manuscripta math. 72, 415 - 423 (1991)

manuscripta mathematica

COHOMOLOGY AND ASYMPTOTIC STABILITY OF 1-DIMENSIONAL CONTINUA

Konstantin Athanassopoulos

We prove that a 1-dimensional continuum carrying a flow without singular points is homeomorphic to the unit circle if its first Čech cohomology group with integer coefficients is isomorphic to $\mathbb{Z}$. As an application of this we obtain that an asymptotically stable invariant 1-dimensional continuum of a flow on a locally compact ANR, which does not contain singular points, must be a periodic orbit.

1. Introduction

Two of the most interesting problems in the theory of dynamical systems are to determine the structure of the limit sets and describe the behavior of the orbits near them. A serious step in this direction is the study of minimal sets, since every compact limit set contains a minimal set.

The original motivation of this note is the problem of finding conditions refering to the behavior of the orbits near a compact minimal set of a flow under which the minimal set is a periodic orbit. G. Allaud and E.S. Thomas have shown in [1; Theorem 3.4] that an almost periodic asymptotically stable compact minimal set of a flow on a regular manifold is a torus. The almost periodicity assumption plays an important role in the proof of this result.

It is shown in paragraph 4 of the present note that every asymptotically stable invariant 1-dimensional continuum A of a flow on a locally compact ANR, which does not contain singular points, is a periodic orbit. This is based on the one hand on the observation that the first Čech cohomology group with integer coefficients of A is isomorphic to $\mathbb{Z}$ and on the other hand on the fact that a 1-dimensional continuum with first Čech cohomology group $\mathbb{Z}$, which carries a flow without singular points, must be homeomorphic to S^1, which is proved in paragraph 3.

The main part of this note was written during my stay in the Mathematical Institute of the Freie Universität Berlin as a fellow of the Alexander von Humboldt Foundation to which I am grateful for its support. I also thank Elmar Vogt for the useful conversations we had and the referee whose comments improved an earlier version of the paper.

2. Preliminaries and notations

Let $(\mathbb{R}, X, \phi)$ denote a (continuous) flow on a metric space X. We shall use the convenient notation $\phi(t,x) = tx$ and $\phi(I \times A) = IA$, if $I \subset \mathbb{R}$ and $A \subset X$. The orbit of the point $x \in X$ is denoted by $C(x)$, the positive limit set by $L^+(x)$ and the negative by $L^-(x)$.

Given $\varepsilon, T > 0$ an (ε,T)-chain from x to y is a pair of finite sets of points $\{x_0, \ldots, x_{p+1}\}$ of X and times $\{t_0, \ldots, t_p\}$ such that $x_0 = x$, $x_{p+1} = y$, $t_j \geqq T$ and $d(t_j x_j, x_{j+1}) < \varepsilon$, $0 \leqq j \leqq p$, where d is the metric in X. A point x is called <u>chain recurrent</u> if for every ε, $T > 0$ there exists an (ε,T)-chain from x to x and the flow on X is called chain recurrent if every point of X is chain recurrent. It is well known [9; Theorem 3.1] that a flow $(\mathbb{R}, X, \phi)$ on a continuum X is chain recurrent if and only if there is a flow on some compact metric space Z and $z \in Z$ such that $(\mathbb{R}, X, \phi)$ is topologically equivalent to the flow on $L^+(z)$.

A flow on a compact metric space X is called <u>minimal</u> if every orbit is dense in X. In this case X is called a minimal set and clearly the flow is chain recurrent.

In the sequel we shall make use of the following representation of the first Čech cohomology group with integer coefficients $\check{H}^1(X)$ of a compact metric space X. Let $[X;S^1]$ denote the set of homotopy classes of continuous maps of X into S^1. The usual multiplication in S^1 induces a group operation in $[X;S^1]$. Let θ be the generator of $\check{H}^1(S^1)$. To every continuous map $f : X \to S^1$ corresponds the cohomology class $f^*(\theta)$ in $\check{H}^1(X)$, which depends only on the homotopy class of f, where f^* is the induced homomorphism in Čech cohomology. The function $\chi : [X;S^1] \to \check{H}^1(X)$ defined in this way is an isomorphism of abelian groups [7; Theorem 8.1]. It follows easily from this representation that $\check{H}^1(X)$ is torsion free [12; p. 409].

