On the existence of absolutely continuous automorphic measures

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Abstract. Let X be a compact metric space and $T : X \to X$ a continuous surjection. We present sufficient conditions which imply the existence of absolutely continuous automorphic measures for T with respect to a given ergodic T-invariant Borel probability measure. The same conditions give measurable or L^∞ solutions of the corresponding cohomological equation.

1. Introduction.

Let X be a compact metric space, $T : X \to X$ a continuous surjection and let $f : X \to \mathbb{R}$ be a continuous function. A Borel probability measure ν on X is called an e^f-automorphic measure for T if ν is equivalent to $T_*\nu$ and $\dfrac{d\nu}{d(T_*\nu)} = e^f$. This kind of measure has been used without a particular name in [4].

In this note we study the existence of absolutely continuous automorphic measures with respect to a given ergodic T-invariant Borel probability measure. We present a sufficient condition for the existence of an absolutely continuous automorphic measure for a continuous surjection. The problem of the existence of an e^f-automorphic measure ν for a homeomorphism T which is absolutely continuous with respect to an ergodic T-invariant Borel probability measure μ is closely related to the existence and regularity properties of solutions of the cohomological equation $f = u - u \circ T$. This relation is explained with details in section 2. If there exists a continuous solution u, then f is called a continuous coboundary. According to the classical Gottschalk-Hedlund theorem (see page 102 in [3]), if T is minimal, then f is a continuous coboundary if and only if there exists $x_0 \in X$ such that

$$\sup\{|\sum_{k=0}^{n-1} f(T^k(x_0))| : n \in \mathbb{N}\} < +\infty.$$

The main result is Theorem 3.5 which can be stated as follows.

Main Theorem. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$*, where* $E_n(f) = e^{S_n(f)}$ *and* $S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k$*, then there exists an* e^f*-automorphic measure* ν *for* T *which is absolutely continuous with respect to* μ*. Moreover,* $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ *and* $-\log(\frac{d\nu}{d\mu})$ *is a measurable solution of the cohomological equation* $f = u - u \circ T$*.* $\square$

In Theorem 3.7 we give sufficient condition for a continuous function $f : X \to \mathbb{R}$ such that $\int_X f d\mu = 0$ to be an $L^\infty(\mu)$ coboundary, with respect to an ergodic T-invariant measure μ, without the minimality assumption for T. This condition is stronger than the one given in the main Theorem 3.5, and the result is obtained by investigating the logarithm of the Radon-Nikodym derivative $\log(\frac{d\nu}{d\mu})$.

2. Automorphic measures.

Let $T : X \to X$ be a continuous surjection, where X is a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. An e^f-automorphic measure for T is a Borel probability measure ν on X such that

$$\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$. Evidently, an e^f-automorphic measure for T is T-quasi-invariant. In case T is a homeomorphism, a e^f-automorphic measure for T is an $e^{-f\circ T^{-1}}$-automorphic measure for T^{-1}.

It is easy to see that if $h : X \to X$ is a homeomorphism and $S = h \circ T \circ h^{-1}$, then $h_*\nu$ is an $e^{f\circ h^{-1}}$-automorphic measure for S for every e^f-automorphic measure ν for T.

In the case of a homeomorphism T the construction of an automorphic measure can be described as follows (see [1]). Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $c = \limsup_{n\to+\infty} \frac{a_n}{n}$. The series $\sum_{n=1}^{\infty} e^{a_n - ns}$ converges for $s > c$, diverges for $s < c$ and we cannot tell for $s = c$, by the root test. There exists a sequence of positive real numbers $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \frac{b_n}{b_{n+1}} = 1$ and the series $\sum_{n=1}^{\infty} b_n e^{a_n - ns}$ converges for $s > c$ and diverges for $s \le c$.

Let $f : X \to \mathbb{R}$ ba a continuous function such that $\int_X f d\mu = 0$ for some ergodic T-invariant Borel probability measure μ. It is well known that the set of points $x \in X$ such that the limit

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x))$$

exists in $\mathbb{R}$ has measure 1 with respect to every T-invariant Borel probability measure, and is therefore non-empty. So there exists a point $x \in X$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x)) = 0.$$

Let $a_n = -\sum_{k=1}^{n} f(T^{-k}(x))$ and $M_s = \sum_{n=1}^{\infty} b_n e^{a_n - ns}$, where $(b_n)_{n\in\mathbb{N}}$ is the corresponding sequence as above. Each accumulation point in the weak* topology as $s \downarrow 0$ of the directed family

$$\mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \delta_{T^{-n}(x)}, \quad s > 0$$

is an e^f-automorphic measure for T.

