h(f_1 e_U, df_2 \otimes e_U)) + h(f_1 e_U, f_2 \nabla e_U) \\ &= \overline{f_2} h(e_U, e_u) df_1 + f_1 \overline{f_2} h(\nabla e_U, e_U) + f_1 h(e_U, e_U) d\overline{f_2} + f_1 \overline{f_2} h(e_U, \nabla e_U) \\ &= f_1 \overline{f_2} h(e_U, e_U) + h(e_U, e_U) d(f_1 \overline{f_2}) = dh(f_1 e_U, f_2 e_U). \quad \square \end{aligned}$$

It is evident from the above that given a hermitian inner product h on the complex line bundle ξ, then a connection ∇ on ξ is hermitian if and only if locally

$$\omega_U + \overline{\omega_U} = d(\log h(e_U, e_U))$$

on every $U \in \mathcal{U}$. If we choose unit local sections, that is $h(e_U, e_U) = 1$ on U, then $\omega_U + \overline{\omega_U} = 0$ and ω_U is purely imaginary. If $L_1 = \{v \in L : h(v, v) = 1\}$, then $(L_1, p|_{L_1}, M, S^1)$ is the associated principle circle bundle to ξ and this is equivalent to saying that the corresponding connection 1-form a on L_1 is purely imaginary.

3.4 Integer cohomology classes in degree 2

Let M be a smooth manifold and $\Omega \in A^2(M)$ be a (real) closed smooth 2-form. In this section we shall be concerned with the problem of finding necessary and sufficient conditions in order the cohomology class $[\Omega] \in H^2(M; \mathbb{R})$ to be equal to $c_1(\xi)$ for some smooth complex line bundle ξ over M. We need to recall the Čech-deRham isomorphism

$$\check{H}^2(\mathcal{U}; \mathbb{R}) \cong \check{H}^2(M; \mathbb{R}) \cong H^2(M; \mathbb{R})$$

in degree 2 for an admissible open cover $\mathcal{U}$ of M.

Since each $U \in \mathcal{U}$ is contractible and Ω is closed, there exists $\omega_U \in A^1(U)$ such that $\Omega|_U = d\omega_U$. If U, $V \in \mathcal{U}$ are such that $U \cap V \neq \varnothing$, there is a smooth function $f_{UV} : U \cap V \to \mathbb{R}$ such that $df_{UV} = \omega_V - \omega_U$ on $U \cap V$, because $d\omega_U = d\omega_V$ on $U \cap V$ and the latter is contractible. If now $W \in \mathcal{U}$ and $U \cap V \cap W \neq \varnothing$, then

$$df_{VW} - df_{UW} + df_{UV} = 0, \quad \text{on } U \cap V \cap W$$

and from the connectivity of $U \cap V \cap W$ there exists $a_{UVW} \in \mathbb{R}$ such that

$$f_{VW} - f_{UW} + f_{UV} = a_{UVW}, \quad \text{on } U \cap V \cap W.$$

It is obvious that $a = (a_{UVW}) \in \check{C}^2(\mathcal{U}; \mathbb{R})$ is a Čech 2-cocycle. In this way one constructs the Čech-deRham isomorphism $H^2(M; \mathbb{R}) \cong \check{H}^2(\mathcal{U}; \mathbb{R})$, which sends $[\Omega]$ to $[a]$. It is well defined because if Ω' is another representative of $[\Omega]$, there exists some smooth 1-form η such that $\Omega' = \Omega + d\eta$. If f'_{UV} are the smooth functions corresponding to Ω', there are $g_U \in C^\infty(U)$ such that $\omega'_U - \omega_U = \eta + dg_U$ and therefore

$$df'_{UV} = df_{UV} + dg_V - dg_U, \quad \text{on } U \cap V,$$

Thus, $\beta_{UV} = f'_{UV} - f_{UV} + g_U - g_V$ is a constant on $U \cap V$. Consequently,

$$a'_{UVW} - a_{UVW} = \beta_{VW} - \beta_{UW} + \beta_{UV}, \quad \text{on } U \cap V \cap W,$$

which means that $a' - a = \delta\beta$, where $\beta = (\beta_{UVW}) \in \check{C}^1(\mathcal{U};\mathbb{R})$.

The inclusion $\epsilon : \mathbb{Z} \to \mathbb{R}$ induces a homomorphism $\epsilon^2 : \check{H}^2(\mathcal{U};\mathbb{Z}) \to \check{H}^2(\mathcal{U};\mathbb{R})$ (and in any other degree). We say that the cohomology class $[\Omega] \in H^2(M;\mathbb{R})$ is integer if there exists some admissible open cover $\mathcal{U}$ of M such that its corresponding Čech class $[a] \in \check{H}^2(\mathcal{U};\mathbb{R})$ under the Čech-deRham isomorphism belongs to the image of ϵ^2, which is equivalent to $f_{VW} - f_{UW} + f_{UV} \in \mathbb{Z}$ for every U, V, $W \in \mathcal{U}$ such that $U \cap V \cap W \neq \varnothing$.

