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Flows with cyclic winding numbers groups

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1. Introduction

One of the classical problems in the theory of dynamical systems is to find conditions which guarantee the existence of periodic orbits. The best known condition concerning orientation preserving homeomorphisms of the unit circle is the rationality of the Poincaré rotation number. One of the results of the present work can be viewed as a generalization of this. The basic notion we use is the asymptotic cycle of a flow due to S. Schwartzman, which in a sense generalizes the notion of rotation number.

Let ϕ be a flow on a compact metrizable space X and suppose that C is a periodic orbit. Let μ be the ϕ-invariant Borel probability measure which is uniformly distributed along C. If the μ-winding numbers group W_μ of ϕ (for definition see section 2) is not trivial, then it is cyclic, that is isomorphic to the integers $\mathbb{Z}$. The question arises whether the converse is true. That is, does the existence of a ϕ-invariant Borel probability measure with cyclic winding numbers group imply the existence of a periodic orbit? The answer to this question is in general negative in phase spaces where the codimension of orbits is greater than 1. For every $n \geqq 3$ there is a closed orientable n-manifold carrying a smooth flow without singular points or periodic orbits which has an invariant Borel probability measure with cyclic winding numbers group. However as we show below, if the phase space X is a 1-dimensional continuum or a closed orientable 2-manifold and the flow has an invariant Borel probability measure μ with cyclic winding numbers group, then every point in the support of μ is periodic provided its positive (or negative) limit set does not contain singular points. Thus a non-singular continuous flow on the 2-torus has a periodic orbit if and only if it has an invariant Borel probability measure with cyclic winding numbers group. This generalizes the condition on the rationality of the Poincaré rotation number for orientation preserving homeomorphisms of the unit circle.

In the sequel given a continuous flow ϕ on a compact metrizable space X we shall denote by $\phi(t,x) = tx$ the translation of the point $x \in X$ along its orbit $C(x)$ in time $t \in \mathbb{R}$. We shall also write $\phi(I \times A) = IA$, for $I \subset \mathbb{R}$ and $A \subset X$. The *positive limit set* of the point $x \in X$ is the closed invariant set

$$L^+(x) = \{y \in X : t_n x \to y,\ t_n \to +\infty\}$$

and the *first prolongational positive limit set* is the set

$$J^+(x) = \{y \in X : t_n x_n \to y,\ x_n \to x,\ t_n \to +\infty\}.$$

The negative versions of these sets are defined by reversing time. The point $x \in X$ is *non-wandering* if $x \in J^+(x)$. A set $A \subset X$ is called *minimal* if it is non-empty, closed, invariant and has no proper subset with these properties.

2. Asymptotic cycles

Let X be a compact metrizable space carrying a continuous flow ϕ. For every continuous function $f: X \to S^1$ there is a continuous function $g: \mathbb{R} \times X \to \mathbb{R}$, called the *1-cocycle* of f such that $f(tx) = f(x)\exp(2\pi i g(t,x))$ and $g(t+s,x) = g(s,tx) + g(t,x)$ for every $x \in X$ and $t, s \in \mathbb{R}$. The Ergodic Theorem of Birkhoff implies that for every ϕ-invariant Borel probability measure μ on X the limit

$$g^*(x) = \lim_{t \to +\infty} \frac{g(t,x)}{t}$$

exists μ-almost for every $x \in X$ and $\int_X g^* d\mu = \int_X g(1,.)\, d\mu$. This integral describes the μ-average rotation of points moving along their orbits with respect to the projection f.

