On the existence of absolutely continuous automorphic measures

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Abstract. Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. We present sufficient conditions which imply the existence of absolutely continuous automorphic measures for $T$ with respect to a given ergodic $T$-invariant Borel probability measure. The same conditions give measurable or $L^\infty$ solutions of the corresponding cohomological equation.

1. Introduction.

Let $X$ be a compact metric space, $T : X \to X$ a continuous surjection and let $f : X \to \mathbb{R}$ be a continuous function. A Borel probability measure $\nu$ on $X$ is called an $e^f$-automorphic measure for $T$ if $\nu$ is equivalent to $T_*\nu$ and $\frac{d\nu}{d(T_*\nu)} = e^f$. This kind of measure has been used without a particular name in [4].

In this note we study the existence of absolutely continuous automorphic measures with respect to a given ergodic $T$-invariant Borel probability measure. We present a sufficient condition for the existence of an absolutely continuous automorphic measure $\nu$ for a continuous surjection. The problem of the existence of an $e^f$-automorphic measure $\nu$ for a homeomorphism $T$ which is absolutely continuous with respect to an ergodic $T$-invariant Borel probability measure $\mu$ is closely related to the existence and regularity properties of solutions of the cohomological equation $f = u - u \circ T$. This relation is explained with details in section 2. If there exists a continuous solution $u$, then $f$ is called a continuous coboundary. According to the classical Gottschalk-Hedlund theorem (see page 102 in [3]), if $T$ is minimal, then $f$ is a continuous coboundary if and only if there exists $x_0 \in X$ such that

$$\sup\{\left|\sum_{k=0}^{n-1} f(T^k(x_0))\right| : n \in \mathbb{N}\} < +\infty.$$ 

The main result is Theorem 3.5 which can be stated as follows.

Main Theorem. Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

1
for every \( n \in \mathbb{N} \), where \( E_n(f) = e^{S_n(f)} \) and \( S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k \), then there exists an \( e^f \)-automorphic measure \( \nu \) for \( T \) which is absolutely continuous with respect to \( \mu \). Moreover, \( \frac{d\nu}{d\mu} \in L^\infty(\mu) \) and \(-\log\left(\frac{d\nu}{d\mu}\right)\) is a measurable solution of the cohomological equation \( f = u - u \circ T \). □

In Theorem 3.7 we give sufficient condition for a continuous function \( f : X \to \mathbb{R} \) such that \( \int_X f \,d\mu = 0 \) to be an \( L^\infty(\mu) \) coboundary, with respect to an ergodic \( T \)-invariant measure \( \mu \), without the minimality assumption for \( T \). This condition is stronger than the one given in the main Theorem 3.5, and the result is obtained by investigating the logarithm of the Radon-Nikodym derivative \( \log\left(\frac{d\nu}{d\mu}\right) \).

### 2. Automorphic measures.

Let \( T : X \to X \) be a continuous surjection, where \( X \) is a compact metric space, and let \( f : X \to \mathbb{R} \) be a continuous function. An \( e^f \)-automorphic measure for \( T \) is a Borel probability measure \( \nu \) on \( X \) such that

\[
\int_X \phi d\nu = \int_X (\phi \circ T)e^f \,d\nu
\]

for every continuous function \( \phi : X \to \mathbb{R} \). Evidently, an \( e^f \)-automorphic measure for \( T \) is \( T \)-quasi-invariant. In case \( T \) is a homeomorphism, an \( e^f \)-automorphic measure for \( T \) is an \( e^{-f \circ T^{-1}} \)-automorphic measure for \( T^{-1} \).

It is easy to see that if \( h : X \to X \) is a homeomorphism and \( S = h \circ T \circ h^{-1} \), then \( h_\ast \nu \) is an \( e^{f \circ h^{-1}} \)-automorphic measure for \( S \) for every \( e^f \)-automorphic measure \( \nu \) for \( T \).

In the case of a homeomorphism \( T \) the construction of an automorphic measure can be described as follows (see [1]). Let \( (a_n)_{n \in \mathbb{N}} \) be a sequence of real numbers and let

\[
c = \limsup_{n \to +\infty} \frac{a_n}{n}
\]

The series \( \sum_{n=1}^{\infty} e^{a_n - ns} \) converges for \( s > c \), diverges for \( s < c \) and we cannot tell for \( s = c \), by the root test. There exists a sequence of positive real numbers \( (b_n)_{n \in \mathbb{N}} \) such that \( \lim_{n \to +\infty} \frac{b_n}{b_{n+1}} = 1 \) and the series \( \sum_{n=1}^{\infty} b_n e^{a_n - ns} \) converges for \( s > c \) and diverges for \( s \leq c \).