3. Flows on 1-dimensional continua

Let $(\mathbb{R}, X, \phi)$ be a flow on a compact metric space X such that not every point is singular. Let S_0 be a local section in X of extent $\varepsilon_0 > 0$, so that ϕ maps $(-\varepsilon_0,\varepsilon_0)\times S_0$ homeomorphically onto an open subset of X. Suppose that S is an open-compact subset of S_0 and $0 < \varepsilon < \varepsilon_0/2$. Then ϕ maps $[-2\varepsilon,2\varepsilon]\times S$ homeomorphically onto a compact neighbourhood of S. To such a local section corresponds a continuous map $f : X \to S^1$ defined by

$$f(x) = \begin{cases} \pi(0) & , \text{ if } x \in X - [0,\varepsilon]S \\ \pi(t/\varepsilon) & , \text{ if } x \in tS \text{ and } 0 \leq t \leq \varepsilon \end{cases}$$

where $\pi : \mathbb{R} \to S^1$ is the canonical covering projection. Clearly, the homotopy class of f does not depend on ε but only on S. The map f is called the cosection map associated to S and was first defined by M.W. Hirsch and C.C. Pugh in [10]. The next Lemma concerning cosection maps is of fundamental importance for what follows.

<u>LEMMA 3.1</u> *Let $(\mathbb{R}, X, \phi)$ be a flow on a compact metric space X and S_0 a local section in X of extent $\varepsilon_0 > 0$. Let P, Q be open-compact subsets of S_0 with $P \cap Q = \emptyset$ and $0 < \varepsilon < \varepsilon_0/2$. Let $h, g : X \to S^1$ be the cosection maps associated to P and Q respectively, as defined above. We assume that the set $S = P \cup Q$ contains a chain recurrent point x_0 and that for some non-zero integers m, n the map $h^m \cdot g^n : X \to S^1$ is nullhomotopic. Then, $m \cdot n < 0$.*

<u>Proof.</u> Since $h^m g^n$ is assumed to be nullhomotopic, there is a

continuous map $\alpha : X \to \mathbb{R}$ such that

$$h^m(x)\cdot g^n(x) = \pi(\alpha(x))$$

for every $x \in X$ [8; Ch. XVII, 5.1]. Let $\beta : X \to \mathbb{R}$ be the function defined by

$$\beta(x) = \begin{cases} 0 & , \text{ if } x \in X - [0,\varepsilon]S \\ mt/\varepsilon & , \text{ if } x \in tP \text{ and } 0 \leq t \leq \varepsilon \\ nt/\varepsilon & , \text{ if } x \in tQ \text{ and } 0 \leq t \leq \varepsilon \end{cases}$$

Obviously, β is everywhere continuous except at the points of εS and $\pi(\alpha(x)) = \pi(\beta(x))$ for every $x \in X$. Moreover, the restriction of β to $[-\varepsilon,\varepsilon]S$ is continuous. Let $\gamma : X \to \mathbb{Z}$ be defined by $\gamma(x) = \alpha(x) - \beta(x)$. Since the restriction of γ to $[-\varepsilon,\varepsilon]S$ is continuous, there is a covering $\{W_1,\ldots,W_q\}$ of S by disjoint, open-closed subsets of S such that γ takes the value k_i on the set $[-\varepsilon,\varepsilon]W_i$, $1 \leq i \leq q$. We may assume that each W_i is contained in P or in Q. On the other hand, if $x_k \to x$ where $x \in \varepsilon W_i$ and $x_k \in X - [0,\varepsilon]S$, then $\alpha(x_k) \to \alpha(x)$ and therefore $\gamma(x_k)$ is eventually constant and equal to $m + k_i$ or $n + k_i$, depending upon whether W_i is contained in P or in Q respectively. This means that for some $0 < \delta < \varepsilon$ we have

$$\gamma(x) = \begin{cases} k_i & , \text{ if } x \in [-\varepsilon,\varepsilon]W_i \\ m + k_i & , \text{ if } x \in (\varepsilon,\varepsilon+\delta]W_i \text{ and } W_i \subset P \\ n + k_i & , \text{ if } x \in (\varepsilon,\varepsilon+\delta]W_i \text{ and } W_i \subset Q \end{cases}$$

for every $x \in [-\varepsilon,\varepsilon+\delta]W_i$, $1 \leq i \leq q$.