If $f_1, f_2 : X \to S^1$ are homotopic continuous functions with cocycles g_1, g_2, respectively, then $\int_X g_1^* d\mu = \int_X g_2^* d\mu$. Since the first Cech cohomology group with integer coefficients $\check{H}^1(X)$ of X is isomorphic to the group of homotopy classes of continuous functions of X to S^1, there is a group homomorphism $A_\mu : \check{H}^1(X) \to \mathbb{R}$ defined by

$$A_\mu[f] = \int_X g(1,.)\, d\mu$$

where g is the 1-cocycle of $f: X \to S^1$ and $[f]$ the homotopy class of f. The homomorphism A_μ was defined by S. Schwartzman [10] and is called the *μ-asymptotic cycle* of the flow. It describes how a μ-average orbit winds around X. The image of A_μ is called the *μ-winding numbers group* of the flow ϕ and will be denoted by W_μ. An exposition of the basic theory of asymptotic cycles with details is given in [1].

2.1. Examples. (a) Let C be a periodic orbit in X of prime period $T > 0$ and let μ be the uniformly distributed Borel probability measure along C. Then,

$$A_\mu[f] = \frac{1}{T}\deg(f \mid C)$$

for every continuous function $f: X \to S^1$. Suppose now that $X = S^1 \times S^1$ the 2-torus and the flow has no singular point. Then C is not nullhomotopic. This means that if l is the longitude and m is the meridian, then C is homotopic to $l^a * m^b$ for some $a, b \in \mathbb{Z}$ not both zero. Let p and q be the projections onto the longitude and the meridian respectively. Every continuous function $f: S^1 \times S^1 \to S^1$ is homotopic to $p^\kappa q^\lambda$ for some $\kappa, \lambda \in \mathbb{Z}$ and we can compute

$$A_\mu[f] = \frac{1}{T}\deg(p^\kappa q^\lambda | l^a * m^b) = \frac{1}{T}(\kappa a + \lambda b).$$

Since a, b are not both zero, we have $W_\mu \cong \mathbb{Z}$.

(b) An irrational flow on the 2-torus of slope $a \in \mathbb{R}\backslash\mathbb{Q}$ is uniquely ergodic the unique invariant Borel probability measure being the Haar measure μ and $W_\mu = \mathbb{Z} + a\mathbb{Z}$.

The following formula is useful in the calculation of winding numbers.

2.2. Lemma. *Let μ be a ϕ-invariant Borel probability measure on X, $f\colon X \to S^1$ a continuous function with 1-cocycle g and $A \subset X$ a closed set with $\mu(A) = 0$. We make the following assumptions*:

(a) *The time derivative $g'(0, x)$ exists for every $x \in X\backslash A$ and is continuous and bounded on $X\backslash A$.*

(b) *For every $x \in X$ the set of times $t \in \mathbb{R}$ such that $tx \in A$ is discrete.*

Then,

$$A_\mu[f] = \int_{X\backslash A} g'(0, x)\, d\mu .$$

Proof. We first observe that $g'(t, x)$ exists for every $t \in \mathbb{R}$ and $x \in X$ such that $tx \in X\backslash A$, and $g'(t, x) = g'(0, tx)$. Let $f'\colon X \to \mathbb{C}$ be the function defined by $f'(x) = 0$, if $x \in A$ and $f'(x) = 2\pi i f(x) g'(0, x)$, if $x \in X\backslash A$. Then, f' is measurable and

$$f'(tx) = 2\pi i f(tx) g'(t, x),$$

if $tx \in X\backslash A$. Thus, from our assumptions follows that

$$g(t, x) = \int_0^t \frac{f'(sx)}{2\pi i f(sx)}\, ds$$

for every $t \in \mathbb{R}$ and $x \in X$. By Fubini's theorem we have

$$A_\mu[f] = \int_X \left(\int_0^1 \frac{f'(sx)}{2\pi i f(sx)}\, ds \right) d\mu = \int_0^1 \left(\int_X \frac{f'(sx)}{2\pi i f(sx)}\, d\mu \right) ds$$
$$= \int_X \frac{f'(x)}{2\pi i f(x)}\, d\mu = \int_{X\backslash A} g'(0, x)\, d\mu .$$

3. Winding numbers groups of suspensions

The suspensions of homeomorphisms are a class of flows whose winding numbers groups can be computed in terms of initial data. Let Y be a compact metrizable space and let $h\colon Y \to Y$ be a homeomorphism. On $[0,1] \times Y$ we consider the equivalence relation

$(1,x) \sim (0,h(x))$, $x \in Y$. The quotient space $X = [0,1] \times Y/\sim$ is compact metrizable and is called the *mapping torus* of h. Let $[s,x]$ denote the class of $(s,x) \in [0,1] \times Y$. The flow on X defined by

$$t[s,x] = [t+s-n, h^n(x)]$$

if $n \leqq t+s < n+1$, is called the *suspension* of h.