Let now \( x \in X \) be such that the limit

\[
c = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^{-k}(x))
\]

exists in \( \mathbb{R} \). It is well known that the set consisting of these points has measure 1 with respect to every \( T \)-invariant Borel probability measure, and is therefore non-empty. Let

\[
a_n = nc - \sum_{k=1}^{n} f(T^{-k}(x)) \quad \text{and} \quad M_s = \sum_{n=1}^{\infty} b_n e^{a_n - ns}, \quad \text{where} \quad (b_n)_{n \in \mathbb{N}} \quad \text{is the corresponding}
\]
sequence as above. Each accumulation point in the weak* topology as s ↓ 0 of the directed family

\[ \mu_s = \frac{1}{M_s} \sum_{n=1}^{\infty} b_n e^{a_n - ns} \delta_{T^{-n}(x)}, \quad s > 0 \]

is an \( e^f\)-automorphic measure for \( T \).

There is a relation between \( e^f\)-automorphic measures for a continuous surjection \( T : X \to X \) of a compact metric space and solvability of the cohomological equation \( f = u - u \circ T \), where \( f : X \to \mathbb{R} \) is continuous.

Let \( \mu \) be any \( T \)-invariant Borel probability measure. If there exists a measurable solution \( u \) of the above cohomological equation defined \( \mu \)-almost everywhere such that \( e^{-u} \in L^1(\mu) \), then there exists an \( e^f\)-automorphic measure \( \nu \) for \( T \) equivalent to \( \mu \) with density

\[ \frac{d\nu}{d\mu} = \frac{e^{-u}}{\int_X e^{-u} d\mu}. \]

Thus, if there exists a continuous solution \( u \), then for every \( T \)-invariant Borel probability measure we get an equivalent \( e^f\)-automorphic measure for \( T \). Moreover, in this case, every \( e^f\)-automorphic measure \( \nu \) for \( T \) is obtained in this way. Indeed, we have

\[ \int_X \phi e^u d\nu = \int_X (\phi \circ T) e^u d\nu \]

for every continuous function \( \phi : X \to \mathbb{R} \), and so the equivalent measure \( \mu \) to \( \nu \) with density

\[ \frac{d\mu}{d\nu} = \frac{e^u}{\int_X e^u d\nu} \]

is \( T \)-invariant. Consequently, if \( f \) is a continuous coboundary, then the \( e^f\)-automorphic measures for \( T \) are in one-to-one correspondence with the \( T \)-invariant Borel probability measures and each \( e^f\)-automorphic measure for \( T \) is equivalent to its corresponding \( T \)-invariant measure.

Conversely, suppose that \( \mu \) is an ergodic \( T \)-invariant Borel probability measure and \( f : X \to \mathbb{R} \) is a continuous function such that \( \int_X f d\mu = 0 \). Suppose further that there exists an \( e^f\)-automorphic measure \( \nu \in \mathcal{M}(X) \) for \( T \) which is absolutely continuous with respect to \( \mu \) and \( g = \frac{d\nu}{d\mu} \). For every measurable set \( A \subset X \) we have

\[ \int_X (\chi_A \circ T)(g \circ T) d\mu = \nu(A) = \int_X (\chi_A \circ T) e^f d\nu = \int_X (\chi_A \circ T) e^f g d\mu \]

and therefore

\[ \int_{T^{-1}(A)} [ge^f - (g \circ T)] d\mu = 0. \]

Since \( \mu \) is \( T \)-invariant, it follows that \( g \circ T = ge^f \mu \)-almost everywhere. The ergodicity of \( \mu \) implies now that \( g > 0 \) \( \mu \)-almost everywhere. So, \( u = -\log g \) is a measurable solution of the cohomological equation \( f = u - u \circ T \). If \( \log g \in L^{\infty}(\mu) \) and \( T \) is
a minimal homeomorphism or $T$ is a locally eventually onto local homeomorphism
and $\mu$ has full support, then there exists some continuous function $u : X \to \mathbb{R}$ such
that $f = u - u \circ T$, by Proposition 4.2 on page 46 in [2] and Theorem 6 in [6], respectively.

