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On minimal sets in 2-manifolds

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1. The structure of limit sets is important to understand the phase portraits of flows. A serious step in this direction is the study of minimal sets, at least because every compact invariant set contains such a set. It is known that a compact minimal set of a C^2-differentiable dynamical system on a 2-manifold M is either a fixed point, a periodic orbit or else all of M, in which case the system is an irrational flow on the torus, see A.J. Schwartz [17] (note that the compactness of the manifold is not necessary, see P. Hartman [12], Ch. VII, 12. 1). Such a minimal set will be called *simple* in the sequel. The assumption in Schwartz's theorem that the dynamical system is at least C^2-differentiable is essential, as it had been shown by A. Denjoy [8].

This paper is concerned with the question of *how the qualitative behavior near a compact minimal set of a continuous but not necessarily differentiable flow affects the structure of this minimal set.* We intend to prove the following

Theorem. *A compact minimal set of a dynamical system on a 2-manifold is simple, if it is either stable or a saddle set.*

This theorem follows from the Propositions in 3 and 5 below, the proof of which are based on two constructions which make possible the use of powerful known results about dynamical systems on compact 2-manifolds, although the manifold in the theorem is not assumed to be compact. The one of the two constructions is based on the notion of the *end point compactification* which is explained in 4.

We note that the first part of the assertion of the above theorem leads to a rather complete description of the D^+-stable (or of characteristic 0^+) dynamical systems on 2-manifolds given in K. Athanassopoulos [2].

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2. Let $(\mathbb{R}, X, \varphi)$ be a dynamical system on the metric space X. We let $\varphi(t, x) = tx$ and we denote by $C(x)$, $C^+(x)$ and $C^-(x)$ the orbit, the positive semiorbit and the negative semiorbit of the point $x \in X$ and by

$$L^+(x) = \{y \in X : t_n x \to y \text{ for some } t_n \to +\infty\},$$

$$J^+(x) = \{y \in X : t_n x_n \to y \text{ for some } x_n \to x \text{ and } t_n \to +\infty\}$$

the positive limit set and the (first) positive prolongational limit set of this point respectively.

We recall that a subset of X is called invariant whenever it is a union of orbits and positively invariant if it is a union of positive semiorbits. A *minimal* set is a non-empty, closed, invariant set which does not have a proper subset with these properties.

2.1. **Definition.** A subset A of X is called *stable* if every neighborhood of A contains a positively invariant neighborhood of A. A stable set is called *asymptotically stable* if its region of attraction is an open neighborhood of it. The region of attraction of A is the set of the points $x \in X$ such that to every neighborhood V of A corresponds a $t > 0$ with $C^+(tx) \subset V$.

2.2. **Definition.** A subset A of X is a *saddle* set if there exists a neighborhood U of A such that every neighborhood V of A contains a point x_V with $C^+(x_V) \not\subset U$ and $C^-(x_V) \not\subset U$ (cf. N.P. Bhatia [3]).

2.3. **Definition.** The dynamical system $(\mathbb{R}, X, \varphi)$ is said to be *parallelizable* if there exists a set $S \subset X$ intersecting every orbit and a homeomorphism $h: X \to \mathbb{R} \times S$ with $h(tx) = (t, x)$ for all $t \in \mathbb{R}$ and $x \in S$. The set S is called *global section* to the flow. Note that there is a continuous function $T: X \to \mathbb{R}$ such that $T(x)x \in S$ for all $x \in X$.

2.4. We recall that for X locally compact and separable the dynamical system $(\mathbb{R}, X, \varphi)$ is parallelizable iff $J^+(x) = \emptyset$ for all $x \in X$, see N.P. Bhatia and G.P. Szegö [4], Ch. IV, 2.6.

2.5. **Remark.** In the sequel we shall use the concept of *local* dynamical system. It is shown however in D.H. Carlson [6] that *a local dynamical system on a metric space can be reparametrized to yield a dynamical system.* We shall use this fact repeatedly.

3. In this section we prove the following

Proposition. *A stable, compact, minimal set of a dynamical system $(\mathbb{R}, M, \varphi)$ on a 2-manifold M is simple.*

Proof. Let A be a non-simple stable compact and minimal subset of M. We shall show in three steps that this leads to a contradiction.

Step 1. Let U be a connected, relatively compact, positively invariant and open neighborhood of A (being connected because it is minimal). To every $x \in U$ corresponds a $t_x \in [-\infty, 0)$ such that $I_x = (t_x, +\infty)$ is the maximal interval with the property $\{tx: t \in I_x\} \subset U$. Putting

$$D = \bigcup_{x \in U} (I_x \times \{x\})$$

and letting $(t, x) \mapsto tx$ we define a local dynamical system on U which reparametrized yields a new dynamical system $(\mathbb{R}, U, \mu)$ on U. It is easy to see that a minimal set in M intersecting U is contained in U and it is a minimal set with respect to $(\mathbb{R}, U, \mu)$ also.

