University of Crete
Department of Mathematics and Applied Mathematics

An introduction to smooth manifolds

Course notes

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Introduction

These notes correspond to the one-semester introductory course on differentiable manifolds that I have taught several times in the graduate program of the Department of Mathematics of the University of Crete. They are written for graduate students who during their undergraduate studies have built a solid background in Algebra and Analysis. More precisely, the reader is required to be familiar with basic Algebra, basic Topology and advanced Calculus of functions of several variables, including the basic theory of Ordinary Differential Equations.

The first two chapters are devoted to the presentation of the basic notions. The third chapter is concerned with the basic theory of Riemannian manifolds, the Levi-Civita connection and the basic theory of geodesics, including geodesic convexity from which the existence of admissible open covers is derived. The fourth, fifth and sixth chapters are concentrated on differential forms and de Rham cohomology. This theory can be considered as a generalization of vector analysis from $\mathbb{R}^3$ to higher dimensional and non-euclidean spaces, on the one hand, and as the geometric viewpoint of the part of Algebraic Topology called (co-)homology theory, on the other. In particular the fifth and sixth chapters are essentially a crash course on Algebraic Topology using Calculus.

K. Athanassopoulos

Contents

Chapter 1

Manifolds

1.1 Topological and smooth manifolds

Problems of Classical Physics lead to the need for the development of differential and integral calculus on subsets of the phase space, like for instance level sets of constant energy, which are not open subsets of any euclidean space. Since differentiability of a function at a given point depends only on its local behaviour near the point, it is reasonable to try to develop differential calculus on topological spaces which are locally like euclidean space.

A topological space M is said to be a *topological n-manifold*, where $n \in \mathbb{Z}^+$, if it is a Hausdorff space with a countable basis for its topology and has the following property: there exists an open cover $\mathcal{U}$ of M every element of which is homeomorphic to some open subset of $\mathbb{R}^n$. Since the topology of M is assumed to have a countable basis, there exists a countable open cover $\mathcal{U}$ of M every element of which is homeomorphic to $\mathbb{R}^n$. If $U \in \mathcal{U}$, a homeomorphism $\phi : U \to \phi(U)$, where $\phi(U)$ is an open subset of $\mathbb{R}^n$, is called a *chart* of M and is usually denoted by (U, ϕ). The non-negative integer n is the *dimension* on M.

A topological manifold is a locally compact space, hence regular, and it follows from Uryshn's theorem that its topology is defined by some metric.

If now $f : M \to \mathbb{R}$ is a continuous function, it is reasonable to call f differentiable at a point $p \in M$, if there exists a chart $\phi : U \to \phi(U) \subset \mathbb{R}^n$ with $p \in U$ such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is differentiable at $\phi(p)$.

$$
\begin{tikzcd}
U \arrow[r, "f"] & \mathbb{R} \\
\phi(U) \arrow[u, "\phi^{-1}"] \arrow[ur, "f\circ\phi^{-1}"'] &
\end{tikzcd}
$$

However, in order such a definition to be good it must be independent of the choice of the chart ϕ. If $\psi : V \to \psi(V) \subset \mathbb{R}^n$ is another chart with $p \in V$, we have

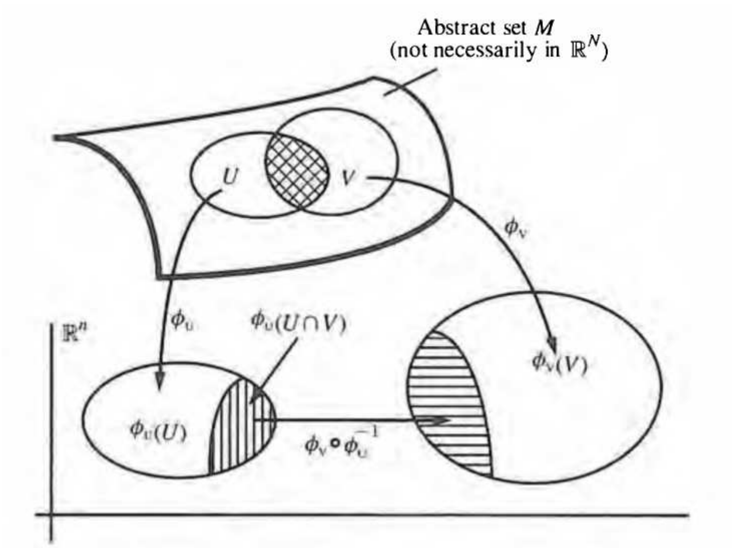
$$f \circ \phi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}).$$

Therefore, in order the differentiability of $f \circ \phi^{-1}$ at $\phi(p)$ to be equivalent to that of $f \circ \psi^{-1}$ at $\psi(p)$ it suffices $\psi \circ \phi^{-1}$ to be differentiable at $\phi(p)$ and $\phi \circ \psi^{-1}$ to be differentiable at $\psi(p)$. We are thus led to the following.

Definition 1.1.1. Two charts (U, ϕ_U) and (V, ϕ_V) of a topological n-manifold M are called *smoothly related* if $U \cap V \neq \varnothing$ and the transition map

$$\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$$

is a smooth diffeomorphism of open subsets of $\mathbb{R}^n$.



Definition 1.1.2. A *smooth atlas* of a topological n-manifold M is a family $\mathcal{A} = \{(U, \phi_U) : U \in \mathcal{U}\}$ consisting of smoothly related charts of M such that $\mathcal{U}$ is an open cover of M.

Two smooth atlases of M are called equivalent if their union is again a smooth atlas. Evidently, this is an equivalence relation on the set of smooth atlases of M. Every smooth atlas is contained in a unique maximal smooth atlas, which is the union of all smooth atlases in its equivalence class.

Definition 1.1.3. A *smooth structure* on a topological n-manifold is a maximal smooth atlas $\mathcal{A}$ of M. In this case the couple $(M, \mathcal{A})$ is called a *smooth n-manifold*. The smooth atlas $\mathcal{A}$ is usually omitted if it is clear which one is considered. The elements of $\mathcal{A}$ are called the smooth charts of M.

It is clear from the above that a smooth structure on a topological manifold can be described by a single, not necessarily maximal, smooth atlas. So, we can describe a smooth structure by defining a smooth atlas of minimum cardinality.

Examples 1.1.4. (a) The trivial example of a smooth n-manifold is an open subset M of $\mathbb{R}^n$, whose smooth structure is defined by the smooth atlas $\mathcal{A} = \{(M, id_M)\}$.