**Remark 2.1.** It is a general fact that every quasi-invariant measure of a homeomorphism
which is is absolutely continuous with respect to an ergodic invariant measure is actually
equivalent to it. Indeed, let $T : X \to X$ be a homeomorphism of a compact metric space
$X$ and $\mu$ be an ergodic $T$-invariant Borel probability measure. If $\nu$ is a $T$-quasi-invariant
Borel probability measure which is absolutely continuous with respect to $\mu$, then $\nu$ is
equivalent to $\mu$. Indeed, let $g = \frac{d\nu}{d\mu}$ and $A = g^{-1}(0)$. If $S = \bigcup_{n \in \mathbb{Z}} T^n(A)$, then $S$ is $T$
-invariant and $\nu(S) = 0$. On the other hand $\mu(X \setminus S) > 0$, and since $\mu$ is ergodic we get
$\mu(S) = 0$, that is $g > 0$ $\mu$-almost everywhere.

3. Absolutely continuous automorphic measures.

Let $X$ be a compact metric space and $\mu \in \mathcal{M}(X)$. The set

$$E_\mu = \{ \nu \in \mathcal{M}(X) : \nu \ll \mu \}$$

is not empty, since it contains $\mu$, and is convex. In general, $E_\mu$ is not a closed subset of
$\mathcal{M}(X)$ with respect to the weak* topology. For example, if we let $\mu$ be the Lebesgue
measure on the unit interval $[0, 1]$ and for $0 < \epsilon < 1$ we let $\mu_\epsilon$ denote the Borel
probability measure on $[0, 1]$ with density $\frac{1}{\epsilon} \chi_{[0, \epsilon]}$, then $\lim_{\epsilon \to 0} \mu_\epsilon$ is the Dirac point measure
at 0.

**Lemma 3.1.** Let $X$ be a compact metric space and $\mu \in \mathcal{M}(X)$. Let $(\nu_n)_{n \in \mathbb{N}}$ be a
sequence in $E_\mu$ converging weakly* to some $\nu \in \mathcal{M}(X)$ and let $f_n = \frac{d\nu_n}{d\mu}$, $n \in \mathbb{N}$. If
there exist non-negative $h$, $g \in L^1(\mu)$ such that $h \leq f_n \leq g$ for every $n \in \mathbb{N}$, then $\nu \in E_\mu$
and $h \leq \frac{d\nu}{d\mu} \leq g$.

**Proof.** Since $\nu$ is a finite measure, there exists a (countable) basis $\mathcal{U}$ of the topology of
$X$ such that $\nu(\partial U) = 0$ for every $U \in \mathcal{U}$. So $\mathcal{U}$ is contained in the algebra

$$C(\nu) = \{ A | A \subset X \text{ Borel and } \nu(\partial A) = 0 \}$$

and since it generates the Borel $\sigma$-algebra of $X$, so does $C(\nu)$. Let now $A \subset X$ be a
Borel set with $\mu(A) = 0$ and $\epsilon > 0$. There exists $0 < \delta < \epsilon$ such that $\int_B g d\mu < \epsilon$ for
every Borel set $B \subset X$ with $\mu(B) < \delta$, because $g \in L^1(\mu)$. There exists some $A_0 \in C(\nu)$
such that $\mu(A \Delta A_0) < \delta$ and $\nu(A \Delta A_0) < \delta$. Thus $\mu(A_0) < \delta$ and $|\nu(A) - \nu(A_0)| < \delta$. By weak* convergence, $\nu(A_0) = \lim_{n \to +\infty} \nu_n(A_0)$ and so there exists some $n_0 \in \mathbb{N}$ such that
$|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \geq n_0$. Therefore,

$$\nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq \int_{A_0} g d\mu + \epsilon < 2\epsilon.$$

4
It follows that $0 \leq \nu(A) < 3\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in E_\mu$.

To prove the last assertion, we note first that there exists a sequence of (finite) partitions $(\mathcal{P}_n)_{n \in \mathbb{N}}$ of $X$ such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, the Borel $\sigma$-algebra of $X$ is generated by $\bigcup_{n=1}^\infty \mathcal{P}_n$ and $\mu(\partial B) = 0$ for every $B \in \mathcal{P}_n$ and $n \in \mathbb{N}$. It can be constructed starting with a countable basis $\{U_n : n \in \mathbb{N}\}$ of the topology of $X$ such that $\mu(\partial U_n) = 0$ for every $n \in \mathbb{N}$ and defining inductively $\mathcal{P}_n$ to be the finite family consisting of Borel sets with positive $\mu$ measure of the form $B \cap U_n$ or $B \cap (X \setminus U_n)$, for $B \in \mathcal{P}_{n-1}$, taking $\mathcal{P}_0 = \{X\}$.