Proposition 3.4.1. *The Chern class $c_1(\xi)$ of a smooth complex line bundle $\xi = (L, p, M)$ over M is integer and actually $c_1(\xi) = -\epsilon^2(c(\xi))$.*

Proof. Let ∇ be any connection on ξ. Let $\mathcal{U}$ be an admissible open cover of M. For each $U \in \mathcal{U}$ let $e_U : U \to L$ be a nowhere vanishing smooth section of ξ and corresponding transition functions $g_{UV} : U \cap V \to S^1$. Let also ω_U be the connection form of ∇ on U with respect to e_U. Then,

$$\omega_V - \omega_U = \frac{dg_{UV}}{g_{UV}}, \quad \text{on } U \cap V$$

and $F^\nabla|_U = d\omega_U$. From Corollary 2.7.4, the Chern class

$$c_1(\xi) = \left[\frac{-1}{2\pi i} F^\nabla\right]$$

is real. Hence there exists a real closed smooth 2-form $F \in A^2(M)$ and a $\mathbb{C}$-valued smooth 1-form η on M such that

$$\frac{1}{2\pi i} F^\nabla = F + d\eta.$$

Since each $U \in \mathcal{U}$ is contractible, there exists $F_U \in A^1(U)$ such that $df_U = F|_U$. If now $g_{UV} = e^{2\pi i f_{UV}}$ on $U \cap V$, then

$$F_V - F_U = \frac{1}{2\pi i}(\omega_V - \omega_U) = \frac{1}{2\pi i} \cdot \frac{dg_{UV}}{g_{UV}} = df_{UV}$$

on $U \cap V$. From the constructions of the Čech-deRham isomorphism and the isomorphism $c : \mathrm{Pic}^\infty(M) \cong \check{H}^2(\mathcal{U};\mathbb{Z})$ follows immediately that $c_1(\xi) = -\epsilon^2(c(\xi))$. □

The preceding Proposition 3.4.1 combined with the Splitting Principle for complex vector bundles implies the following corollary.

Corollary 3.4.2. *If $\xi = (E, p, M)$ is a smooth complex vector bundle over a smooth manifold M, then the Chern classes $c_k(\xi)$, $k \in \mathbb{Z}^+$, of ξ are integer.* □

Corollary 3.4.3. *If ξ_1 and ξ_2 are two smooth complex line bundles over the same smooth manifold, then $c_1(\xi_1 \otimes \xi_2) = c_1(\xi_1) + c_1(\xi_2)$.* □

A combination of the Splitting Principle and Proposition 3.4.1 also gives the following important property of the total Chern character which says that it is a ring homomorphism from the K-ring of a smooth manifold to its cohomology ring with rational coefficients.

Corollary 3.4.4. *If ξ and ζ are two smooth complex vector bundles over the smooth manifold M, then $ch(\xi \otimes \zeta) = ch(\xi) \wedge ch(\zeta)$.*

Proof. If ξ has rank n and ζ has rank m, then there are smooth complex line bundles $\xi_1, ..., \xi_n, \zeta_1, ..., \zeta_m$ over M such that $\xi \cong \xi_1 \oplus \cdots \oplus \xi_n$ and $\zeta \cong \zeta_1 \oplus \cdots \oplus \zeta_m$. Thus, $\xi \otimes \zeta \cong \bigoplus_{k,l} \xi_k \otimes \zeta_l$ and

$$ch(\xi \otimes \zeta) = \sum_{k,l} ch(\xi_k \otimes \zeta_l) = \sum_{k,l} e^{c_1(\xi_k \otimes \zeta_l)}$$

$$= \sum_{k,l} e^{c_1(\xi_k) + c_1(\zeta_l)} = \left(\sum_{k=1}^{n} e^{c_1(\xi_k)}\right) \wedge \left(\sum_{l=1}^{m} e^{c_1(\zeta_l)}\right) = ch(\xi) \wedge ch(\zeta)$$

from Proposition 2.4.5(a) and Corollary 3.4.3. □

The converse of Proposition 3.4.1 also holds.

Theorem 3.4.5. (B. Kostant) *Let M be a smooth manifold and $\Omega \in A^2(M)$ a real closed smooth 2-form on M. The cohomology class $[\Omega]$ is integer if and only if $2\pi i\Omega$ is the curvature form of a hermitian connection on some smooth complex line bundle over M.*

Proof. Only the direct needs proof, as the converse is Proposition 3.4.1. So, let $[\Omega]$ be integer. Using the same notation as in the beginning of this section with respect to an admissible open cover $\mathcal{U}$ of M, we have

$$f_{VW} - f_{UW} + f_{UV} \in \mathbb{Z}, \quad \text{on } U \cap V \cap W.$$

Putting $g_{UV} = e^{2\pi i f_{UV}}$, for U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$, we have $g_{UV} = g_{VU}^{-1}$, since $f_{UU} \in \mathbb{Z}$, and $g_{UV} g_{VW} = g_{UW}$. As in the last part of the proof of Theorem 3.1.1, there exists a smooth complex line bundle $\xi = (L, p, M)$ with transition functions g_{UV} with respect to $\mathcal{U}$. Since

$$\omega_V - \omega_U = df_{UV} = \frac{1}{2\pi i} \cdot \frac{dg_{UV}}{g_{UV}},$$

there exists a connection ∇ on ξ with curvature form $2\pi i\Omega$. It remains to show that there is an invariant hermitian inner product on ξ. We consider the hermitian inner product h defined by

$$h(h_U^{-1}(x,z_1), h_U^{-1}(x,z_2)) = z_1\overline{z_2}$$

where h_U is a trivialization of $\xi|_U$. This defines h globally, because $|g_{UV}| = 1$. In order to show that ∇ is hermitian with respect to h, it suffices to check that

$$2\pi i\omega_U + \overline{2\pi i\omega_U} = d(\log h(e_U, e_U))$$

where $e_U = h_U^{-1}(.,1)$ for every $U \in \mathcal{U}$. But this is trivial since both sides are equal to zero, the left hand side of this equality being zero because $\omega_U = \Omega|_U$ is real. $\square$