There is a relation between e^f-automorphic measures for a continuous surjection $T : X \to X$ of a compact metric space and solvability of the cohomological equation $f = u - u \circ T$, where $f : X \to \mathbb{R}$ is continuous.

Let μ be any T-invariant Borel probability measure. If there exists a measurable solution u of the above cohomological equation defined μ-almost everywhere such that $e^{-u} \in L^1(\mu)$, then there exists an e^f-automorphic measure ν for T equivalent to μ with density

$$\frac{d\nu}{d\mu} = \frac{e^{-u}}{\displaystyle\int_X e^{-u} d\mu}.$$

Thus, if there exists a continuous solution u, then for every T-invariant Borel probability measure we get an equivalent e^f-automorphic measure for T. Moreover, in this case, every e^f-automorphic measure ν for T is obtained in this way. Indeed, we have

$$\int_X \phi e^u d\nu = \int_X (\phi \circ T) e^u d\nu$$

for every continuous function $\phi : X \to \mathbb{R}$, and so the equivalent measure μ to ν with density

$$\frac{d\mu}{d\nu} = \frac{e^u}{\displaystyle\int_X e^u d\nu}$$

is T-invariant. Consequently, if f is a continuous coboundary, then the e^f-automorphic measures for T are in one-to-one correspondence with the T-invariant Borel probability measures and each e^f-automorphic measure for T is equivalent to its corresponding T-invariant measure.

Conversely, suppose that μ is an ergodic T-invariant Borel probability measure and $f : X \to \mathbb{R}$ is a continuous function such that $\int_X f d\mu = 0$. Suppose further that there exists an e^f-automorphic measure $\nu \in \mathcal{M}(X)$ for T which is absolutely continuuous with respect to μ and $g = \frac{d\nu}{d\mu}$. For every measurable set $A \subset X$ we have

$$\int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T) e^f d\nu = \int_X (\chi_A \circ T) e^f g d\mu$$

and therefore

$$\int_{T^{-1}(A)} [g e^f - (g \circ T)] d\mu = 0.$$

Since μ is T-invariant, it follows that $g \circ T = g e^f$ μ-almost everywhere. The ergodicity of μ implies now that $g > 0$ μ-almost everywhere. So, $u = -\log g$ is a measurable solution of the cohomological equation $f = u - u \circ T$. If $\log g \in L^\infty(\mu)$ and T is a minimal homeomorphism or T is a locally eventually onto local homeomorphism and μ has full support, then there exists some continuous function $u : X \to \mathbb{R}$ such that $f = u - u\circ T$, by Proposition 4.2 on page 46 in [2] and Theorem 6 in [6], respectively.

Remark 2.1. It is a general fact that every quasi-invariant measure of a homeomorphism which is is absolutely continuous with respect to an ergodic invariant measure is actually equivalent to it. Indeed, let $T : X \to X$ be a homeomorphism of a compact metric space X and μ be an ergodic T-invariant Borel probability measure. If ν is a T-quasi-invariant Borel probability measure which is absolutely continuous with respect to μ, then ν is equivalent to μ. Indeed, let $g = \dfrac{d\nu}{d\mu}$ and $A = g^{-1}(0)$. If $S = \bigcup_{n\in\mathbb{Z}} T^n(A)$, then S is T-invariant and $\nu(S) = 0$. On the other hand $\mu(X \setminus S) > 0$, and since μ is ergodic we get $\mu(S) = 0$, that is $g > 0$ μ-almost everywhere.