If v is an h-invariant Borel probability measure on Y and λ is the Lebesgue measure on $[0,1]$, then the product measure $\lambda \times v$ induces a Borel probability measure on X which is invariant by the suspension of h. It is easy to see that the converse is true. That is, every invariant by the suspension of h Borel probability measure on X is of this form.

In order to compute the winding numbers groups of the suspension of h we need to know the relation between the integral first Cech cohomology groups of Y and X. Let $C(Y,\mathbb{Z})$ be the group of integer valued continuous functions on Y. Let $\gamma : C(Y,\mathbb{Z}) \to \check{H}^1(X)$ be defined by $\gamma(\psi) = [f]$, where $f: X \to S^1$ is the continuous function defined by $f[t,x] = \exp(2\pi i t \psi(x))$ and $j^*: \check{H}^1(X) \to \check{H}^1(Y)$ be the homomorphism induced by the inclusion $j: Y \to X$ with $j(x) = [0,x]$. Then one can easily verify that the sequence

$$C(Y,\mathbb{Z}) \xrightarrow{h^*-\mathrm{id}} C(Y,\mathbb{Z}) \xrightarrow{\gamma} \check{H}^1(X) \xrightarrow{j^*} \check{H}^1(Y) \xrightarrow{h^*-\mathrm{id}} \check{H}^1(Y)$$

is exact.

If now v is an h-invariant Borel probability measure on Y and μ is the corresponding invariant measure on X, then

$$W_\mu = \{\int_Y \psi\, dv : \psi \in Log(Y,h)\}$$

where $Log(Y,h)$ is the set of continuous functions $\psi: Y \to \mathbb{R}$ satisfying $f \circ h = f \exp(2\pi i \psi)$ for some continuous function $f: Y \to S^1$ ([1] or [8], Appendix).

If Y is in addition connected, then W_μ can also be described through the rotation number map of h with respect to v. Let $f: Y \to S^1$ be a continuous function such that $[f] \in \mathrm{Ker}(h^* - \mathrm{id})$. There is a continuous function $\psi: Y \to \mathbb{R}$ such that $f \circ h = f \exp(2\pi i \psi)$ and any two such functions differ by an integer. One can easily see that there is a well defined group homomorphism $R_v : \mathrm{Ker}(h^* - \mathrm{id}) \to S^1$ with

$$R_v[f] = \exp(2\pi i \int_Y \psi\, dv)$$

called the *rotation number map* of h. It is clear that we have a commutative diagram

$$\begin{array}{ccc} \check{H}^1(X) & \xrightarrow{A_\mu} & \mathbb{R} \\ j^* \downarrow & & \exp \downarrow \\ \mathrm{Ker}(h^* - \mathrm{id}) & \xrightarrow{R_v} & S^1 \end{array}$$

and therefore $W_\mu = \exp^{-1}(\mathrm{Im}\, R_v)$. Thus we arrive at the following.