Since γ is continuous on $X - [0,\varepsilon+(\delta/2)]S$, the family of sets $\{(-\varepsilon,\varepsilon+\delta)W_i : 1 \leq i \leq q\} \cup \{\gamma^{-1}(k) \cap (X - [0,\varepsilon+(\delta/2)]S) : k \in \mathbb{Z}\}$ is an open covering of X for which there is a Lebesgue number $\eta > 0$ so small that the ball of radius η centered at x_0 is contained in some set $(-\varepsilon,\varepsilon)W_i$. By chain recurrence, there are in X points $x_1,\ldots,x_{p+1}$ and times $t_0,\ldots,t_p$ such that $x_{p+1} = x_0$, $t_j \geq 2\varepsilon$ and $d(t_jx_j, x_{j+1}) < \eta$, $0 \leq j \leq p$. Note that since $t_0 \geq 2\varepsilon$, the orbit segment $[0,t_0]x_0$ crosses εS and hence $\gamma(t_0x_0) = \gamma(x_0) + \lambda_0$, where λ_0 is a sum each term of which is equal to m or n. If $1 \leq j \leq p$, then $\gamma(t_j x_j) = \gamma(x_j)$ or $\gamma(x_j) + \lambda_j$, where λ_j is a sum like λ_0. If $\gamma(t_j x_j) \neq \gamma(x_{j+1})$, then both points $t_j x_j$ and x_{j+1} belong to some $(-\varepsilon, \varepsilon + \delta)W_i$ and either $t_j x_j \in (-\varepsilon, \varepsilon]W_i$ and $x_{j+1} \in (\varepsilon, \varepsilon + \delta)W_i$ or $x_{j+1} \in (-\varepsilon, \varepsilon]W_i$ and $t_j x_j \in (\varepsilon, \varepsilon + \delta)W_i$. In the first case we have $\gamma(x_{j+1}) = \gamma(t_j x_j) + \lambda$, where λ is equal to m or to n and therefore $\gamma(x_{j+1}) = \gamma(x_j) + \mu_j$, where μ_j is a sum like λ_0. In the second case we have $\gamma(x_{j+1}) = \gamma(t_j x_j) - \lambda$ and the last term of λ_j is λ. Hence $\gamma(x_{j+1}) = \gamma(x_j) + \mu_j$, where μ_j is zero or a sum like λ_0. We conclude that there are integers $\mu_0, \dots, \mu_p$ such that each μ_j is zero or a sum each term of which is m or n, the later occuring at least for μ_0, and $\gamma(x_{j+1}) = \gamma(x_j) + \mu_j$, $0 \leq j \leq p$. Since $x_0 = x_{p+1}$ we have that $\mu_0 + \dots + \mu_p = 0$. It follows that $m \cdot n < 0$.

Using Lemma 3.1 we may now reprove the main result of [10].

<u>THEOREM 3.2 Let $(\mathbb{R}, X, \phi)$ be a flow on a compact metric space X and S_0 be a local section in X. If S is an open-compact subset of S_0 which contains a chain recurrent point, then the cosection map associated to S defines a non-zero element of $\check{H}^1(X)$.</u>

<u>Proof.</u> Taking in Lemma 3.1 $P = S$, $Q = \emptyset$ and $m = n = 1$ we conclude that the cosection map $h : X \to S^1$ associated to S cannot be null-homotopic. Hence h defines a non-zero element of $H^1(X)$.

<u>THEOREM 3.3 Let $(\mathbb{R}, X, \phi)$ be a flow on a 1-dimensional compact metric space X. If there is a nonsingular chain recurrent point x in X, then $\check{H}^1(X)$ is not trivial.</u>

<u>Proof.</u> Since x is nonsingular, there exists $\varepsilon_0 > 0$ and a locally compact local section S_0 containing x, so that ϕ maps $(-\varepsilon_0, \varepsilon_0) \times S_0$ homeomorphically onto an open neighbourhood of x [4; Lemma 1]. It follows that $\dim S_0 = 0$. Thus, x has an open-compact neighbourhood S in S_0. According to Theorem 3.2 the cosection map associated to S defines a non-zero element of $\check{H}^1(X)$.

<u>PROPOSITION 3.4 Let $(\mathbb{R}, X, \phi)$ be a chain recurrent flow on a 1-dimensional continuum X. If $\check{H}^1(X) \cong \mathbb{Z}$, then for every nonsingular point $x \in X$ there exists $\varepsilon > 0$ such that the orbit segment</u>

$(-\varepsilon,\varepsilon)x$ _is an open neighbourhood of_ x _in_ X.