3. Absolutely continuous automorphic measures.

Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. The set

$$E_\mu = \{\nu \in \mathcal{M}(X) : \nu \ll \mu\}$$

is not empty, since it contains μ, and is convex. In general, E_μ is not a closed subset of $\mathcal{M}(X)$ with respect to the weak* topology. For example, if we let μ be the Lebesgue measure on the unit interval $[0, 1]$ and for $0 < \epsilon < 1$ we let μ_ϵ denote the Borel probability measure on $[0, 1]$ with density $\dfrac{1}{\epsilon}\chi_{[0,\epsilon]}$, then $\lim_{\epsilon\to 0} \mu_\epsilon$ is the Dirac point measure at 0.

Lemma 3.1. *Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence in E_μ converging weakly* to some $\nu \in \mathcal{M}(X)$ and let $f_n = \dfrac{d\nu_n}{d\mu}$, $n \in \mathbb{N}$. If there exist non-negative h, $g \in L^1(\mu)$ such that $h \le f_n \le g$ for every $n \in \mathbb{N}$, then $\nu \in E_\mu$ and $h \le \dfrac{d\nu}{d\mu} \le g$.*

Proof. Since ν is a finite measure, there exists a (countable) basis $\mathcal{U}$ of the topology of X such that $\nu(\partial U) = 0$ for every $U \in \mathcal{U}$. So $\mathcal{U}$ is contained in the algebra

$$\mathcal{C}(\nu) = \{A | A \subset X \text{ Borel and } \nu(\partial A) = 0\}$$

and since it generates the Borel σ-algebra of X, so does $\mathcal{C}(\nu)$. Let now $A \subset X$ be a Borel set with $\mu(A) = 0$ and $\epsilon > 0$. There exists $0 < \delta < \epsilon$ such that $\int_B g d\mu < \epsilon$ for every Borel set $B \subset X$ with $\mu(B) < \delta$, because $g \in L^1(\mu)$. There exists some $A_0 \in \mathcal{C}(\nu)$ such that $\mu(A \triangle A_0) < \delta$ and $\nu(A\triangle A_0) < \delta$. Thus $\mu(A_0) < \delta$ and $|\nu(A) - \nu(A_0)| < \delta$. By weak* convergence, $\nu(A_0) = \lim_{n\to+\infty} \nu_n(A_0)$ and so there exists some $n_0 \in \mathbb{N}$ such that $|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \ge n_0$. Therefore,

$$\nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \le \int_{A_0} g d\mu + \epsilon < 2\epsilon.$$

It follows that $0 \le \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in E_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of X such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, the Borel σ-algebra of X is generated by $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ and $\mu(\partial B) = 0$ for every $B \in \mathcal{P}_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of X such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively $\mathcal{P}_n$ to be the finite family consisting of Borel sets with positive μ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in \mathcal{P}_{n-1}$, taking $\mathcal{P}_0 = \{X\}$.

Let $\mathcal{P}_n(x)$ denote the element of $\mathcal{P}_n$ which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n\to+\infty} \frac{\nu(\mathcal{P}_n(x))}{\mu(\mathcal{P}_n(x))},$$

μ-almost everywhere on X and in $L^1(\mu)$ (see page 8 in [5]). On the other hand, by the weak* convergence and since $\nu \in E_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k}\mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \le \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} < \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} f_{n_k} d\mu \le \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} g d\mu.$$

Since

$$\lim_{k\to+\infty} \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} g d\mu = g(x)$$

μ-almost everywhere on X and in $L^1(\mu)$, it follows that $0 \le \frac{d\nu}{d\mu}(x) \le g(x)$ μ-almost everywhere on X.

Similarly, from

$$\frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} > -\frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} f_{n_k} d\mu \ge -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))}\int_{\mathcal{P}_k(x)} h d\mu$$

follows that $h(x) \le \frac{d\nu}{d\mu}(x)$ μ-almost everywhere on X. $\Box$

Let X be a compact metric space and $T : X \to X$ a continuous surjection. For any continuous function $f : X \to \mathbb{R}$ we put $S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k$ and $E_n(f) = e^{S_n(f)}$. Let $M_n = \sup\{S_n(f)(x) : x \in X\}$ and $L_n = \inf\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{N}$. Since $S_n(f) \circ T = S_{n+1}(f) + f$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T)e^{-f} - g_n = E_n(f) - e^{-f}.$$

Let now $\mu \in \mathcal{M}(X)$ be T-invariant and suppose that $\int_X f d\mu = 0$. So, $L_n \leq 0 \leq M_n$ for every $n \in \mathbb{N}$. Putting $h_n = \dfrac{g_n}{\int_X g_n d\mu}$, we get

$$(h_n \circ T) - h_n e^f = \frac{e^f - e^{-S_n(f)}}{e^{-S_n(f)} \int_X g_n d\mu},$$

for every $n \in \mathbb{N}$.