Also, if M is a smooth manifold, then any open set $X \subset M$ is a smooth manifold. If $\mathcal{A}$ is the smooth structure of M, the smooth structure of X is

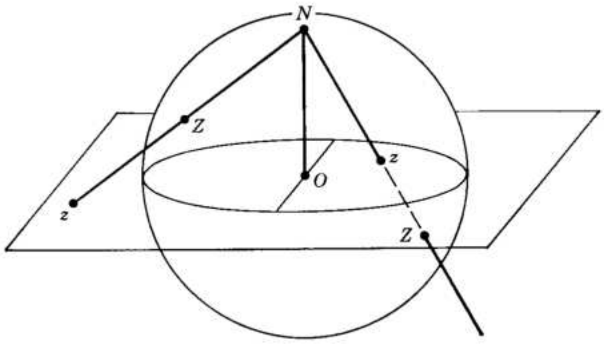
$$\mathcal{A}|_X = \{(X \cap U, \phi|_{X\cap U}) : (U, \phi) \in \mathcal{A}\}.$$

(b) The n-sphere $S^n_R = \{Z \in \mathbb{R}^{n+1} : \|Z\| = R\}$ of radius $R > 0$ is a smooth n-manifold. Its smooth structure is defined by the smooth atlas consisting of the stereographic projections with respect to the north and the south poles. More precisely, the stereographic projection with respect to the north pole is the homeomorphism $\pi_+ : S^n_R \setminus \{Re_{n+1}\} \to \mathbb{R}^n$ defined by

$$\pi_+(Z_1, ..., Z_n, Z_{n+1}) = \frac{R}{R - Z_{n+1}} \cdot (Z_1, ..., Z_n)$$

and the stereographic projection with respect to the south pole is the homeomorphism $\pi_- : S^n_R \setminus \{-Re_{n+1}\} \to \mathbb{R}^n$ defined by

$$\pi_-(Z_1, ..., Z_n, Z_{n+1}) = \frac{R}{R + Z_{n+1}} \cdot (Z_1, ..., Z_n).$$



Since the inverse π_+^{-1} is given by the formula

$$\pi_+^{-1}(z_1, ..., z_n) = \left(\frac{2R^2 z_1}{R^2 + \sum_{j=1}^n z_j^2}, ..., \frac{2R^2 z_n}{R^2 + \sum_{j=1}^n z_j^2}, \frac{2\left(-R^2 + \sum_{j=1}^n z_j^2\right)}{R^2 + \sum_{j=1}^n z_j^2} \right),$$

the transition map $\pi_- \circ \pi_+^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by

$$(\pi_- \circ \pi_+^{-1})(z) = \frac{R^2}{\|z\|^2} \cdot z.$$

In other words, $\pi_- \circ \pi_+^{-1}$ is the inversion with respect to S^{n-1}_R and is of course a smooth diffeomorphism. The standard smooth structure of S^n_R is defined by the smooth atlas $\mathcal{A} = \{(S^n_R \setminus \{Re_{n+1}\}, \pi_+), (S^n_R \setminus \{-Re_{n+1}\}, \pi_-)\}$. In case $R = 1$, we usually write S^n instead of S^n_1.

(c) If M_1 is a smooth n_1-manifold and M_2 is a smooth n_2-manifold, then their product $M_1 \times M_2$ is a smooth $(n_1 + n_2)$-manifold. Indeed, if $\mathcal{A}_j$ is a smooth atlas of M_j, $j = 1, 2$, then

$$\mathcal{A} = \{(U \times V, \phi \times \psi) : (U, \phi) \in \mathcal{A}_1, \quad (V, \psi) \in \mathcal{A}_2\}$$

is a smooth atlas of $M_1 \times M_2$.

In particular, the n-dimensional torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times) is a smooth n-manifold.

(d) The *complex projective space* $\mathbb{C}P^n$, $n \in \mathbb{Z}^+$, is the quotient space of the equivalence relation $\sim$ in $\mathbb{C}^{n+1} \setminus \{0\}$ such that $z \sim w$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ with $w = \lambda z$. In other words, the equivalence classes of $\sim$ are the complex 1-dimensional linear subspaces of $\mathbb{C}^{n+1}$ minus $0 \in \mathbb{C}^{n+1}$. Alternatively, $\mathbb{C}P^n$, can be defined as the quotient space of the equivalence relation $\sim$ on S^{2n+1} such that $z \sim w$ if and only if there exists $\lambda \in S^1$ with $w = \lambda z$. Thus, $\mathbb{C}P^n$ is the orbit space of the continuous action of the unit circle S^1 on the $(2n+1)$-sphere S^{2n+1} by scalar multiplication, whose orbits are great circles. The quotient map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a continuous, open, surjection and is called the *Hopf map*. We usually write $\pi(z_0, z_1, ..., z_n) = [z_0, z_1, ..., z_n]$ and call the complex numbers z_0, z_1,..., z_n the homogeneous coordinates od the point $[z_0, z_1, ..., z_n] \in \mathbb{C}P^n$. Obviously, $[z_0, z_1, ..., z_n] = [w_0, w_1, ..., w_n]$ if and only if

$$\begin{vmatrix} z_j & w_j \\ z_k & w_k \end{vmatrix} = 0$$

for every j, $k = 0, 1, ..., n$.

If $[z_0, z_1, ..., z_n] \neq [w_0, w_1, ..., w_n]$, there exist $0 \leq j, k \leq n$ such that $z_j w_k \neq z_k w_j$. The sets

$$U = \{[u_0, u_1, ..., u_n] \in \mathbb{C}P^n : |u_k z_j - u_j z_k| < |u_k w_j - u_j w_k|\},$$

$$W = \{[u_0, u_1, ..., u_n] \in \mathbb{C}P^n : |u_k z_j - u_j z_k| > |u_k w_j - u_j w_k|\}$$

are open, disjoint and $[z_0, z_1, ..., z_n] \in U$, $[w_0, w_1, ..., w_n] \in W$. This shows that $\mathbb{C}P^n$ is a Hausdorff space. Since the Hopf map is a continuous, open surjection, $\mathbb{C}P^n$ is a connected, compact space with a countable basis for its topology, hence metrizable.

For every integer $0 \leq k \leq n$ the set

$$U_k = \{[z_0, z_1, ..., z_n] \in \mathbb{C}P^n : z_k \neq 0\}$$

is open and the map $\phi_k : U_k \to \mathbb{C}^n$ with

$$\phi_k([z_0, z_1, ..., z_n]) = (\frac{z_0}{z_k}, ..., \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, ..., \frac{z_n}{z_k})$$

is a homeomorphism whose inverse is given by

$$\phi_k^{-1}(t_1, ..., t_n) = [t_1, ..., t_{k-1}, 1, t_k, ..., t_n].$$

Thus, $\mathbb{C}P^n$ is a topological $2n$-manifold, since

$$\mathbb{C}P^n = U_0 \cup U_1 \cup \cdots \cup U_n.$$

Moreover, if $U_j \cap U_k \neq \varnothing$ and $j \neq k$, then

$$\phi_k(U_j \cap U_k) = \begin{cases} \{(t_1, ..t_n) \in \mathbb{C}^n : t_j \neq 0\} & \text{if } j < k \\ \{(t_1, ..t_n) \in \mathbb{C}^n : t_{j-1} \neq 0\} & \text{if } j > k. \end{cases}$$

Thus, for $j < k$ we have

$$(\phi_j \circ \phi_k^{-1})(t_1, ..., t_n) = (\frac{t_1}{t_j}, ..., \frac{t_{j-1}}{t_j}, \frac{t_{j+1}}{t_j}, ..., \frac{t_{k-1}}{t_j}, \frac{1}{t_j}, \frac{t_k}{t_j}, ..., \frac{t_n}{t_j})$$

and for $j > k$ we have

$$(\phi_j \circ \phi_k^{-1})(t_1, ..., t_n) = (\frac{t_1}{t_{j-1}}, ..., \frac{t_{k-1}}{t_{j-1}}, \frac{1}{t_{j-1}}, \frac{t_k}{t_{j-1}}, ..., \frac{t_{j-2}}{t_{j-1}}, \frac{t_j}{t_{j-1}}, , ..., \frac{t_n}{t_{j-1}}).$$

So, $\mathcal{A} = \{(U_k, \phi_k) : k = 0, 1, ...n\}$ is a smooth atlas which defines a smooth structure and is called the canonical atlas of $\mathbb{C}P^n$.