Proof. Let S_0 be a locally compact local section at x. Then $\dim S_0 = 0$. Let S, Q be open-compact neighbourhoods of x in S_0 such that Q is properly contained in S. If f, g and h are the cosection maps associated to S, Q and $S-Q$ respectively, then $f = g \cdot h$. By Theorem 3.2 h is not nullhomotopic. Therefore, f and g are not homotopic to each other and represent different elements of $\check{H}^1(X)$. Moreover, f^k is not homotopic to g^k for every integer k, because $\check{H}^1(X)$ is torsion free. Since we assume that $\check{H}^1(X) \cong \mathbb{Z}$, the cosection maps f, g correspond to integers n, m respectively. We shall show that $0 < |m| < |n|$.

We first observe that $n \cdot m \neq 0$ and $n \neq m$. From the commutativity of the multiplication of integers follows that f^m is homotopic to g^n or equivalently that $f^m \cdot g^{-n} : X \to S^1$ is nullhomotopic. However,

$$f^m \cdot g^{-n} = (f \cdot g^{-1})^m \cdot g^{m-n} = h^m \cdot g^{m-n} .$$

It follows from Lemma 3.1 that $m \cdot (m-n) < 0$, that is $0 < |m| < |n|$.

Now let $\{ S_k : k \in \mathbb{N} \}$ be a decreasing neighbourhood basis of x in S_0 consisting of open-compact sets. In order to prove the proposition, it suffices to show that this neighbourhood basis is eventually constant. Let f_k be the cosection map associated to S_k. Since $\check{H}^1(X) \cong \mathbb{Z}$, the homotopy class of f_k corresponds to a non-zero integer n_k. If each S_{k+1} was properly contained in S_k, then by what we have shown above the sequence $\{ |n_k| : k \in \mathbb{N} \}$ would be strictly decreasing, which is impossible.

<u>THEOREM 3.5</u> _Let_ $(\mathbb{R}, X, \phi)$ _be a flow without singular points on a 1-dimensional continuum_ X. _If_ $\check{H}^1(X) \cong \mathbb{Z}$, _then_ X _is homeomorphic to_ S^1.

Proof. Let $x \in X$ and $A = L^+(x)$. Since A is compact and there are no singular points, A is a 1-dimensional continuum. The flow on A is chain recurrent and hence $\check{H}^1(A)$ is not trivial by Theorem 3.3. On the other hand, the inclusion $i : A \subset X$ induces an epimorphism $i^* : \check{H}^1(X) \to \check{H}^1(A)$ [5]. It follows that $\check{H}^1(A) \cong \mathbb{Z}$ and therefore A is a periodic orbit by Proposition 3.4. Similarly, $L^-(x)$ is a periodic orbit. We conclude that the orbit $C(x)$ must be periodic, because otherwise we have $\check{H}^1(\overline{C(x)}) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is impossible [5]. We have thus proved that every orbit in X is periodic. By Proposition 3.4, X must be homeomorphic to S^1.

EXAMPLES 3.6 (a) It is clear that Theorem 3.5 is not true without the assumption $\check{H}^1(X) \cong \mathbb{Z}$. It is however noteworthy that there are 1-dimensional continua X with $\check{H}^1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and carrying flows without singular points which differ drastically from each other. Suppose that X consists of a circle together with an orbit that spirals in positive and negative time against the circle or two circles with an orbit whose positive resp. negative limit set is the first resp. second circle. In both cases $\check{H}^1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. There is another example completely different from the above. More precisely, there is a flow on the torus T^2 with a nonperiodic 1-dimensional minimal set X such that $T^2 - X$ is homeomorphic to $\mathbb{R}^2$ [6]. By duality we have again $\check{H}^1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(b) If the assumption of the non-existence of singular points in Theorem 3.5 is removed, then the conclusion fails. Consider the flow on the closed unit disc D^2 which fixes the origin and such that every orbit in the interior of D^2 tends in negative time to the origin and in positive time spirals against the boundary which is a periodic orbit. If $x \in \operatorname{int} D^2$, then $X = \overline{C(x)}$ is a 1-dimensional continuum with $\check{H}^1(X) \cong \mathbb{Z}$, which is not homeomorphic to S^1 and carries a flow with one singular point.