Suppose that there exists a positive $h \in L^1(\mu)$ such that $E_n(f) \leq h \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$. Then also $0 \leq h_n \leq h$ for $n \in \mathbb{N}$. If ν_n denotes the element of E_μ with $h_n = \dfrac{d\nu_n}{d\mu}$, then $\overline{\{\nu_n : n \in \mathbb{N}\}} \subset E_\mu$, by Lemma 3.1.

Proposition 3.2. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be T-invariant and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that*

(i) there exists a positive $h \in L^1(\mu)$ such that $E_n(f) \leq h \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and

(ii) the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}$, $n \in \mathbb{N}$, is unbounded.

Then there exists an e^f-automorphic measure for T which is absolutely continuous with respect to μ.

Proof. Using the above notations, it suffices to prove that there exists a sequence of positive integers $n_j \to +\infty$ such that $\lim_{j \to +\infty} \big((h_{n_j} \circ T) - h_{n_j} e^f\big) = 0$ μ-almost everywhere on X. Indeed, passing to a subsequence if necessary, there exists $\nu \in E_\mu$ such that $\nu = \lim_{j \to +\infty} \nu_{n_j}$, by Lemma 3.1. Since μ is T-invariant, for every continuous function $\phi : X \to \mathbb{R}$ we have

$$\int_X \big(\phi - (\phi \circ T) e^f\big) d\nu = \lim_{j \to +\infty} \int_X (\phi \circ T)\big((h_{n_j} \circ T) - h_{n_j} e^f\big) d\mu = 0,$$

by dominated convergence, because

$$|(\phi \circ T)((h_n \circ T) - h_n e^f)| \leq \|\phi\|((h \circ T) + h e^f) \in L^1(\mu).$$

Since

$$|(h_n \circ T) - h_n e^f| = e^f \frac{|E_n(f) - e^{-f}|}{\int_X g_n d\mu},$$

we need only prove that there exist $n_j \to +\infty$ such that

$$\lim_{j \to +\infty} \mu(\{x \in X : |E_{n_j}(f)(x) - e^{-f(x)}| \geq \delta \int_X g_{n_j} d\mu\}) = 0$$

for every $\delta > 0$. Let

$$A_{n,\delta} = \{x \in X : E_n(f)(x) \geq e^{-f(x)} + \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x)\}, \quad \text{and}$$

$$A'_{n,\delta} = \{x \in X : E_n(f)(x) \leq e^{-f(x)} - \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x)\}.$$

Our assumption (i) implies that it suffices to prove the existence of a sequence of positive integers $n_j \to +\infty$ such that $\lim_{j\to+\infty} \mu(A_{n_j,\delta}) = \lim_{j\to+\infty} \mu(A'_{n_j,\delta}) = 0$ for every $\delta > 0$.

For every $x \in A_{n,\delta}$ we have

$$\frac{h(x)}{\delta} > e^{-M_n} \sum_{k=0}^{n-1} E_k(f)(x)$$

and integrating over $A_{n,\delta}$ we obtain

$$\frac{1}{\delta} \int_X h d\mu \geq \mu(A_{n,\delta}) e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}.$$

Similarly, for every $x \in A'_{n,\delta}$ we have

$$\sum_{k=0}^{n-1} E_k(f)(x) < \frac{h(x)}{\delta} e^{-f(x)}$$

and integrating over $A'_{n,\delta}$ we get

$$\mu(A'_{n,\delta}) \sum_{k=0}^{n-1} e^{L_k} \leq \frac{1}{\delta} \int_X h e^{-f} d\mu.$$

Our assumption (ii) means that there exist $n_j \to +\infty$ such that $e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{L_k} \to +\infty$, and therefore we also have $\sum_{k=0}^{n_j-1} e^{L_k} \to +\infty$, because $L_n \leq 0 \leq M_n$. Consequently, $\lim_{j\to+\infty} \mu(A_{n_j,\delta}) = \lim_{j\to+\infty} \mu(A'_{n_j,\delta}) = 0$. □

In the next proposition we make a more restrictive assumption (i) and a weaker assummption (ii).