(e) The *real projective space* $\mathbb{R}P^n$, $n \in \mathbb{Z}^+$, is defined in the same way simply by replacing the field $\mathbb{C}$ with the field $\mathbb{R}$. Now $\mathbb{R}P^n$ is the quotient space of the equivalence relation $\sim$ on S^n such that $x \sim -x$ for every $x \in S^n$. Again $\mathbb{R}P^n$ is a connected, compact metrizable space and a smooth n-manifold.

Definition 1.1.5. Let M be a smooth m-manifold and N be a smooth n-manifold. A continuous map $f : M \to N$ is clalled *smooth* if for every $p \in M$ there exist a smooth chart (U, ϕ) of M and smooth chart (V, ψ) of N such that $p \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is a smooth map of open subsets of euclidean spaces. We call $\psi \circ f \circ \phi^{-1}$ the local representation of f with respect to the smooth charts (U, ϕ) and (V, ψ).

The sbove definition is independent of the choice of the smooth charts (U, ϕ) and (V, ψ), because if (U_1, ϕ_1) and (V_1, ψ_1) is another pair of smooth charts with $p \in U_1$ and $f(U_1) \subset V_1$, then

$$\psi_1 \circ f \circ \phi_1^{-1} = (\psi_1 \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \phi_1^{-1})$$

and thus $\psi \circ f \circ \phi^{-1}$ is smooth if and only if $\psi_1 \circ f \circ \phi_1^{-1}$.

The class of smooth manifolds are the objects of a category whose morphisms are the smooth maps between smooth manifolds. The isomorphisms in the category are called diffeomorphisms. More precisely, a smooth map $f : M \to N$ as in Definition 1.1.5 is called a *smooth diffeomorphism* if there exists a smooth map $g : N \to M$ such that $g \circ f = id_M$ and $f \circ g = id_N$.

Definition 1.1.6. Two smooth manifolds M and N are called *(smoothly) diffeomorphic* if there exists a smooth diffeomorphism $f : M \to N$.

Obviously, two diffeomorphic manifolds must have the same dimension. If (U, ϕ) is a smooth chart of a smooth manifold M, then $\phi : U \to \phi(U)$ is a smooth diffeomorphism.

It is not true in general that any topological manifold admits a smooth structure. Also, a topological manifold may carry many non-diffeomorphic smooth structures (with the same underlying topology). J. Milnor proved in 1956 that on the 7-sphere S^7 there are non-diffeomorphic smooth structures. His work was the birth of Differential Topology. In 1982 S. Donaldson showed that already on $\mathbb{R}^4$ there exist uncountably many non-diffeomorphic smooth structures. On any topological n-manifold for $n = 1, 2, 3$ there exists a unique up to diffeomorphism smooth structure. In dimension 1 this is easy to prove. In dimension 2 this follows from the classification of topological surfaces and the uniformization theorem. In dimension 3 it was proved by J. Munkres in 1960. In both cases of dimensions 2 and 3 an important step in the proof is the non-trivial fact that topological 2- and 3-manifolds can be triangulated.

1.2 The tangent space

In order to define the derivative of a smooth map between manifolds, we shall describe the derivative of a map defined on a open subset of euclidean space in a suitable way that it can be carried over to smooth manifolds.

Let $A \subset \mathbb{R}^n$ be an open set and $p = (p^1, ..., p^n) \in A$. We denote by $C^\infty(A, p)$ the set of smooth real functions defined on some open neighbourhood of p contained in A. Let also

$$S(A, p) = \{\gamma | \gamma : (-\epsilon, \epsilon) \to A \quad \text{is smooth for some} \quad \epsilon > 0, \quad \text{with} \quad \gamma(0) = p\}.$$

Two curves γ_1, $\gamma_2 \in S(A, p)$ are tangent at p if and only if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every $f \in C^\infty(A, p)$. Tangency at p is an equivalence relation $\sim_p$ on $S(A, p)$. The quotient set $T_pA = S(A, p)/ \sim_p$ is called the *tangent space* of A at p and carries a vector space structure which is defined as follows. If $[\gamma_1]_p$, $[\gamma_2]_p \in T_pA$ and λ_1, $\lambda_2 \in \mathbb{R}$, then $\lambda_1[\gamma_1]_p + \lambda_2[\gamma_2]_p$ is the element of T_pA represented by

$$\gamma(t) = \lambda_1\gamma_1(t) + \lambda_2\gamma_2(t) - (\lambda_1 + \lambda_2 - 1)p.$$

The zero element of T_pA is represented by the constant curve at p. The elements of T_pA are called *tangent vectors* of A at p. If $\gamma_j(t) = p + te_j$, $j = 1, 2, ..., n$, then $\{[\gamma_1]_p, [\gamma_2]_p, ...[\gamma_n]_p\}$ is a basis of T_pA.

We shall give an alternative "algebraic" description of the tangent space. To every tangent vector $[\gamma]_p \in T_pA$ corresponds a linear operator $D_{[\gamma]_p} : C^\infty(A, p) \to \mathbb{R}$ which is defined by

$$D_{[\gamma]_p}(f) = (f \circ \gamma)'(0).$$

This is a fancy way to consider the directional derivative with respect to the velocity of γ at p. Recall that two functions f, $g \in C^\infty(A, p)$ are said to define the same germ at p if they agree on some small neighbourhood of p and this is an equivalence relation on $C^\infty(A, p)$ whose classes are called the germs of smooth functions at p. Note that if two functions f, $g \in C^\infty(A, p)$ define the same germ at p, then $D_{[\gamma]_p}(f) = D_{[\gamma]_p}(g)$.

The set $\mathcal{G}_p$ of germs of smooth functions at p can be endowed with the structure of a commutative, associative real algebra with a unity in the obvious way. The unity is the germ of the constant function with value 1. It is evident now that to every tangent vector $[\gamma]_p \in T_pA$ corresponds a linear operator $D_{[\gamma]_p} : \mathcal{G}_p \to \mathbb{R}$, as above, and this correspondence is injective by definition. Moreover, it satisfies the Leibniz rule for the derivative of a product of functions. Thus, we are led to the following.

Definition 1.2.1. A *derivation* on the algebra $\mathcal{G}_p$ of germs of smooth functions at p is a linear operator $D : \mathcal{G}_p \to \mathbb{R}$ which satisfies the Leibniz rule

$$D(\alpha \cdot \beta) = e_p(\beta)D(a) + e_p(\alpha)D(\beta)$$

for every α, $\beta \in \mathcal{G}_p$, where $e_p : \mathcal{G}_p \to \mathbb{R}$ denotes the evaluation at p.

