COHOMOLOGY AND ASYMPTOTIC STABILITY OF 1-DIMENSIONAL CONTINUA

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We prove that a 1-dimensional continuum carrying a flow without singular points is homeomorphic to the unit circle if its first Čech cohomology group with integer coefficients is isomorphic to $\mathbb{Z}$. As an application of this we obtain that an asymptotically stable invariant 1-dimensional continuum of a flow on a locally compact ANR, which does not contain singular points, must be a periodic orbit.

1. Introduction

Two of the most interesting problems in the theory of dynamical systems are to determine the structure of the limit sets and describe the behavior of the orbits near them. A serious step in this direction is the study of minimal sets, since every compact limit set contains a minimal set.

The original motivation of this note is the problem of finding conditions referring to the behavior of the orbits near a compact minimal set of a flow under which the minimal set is a periodic orbit. G. Allaud and E.S. Thomas have shown in [1; Theorem 3.4] that an almost periodic asymptotically stable compact minimal set of a flow on a regular manifold is a torus. The almost periodicity assumption plays an important role in the proof of this result.

It is shown in paragraph 4 of the present note that every asymptotically stable invariant 1-dimensional continuum $A$ of a flow on a locally compact ANR, which does not contain singular points, is a
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periodic orbit. This is based on the one hand on the observation that
the first Čech cohomology group with integer coefficients of \( A \) is
isomorphic to \( \mathbb{Z} \) and on the other hand on the fact that a 1-dimen-
sional continuum with first Čech cohomology group \( \mathbb{Z} \), which carries
a flow without singular points, must be homeomorphic to \( S^1 \), which is
proved in paragraph 3.

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2. Preliminaries and notations

Let \((R, X, \phi)\) denote a (continuous) flow on a metric space \(X\).
We shall use the convenient notation \(\phi(t,x) = tx\) and \(\phi(I \times A) = I A\), if
\(I \subset R\) and \(A \subset X\). The orbit of the point \(x \in X\) is denoted by
\(C(x)\), the positive limit set by \(L^+(x)\) and the negative by \(L^-(x)\).

Given \(\varepsilon, T > 0\) an \((\varepsilon, T)\)-chain from \(x\) to \(y\) is a pair of
finite sets of points \(\{x_0, \ldots, x_{p+1}\}\) of \(X\) and times \(\{t_0, \ldots, t_p\}\)
such that \(x_0 = x\), \(x_{p+1} = y\), \(t_j \geq T\) and \(d(t_j x_j, x_{j+1}) < \varepsilon\), \(0 \leq j \leq p\),
where \(d\) is the metric in \(X\). A point \(x\) is called chain recurrent
if for every \(\varepsilon, T > 0\) there exists an \((\varepsilon, T)\)-chain from \(x\) to \(x\)
and the flow on \(X\) is called chain recurrent if every point of \(X\) is
chain recurrent. It is well known \([9; Theorem 3.1]\) that a flow
\((R, X, \phi)\) on a continuum \(X\) is chain recurrent if and only if there
is a flow on some compact metric space \(Z\) and \(z \in Z\) such that
\((R, X, \phi)\) is topologically equivalent to the flow on \(L^+(z)\).

A flow on a compact metric space \(X\) is called minimal if every
orbit is dense in \(X\). In this case \(X\) is called a minimal set and
clearly the flow is chain recurrent.

In the sequel we shall make use of the following representation of
the first Čech cohomology group with integer coefficients \(\tilde{H}^1(X)\) of
a compact metric space $X$. Let $[X; S^1]$ denote the set of homotopy classes of continuous maps of $X$ into $S^1$. The usual multiplication in $S^1$ induces a group operation in $[X; S^1]$. Let $\theta$ be the generator of $\tilde{H}^1(S^1)$. To every continuous map $f : X \to S^1$ corresponds the cohomology class $f^*(-\theta)$ in $\tilde{H}^1(X)$, which depends only on the homotopy class of $f$, where $f^*$ is the induced homomorphism in Čech cohomology. The function $\chi : [X; S^1] \to \tilde{H}^1(X)$ defined in this way is an isomorphism of abelian groups [7; Theorem 8.1]. It follows easily from this representation that $\tilde{H}^1(X)$ is torsion free [12; p. 409].