Proposition 3.3. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be T-invariant and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that*

(i) there exists a conatsnt $c \geq 1$ such that $E_n(f) \leq c \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and

(ii) the sequence $e^{-M_n}\sum_{k=0}^{n-1}e^{M_k}$*,* $n\in\mathbb{N}$*, is unbounded,*

Then there exists an e^f*-automorphic measure for* T *which is absolutely continuous with respect to* μ*.*

Proof. Our assumption (ii) means that there exists a sequence of positive integers $n_j\to+\infty$ such that $e^{-M_{n_j}}\sum_{k=0}^{n_j-1}e^{M_k}\to+\infty$, as $j\to+\infty$. Using the same notations as above we have $\int_X g_{n_j}d\mu\to+\infty$ and

$$e^{-S_{n_j}}\int_X g_{n_j}d\mu\geq\frac{1}{c}\cdot e^{-M_{n_j}}\sum_{k=0}^{n_j-1}e^{M_k}\to+\infty,$$

as $j\to+\infty$, by our assumptions. Therefore, $\lim_{j\to+\infty}\left((h_{n_j}\circ T)-h_{n_j}e^f\right)=0$ uniformly on X and as in the proof of Proposition 3.2, every $\nu\in\overline{\{\nu_{n_j}:j\in\mathbb{N}\}}$ is e^f-automorphic measure for T that is absolutely continuous with respect to μ. $\square$

As the following Lemma shows, if in Proposition 3.3 the T-invariant measure $\mu\in\mathcal{M}(X)$ is ergodic, then condition (ii) is implied by condition (i).

Lemma 3.4. *Let* X *be a compact metric space and* $T:X\to X$ *a continuous surjection. Let* $\mu\in\mathcal{M}(X)$ *be an ergodic* T*-invariant measure and let* $f:X\to\mathbb{R}$ *be a continuous function such that* $\int_X fd\mu=0$*. Suppose that there exists a constant* $c\geq 1$ *such that*

$$E_n(f)\leq c\int_X E_n(f)d\mu$$

for every $n\in\mathbb{N}$*.*

(a) If $A_n=\{x\in X:S_n(x)>M_n-\log c-1\}$*,* $n\in\mathbb{N}$*, then* $\mu(A_n)\geq\frac{e-1}{ec-1}$ *for* $n\in\mathbb{N}$*.*

(b) For every $N\in\mathbb{N}$ *there exists* $n\in\mathbb{N}$ *such that* $M_{n+j}\leq M_n+1$ *for all* $0\leq j\leq N$*.*

(c) The sequence $e^{-M_n}\sum_{k=0}^{n-1}e^{M_k}$*,* $n\in\mathbb{N}$*, is unbounded.*

Proof. (a) From our assumption we have

$$e^{M_n-\log c}\leq\int_X E_n(f)d\mu\leq e^{M_n}\mu(A_n)+e^{M_n-\log c-1}\mu(X\setminus A_n),$$

from which the required inequality follows.

(b) We proceed to prove the assertion by contradiction assuming that there exists some $N\in\mathbb{N}$ such that for every $n\in\mathbb{N}$ there exists $1\leq j_n\leq N$ such that $M_{n+j_n}>M_n+1$. Inductively, if we put $n_k=1+j_1+\cdots+j_k$, then $M_{n_k}>M_1+k$ and $1+k\leq n_k\leq 1+kN$ for every $k\in\mathbb{N}$. Therefore,

$$\frac{M_{n_k}}{n_k}>\frac{1}{N+1}$$

for every $k \in \mathbb{N}$. If now $k_0 \in \mathbb{N}$ is such that $\left|\dfrac{\log c - 1}{n_k}\right| < \dfrac{1}{2(N+1)}$ for $k \geq k_0$, then for $x \in A_n$ we have

$$\frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N+1)}$$

and by (a) we get

$$\mu(\{x \in X : \frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N+1)}\}) \geq \frac{e-1}{ec-1} > 0$$

for every $k \geq k_0$. Hence the sequence $(\frac{1}{n} S_n)_{n \in \mathbb{N}}$ does not converge in measure to zero. This contradicts the Ergodic Theorem of Birkhoff, since we assume that μ is an ergodic T-invariant Borel probability measure.