Let $\mathcal{P}_n(x)$ denote the element of $\mathcal{P}_n$ which contains $x \in X$. Then,

$$\frac{d\nu}{d\mu}(x) = \lim_{n \to +\infty} \frac{\nu(\mathcal{P}_n(x))}{\mu(\mathcal{P}_n(x))},$$

$\mu$-almost everywhere on $X$ and in $L^1(\mu)$ (see page 8 in [5]). On the other hand, by the weak* convergence and since $\nu \in E_\mu$, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k}\mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \leq \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} < \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \leq \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} g d\mu.$$

Since

$$\lim_{k \to +\infty} \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} g d\mu = g(x)$$

$\mu$-almost everywhere on $X$ and in $L^1(\mu)$, it follows that $0 \leq \frac{d\nu}{d\mu}(x) \leq g(x)$ $\mu$-almost everywhere on $X$.

Similarly, from

$$\frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} > -\frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \geq -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} h d\mu$$

follows that $h(x) \leq \frac{d\nu}{d\mu}(x)$ $\mu$-almost everywhere on $X$. $\square$

Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. For any continuous function $f : X \to \mathbb{R}$ we put $S_n(f) = -\sum_{k=0}^{n-1} f \circ T^k$ and $E_n(f) = e^{S_n(f)}$.

Let $M_n = \sup\{S_n(f)(x) : x \in X\}$ and $L_n = \inf\{S_n(f)(x) : x \in X\}$, $n \in \mathbb{N}$. Since $S_n(f) \circ T = S_{n+1}(f) + f$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T)e^{-f} - g_n = E_n(f) - e^{-f}.$$
Let now \( \mu \in \mathcal{M}(X) \) be \( T \)-invariant and suppose that \( \int_X f d\mu = 0 \). So, \( L_n \leq 0 \leq M_n \) for every \( n \in \mathbb{N} \). Putting \( h_n = \frac{g_n}{\int_X g_n d\mu} \), we get

\[
(h_n \circ T) - h_ne^f = \frac{e^f - e^{-S_n(f)}}{e^{-S_n(f)}} \int_X g_n d\mu,
\]

for every \( n \in \mathbb{N} \).

Suppose that there exists a positive \( h \in L^1(\mu) \) such that \( E_n(f) \leq h \int_X E_n(f) d\mu \) for every \( n \in \mathbb{N} \). Then also \( 0 \leq h_n \leq h \) for \( n \in \mathbb{N} \). If \( \nu_n \) denotes the element of \( E_\mu \) with \( h_n = \frac{d\nu_n}{d\mu} \), then \( \{ \nu_n : n \in \mathbb{N} \} \subset E_\mu \), by Lemma 3.1.

**Proposition 3.2.** Let \( X \) be a compact metric space and \( T : X \to X \) a continuous surjection. Let \( \mu \in \mathcal{M}(X) \) be \( T \)-invariant and let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f d\mu = 0 \). Suppose that

(i) there exists a positive \( h \in L^1(\mu) \) such that \( E_n(f) \leq h \int_X E_n(f) d\mu \) for every \( n \in \mathbb{N} \), and

(ii) the sequence \( e^{-M_n} \sum_{k=0}^{n-1} e^{L_k} \), \( n \in \mathbb{N} \), is unbounded.

Then there exists an \( e^f \)-automorphic measure for \( T \) which is absolutely continuous with respect to \( \mu \).