3. Flows on 1-dimensional continua

Let $(\mathbb{R}, X, \phi)$ be a flow on a compact metric space $X$ such that not every point is singular. Let $S_0$ be a local section in $X$ of extent $\varepsilon_0 > 0$, so that $\phi$ maps $(-\varepsilon_0, \varepsilon_0) \times S_0$ homeomorphically onto an open subset of $X$. Suppose that $S$ is an open-compact subset of $S_0$ and $0 < \varepsilon < \varepsilon_0 / 2$. Then $\phi$ maps $[-2\varepsilon, 2\varepsilon] \times S$ homeomorphically onto a compact neighbourhood of $S$. To such a local section corresponds a continuous map $f : X \to S^1$ defined by

$$f(x) = \begin{cases} \pi(0), & \text{if } x \in X - [0, \varepsilon)S \\ \pi(t/\varepsilon), & \text{if } x \in tS \text{ and } 0 \leq t \leq \varepsilon \end{cases}$$

where $\pi : \mathbb{R} \to S^1$ is the canonical covering projection. Clearly, the homotopy class of $f$ does not depend on $\varepsilon$ but only on $S$. The map $f$ is called the cosection map associated to $S$ and was first defined by M.W. Hirsch and C.C. Pugh in [10]. The next Lemma concerning cosection maps is of fundamental importance for what follows.

**Lemma 3.1.** Let $(\mathbb{R}, X, \phi)$ be a flow on a compact metric space $X$ and $S_0$ a local section in $X$ of extent $\varepsilon_0 > 0$. Let $P, Q$ be open-compact subsets of $S_0$ with $P \cap Q = \emptyset$ and $0 < \varepsilon < \varepsilon_0 / 2$. Let $h, g : X \to S^1$ be the cosection maps associated to $P$ and $Q$ respectively, as defined above. We assume that the set $S = P \cup Q$ contains a chain recurrent point $x_0$ and that for some non-zero integers $m, n$ the map $h^m g^n : X \to S^1$ is nullhomotopic. Then, $m n < 0$.

**Proof.** Since $h^m g^n$ is assumed to be nullhomotopic, there is a
continuous map \( \alpha : X \to \mathbb{R} \) such that

\[
h^m(x) \cdot g^n(x) = \pi(\alpha(x))
\]

for every \( x \in X \) [8; Ch. XVII, 5.1]. Let \( \beta : X \to \mathbb{R} \) be the function defined by

\[
\beta(x) = \begin{cases} 
0 & \text{if } x \in X-[0,\varepsilon]S \\
\frac{mt}{\varepsilon} & \text{if } x \in tP \text{ and } 0 \leq t \leq \varepsilon \\
\frac{nt}{\varepsilon} & \text{if } x \in tQ \text{ and } 0 \leq t \leq \varepsilon 
\end{cases}
\]

Obviously, \( \beta \) is everywhere continuous except at the points of \( \varepsilon S \) and \( \pi(\alpha(x)) = \pi(\beta(x)) \) for every \( x \in X \). Moreover, the restriction of \( \beta \) to \([-\varepsilon,\varepsilon]S \) is continuous. Let \( \gamma : X \to \mathbb{Z} \) be defined by

\[
\gamma(x) = \alpha(x) - \beta(x)
\]

Since the restriction of \( \gamma \) to \([-\varepsilon,\varepsilon]S \) is continuous, there is a covering \( \{W_1, \ldots, W_q\} \) of \( S \) by disjoint, open-closed subsets of \( S \) such that \( \gamma \) takes the value \( k_i \) on the set \([-\varepsilon,\varepsilon]W_i \), \( 1 \leq i \leq q \). We may assume that each \( W_i \) is contained in \( P \) or in \( Q \). On the other hand, if \( x_k \to x \) where \( x \in \varepsilon W_i \) and \( x_k \in X-[0,\varepsilon]S \), then \( \alpha(x_k) \to \alpha(x) \) and therefore \( \gamma(x_k) \) is eventually constant and equal to \( m+k_i \) or \( n+k_i \), depending upon whether \( W_i \) is contained in \( P \) or in \( Q \) respectively. This means that for some \( 0 < \delta < \varepsilon \) we have

\[
\gamma(x) = \begin{cases} 
k_i & \text{if } x \in [-\varepsilon,\varepsilon]W_i \\
m+k_i & \text{if } x \in (\varepsilon,\varepsilon+\delta]W_i \text{ and } W_i \subseteq P \\
n+k_i & \text{if } x \in (\varepsilon,\varepsilon+\delta]W_i \text{ and } W_i \subseteq Q
\end{cases}
\]

for every \( x \in [-\varepsilon,\varepsilon+\delta]W_i \), \( 1 \leq i \leq q \).