A derivation of $\mathcal{G}_p$ vanishes on the germs of constant functions, because

$$D(1) = D(1 \cdot 1) = 1 \cdot D(1) + 1 \cdot D(1) = 2D(1).$$

The set T_p of all derivations of $\mathcal{G}_p$ is obviously a linear subspace of the algebraic dual of the vector space $\mathcal{G}_p$ and the map $F : T_pA \to T_p$ defined by

$$F([\gamma]_p) = D_{[\gamma]_p}$$

is a linear monomorphism, because if $D_{j,p} = F([\gamma_j]_p)$, then

$$D_{j,p}(f) = \frac{\partial f}{\partial x^j}(p)$$

for $j = 1, 2, ..., n$ and the set $\{D_{1,p}, D_{2,p}, ..., D_{n,p}\}$ is linearly independent, since

$$D_{j,p}(x^k) = \delta_{jk}$$

where $x^k : \mathbb{R}^n \to \mathbb{R}$ denotes the projection onto the k-th coordinate.

It is a non-trivial fact that F is actually a linear isomorphism. Its proof is based on the following lemma from advanced calculus.

Lemma 1.2.2. *For every $f \in C^\infty(A, p)$ there exist $g_1,..., g_n \in C^\infty(A, p)$ and a convex open neighbourhood W of p such that*

$$f(x) = f(p) + \sum_{k=1}^{n} (x^k - p^k)g_k(x)$$

for every $x = (x^1, ..., x^n) \in W$, and

$$g_k(p) = \frac{\partial f}{\partial x^k}(p)$$

for every $k = 1, 2, ..., n$.

Proof. Let W be any convex open neighbourhood of p on which f is defined and let

$$g_k(x) = \int_0^1 \frac{\partial f}{\partial x^k}(tx + (1 - t)p)dt$$

for every $x = (x^1, ..., x^n) \in W$ and $k = 1, 2, ...n$. From the Fundamental Theorem of Calculus and the chain rule we have

$$f(x) - f(p) = \int_0^1 \frac{d}{dt}(f(tx + (1 - t)p))dt$$

$$= \int_0^1 \left[\sum_{k=1}^n (x^k - p^k) \frac{\partial f}{\partial x^k}(tx + (1-t)p) \right] dt = \sum_{k=1}^n (x^k - p^k) g_k(x).$$

The rest is obvious. □

Proposition 1.2.3. *The set $\{D_{1,p}, D_{2,p}, ..., D_{n,p}\}$ is a basis of T_p and therefore F is a linear isomorphism.*

Proof. It suffices to prove that $\{D_{1,p}, D_{2,p}, ..., D_{n,p}\}$ generates T_p. Let $D \in T_p$ and $a_k = D(x^k)$, $k = 1, 2, ..., n$. For every $f \in C^\infty(A, p)$ we apply Lemma 1.2.2 and then we have

$$D(f) = D(f(p)) + \sum_{k=1}^n D((x^k - x^k(p))g_k) = \sum_{k=1}^n D(x^k) g_k(p) + \sum_{k=1}^n (x^k(p) - x^k(p)) D(g)$$

$$= \sum_{k=1}^n a_k \frac{\partial f}{\partial x^k}(p) = \left(\sum_{k=1}^n a_k D_{k,p} \right)(f). \quad \square$$

Thus, henceforth we shall identify the linear space T_p with T_pA.

Let now $f = (f_1, f_2, ..., f_m) : A \to \mathbb{R}^m$ be a smooth map. The linear map $f_* : T_pA \to T_{f(p)}\mathbb{R}^m$ defined by

$$f_*([\gamma]_p) = [f \circ \gamma]_{f(p)}$$

is just the derivative of f at p, since $(f \circ \gamma)'(0) = Df(p) \cdot \gamma'(0)$ for every $\gamma \in S(A, p)$. This is a convenient way to consider the derivative of a smooth function that can be carried over to smooth manifolds.

Let M be a smooth n-manifold and $p \in M$. We can define

$$S(M, p) = \{\gamma | \gamma : (-\epsilon, \epsilon) \to M \quad \text{is smooth for some} \quad \epsilon > 0, \quad \text{with} \quad \gamma(0) = p\}$$

and consider the set $C^\infty(M, p)$ of smooth real functions defined on some open neighbourhood of p in M. As before we call γ_1, $\gamma_2 \in S(M, p)$ tangent at p if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every $f \in C^\infty(M, p)$ and define the *tangent space* T_pM of M at p to be the quotient set of this equivalence relation. Let (U, ϕ_U) be a smooth chart of M such that $p \in U$. The map $\widetilde{\phi_U} : T_pM \to T_{\phi_U(p)}\phi(U)$ defined by $\widetilde{\phi_U}([\gamma]_p) = [\phi_U \circ \gamma]_{\phi_U(p)}$ is a bijection whose inverse is given by $\widetilde{\phi_U}^{-1}([\zeta]_{\phi_U(p)}) = [\phi_U^{-1} \circ \zeta]_p$. We transfer the vector space structure of $T_{\phi_U(p)}\phi_U(U)$ to T_pM so that $\widetilde{\phi_U}$ becomes a linear isomorphism. This vector space structure does not depend on the choice of the smooth chart (U, ϕ_U), because if (V, ϕ_V) is another smooth chart of M with $p \in V$, then $\widetilde{\phi_U} \circ \widetilde{\phi_V}^{-1} = (\phi_U \circ \phi_V^{-1})_{*\phi_V(p)}$ is a linear isomorphism, since it is the derivative at $\phi_V(p)$ of the transition map $\phi_U \circ \phi_V^{-1}$, which is a smooth diffeomorphism.

$$\begin{tikzcd}
T_pM \arrow[r, "id"] \arrow[d, "\widetilde{\phi_V}"'] & T_pM \arrow[d, "\widetilde{\phi_U}"] \\
T_{\phi_V(p)}\phi_V(V) \arrow[r, "(\phi_U \circ \phi_V^{-1})_{*\phi_V(p)}"] & T_{\phi_U(p)}\phi_U(U)
\end{tikzcd}$$

The elements of the tangent space T_pM are called *tangent vectors* of M at the point p. From the above discussion, the tangent vectors of M at p can be considered as derivations of the algebra of germs $\mathcal{G}_p(M)$ of real smooth functions defined on some open neighbourhood of p in M. If (U, ϕ_U) is a smooth chart of M, where $\phi_U = (x^1, x^2, ..., x^n)$, and

$$\left(\frac{\partial}{\partial x^j}\right)_p = \widetilde{\phi_U}^{-1}(D_{j,\phi_U(p)})$$

for $j = 1, 2, ..., n$, then the set of tangent vectors

$$\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, ..., \left(\frac{\partial}{\partial x^n}\right)_p\right\}$$

is a basis of T_pM which depends on ϕ_U and is called the *canonical basis* of T_pM with respect to the chart ϕ_U.

If now $f : M \to P$ is a smooth map into a smooth m-manifold P, the *derivative* of f at the point $p \in M$ is defined to be the linear map $f_{*p} : T_pM \to T_{f(p)}P$ with

$$f_{*p}([\gamma]_p) = [f \circ \gamma]_{f)p)}$$

for every $[\gamma]_p \in T_pM$. In particular, $\widetilde{\phi_U} = (\phi_U)_{*p}$ by definition.