3.1. Proposition. *Let Y be a connected, compact, metrizable space and $h: Y \to Y$ a homeomorphism. Let μ be an invariant by the suspension of h Borel probability measure on the mapping torus X of h corresponding to the h-invariant Borel probability measure ν on Y. If the rotation number map R_ν is trivial or its range is a finite cyclic subgroup of S^1, then $W_\mu \cong \mathbb{Z}$.*

It is well known ([9]) that for any $k \in \mathbb{N}$ there is a diffeomorphism $h: S^{2k+1} \to S^{2k+1}$ which is isotopic to the identity and has no fixed or periodic point. The mapping torus of h is (diffeomorphic to) $S^1 \times S^{2k+1}$. If μ is any invariant by the suspension of h Borel probability measure, then $W_\mu \cong \mathbb{Z}$, by Proposition 3.1. The product of the suspension of h with the trivial flow on S^1 is a flow without singular points or periodic orbits on $S^1 \times S^1 \times S^{2k+1}$. If λ is any Borel probability measure on S^1, then $\lambda \times \mu$ is an invariant Borel probability measure on $S^1 \times S^1 \times S^{2k+1}$ with winding numbers group isomorphic to W_μ. The winding numbers group of any invariant Borel probability measure on $S^1 \times S^1 \times S^{2k+1}$ is either trivial or cyclic. So for any integer $n \geqq 4$ there is a smooth flow on a closed orientable n-manifold without singular points or periodic orbits which has an invariant Borel probability measure with cyclic winding numbers group.

There is also such a flow on the 3-torus T^3 that is the suspension of a diffeomorphism of the 2-torus. Let $0 < a < 1$ be an irrational number and $\tilde{\xi}$ the smooth planar vector field defined by

$$\tilde{\xi}(x_1, x_2) = (a \sin 2\pi x_2, \cos^2 2\pi x_2).$$

Clearly $\tilde{\xi}$ is $\mathbb{Z}^2$-invariant and induces a smooth vector field ξ on T^2 with exactly two periodic orbits, the circles C_1 where $x_2 = 1/4$ and C_2 where $x_2 = 3/4$. Every other orbit of ξ is wandering and has limit sets in $C_1 \cup C_2$. Let $h: T^2 \to T^2$ be the time one map of the flow of ξ. Then h leaves C_1 and C_2 invariant and is a rotation by an angle $2\pi a$ on C_1 and a rotation by an angle $-2\pi a$ on C_2. Since a is irrational, h has no fixed or periodic point. Let ν be the h-invariant Borel probability measure on T^2 which is concentrated and uniformly distributed on C_1 and C_2 with density $1/2$ on each of them. Since the ν-mean rotation of points via h projected on the meridian is zero and projected on the longitude is $(1/2)(2\pi a - 2\pi a) = 0$, we conclude that $R_\nu = 0$. Hence the suspension of h is a smooth flow on T^3 without fixed or periodic points and by Proposition 3.1 $W_\mu \cong \mathbb{Z}$, where μ is the invariant Borel probability measure on T^3 corresponding to ν. Summarizing we have:

3.2. Proposition. *For any $n \geqq 3$ there is a smooth flow on a closed orientable n-manifold which has no singular point or periodic orbit and has an invariant Borel probability measure with cyclic winding numbers group.*

4. Periodic orbits in 1-dimensional continua

Let X be a compact metrizable space carrying a continuous flow ϕ. A set $S \subset X$ is called *local section* to the flow of extent $\varepsilon > 0$ if the flow ϕ maps $(-\varepsilon, \varepsilon) \times S$ homeomorphically onto an open subset of X. It is known that every non-singular point is contained in a locally compact local section ([3]).

To every closed local section S of extent $2\varepsilon > 0$ corresponds the continuous function $f_S: X \to S^1$ defined by

$$f_S(x)=\begin{cases}1, & \text{if } x\in X\backslash[0,\varepsilon]\,S,\\ \exp(2\pi it/\varepsilon), & \text{if } x\in tS,\ 0\leqq t\leqq\varepsilon,\end{cases}$$

which is called the *cosection map* of S. It is clear that the homotopy class of f does not depend on ε but only on S. The class $[f_S]\in \check{H}^1(X)$ is called the *flow class* of S. An equivalent definition of the flow class was first given in [10].