Since \( \gamma \) is continuous on \( X-[0,\varepsilon+(\delta/2)]S \), the family of sets \( \{(-\varepsilon,\varepsilon+\delta)W_i : 1 \leq i \leq q\} \cup \{(\gamma^{-1}(k) \cap (X-[0,\varepsilon+(\delta/2)]S) : k \in \mathbb{Z}\} \) is an open covering of \( X \) for which there is a Lebesgue number \( \eta > 0 \) so small that the ball of radius \( \eta \) centered at \( x_0 \) is contained in some set \((-\varepsilon,\varepsilon)W_i \). By chain recurrence, there are in \( X \) points \( x_1, \ldots, x_{p+1} \) and times \( t_0, \ldots, t_p \) such that \( x_{p+1} = x_0 \), \( t_j \geq 2\varepsilon \) and \( d(t_jx_j, x_{j+1}) < \eta \), \( 0 \leq j \leq p \). Note that since \( t_0 \geq 2\varepsilon \), the orbit segment \([0,t_0)x_0 \) crosses \( \varepsilon S \) and hence \( \gamma(t_0x_0) = \gamma(x_0) + \lambda_0 \), where \( \lambda_0 \) is a sum each term of which is equal to \( m \) or \( n \). If \( 1 \leq j \leq p \),
then \( y(t_jx_j) = y(x_j) + \lambda_j \), where \( \lambda_j \) is a sum like \( \lambda_0 \).
If \( y(t_jx_j) \neq y(x_{j+1}) \), then both points \( t_jx_j \) and \( x_{j+1} \) belong to some \((-\epsilon, \epsilon + \delta)W_i\) and either \( t_jx_j \in (-\epsilon, \epsilon]W_i \) and \( x_{j+1} \in (\epsilon, \epsilon + \delta)W_i \) or \( x_{j+1} \in (-\epsilon, \epsilon]W_i \) and \( t_jx_j \in (\epsilon, \epsilon + \delta)W_i \). In the first case we have \( y(x_{j+1}) = y(t_jx_j) + \lambda \), where \( \lambda \) is equal to \( m \) or to \( n \) and therefore \( y(x_{j+1}) = y(x_j) + \mu_j \), where \( \mu_j \) is a sum like \( \lambda_0 \). In the second case we have \( y(x_{j+1}) = y(t_jx_j) - \lambda \) and the last term of \( \lambda_j \) is \( \lambda \). Hence \( y(x_{j+1}) = y(x_j) + \mu_j \), where \( \mu_j \) is zero or a sum like \( \lambda_0 \). We conclude that there are integers \( \mu_0, \ldots, \mu_p \) such that each \( \mu_j \) is zero or a sum of which is \( m \) or \( n \), the later occurring at least for \( \mu_0 \), and \( y(x_{j+1}) = y(x_j) + \mu_j \), \( 0 \leq j \leq p \). Since \( x_0 = x_{p+1} \) we have that \( \mu_0 + \ldots + \mu_p = 0 \). It follows that \( m - n < 0 \).

Using Lemma 3.1 we may now reprove the main result of [10].

**Theorem 3.2** Let \((\mathbb{R}, X, \phi)\) be a flow on a compact metric space \( X \) and \( S_0 \) be a local section in \( X \). If \( S \) is an open-compact subset of \( S_0 \) which contains a chain recurrent point, then the cosection map associated to \( S \) defines a non-zero element of \( \tilde{H}^1(X) \).

**Proof.** Taking in Lemma 3.1 \( P = S, Q = \emptyset \) and \( m = n = 1 \) we conclude that the cosection map \( h : X + S^1 \) associated to \( S \) cannot be null-homotopic. Hence \( h \) defines a non-zero element of \( \tilde{H}^1(X) \).