Let (U, ϕ) be a smooth chart of M with $p \in U$ and (W, ψ) be a smooth chart of P with $f(U) \subset W$. If $\phi = (x^1, x^2, ..., x^n)$ and $\psi = (y^1, y^2, ..., y^m)$, then

$$\psi_{*f(p)}\left(f_{*p}\left(\left(\frac{\partial}{\partial x^j}\right)_p\right)\right) = (\psi \circ f \circ \phi^{-1})_{*\phi(p)}(D_{j,\phi(p)})$$

for $j = 1, 2, ..., n$ and therefore the matrix of f_{*p} with respect to the ordered basis

$$\left[\left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, ..., \left(\frac{\partial}{\partial x^n}\right)_p\right]$$

of T_pM and

$$\left[\left(\frac{\partial}{\partial y^1}\right)_p, \left(\frac{\partial}{\partial y^2}\right)_p, ..., \left(\frac{\partial}{\partial y^m}\right)_p\right]$$

of $T_{f(p)}P$ is the Jacobian matrix at $\phi(p)$ of the local representation $\psi \circ f \circ \phi^{-1}$ of f.

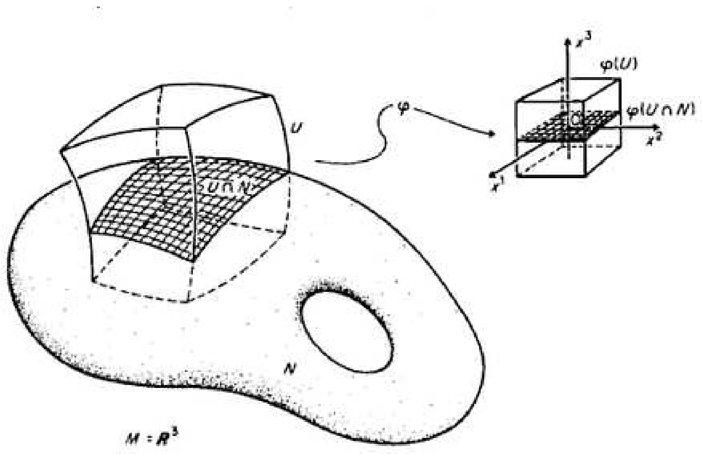
$$\begin{tikzcd}
T_pM \arrow[r, "f_{*p}"] \arrow[d, "\phi_{*p}"'] & T_{f(p)}P \arrow[d, "\psi_{*f(p)}"] \\
T_{\phi(p)}\phi(U) \arrow[r, "(\psi\circ f\circ\phi^{-1})_{*\phi(p)}"] & T_{\psi(f(p))}\psi(W)
\end{tikzcd}$$

1.3 Submanifolds

Let M be a smooth m-manifold and $0 \leq n \leq m$ be an integer. A set $N \subset M$ is said to be a *(regular or embedded) n-dimensional smooth submanifold* of M if for every $p \in N$ there exists smooth chart (U, ϕ) of M such that $p \in N$ and

$$\phi(N \cap U) = Q \cap (\mathbb{R}^n \times \{0\})$$

for some open set $Q \subset \mathbb{R}^m$. The smooth chart (U, ϕ) of M is said to be N-*straightening.*



If we denote by $\pi : \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ the projection onto the first n coordinates and by $i : \mathbb{R}^n \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^m$ the inclusion, then the map

$$(\pi \circ |_{N \cap U})^{-1} = \phi^{-1} \circ i : i^{-1}(Q) \to M$$

is smooth and is usually called local parametrization of N.

Obviously, a n-dimensional smooth submanifold N of M is a topological n-manifold, with respect to the subspace topology which it inherits from M. Moreover,

$$\mathcal{A}|_N = \{(N \cap U, \pi \circ \phi|_{N \cap U}) : (U, \phi) \quad \text{is a } N\text{-straightening smooth chart of } M\}$$

is a smooth atlas of N. If (U, ϕ) and (V, ψ) are two N-straightening smooth charts of M with $N \cap U \cap V \neq \varnothing$, the transition map of the corresponding elements of $\mathcal{A}|_N$ is $\pi \circ (\psi \circ \phi^{-1}) \circ i$ defined on an open subset of $\mathbb{R}^n$. Thus N becomes a smooth n-manifold.

The local representation of the inclusion $i_N : N \hookrightarrow M$ with respect to a N-straightening smooth chart (U, ϕ) of M and the corresponding smooth chart of N in $\mathcal{A}|_N$, as above, is

$$\phi \circ i_N \circ (\pi \circ \phi|_{N \cap U})^{-1} = i|_{i^{-1}(Q)} : i^{-1}(Q) \to \mathbb{R}^m.$$

Therefore, i_N is smooth and its derivative at every point of N is a linear monomorphism. Generalizing, we give the following.

Definition 1.3.1. Let N be a smooth n-manifold and M be a smooth m-manifold, with $n \leq m$. A smooth map $f : N \to M$ is called *immersion* if its derivative $f_{*q} : T_q N \to T_{f(q)} M$ is a linear monomorphism for every $q \in N$. If moreover f is a topological embedding, then f is called a *smooth embedding*.

Perhaps the most important examples of submanifolds are the level sets of smooth maps. Conditions which ensure that this kind of subsets of a given smooth manifold are smooth submanifolds are provided from the Implicit Function Theorem or the more general Constant Rank Theorem of advanced calculus, which we shall prove as a consequence of the Inverse Map Theorem.

Theorem 1.3.2. *Let $A \subset \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}^m$ be a smooth map. If $p \in A$ and the Jacobian matrix $Df(x)$ has constant rank k for every x in some open neighbourhood of p in A, then there exist an open neighbourhood $U \subset A$ of p and a smooth diffeomorphism $\phi : U \to \phi(U)$ onto an open set $\phi(U) \subset \mathbb{R}^n$, and an open neighbourhood V of $f(p)$ and a smooth diffeomorphism $\psi : V \to \psi(V)$ onto an open set $\psi(V) \subset \mathbb{R}^m$ such that the smooth map*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \subset \mathbb{R}^m$$

is given by the formula

$$(\psi \circ f \circ \phi^{-1})(x^1, ..., x^k, x^{k+1}, ..., x^n) = (x^1, ..., x^k, 0, ..., 0)$$

for every $(x^1, .., x^n) \in \phi(U)$.

Proof. Up to translations and linear isomorphisms of $\mathbb{R}^n$ and $\mathbb{R}^m$, which are of course diffeomorphisms, we may assume that $p = 0$, $f(p) = 0$ and

$$\begin{vmatrix} \frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^k} \\ \frac{\partial f_2}{\partial x^1} & \cdots & \frac{\partial f_2}{\partial x^k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x^1} & \cdots & \frac{\partial f_k}{\partial x^k} \end{vmatrix} \neq 0$$

on an open neighbourhood $A_0 \subset A$ of 0, where $f = (f_1, ..., f_k, f_{k+1}, ..., f_m)$.

We consider the smooth map $F : A_0 \to \mathbb{R}^n$ defined by

$$F(x^1, ..., x^n) = (f_1(x^1, ..., x^n), ..., f_k(x^1, ..., x^n), x^{k+1}, ..., x^n).$$

Then, $F(0) = 0$ and

$$\det DF(0) = \begin{vmatrix} \frac{\partial f_1}{\partial x^1}(0) & \cdots & \frac{\partial f_1}{\partial x^k}(0) \\ \frac{\partial f_2}{\partial x^1}(0) & \cdots & \frac{\partial f_2}{\partial x^k}(0) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x^1}(0) & \cdots & \frac{\partial f_k}{\partial x^k}(0) \end{vmatrix} \neq 0.$$

Applying the Inverse Map Theorem, there exist an open neighbourhood $U_0 \subset A_0$ of 0 such that $F(U_0)$ is an open subset of $\mathbb{R}^n$ and $\phi = F|_{U_0}$ is a smooth diffeomorphism.