If μ is any ϕ-invariant Borel probability measure on X then $\mu(S)=0$. Using Lemma 2.2 we compute

$$A_\mu[f_S]=\int_{[0,\varepsilon]S}\frac{2\pi i}{\varepsilon}\frac{f_S(x)}{2\pi i f_S(x)}\,d\mu=\frac{1}{\varepsilon}\mu([0,\varepsilon]\,S)\,.$$

Suppose that X is a 1-dimensional continuum. Every non-singular point $x\in X$ is contained in a 0-dimensional locally compact local section S. Hence S is totally disconnected and x has a neighbourhood basis on S consisting of open-compact sets in S. Clearly, each member of such a basis is a closed local section at x.

4.1. Proposition. *Let X be a 1-dimensional continuum carrying a continuous flow ϕ and $S\subset X$ be a closed local section. Let μ be a ϕ-invariant Borel probability measure on X with support $|\mu|$. If $W_\mu\cong\mathbb{Z}$, then $S\cap|\mu|$ is a finite set, possibly empty.*

Proof. Let $x\in S\cap|\mu|$ and $\{S_n : n\in\mathbb{N}\}$ be a decreasing neighbourhood basis of x in S, consisting of open-compact subsets of S. If $f_n : X\to S^1$ is the cosection map of S_n, then

$$A_\mu[f_n]=\frac{1}{\varepsilon}\mu([0,\varepsilon]\,S_n)\,.$$

Since $W_\mu\cong\mathbb{Z}$ and $\{A_\mu[f_n] : n\in\mathbb{N}\}$ is a decreasing sequence of non-negative elements of W_μ, there is some $N\in\mathbb{N}$ such that $\mu([0,\varepsilon]\,S_n)=\mu([0,\varepsilon]\,S_N)$ for every $n\geqq N$. It follows that $\mu([0,\varepsilon](S_N\backslash\{x\}))=0$ and also $\mu([-\varepsilon,\varepsilon](S_N\backslash\{x\}))=0$, because μ is ϕ-invariant. Thus, $S_N\cap|\mu|=\{x\}$. This shows that $S\cap|\mu|$ is discrete, hence finite by compactness.

4.2. Theorem. *Let X be a 1-dimensional continuum carrying a continuous flow ϕ. If μ is a ϕ-invariant Borel probability measure on X such that $W_\mu\cong\mathbb{Z}$, then every point in $|\mu|$ is either periodic or its positive (and negative) limit set consists of singular points.*

Proof. Let $x\in|\mu|$. By compactness, the positive limit set $L^+(x)$ is a non-empty subset of $|\mu|$, because $|\mu|$ is closed invariant. Suppose that $y\in L^+(x)$ is a non-singular point. Since X is 1-dimensional, there is a closed local section S at y. There is also a sequence $t_n\to+\infty$ such that $t_n x\to y$ and $t_n x\in S\cap|\mu|$ for every $n\in\mathbb{N}$. Since $S\cap|\mu|$ is finite by Proposition 4.1, we conclude that the orbit of x must be periodic.

This of course may be considered as a criterion for the existence of periodic orbits. In other words we have:

4.3. Corollary. *Let X be a 1-dimensional continuum carrying a non-singular flow ϕ. The following are equivalent:*

(i) *There is a periodic orbit in X.*

(ii) *There is a ϕ-invariant Borel probability measure μ on X such that $W_\mu \cong \mathbb{Z}$.*

In fact the support of μ in Corollary 4.3 is a finite union of periodic orbits.

4.4. Corollary. *If X is a 1-dimensional continuum carrying a non-periodic minimal flow ϕ, then W_μ is a countable dense subgroup of $\mathbb{R}$ for every ϕ-invariant Borel probability measure μ on X.*

5. Winding numbers and Poincaré-Bendixson theory on 2-manifolds

Let M be a closed orientable 2-manifold carrying a continuous flow ϕ. Every non-singular point $x \in M$ is contained in a flow box. That is, there exists a local section through x which is homeomorphic to an open interval ([3] or [5], Ch. VII, 1.6). In fact one can easily see that any arc of finite time on a non-singular orbit can be enclosed in a flow box.