**Theorem 3.3** Let \((\mathbb{R}, X, \phi)\) be a flow on a 1-dimensional compact metric space \( X \). If there is a nonsingular chain recurrent point \( x \) in \( X \), then \( \tilde{H}^1(X) \) is not trivial.

**Proof.** Since \( x \) is nonsingular, there exists \( \epsilon_0 > 0 \) and a locally compact local section \( S_0 \) containing \( x \), so that \( \phi \) maps \( (-\epsilon_0, \epsilon_0) \times S_0 \) homeomorphically onto an open neighbourhood of \( x \) [4; Lemma 1]. It follows that \( \dim S_0 = 0 \). Thus, \( x \) has an open-compact neighbourhood \( S \) in \( S_0 \). According to Theorem 3.2 the cosection map associated to \( S \) defines a non-zero element of \( \tilde{H}^1(X) \).

**Proposition 3.4** Let \((\mathbb{R}, X, \phi)\) be a chain recurrent flow on a 1-dimensional continuum \( X \). If \( \tilde{H}^1(X) \neq \mathbb{Z} \), then for every nonsingular point \( x \in X \) there exists \( \epsilon > 0 \) such that the orbit segment...
(-ε,ε)x is an open neighbourhood of x in X.

Proof. Let $S_0$ be a locally compact local section at x. Then $\dim S_0 = 0$. Let $S, Q$ be open-compact neighbourhoods of x in $S_0$ such that $Q$ is properly contained in $S$. If $f, g$ and $h$ are the cosection maps associated to $S, Q$ and $S-Q$ respectively, then $f = gh$. By Theorem 3.2 $h$ is not nullhomotopic. Therefore, $f$ and $g$ are not homotopic to each other and represent different elements of $\tilde{H}^1(X)$. Moreover, $f^k$ is not homotopic to $g^k$ for every integer $k$, because $\tilde{H}^1(X)$ is torsion free. Since we assume that $\tilde{H}^1(X) \cong \mathbb{Z}$, the cosection maps $f, g$ correspond to integers $n, m$ respectively. We shall show that $0 < |m| < |n|$.

We first observe that $n \cdot m \neq 0$ and $n \neq m$. From the commutativity of the multiplication of integers follows that $f^m$ is homotopic to $g^n$ or equivalently that $f^m g^{-n} : X \to S^1$ is nullhomotopic. However,

$$f^m g^{-n} = (f^m g^{-n})^m g^m = h^m g^m = h^m g^m = h^m g^m g^{-1} = f^m g^{-n}.$$

It follows from Lemma 3.1 that $m \cdot (m-n) < 0$, that is $0 < |m| < |n|$.

Now let $\{ S_k : k \in \mathbb{N} \}$ be a decreasing neighbourhood basis of x in $S_0$ consisting of open-compact sets. In order to prove the proposition, it suffices to show that this neighbourhood basis is eventually constant. Let $f_k$ be the cosection map associated to $S_k$. Since $\tilde{H}^1(X) \cong \mathbb{Z}$, the homotopy class of $f_k$ corresponds to a non-zero integer $n_k$. If each $S_{k+1}$ was properly contained in $S_k$, then by what we have shown above the sequence $\{ |n_k| : k \in \mathbb{N} \}$ would be strictly decreasing, which is impossible.

**THEOREM 3.5** Let $(\mathbb{R}, X, \phi)$ be a flow without singular points on a 1-dimensional continuum $X$. If $\tilde{H}^1(X) \cong \mathbb{Z}$, then $X$ is homeomorphic to $S^1$.

Proof. Let $x \in X$ and $A = L^+(x)$. Since $A$ is compact and there are no singular points, $A$ is a 1-dimensional continuum. The flow on $A$ is chain recurrent and hence $\tilde{H}^1(A)$ is not trivial by Theorem 3.3. On the other hand, the inclusion $i : A \subset X$ induces an epimorphism $i^* : \tilde{H}^1(X) \to \tilde{H}^1(A)$ [5]. It follows that $\tilde{H}^1(A) \cong \mathbb{Z}$ and therefore $A$ is a periodic orbit by Proposition 3.4. Similarly, $L(x)$ is a
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periodic orbit. We conclude that the orbit $C(x)$ must be periodic, because otherwise we have $\tilde{H}^1(C(x)) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is impossible [5]. We have thus proved that every orbit in $X$ is periodic. By Proposition 3.4, $X$ must be homeomorphic to $S^1$.