4. Asymptotically stable 1-dimensional continua

Let $(\mathbb{R}, M, \phi)$ be a flow with M a locally compact ANR (for metrizable spaces) and let $A \subset M$ be an asymptotically stable compact invariant set. The region of attraction E of A is an open invariant neighbourhood of A and there exists a strictly decreasing uniformly unbounded Lyapunov function $F : E \to \mathbb{R}^+$ for A [2; Ch. V, Theorem 2.9]. For any $c > 0$ the set $B = F^{-1}([0,c])$ is a positively invariant compact retract of E. Since E is open in M, the set B is a compact ANR. Therefore, the Čech cohomology of B coincides with the singular cohomology of B and is finitely generated [11; Ch. IV, Section 7]. For $0 < b < c$, the inclusion $F^{-1}([0,b]) \subset F^{-1}([0,c])$ induces an isomorphism in cohomology. From the continuity property of the Čech cohomology follows that $\check{H}^*(A)$ is isomorphic to $H^*(B)$. Hence $\check{H}^*(A)$ is finitely generated.

LEMMA 4.1 *If A does not contain singular points, then*

$$\sum_{q=0}^{\infty} (-1)^q \operatorname{rank} \check{H}^q(A) = 0 \quad .$$

Proof. For each $t > 0$ the continuous map $\phi_t : M \to M$ defined by $\phi_t(x) = tx$ is homotopic to the identity and sends B into B, where B is the above defined set. Since A is compact and does not contain singular points, the periods of the periodic points in A are bounded away from zero [2; Ch. V, Lemma 3.7]. Let $t > 0$ be smaller than any period of a periodic point in A. Then ϕ_t has no fixed points in B. Since B is a compact ANR, the conclusion follows from the Lefschetz fixed point theorem for compact ANRs [3; Ch. III, Theorem 2].

THEOREM 4.2 *Let $(\mathbb{R}, M, \phi)$ be a flow with M a locally compact ANR and $A \subset M$ an asymptotically stable 1-dimensional invariant continuum. If A does not contain singular points, then A is a periodic orbit.*

Proof. Since A is 1-dimensional, $\check{H}^q(A) = 0$, for $q > 1$. Thus, from Lemma 4.1 we have $\operatorname{rank} \check{H}^1(A) = 1$. Since $\check{H}^1(A)$ is a finitely generated, torsion free abelian group, it must be isomorphic to $\mathbb{Z}$. According now to Theorem 3.5, A is a periodic orbit.

REFERENCES

1. ALLAUD,G., THOMAS,E.S.: Almost periodic minimal sets. J. Differ. Equations. 15, 158 - 171 (1974)
2. BHATIA,N.P.,SZEGŐ,G.P.: Stability Theory of Dynamical Systems. Berlin: Springer 1970
3. BROWN,R.F.: The Lefschetz Fixed Point Theorem. Illinois: Scott-Foresman 1971
4. CHEWNING,W.C., OWEN,R.S.: Local sections of flows on manifolds. Proc. Amer. Math. Soc. 49, 71 - 77 (1975)
5. COHEN:H.: A cohomological definition of dimension for locally compact Hausdorff spaces. Duke Math. J. 21, 209 - 224 (1954)

6. DENJOY,A.: Sur les courbes définies par les équations différentielles á la surface du tore. J. Math. Pures Appl. 11, 333 - 375 (1932)

7. DOWKER,C.H.: Mapping theorems for noncompact spaces. Amer. J. Math. 69, 200 - 247 (1947)

8. DUGUNJI,J.: Topology. Boston: Allyn - Bacon 1966

9. FRANKE,J., SELGRADE,J.: Abstract ω - limit sets, chain recurrent sets and basic sets for flows. Proc. Amer. Math. Soc. 60, 309 - 316 (1976)

10. HIRSCH,M.W., PUGH,C.C.: Cohomology of chain recurrent sets. Ergodic Th. Dyn. Sys. 8, 73 - 80 (1988)

11. HU,S.T.: Theory of Retracts. Detroit: Wayne State Univ. Press 1965

12. KURATOWSKI,K.: Topology, vol. 2. New York/London: Academic Press 1968

Department of Mathematics,
University of Crete,
GR - 71409 Iraklion,
Greece

(Received January 4, 1991;
in revised form May 15, 1991)