**Proof.** Using the above notations, it suffices to prove that there exists a sequence of positive integers \( n_j \to +\infty \) such that \( \lim_{j \to +\infty} (h_{n_j} \circ T) - h_{n_j} e^f = 0 \) \( \mu \)-almost everywhere on \( X \). Indeed, passing to a subsequence if necessary, there exists \( \nu \in E_\mu \) such that \( \nu = \lim_{j \to +\infty} \nu_{n_j} \), by Lemma 3.1. Since \( \mu \) is \( T \)-invariant, for every continuous function \( \phi : X \to \mathbb{R} \) we have

\[
\int_X (\phi - (\phi \circ T)e^f) d\nu = \lim_{j \to +\infty} \int_X (\phi \circ T)((h_{n_j} \circ T) - h_{n_j} e^f) d\mu = 0,
\]

by dominated convergence, because

\[
|(\phi \circ T)((h_n \circ T) - h_n e^f)| \leq \|\phi\|((h \circ T) + he^f) \in L^1(\mu).
\]

Since

\[
|(h_n \circ T) - h_n e^f| = e^f \frac{|E_n(f) - e^{-f}|}{\int_X g_n d\mu},
\]

we need only prove that there exist \( n_j \to +\infty \) such that

\[
\lim_{j \to +\infty} \mu(\{ x \in X : |E_{n_j}(f)(x) - e^{-f(x)}| \geq \delta \int_X g_{n_j} d\mu \}) = 0
\]

for some \( \delta > 0 \).
for every $\delta > 0$. Let

$$A_{n,\delta} = \{ x \in X : E_n(f)(x) \geq e^{-f(x)} + \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x) \},$$

and

$$A'_{n,\delta} = \{ x \in X : E_n(f)(x) \leq e^{-f(x)} - \frac{\delta}{h(x)} \sum_{k=0}^{n-1} E_k(f)(x) \}.$$

Our assumption (i) implies that it suffices to prove the existence of a sequence of positive integers $n_j \to +\infty$ such that

$$\lim_{j \to +\infty} \mu(A_{n_j,\delta}) = \lim_{j \to +\infty} \mu(A'_{n_j,\delta}) = 0 \text{ for every } \delta > 0.

For every $x \in A_{n,\delta}$ we have

$$\frac{h(x)}{\delta} > e^{-M_n} \sum_{k=0}^{n-1} E_k(f)(x)$$

and integrating over $A_{n,\delta}$ we obtain

$$\frac{1}{\delta} \int_X h d\mu \geq \mu(A_{n,\delta}) e^{-M_n} \sum_{k=0}^{n-1} e^{L_k}.$$}

Similarly, for every $x \in A'_{n,\delta}$ we have

$$\sum_{k=0}^{n-1} E_k(f)(x) < \frac{h(x)}{\delta} e^{-f(x)}$$

and integrating over $A'_{n,\delta}$ we get

$$\mu(A'_{n,\delta}) \sum_{k=0}^{n-1} e^{L_k} \leq \frac{1}{\delta} \int_X h e^{-f} d\mu.$$}

Our assumption (ii) means that there exist $n_j \to +\infty$ such that $e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{L_k} \to +\infty$, and therefore we also have $\sum_{k=0}^{n_j} e^{L_k} \to +\infty$, because $L_n \leq 0 \leq M_n$. Consequently, we have

$$\lim_{j \to +\infty} \mu(A_{n_j,\delta}) = \lim_{j \to +\infty} \mu(A'_{n_j,\delta}) = 0. \quad \square$$

In the next proposition we make a more restrictive assumption (i) and a weaker assumption (ii).

**Proposition 3.3.** Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be $T$-invariant and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that

(i) there exists a constant $c \geq 1$ such that $E_n(f) \leq c \int_X E_n(f) d\mu$ for every $n \in \mathbb{N}$, and

(ii) \[ \lim_{j \to +\infty} \mu(A_{n_j,\delta}) = \lim_{j \to +\infty} \mu(A'_{n_j,\delta}) = 0. \]
(ii) the sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}$, $n \in \mathbb{N}$, is unbounded.

Then there exists an $e^f$-automorphic measure for $T$ which is absolutely continuous with respect to $\mu$.

Proof. Our assumption (ii) means that there exists a sequence of positive integers $n_j \to +\infty$ such that $e^{-M_n} \sum_{k=0}^{n_j-1} e^{M_k} \to +\infty$, as $j \to +\infty$. Using the same notations as above we have

$$\int_X g_{n_j} d\mu \to +\infty$$

and

$$e^{-S_{n_j}} \int_X g_{n_j} d\mu \geq \frac{1}{c} \cdot e^{-M_{n_j}} \sum_{k=0}^{n_j-1} e^{M_k} \to +\infty,$$

as $j \to +\infty$, by our assumptions. Therefore, $\lim_{j \to +\infty} (h_{n_j} \circ T - h_{n_j} e^f) = 0$ uniformly on $X$ and as in the proof of Proposition 3.2, every $\nu \in \{\nu_{n_j} : j \in \mathbb{N}\}$ is $e^f$-automorphic measure for $T$ that is absolutely continuous with respect to $\mu$. $\square$

As the following Lemma shows, if in Proposition 3.3 the $T$-invariant measure $\mu \in \mathcal{M}(X)$ is ergodic, then condition (ii) is implied by condition (i).