Shrinking, we can take U_0 such that $\phi(U_0)$ is an open cube in $\mathbb{R}^n$ with center 0. Now there exist smooth functions g_{k+1},..., $g_m : \phi(U_0) \to \mathbb{R}$ such that

$$(f \circ \phi^{-1})(z^1, ..., z^n) = (z^1, ..., z^k, g_{k+1}(z^1, ..., z^n), ..., g_m(z^1, ..., z^n))$$

for every $(z^1, ..., z^n) \in \phi(U_0)$ and $g_{k+1}(0) = \cdots = g_m(0) = 0$. Moreover,

$$Df(\phi^{-1}(z)) \cdot D(\phi^{-1})(z) = D(f \circ \phi^{-1})(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ * & * & \cdots & * & \frac{\partial g_{k+1}}{\partial x^{k+1}}(z) & \cdots & \frac{\partial g_{k+1}}{\partial x^n}(z) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & \frac{\partial g_m}{\partial x^{k+1}}(z) & \cdots & \frac{\partial g_m}{\partial x^n}(z) \end{pmatrix}$$

for every $z = (z^1, ..., z^n) \in \phi(U_0)$. Since $Df(\phi^{-1}(z))$ has constant rank k and $D(\phi^{-1})(z)$ is invertible for every $z = (z^1, ..., z^n) \in \phi(U_0)$, we must have

$$\frac{\partial g_j}{\partial x^l} = 0$$

on $\phi(U_0)$ for every $j = k+1, ..., m$ and $l = k+1, ..., n$. This implies that the smooth functions g_{k+1},..., g_m do not depend on the variables x^{k+1},.., x^n and descent to smooth functions (again denoted by) g_{k+1},..., $g_m : P \to \mathbb{R}$, where the open cube $P \subset \mathbb{R}^k$ is the image of $\phi(U_0)$ under the projection onto the first k coordinates.

If now $\psi : P \times \mathbb{R}^{m-k} \to \mathbb{R}^m$ is the smooth map defined by

$$\psi(y^1, ..., y^m) = (y^1, ..., y^k, y^{k+1} - g_{k+1}(y^1, ..., y^k), ..., y^m - g_m(y^1, ..., y^k)),$$

then

$$D\psi(0) = \begin{pmatrix} I_k & 0 \\ * & I_{m-k} \end{pmatrix}$$

and by the Inverse Map Theorem there exists an open neighbourhood V of 0 in $\mathbb{R}^m$ such that $\psi(V)$ is an open neighbourhood of $\psi(0) = 0$ and $\psi|_V$ is a smooth diffeomorphism. Let $U \subset U_0$ be an open neighbourhood of 0 such that $f(U) \subset V$. Then,

$$(\psi \circ f \circ \phi^{-1})(z^1, ..., z^k, z^{k+1}, ..., z^n) = (z^1, ..., z^k, 0, ..., 0)$$

for every $(z^1, .., z^n) \in \phi(U)$. □

Corollary 1.3.3. *Let N be a smooth n-manifold, M be a smooth m-manifold, with $n \leq m$, and let $f : N \to M$ be an immersion. Then, for every $p \in N$ there exist a smooth chart (U, ϕ) of N with $p \in U$ and a smooth chart (V, ψ) of M with $f(U) \subset V$ such that the corresponding local representation of f is*

$$(\psi \circ f \circ \phi^{-1})(x^1, ..., x^n) = (x^1, ..., x^n, 0, ..., 0). \quad \square$$

Corollary 1.3.4. *Let N be a smooth n-manifold and M be a smooth m-manifold, with $n \leq m$. If $f : N \to M$ is a smooth embedding, then $f(N)$ is a n-dimensional* *smooth submanifold of M.* □

Let M be a smooth m-manifold, P be a smooth n-manifold, with $n \leq m$, and let $f : M \to P$ be a smooth map. We call $p \in M$ a *critical point* of f if the derivative $f_{*p} : T_pM \to T_{f(p)}P$ is not a linear epimorphism. Note that if $p \in M$ is a non-critical point of f, then f_{*q} has constant maximal rank n for every point q in some open neighbourhood of p in M. A point $c \in P$ is called a *regular value* of f if the level set $f^{-1}(c)$ does not contain any critical point of f.

Corollary 1.3.5. *Let M be a smooth m-manifold, P be a smooth n-manifold, with $n \leq m$, and let $f : M \to P$ be a smooth map. If $c \in P$ is a regular value of f, then the level set $f^{-1}(c)$ is a $(m-n)$-dimensional smooth submanifold of M, if non-empty.*

Proof. By Theorem 1.3.2, for every point $p \in f^{-1}(c)$ there exists a smooth chart (U, ϕ) of M with $p \in U$ and a smooth chart (V, ψ) of P with $f(U) \subset V$ such that the corresponding local representation of f is

$$(\psi \circ f \circ \phi^{-1})(x^1, ..., x^m) = (x^1, ..., x^n)$$

for every $(x^1, .., x^m) \in \phi(U)$. Now we have

$$\phi(f^{-1}(c) \cap U) = \phi(U) \cap (\{\psi(c)\} \times \mathbb{R}^{m-n})$$

and therefore (U, ϕ) is a $f^{-1}(c)$-straightening chart of M. □

Definition 1.3.6. Let M be a smooth m-manifold and P be a smooth n-manifold, with $n \leq m$. A smooth map $f : M \to P$ onto P is called *submersion* if its derivative $f_{*p} : T_pM \to T_{f(p)}P$ is a linear epimorphism for every $p \in M$.

Thus, if $f : M \to P$ is a submersion, then $f^{-1}(c)$ is a $(m - n)$-dimensional smooth submanifold of M for every $c \in P$.

Example 1.3.7. The determinant is a smooth function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ and the general linear group $GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$ is an open subset of $\mathbb{R}^{n\times n}$. Let $A \in GL(n, \mathbb{R})$ and $\gamma(t) = (1 + t)A$. Then, $\gamma(0) = A$ and

$$(\det)_{*A}([\gamma]_A) = [\det \circ\gamma]_{\det A}.$$

Also, $(\det \circ\gamma)(t) = (1 + t)^n \det A$, and so $(\det \circ\gamma)'(0) = n \det A \neq 0$. This implies that $(\det)_{*A}$ is non-zero, and hence an epimorphism. This shows that $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is a submersion. In particular, the special linear group $SL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$ is a $(n^2 - 1)$-dimensional smooth submanifold of $\mathbb{R}^{n\times n}$.

1.4 Smooth partitions of unity

Our requirement a smooth manifold to have a countable basis for its topology implies the existence of technically very useful families of smooth functions, the construction of which will be the subject of this section.

Definition 1.4.1. Let M be a smooth manifold and let $\mathcal{U}$ be an open cover of M. A *smooth partition of unity* subordinated to $\mathcal{U}$ is a family of smooth functions $f_U : M \to [0,1]$, $U \in \mathcal{U}$, with the following properties:
(i) $\text{supp} f_U = \overline{\{p \in M : f_U(p) \neq 0\}} \subset U$ for every $U \in \mathcal{U}$.
(ii) The family $\{\text{supp} f_U : U \in \mathcal{U}\}$ of closed subsets of M is a locally finite cover of M.
(iii) $\sum_{U \in \mathcal{U}} f_U(p) = 1$ for every $p \in M$.