EXAMPLES 3.6 (a) It is clear that Theorem 3.5 is not true without the assumption $\tilde{H}^1(x) \cong \mathbb{Z}$. It is however noteworthy that there are 1-dimensional continua $X$ with $\tilde{H}^1(x) \cong \mathbb{Z} \oplus \mathbb{Z}$ and carrying flows without singular points which differ drastically from each other. Suppose that $X$ consists of a circle together with an orbit that spirals in positive and negative time against the circle or two circles with an orbit whose positive resp. negative limit set is the first resp. second circle. In both cases $\tilde{H}^1(x) \cong \mathbb{Z} \oplus \mathbb{Z}$. There is another example completely different from the above. More precisely, there is a flow on the torus $T^2$ with a nonperiodic 1-dimensional minimal set $X$ such that $T^2 - X$ is homeomorphic to $\mathbb{R}^2$ [6]. By duality we have again $\tilde{H}^1(x) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(b) If the assumption of the non-existence of singular points in Theorem 3.5 is removed, then the conclusion fails. Consider the flow on the closed unit disc $D^2$ which fixes the origin and such that every orbit in the interior of $D^2$ tends in negative time to the origin and in positive time spirals against the boundary which is a periodic orbit. If $x \in \text{int}D^2$, then $X = \overline{C(x)}$ is a 1-dimensional continuum with $\tilde{H}^1(x) \cong \mathbb{Z}$, which is not homeomorphic to $S^1$ and carries a flow with one singular point.

4. Asymptotically stable 1-dimensional continua

Let $(R, M, \phi)$ be a flow with $M$ a locally compact ANR (for metrizable spaces) and let $A \subset M$ be an asymptotically stable compact invariant set. The region of attraction $E$ of $A$ is an open invariant neighbourhood of $A$ and there exists a strictly decreasing uniformly unbounded Lyapunov function $F : E \to R^+$ for $A$ [2; Ch. V, Theorem 2.9]. For any $c > 0$ the set $B = F^{-1}((0,c])$ is a positively invariant compact retract of $E$. Since $E$ is open in $M$, the set $B$ is a compact ANR. Therefore, the Čech cohomology of $B$ coincides with the singular cohomology of $B$ and is finitely generated [11; Ch. IV,
Section 7. For $0 < b < c$, the inclusion $F^{-1}([0,b]) \subset F^{-1}([0,c])$ induces an isomorphism in cohomology. From the continuity property of the Čech cohomology follows that $\check{H}^*(A)$ is isomorphic to $\check{H}^*(B)$. Hence $\check{H}^*(A)$ is finitely generated.

**LEMMA 4.1** If $A$ does not contain singular points, then

$$\sum_{q=0}^{\infty} (-1)^q \text{rank} \check{H}^q(A) = 0 .$$

**Proof.** For each $t > 0$ the continuous map $\phi_t : M \to M$ defined by $\phi_t(x) = tx$ is homotopic to the identity and sends $B$ into $B$, where $B$ is the above defined set. Since $A$ is compact and does not contain singular points, the periods of the periodic points in $A$ are bounded away from zero [2; Ch. V, Lemma 3.7]. Let $t > 0$ be smaller than any period of a periodic point in $A$. Then $\phi_t$ has no fixed points in $B$. Since $B$ is a compact ANR, the conclusion follows from the Lefschetz fixed point theorem for compact ANRs [3; Ch. III, Theorem 2].

**THEOREM 4.2** Let $(\mathbb{R}, M, \phi)$ be a flow with $M$ a locally compact ANR and $A \subset M$ an asymptotically stable 1-dimensional invariant continuum. If $A$ does not contain singular points, then $A$ is a periodic orbit.

**Proof.** Since $A$ is 1-dimensional, $\check{H}^q(A) = 0$, for $q > 1$. Thus, from Lemma 4.1 we have $\text{rank} \check{H}^1(A) = 1$. Since $\check{H}^1(A)$ is a finitely generated, torsion free abelian group, it must be isomorphic to $\mathbb{Z}$. According now to Theorem 3.5, $A$ is a periodic orbit.

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