Considered as a 2-manifold U has finite genus, because (being relatively compact) it contains finitely many "handles" of M (cf. 4). Hence we may apply the Lemmas 5 and 6 of E. Lima [14] to conclude that

(1) there are at most $2g-1$ non-simple minimal sets in U, where g is the genus of U, and

(2) A is isolated from periodic orbits.

So (the set of fixed points being closed) we may assume that A is the only minimal set of the dynamical system $(\mathbb{R}, M, \varphi)$ contained in U. Assuming this and denoting by $L^+(x)$ the positive limit set with respect to the dynamical system $(\mathbb{R}, U, \mu)$ which is compact and therefore contains a minimal set, we see that $A \subset L^+(x)$ for every $x \in U$. Consequently for every $x \in U$ and every positively invariant neighborhood $V \subset U$ of A there exists some $t > 0$ with $C^+(tx) \subset V$, from which follows $L^+(x) \subset A$. Thus $L^+(x) = A$ for all $x \in U$, and A is asymptotically stable.

Step 2. Let $J^+(x)$ denote the positive prolongational limit set of $x \in U$ with respect to the dynamical system $(\mathbb{R}, U, \mu)$. By O. Hajek [11], Th. 4, we have $\emptyset \neq J^+(x) \subset A$ for all $x \in U - A$. Therefore the restricted dynamical on $U - A$ is parallelizable (cf. 2.3 and 2.4). Hence it has a global section, S, which is compact, because A is compact. By O. Hajek [10], Ch. VII, 1.6, the connected components of S are simple closed curves.

Let $N = A \cup \{tx: t \geqq 0, x \in S\}$. Let $\{V_i: i \in I\}$ be an open covering of N. Since the sets A and S are compact and $L^+(x) = A$ for all $x \in S$, there is some $t > 0$ such that $A \cup \{rx: r \geqq t, x \in S\}$ is contained in the union of finitely many V_i's. From this and the compactness of the set $\{rx: r \in [0, t], x \in S\}$ follows that there exists a finite subcovering of the above covering of N, which is therefore compact.

Since the dynamical system in $U - A$ is parallelizable and every component of S is a simple closed curve, N is a connected compact 2-manifold with boundary S and contains A in its interior. Let $N \times \{0\}$ and $N \times \{1\}$ be two disjoint copies of N. In their free union we consider the equivalence relation $\sim$ generated by $(s, 0) \sim (s, 1)$ when $s \in S$. For $x \in N$ we denote by $[(x, 0)]$, $[(x, 1)]$ the corresponding classes. The quotient space is N^*. According to M. Brown [5], Th. 2, N^* is a compact 2-manifold. On it we define a dynamical system, roughly speaking, by reversing the orientation of the orbits in the copy $N \times \{1\}$ of N. Precisely, using the function $T: X \to \mathbb{R}$ indicated in 2.3, we define the dynamical system $(\mathbb{R}, N^*, f)$ as follows.

For $x \in A$ we let $f(t, [(x, 0)]) = [(tx, 0)]$ and $f(t, [(x, 1)]) = [(-tx, 1)]$. For $x \in N - A$ we let

$$f(t, [(x, 0)]) = [(tx, 0)], \qquad t \geqq T(x),$$
$$f(t, [(x, 0)]) = [((2T(x) - t)x, 1)], \qquad t \leqq T(x),$$
$$f(t, [(x, 1)]) = [((2T(x) + t)x, 0)], \qquad t \geqq -T(x),$$
$$f(t, [(x, 1)]) = [(-tx, 1)], \qquad t \leqq -T(x).$$

Step 3. This dynamical system has exactly two minimal sets which are copies of A and hence non-simple, while the other orbits are homeomorphic to $\mathbb{R}$. It follows that this dynamical system has no fixed points. Therefore N^* cannot be any other 2-manifold than the torus or the Klein bottle. Because otherwise every map $h: N^* \to N^*$ has a fixed point (cf. W. Franz [9], 23. 12). So in particular, to any sequence $t_n \to 0$ corresponds a sequence x_n^*, $n \in \mathbb{N}$, of fixed points of the maps defined by $x^* \mapsto t_n x^*$ for $x^* \in N^*$. Since N^* is compact, we may assume $x_n^* \to y^*$ for some $y^* \in N^*$ which is a fixed point of the flow, by N.P. Bhatia and G.P. Szegö [4], Ch. V, 3. 7.

However a dynamical system on the Klein bottle without fixed points must have a periodic orbit, by H. Kneser [13]. On the other hand, if N^* is the torus, then we have a dynamical system on the torus with two non-simple minimal sets. This contradicts to (1) from the Step 2, and the Proposition is proved.

4. For the reader's convenience we shall give in this section an outline of some facts about the end point compactification of a manifold. We shall not go into details, because in the sequel we only consider the case where X is a 2-manifold with finite genus and finitely many ends, in which case the details are easily seen.