Recall that a family $\mathcal{F}$ of subsets of a topological space X is called *locally finite* if every point $x \in X$ has an open neighbourhood V in X such that the set

$$\{F \in \mathcal{F} : F \cap V \neq \varnothing\}$$

is finite. A family $\mathcal{S}$ of subsets of X is called a *refinement* of $\mathcal{F}$ if for every $F \in \mathcal{F}$ there exists some $S \in \mathcal{S}$ such that $S \subset F$.

In order to prove the existence of smooth partitions of unity we shall need some preliminary lemmas. In the sequel we shall denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$.

Lemma 1.4.2. *For every $0 < \rho < r$ there exists a smooth function $f : \mathbb{R}^n \to [0,1]$ such that $\overline{B(0,\rho)} \subset f^{-1}(1)$ and $\mathbb{R}^n \setminus B(0,r) \subset f^{-1}(0)$.*

Proof. It suffices to consider the smooth function $g : \mathbb{R} \to \mathbb{R}$ with

$$g(t) = \begin{cases} e^{-\frac{1}{t}}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0 \end{cases}$$

and take $f : \mathbb{R}^n \to [0,1]$ defined by

$$f(x) = \frac{g(r^2 - \|x\|^2)}{g(r^2 - \|x\|^2) + g(\|x\|^2 - \rho^2)}. \quad \square$$

Functions like f in Lemma 1.4.2 are usually called bump functions.

Lemma 1.4.3. *Let M be a smooth n-manifold and let $\mathcal{U}$ be an open cover of M. There exists a countable smooth atlas $\mathcal{A}$ of M with the following properties:*
(a) The open cover $\mathcal{V} = \{V : (V, \phi_V) \in \mathcal{A}\}$ is a locally finite refinement of $\mathcal{U}$.
(b) $\phi_V(V) = B(0,3) \subset \mathbb{R}^n$, for every $(V, \phi_V) \in \mathcal{A}$.
(c) $\{\phi_V^{-1}(B(0,1)) : (V, \phi_V) \in \mathcal{A}\}$ is an open cover of M.

Proof. There exists a countable open cover $\{A_k : k \in \mathbb{N}\}$ of M such that $\overline{A_k} \subset A_{k+1}$ and $\overline{A_k}$ is compact for every $k \in N$, because M is locally compact and its topology has a countable basis. This sort of cover can be constructed inductively, starting with any countable open cover $\{C_k : k \in \mathbb{N}\}$ such that $\overline{C_k}$ is compact for every $k \in N$. First we choose any open set $A_1 \subset M$ with compact closure such that

$\overline{C_1} \subset A_1$ and once A_{k-1} has been defined we choose $A_k \subset M$ to be any open set with compact closure such that $\overline{A_{k-1}} \cup C_k \subset A_k$.

The set $\overline{A_{k+1}} \setminus A_k$ is compact and contained in the open set $A_{k+2} \setminus \overline{A_{k-1}}$. For every $p \in \overline{A_{k+1}} \setminus A_k$ there exist $U_p \in \mathcal{U}$ and a smooth chart $(V_{k,p}, \phi_{V_{k,p}})$ of M such that $p \in V_{k,p} \subset U_p \cap A_{k+2} \setminus \overline{A_{k-1}}$ and $\phi_{V_{k,p}}(V_{k,p}) = B(0,3)$ with $\phi_{V_{k,p}}(p) = 0$. By compactness of $\overline{A_{k+1}} \setminus A_k$, there exist p_1,...,$p_{m_k} \in \overline{A_{k+1}} \setminus A_k$, for some $m_k \in \mathbb{N}$, such that

$$\overline{A_{k+1}} \setminus A_k \subset \phi^{-1}_{V_{k,p_1}}(B(0,1)) \cup \cdots \cup \phi^{-1}_{V_{k,p_{m_k}}}(B(0,1)).$$

It suffices now to take

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \{(V_{k,p_1}, \phi_{V_{k,p_1}}), ..., (V_{k,p_{m_k}}, \phi_{V_{k,p_{m_k}}})\}. \quad \square$$

Theorem 1.4.4. *If M is a smooth n-manifold and $\mathcal{U}$ is an open cover of M, then there exists a smooth partition of unity subordinated to $\mathcal{U}$.*

Proof. Let $\mathcal{A}$ be the smooth atlas of M provided by Lemma 1.4.3. By Lemma 1.4.2, there exists a smooth function $f : \mathbb{R}^n \to [0,1]$ such that $\overline{B(0,1)} \subset f^{-1}(1)$ and $\mathbb{R}^n \setminus B(0,2) \subset f^{-1}(0)$. For every $(V, \phi_V) \in \mathcal{A}$ we consider the smooth function $g_V : M \to [0,1]$ defined by

$$g_V(p) = \begin{cases} f(\phi_V(p)), & \text{if } p \in V, \\ 0, & \text{if } p \in M \setminus V. \end{cases}$$

According to Lemma 1.4.3, $\mathcal{V} = \{V : (V, \phi_V) \in \mathcal{A}\}$ is a locally finite open cover of M. So the function $\sum_{V \in \mathcal{V}} g_V : M \to [0, +\infty)$ is well defined and smooth. Since $\mathcal{V}$ is also a refinement of $\mathcal{U}$, there exists a function $\sigma : \mathcal{V} \to \mathcal{U}$ such that $V \subset \sigma(V)$ for every $V \in \mathcal{V}$. For every $U \in \mathcal{U}$ we define now

$$f_U = \frac{1}{\sum_{V \in \mathcal{V}} g_V} \cdot \sum_{\sigma(V)=U} g_V : M \to [0,1].$$

In case $\sigma^{-1}(U) = \varnothing$ we put $f_U = 0$. It follows from Lemma 1.4.3(c) that f_U is a well defined smooth function Obviously,

$$\mathrm{supp} f_U \subset \bigcup_{\sigma(V)=U} \mathrm{supp} g_V \subset \bigcup_{\sigma(V)=U} V \subset U.$$

and $\{\mathrm{supp} f_U : U \in \mathcal{U}\}$ is locally finite, because $\mathcal{V}$ is locally finite. Finally,

$$\sum_{U \in \mathcal{U}} f_U = \frac{1}{\sum_{V \in \mathcal{V}} g_V} \cdot \sum_{U \in \mathcal{U}} \sum_{\sigma(V)=U} g_V = \frac{1}{\sum_{V \in \mathcal{V}} g_V} \cdot \sum_{V \in \mathcal{V}} g_V = 1. \quad \square$$

Corollary 1.4.5. *Let M be a smooth manifold and $F \subset A \subset M$, where F is closed in M and A is open in M. Then, then exists a smooth function $f : M \to [0,1]$ such that $F \subset f^{-1}(1)$ and $M \setminus A \subset f^{-1}(0)$.*

Proof. From Theorem 1.4.4, there exists a smooth partition of unity $\{f_{M\setminus F}, f_A\}$ subordinated to the open cover $\{M \setminus F, A\}$ of M. It suffices to take $f = f_A$. □

As an application of the existence of smooth partitions of unity we shall give a partial answer to the following question. Is a smooth manifold diffeomorphic to a smooth submanifold of some $\mathbb{R}^N$ for sufficiently large $N \in \mathbb{N}$ and what is the minimum value of N for which this is possible?