A minimal set $A \subset M$ is called *trivial* if A is either a singular point, a periodic orbit or $A = M$. In the last case M is homeomorphic to the 2-torus and the flow topologically equivalent to an irrational flow, that is the suspension of an irrational rotation of S^1. A non-trivial minimal set in M is a nowhere locally connected 1-dimensional continuum and has an open invariant neighbourhood E in M such that every point in $E\backslash A$ is wandering ([2]). So, if μ is a ϕ-invariant Borel probability measure on M with support $|\mu|$ and $|\mu| \cap E \neq \emptyset$, then $|\mu| \cap E = A$, by Poincaré Recurrence ([7], Theorem 4.1.19).

5.1. Theorem. *Let M be a closed orientable 2-manifold carrying a continuous flow ϕ and let μ be a ϕ-invariant Borel probability measure on M. If $W_\mu \cong \mathbb{Z}$, then every minimal set in the support of μ is trivial.*

Proof. Suppose that there is a non-trivial minimal set A in $|\mu|$. Let $S \subset E$ be a local section of extent $\varepsilon > 0$ which is homeomorphic to an open interval and $S \cap A \neq \emptyset$. Since A is non-trivial, we may choose S so that $S \cap A$ is a Cantor set. Let "$<$" denote an orientation of S and let $x = \sup(S \cap A)$ with respect to this orientation. The set

$$I = \{z \in S : z > x\}$$

is an open segment on S and $I \cap A = \emptyset$. Let $S_n = (a_n, b_n)$, $n \in \mathbb{N}$, be a neighbourhood basis of x in S consisting of open segments in S with endpoints in $S\backslash A$. It is well known that there is a simple closed curve through x which is a closed local section to the flow ([4]). We shall need a slight modification of the construction given in [4]. Since A is non-trivial, for every $n \in \mathbb{N}$ the orbit of x returns infinitely many times in S_n. Let $t_n > 0$ be the first time the orbit of x returns to S_n and $y_n = t_n x$. Then $y_n < x$. The orbit segment $[0, t_n + \varepsilon]\, x$ can be enclosed in a flow box. More precisely, there is an open segment $I_n \subset S_n$ containing x and $\theta > 0$ such that $B_n = (-\theta, t_n + \varepsilon + \theta)\, I_n$ is a flow box. Let $z_n \in (S\backslash A) \cap [y_n, x]$, where $[y_n, x]$ denotes the closed segment on S with endpoints y_n and x. By construction, the sequence $\{z_n : n \in \mathbb{N}\}$ converges to x on S and the open segments (z_n, b_n), $n \in \mathbb{N}$, are a neighbourhood basis of x in S. If T_n is an arc in B_n from x to z_n transverse to the flow, then $C_n = [z_n, x] \cup T_n$ is a simple closed curve which is a local section of some extent $0 < 2\varepsilon_n < \varepsilon$.

Note that $C_n \cap A \subset [z_n, x]$ since $x = \sup(S \cap A)$. If now $f_n : M \to S^1$ is the cosection map of C_n, then from Lemma 2.2 we have

$$A_\mu[f_n] = \frac{1}{\varepsilon_n}\mu([0,\varepsilon_n]C_n) = \frac{1}{\varepsilon_n}\mu([0,\varepsilon_n]([z_n,x]\cap A))$$

$$= \frac{1}{\varepsilon}\mu([0,\varepsilon]([z_n,x]\cap A)) = \frac{1}{\varepsilon}\mu([0,\varepsilon]([z_n,x]))$$

the third equality being due to the ϕ-invariance of μ. Our assumption $W_\mu \cong \mathbb{Z}$ yields as in the proof of Proposition 4.1 that there is some $N \in \mathbb{N}$ such that $\mu([-\varepsilon,\varepsilon]([z_N,x])) = 0$. However, this can happen only if $|\mu| \cap (z_N, b_N) = \{x\}$, which contradicts our assumption that A is a non-trivial minimal set.