(c) Suppose on the contrary that there esists a real number $a > 0$ such that $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k} \leq a$, for every $n \in \mathbb{N}$. By (b), for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$, and so

$$\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} \leq a \sum_{j=0}^{N} e^{M_{n+j}} \leq eae^{M_n} + a \left(\sum_{k=0}^{n+N-1} e^{M_k} - \sum_{k=0}^{n-1} e^{M_k} \right)$$

$$\leq ea(1+a)e^{M_n} - a \sum_{k=0}^{n-1} e^{M_k}.$$

Substituting

$$\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} = (N+1) \sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N-i) e^{M_{n+i}},$$

we arrive at

$$(N+1+a) \sum_{k=0}^{n-1} e^{M_k} + Ne^{M_n} + \sum_{i=1}^{N-1} (N-i) e^{M_{n+i}} \leq ea(1+a)e^{M_n}$$

and therefore $N \leq ea(1+a)$ for every $N \in \mathbb{N}$, contradiction. $\square$

The above immediately imply the following theorem which is the main result of this note.

Theorem 3.5. *Let X be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, then there exists an e^f-automorphic measure ν for T which is absolutely continuous with respect to μ. Moreover, $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ and $-\log(\frac{d\nu}{d\mu})$ is a measurable solution of the cohomological equation $f = u - u \circ T$. $\square$

Corollary 3.6. *Let X be a compact metric space and $T : X \to X$ a continuous surjection which is a locally eventually onto local homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that*

$$\frac{1}{c}\int_X E_n(f)d\mu \leq E_n(f) \leq c\int_X E_n(f)d\mu$$

for every $n \in \mathbb{N}$, then there exists an e^f-automorphic measure ν for T which is absolutely continuous with respect to μ. Moreover, $-\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu)$ and in case μ has full support the cohomological equation $f = u - u \circ T$ has a continuous solution. $\square$

If X is a compact metric space and $T : X \to X$ is a homeomorphism, for any continuous function $f : X \to \mathbb{R}$ we put

$$E_n(f) = \begin{cases} \exp \sum_{k=1}^{n} f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp\,(-\sum_{k=0}^{|n|-1} f \circ T^k), & \text{if } n < 0. \end{cases}$$

As before we also put $S_n(f) = \log E_n(f)$ and $M_n = \sup\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{Z}$.

Let now $\mu \in \mathcal{M}(X)$ be T-invariant and suppose that $\int_X f d\mu = 0$. Then, $M_n \geq 0$ for every $n \in \mathbb{Z}$. Since $S_n(f) \circ T^{-1} = S_{n+1}(f) - f \circ T^{-1}$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T^{-1})e^{f\circ T^{-1}} - g_n = E_n(f) - 1.$$

Putting $h_n = \dfrac{g_n}{\int_X g_n d\mu}$, we get

$$(h_n \circ T^{-1})e^{f\circ T^{-1}} - h_n = \frac{1 - e^{-S_n(f)}}{e^{-S_n(f)}\int_X g_n d\mu},$$

for every $n \in \mathbb{N}$. So the same reasoning as above and Lemma 3.1 give the following.

Theorem 3.7. *Let X be a compact metric space and $T : X \to X$ a homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$.*

(a) If there exists a constant $c \geq 1$ such that

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then there exists an e^f-automorphic measure ν for T which is equivalent to μ such that $\frac{d\nu}{d\mu} \in L^\infty(\mu)$.

(b) Moreover, if

$$\frac{1}{c}\int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$), then $\log(\frac{d\nu}{d\mu}) \in L^\infty(\mu)$. $\square$

Combining Theorem 3.7 with section 2 we get the following.

Corollary 3.8. *Let X be a compact metric space and $T : X \to X$ a minimal homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic T-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Then the following assertions are equivalent.*

(i) f is a continuous coboundary.

(ii) There exists a constant $c \geq 1$ such that

$$\frac{1}{c}\int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$ (or $-n \in \mathbb{N}$). $\square$

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