**Lemma 3.4.** Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. Suppose that there exists a constant $c \geq 1$ such that

$$E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$.

(a) If $A_n = \{x \in X : S_n(x) > M_n - \log c - 1\}$, $n \in \mathbb{N}$, then $\mu(A_n) \geq \frac{e^{-1}}{ec - 1}$ for $n \in \mathbb{N}$.

(b) For every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_{n+j} \leq M_n + 1$ for all $0 \leq j \leq N$.

(c) The sequence $e^{-M_n} \sum_{k=0}^{n-1} e^{M_k}$, $n \in \mathbb{N}$, is unbounded.

Proof. (a) From our assumption we have

$$e^{M_n - \log c} \leq \int_X E_n(f) d\mu \leq e^{M_n} \mu(A_n) + e^{M_n - \log c - 1} \mu(X \setminus A_n),$$

from which the required inequality follows.

(b) We proceed to prove the assertion by contradiction assuming that there exists some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $1 \leq j_n \leq N$ such that $M_{n+j_n} > M_n + 1$. Inductively, if we put $n_k = 1 + j_1 + \cdots + j_k$, then $M_{n_k} > M_1 + k$ and $1 + k \leq n_k \leq 1 + kN$ for every $k \in \mathbb{N}$. Therefore,

$$\frac{M_{n_k}}{n_k} > \frac{1}{N + 1}$$
for every \( k \in \mathbb{N} \). If now \( k_0 \in \mathbb{N} \) is such that 
\[
\left| \frac{\log c - 1}{n_k} \right| < \frac{1}{2(N + 1)} \text{ for } k \geq k_0,
\]
then for \( x \in A_n \) we have
\[
\frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N + 1)}
\]
and by (a) we get
\[
\mu(\{ x \in X : \frac{1}{n_k} S_{n_k}(x) > \frac{1}{2(N + 1)} \}) \geq \frac{e^{-1}}{e c - 1} > 0
\]
for every \( k \geq k_0 \). Hence the sequence \( (\frac{1}{n} S_n)_{n \in \mathbb{N}} \) does not converge in measure to zero.

This contradicts the Ergodic Theorem of Birkhoff, since we assume that \( \mu \) is an ergodic \( T \)-invariant Borel probability measure.

(c) Suppose on the contrary that there exists a real number \( a > 0 \) such that
\[
e^{-M_n} \sum_{k=0}^{n-1} e^{M_k} \leq a, \text{ for every } n \in \mathbb{N}.
\]
By (b), for every \( N \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that \( M_{n+j} \leq M_n + 1 \) for all \( 0 \leq j \leq N \), and so
\[
\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} \leq a \sum_{j=0}^{N} e^{M_{n+j}} \leq a e^{M_n} \sum_{k=0}^{n-1} e^{M_k} - \left( \sum_{k=0}^{n-1} e^{M_k} \right)
\]
\[
\leq e a (1 + a) e^{M_n} - a \sum_{k=0}^{n-1} e^{M_k}.
\]
Substituting
\[
\sum_{j=0}^{N} \sum_{k=0}^{n+j-1} e^{M_k} = (N + 1) \sum_{k=0}^{n-1} e^{M_k} + N e^{M_n} + \sum_{i=1}^{N-1} (N - i) e^{M_{n+i}},
\]
we arrive at
\[
(N + 1 + a) \sum_{k=0}^{n-1} e^{M_k} + N e^{M_n} + \sum_{i=1}^{N-1} (N - i) e^{M_{n+i}} \leq e a (1 + a) e^{M_n}
\]
and therefore \( N \leq e a (1 + a) \) for every \( N \in \mathbb{N} \), contradiction. \( \Box \)

The above immediately imply the following theorem which is the main result of this note.