A definition of the *end point compactification*, X^+, of a manifold is as follows: let βX be the Stone-Čech compactification of X; then X^+ is the quotient space of βX by the equivalence relation identifying a connected component of $\beta X - X$ to a point. Thus the remainder $X^+ - X$, i.e., the space of the *ends* of X, is totally disconnected, and X^+ is the "maximal" compactification of X with this property (cf. H. Abels [1], 2 and J.R. McCartney [7], 3). An *equivalent definition* of the end point compactification of X is given in F. Raymond [15], 1. We adopt this definition here. Because of the Theorem in F. Raymond [15], 1. 8, we can adopt the proof of the Satz in H. Abels [1], 2. 3 to conclude that, *given a dynamical system* $(\mathbb{R}, X, \varphi)$ *there always exists a (continuous) extension of it to a dynamical system* $(\mathbb{R}, X^+, \varphi^+)$ *on* X^+.

Especially, if $X = M$ is a non-compact 2-manifold, then M^+ can be described as follows. According to I. Richards [16], M can be constructed from the 2-sphere, S^2, by removing a totally disconnected closed set, N, and the interiors of a sequence of non-overlapping discs $D_n \subset S^2 - N$, $n \in \mathbb{N}$, which tend to N in the sense that they are contained in every neighborhood of N except finitely many, and then identifying the boundaries of these discs in pairs to form "handles", where D_n with itself is a possible pair. The number of these "handles" is the *genus* of M. Adding the space N to M we obtain a compactification, M', of M, because the discs D_n tend to N in S^2. Since M is a 2-manifold and N is totally disconnected, M' fulfils the assumptions of J.R. McCartney [7], 3. 12. Therefore $M' = M^+$.

The ends of a non-compact 2-manifold, M, are divided in two classes: the *planar* ends which have neighborhoods in M^+ homeomorphic to $\mathbb{R}^2$, and the *non-planar* ones every neighborhood of which contains "handles". Therefore: *the end point compactification of a non-compact 2-manifold, M, is a compact 2-manifold iff M has finite genus* (cf. also F. Raymond [15], 5).

5. In this section we prove the following

Proposition. *A compact, minimal, saddle set of a dynamical system on a 2-manifold, M, is either a fixed point or a periodic orbit.*

Proof. Let $A \subset M$ be a compact, minimal, saddle set, and let U be a relatively compact and connected neighborhood of A as in the definition of a saddle set in 2. 2. Arguing as in the Step 1 of the proof of the Proposition in 3, we associate to each $x \in U$ a maximal interval $I_x = (a_x, b_x)$ such that $\{tx: t \in I_x\} \subset U$. Moreover we can define a dynamical system on U^+, the end point compactification of U, extending the reparametrized restriction on U of the considered dynamical system on M (cf. 2. 5 and 4). Choosing U such that $\bar{U}$, the closure of U, is a compact 2-manifold with boundary, we may assume that the ends of U are finitely many (they correspond to the boundary components of U).

Let x_n, $n \in \mathbb{N}$, be a sequence of points with disjoint orbits such that $x_n \to x_0 \in A$ and $C^+(x_n) \not\subset U$, $C^-(x_n) \not\subset U$. Such a sequence exists, because of 2. 2 and the fact that A is compact. Then $a_{x_n} x_n$, $b_{x_n} x_n \in \partial U$, the boundary of U. Since the dynamical system on U^+ is the extension of that on U and the ends of U are finitely many, passing to a subsequence if necessary, we may assume that there are two ends, say p and q, such that $L^+(x_n) = \{p\}$ and $L^-(x_n) = \{q\}$ for $n \in \mathbb{N}$.

So the situation is reduced to a dynamical system on the compact manifold U^+ (cf. 4) such that the limit sets of the points x_n have the above property. For $n \neq 1$ the sets $C_n = C(x_n) \cup C(x_1) \cup \{p, q\}$ are simple closed curves. Assuming that A is non simple, we can arrive at a contradiction by induction on the genus of U^+, using the curves C_n for $n \neq 1$ in the same way as periodic orbits are used in the proof of the Lemma 6 in E. Lima [14]. Thus A is simple, and the assertion follows.

Remark. The method of Lemma 6 in E. Lima [14] is of a local nature. In the above proof in order to apply Lima's method we have constructed the closed curves C_n in U^+. This construction relatively to U is of a non local character. This "local-global" combination seems to be applicable in some other situations also. For instance, let A be a not necessarily compact and minimal set of a dynamical system on a 2-manifold, M, of finite genus and with finitely many ends. Let $x_n \in M - A$, $n \in \mathbb{N}$, be a sequence with $L^+(x_n) = L^-(x_n) = \emptyset$ and $x_n \to x_0 \in A$. Then we can prove in an analogous way as above that A is a single orbit.

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