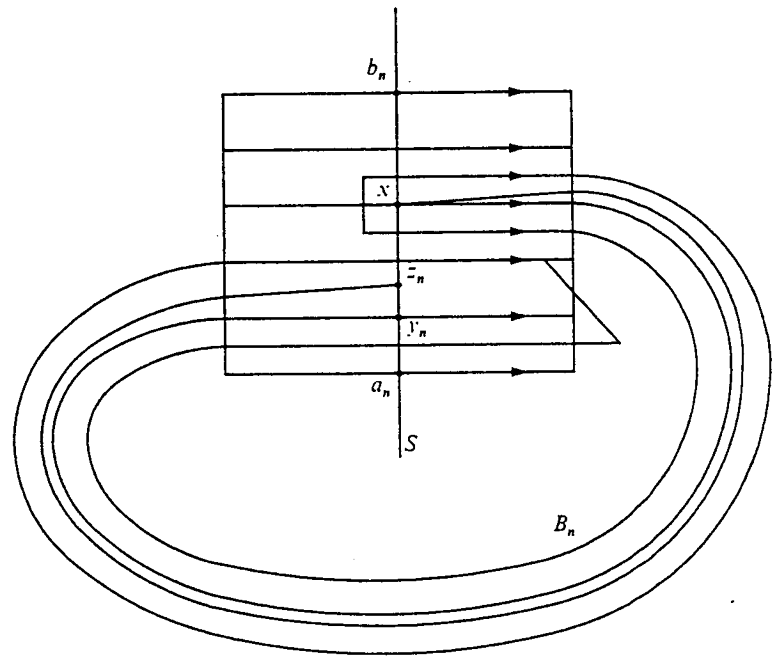

Figure 1

5.2. Lemma. *Let M be closed orientable 2-manifold carrying a continuous flow. If $x \in M$ is a non-periodic point such that $L^+(x)$ is a periodic orbit, then $J^+(x) = L^+(x)$ and therefore x is wandering.*

Proof. Since M is orientable and $L^+(x)$ is a periodic orbit, there is an open set V homeomorphic to the standard annulus $(-1,1)\times S^1$ so that $L^+(x)$ corresponds to $\{0\}\times S^1$. Hence $L^+(x)$ separates V into two connected components V_1 and V_2. We may assume that $x \in V_1$. From the orientability of M and [6], Ch. VII, 5.1, follows that $L^+(x)$ is globally positively asymptotically stable with respect to the restricted flow on the locally compact subspace $\mathbb{R}V_1 \cup L^+(x)$. Thus the flow on $\mathbb{R}V_1$ is parallelizable and admits a global compact section which is a simple closed curve ([5], Ch. VII, 1.6). That is, $\mathbb{R}V_1$ is an annulus in M. Since M is compact connected and $\mathbb{R}V_1$ open in M, the boundary of $\mathbb{R}V_1$ in M has at most two connected components. One of them is $L^+(x)$. Since the flow in $\mathbb{R}V_1$ is parallelizable and M is compact, $J^+(x)$ is a compact connected subset of the boundary of $\mathbb{R}V_1$. Since $L^+(x) \subset J^+(x)$, we must necessarily have $J^+(x) = L^+(x)$.

5.3. Corollary. *Let M be a closed orientable 2-manifold carrying a continuous flow with an invariant Borel probability measure μ such that $W_\mu \cong \mathbb{Z}$. If the positive (or negative) limit set of a point $x \in |\mu|$ does not contain singular points, then x is periodic.*

Proof. Suppose that $L^+(x)$ contains no singular point. The standard trapping argument and Theorem 5.1 yield that $L^+(x)$ is a periodic orbit. If $x \notin L^+(x)$, then x is wandering by Lemma 5.2. As this is impossible by Poincaré Recurrence, x must be periodic.

5.4. Corollary. *Let ϕ be a non-singular continuous flow on the 2-torus. The following are equivalent:*

(i) *There is a periodic orbit.*

(ii) *There is a ϕ-invariant Borel probability measure μ with $W_\mu \cong \mathbb{Z}$.*

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