**Theorem 3.5.** Let \( X \) be a compact metric space and \( T : X \to X \) a continuous surjection. Let \( \mu \in \mathcal{M}(X) \) be an ergodic \( T \)-invariant measure and let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f d\mu = 0 \). If there exists a constant \( c \geq 1 \) such that
\[
E_n(f) \leq c \int_X E_n(f) d\mu
\]
for every $n \in \mathbb{N}$, then there exists an $e^f$-automorphic measure $\nu$ for $T$ which is absolutely continuous with respect to $\mu$. Moreover, $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ and $-\log\left(\frac{d\nu}{d\mu}\right)$ is a measurable solution of the cohomological equation $f = u - u \circ T$. □

**Corollary 3.6.** Let $X$ be a compact metric space and $T : X \to X$ a continuous surjection which is a locally eventually onto local homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. If there exists a constant $c \geq 1$ such that

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu$$

for every $n \in \mathbb{N}$, then there exists an $e^f$-automorphic measure $\nu$ for $T$ which is absolutely continuous with respect to $\mu$. Moreover, $-\log\left(\frac{d\nu}{d\mu}\right) \in L^\infty(\mu)$ and in case $\mu$ has full support the cohomological equation $f = u - u \circ T$ has a continuous solution. □

If $X$ is a compact metric space and $T : X \to X$ is a homeomorphism, for any continuous function $f : X \to \mathbb{R}$ we put

$$E_n(f) = \begin{cases} 
\exp \sum_{k=1}^{n} f \circ T^{-k}, & \text{if } n > 0, \\
1, & \text{if } n = 0, \\
\exp (- \sum_{k=0}^{\lfloor n \rfloor - 1} f \circ T^k), & \text{if } n < 0.
\end{cases}$$

As before we also put $S_n(f) = \log E_n(f)$ and $M_n = \sup \{ S_n(f)(x) : x \in X \}$, $n \in \mathbb{Z}$.

Let now $\mu \in \mathcal{M}(X)$ be $T$-invariant and suppose that $\int_X f d\mu = 0$. Then, $M_n \geq 0$ for every $n \in \mathbb{Z}$. Since $S_n(f) \circ T^{-1} = S_{n+1}(f) - f \circ T^{-1}$ for $n \in \mathbb{N}$, if $g_n = \sum_{k=0}^{n-1} E_k(f)$, then we have

$$(g_n \circ T^{-1}) e^{f \circ T^{-1}} - g_n = E_n(f) - 1.$$

Putting $h_n = \int_X \frac{g_n}{g_n d\mu}$, we get

$$(h_n \circ T^{-1}) e^{f \circ T^{-1}} - h_n = \frac{1 - e^{-S_n(f)}}{e^{-S_n(f)} \int_X g_n d\mu},$$

for every $n \in \mathbb{N}$. So the same reasoning as above and Lemma 3.1 give the following.

**Theorem 3.7.** Let $X$ be a compact metric space and $T : X \to X$ a homeomorphism. Let $\mu \in \mathcal{M}(X)$ be an ergodic $T$-invariant measure and let $f : X \to \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$.
(a) If there exists a constant \( c \geq 1 \) such that

\[ E_n(f) \leq c \int_X E_n(f) d\mu \]

for every \( n \in \mathbb{N} \) (or \( -n \in \mathbb{N} \)), then there exists an \( \mathcal{L}^1 \)-automorphic measure \( \nu \) for \( T \) which is equivalent to \( \mu \) such that \( \frac{d\nu}{d\mu} \in L^\infty(\mu) \).

(b) Moreover, if

\[ \frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu \]

for every \( n \in \mathbb{N} \) (or \( -n \in \mathbb{N} \)), then \( \log(\frac{d\nu}{d\mu}) \in L^\infty(\mu) \). □

Combining Theorem 3.7 with section 2 we get the following.

**Corollary 3.8.** Let \( X \) be a compact metric space and \( T : X \to X \) a minimal homeomorphism. Let \( \mu \in \mathcal{M}(X) \) be an ergodic \( T \)-invariant measure and let \( f : X \to \mathbb{R} \) be a continuous function such that \( \int_X f d\mu = 0 \). Then the following assertions are equivalent.

(i) \( f \) is a continuous coboundary.

(ii) There exists a constant \( c \geq 1 \) such that

\[ \frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f) \leq c \int_X E_n(f) d\mu \]

for every \( n \in \mathbb{N} \) (or \( -n \in \mathbb{N} \)). □

**References**