Theorem 1.4.6. *If M is a compact smooth n-manifold, there exist $N \in \mathbb{N}$ and a smooth embedding $g : M \to \mathbb{R}^N$.*

Proof. From Lemma 1.4.3 and the compactness of M, there exist some $m \in \mathbb{N}$, a finite family $\{(U_i, \phi_i) : 1 \le i \le m\}$ of smooth charts of M and a finite family $\{V_i : 1 \le i \le m\}$ of open subsets of M such that $\overline{V}_i \subset U_i$ for all $1 \le i \le m$ and

$$M = U_1 \cup \cdots \cup U_m = V_1 \cup \cdots \cup V_m.$$

For each $1 \le i \le m$ there exists a smooth function $f_i : M \to [0, 1]$ such that $\overline{V}_i \subset f_i^{-1}(1)$ and $\mathrm{supp} f_i \subset U_i$, from Corollary 1.4.5. The map $\psi_i : M \to \mathbb{R}^n$ defined by

$$\psi_i(p) = \begin{cases} f_i(p)\phi_i(p), & \text{if } p \in U_i, \\ 0, & \text{otherwise,} \end{cases}$$

is smooth. The map $g : M \to (\mathbb{R}^n)^m \times \mathbb{R}^m$ defined by

$$g(p) = (\psi_1(p), ..., \psi_m(p), f_1(p), ..., f_m(p))$$

is smooth and actually an immersion, because for every $p \in M$ there exists some $1 \le i \le m$ with $p \in V_i$ and $\psi_i|_{V_i} = \phi_i|_{V_i}$ maps V_i diffeomorphically onto an open subset of $\mathbb{R}^n$. To see that g is injective, let p, $q \in M$ be such that $g(p) = g(q)$. Then, $\psi_i(p) = \psi_i(q)$ and $f_i(p) = f_i(q)$ for every $1 \le i \le m$. There exists however some $1 \le j \le m$ with $p \in V_j$ and so $f_j(q) = f_j(p) = 1$. Therefore, $q \in U_j$ and $\phi_j(p) = \psi_j(p) = \psi_j(q) = \phi_j(q)$, hence $p = q$. Finally, g is a topological embedding, since M is compact. □

It has been proved by H. Whitney that a compact smooth n-manifold can be smoothly embedded in $\mathbb{R}^{2n}$. Also any smooth n-manifold can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset. The presentation of these topics are beyond the scope of these notes.

1.5 Exercises

1. On $\mathbb{R}$ we consider the smooth structure $\mathcal{B}$ defined by the smooth atlas $\{(\mathbb{R}, \psi)\}$, where $\psi : \mathbb{R} \to \mathbb{R}$ is the map $\psi(t) = t^3$. Let $\mathcal{A}$ denote the standard smooth structure of $\mathbb{R}$.
(a) Prove that $\mathcal{A} \ne \mathcal{B}$.
(b) Prove that $id : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is not a smooth diffeomorphism.

(c) Are the smooth 1-manifolds $(\mathbb{R}, \mathcal{A})$, $(\mathbb{R}, \mathcal{B})$ diffeomorphic?

2. For every $t > 0$ we consider the map $h_t : \mathbb{R} \to \mathbb{R}$ with $h_t(x) = x$, if $x \leq 0$ and $h_t(x) = tx$, if $x \geq 0$. Let $\mathcal{A}_t$ be the smooth structure on $\mathbb{R}$ defined by the smooth atlas $\{(\mathbb{R}, h_t)\}$, $t > 0$.
(a) Prove that $\mathcal{A}_t \neq \mathcal{A}_s$ for $t \neq s$.
(b) Are the smooth 1-manifolds $(\mathbb{R}, \mathcal{A}_t)$ and $(\mathbb{R}, \mathcal{A}_s)$ diffeomorphic for all t, $s > 0$?

3. Let $U_i^+ = \{(x_1, ..., x_{n+1}) \in S^n : x_i > 0\}$, $U_i^- = \{(x_1, ..., x_{n+1}) \in S^n : x_i < 0\}$, and let $h_i^{\pm} : U_i^{\pm} \to \mathbb{R}^n$ be the map with

$$h_i^{\pm}(x_1, ..., x_{n+1}) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}), \qquad 1 \leq i \leq n+1.$$

(a) Prove that $\mathcal{B} = \{(U_i^{\pm}, h_i^{\pm}) : 1 \leq i \leq n+1\}$ is a smooth atlas on S^n.
(b) Prove that $\mathcal{B}$ is equivalent to the smooth atlas

$$\mathcal{A} = \{(S^n \setminus \{e_{n+1}\}, \pi_+), (S^n \setminus \{-e_{n+1}\}, \pi_-)\},$$

where $\pi_{\pm} : S^n \setminus \{\pm e_{n+1}\} \to \mathbb{R}^n$ is the stereographic projection.

4. Let $(V, \langle , \rangle)$ be a finite dimensional inner product real vector space and let

$$S(V) = \{x \in V : \|x\| = 1\},$$

where $\|x\| = \langle x, x\rangle^{1/2}$.
(a) If $p \in S(V)$, prove that for every $x \in S(V) \setminus \{p\}$ the intersection point of the line through p and x with the orthogonal complement $\langle p\rangle^{\perp}$ is

$$\phi(x) = \frac{x - \langle x, p\rangle p}{1 - \langle x, p\rangle}.$$

The map $\phi : S(V) \setminus \{p\} \to \langle p\rangle^{\perp}$ is the stereographic projection with respect to p.
(b) Compute $\phi^{-1} : \langle p\rangle^{\perp} \to S(V) \setminus \{p\}$.
(c) If $\psi : S(V) \setminus \{-p\} \to \langle p\rangle^{\perp}$ is the stereographic projection with respect to $-p$, compute $\psi \circ \phi^{-1} : \langle p\rangle^{\perp} \to \langle p\rangle^{\perp}$.

5. Consider the canonical smooth atlas $\{(U_0, \phi_0), (U_1, \phi_1)\}$ of $\mathbb{C}P^1$ and observe that $\mathbb{C}P^1 \setminus U_0 = \{[0, 1]\}$ and $\mathbb{C}P^1 \setminus U_1 = \{[1, 0]\}$. Prove that $g : \mathbb{C}P^1 \to S^2$ defined by

$$g[z_0, z_1] = \begin{cases} (\pi_+^{-1} \circ \phi_0)[z_0, z_1], & \text{if } z_0 \neq 0 \\ (0, 0, 1), & \text{if } z_0 = 0. \end{cases}$$

is a smooth diffeomorphism, where $\pi_+ : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$ denotes the stereographic projection with respect to the north pole.

6. Let X be a Hausdorff topological space and $H(X)$ be the group of the homeomorphisms of X onto itself. A subgroup G of $H(X)$ defines on X the following equivalence relation: $x \sim y$ if and only if there exists some $g \in G$ with $y = g(x)$.

The equivalence classes are called the orbits of G. Let $\pi : X \to X/G$ denote the quotient map. We say that G acts properly discontinuously on X if every point $x \in X$ has some open neighbourhood U in X such that $U \cap g(U) = \varnothing$, for every $g \in G$, $g \neq id$.
(a) If G acts properly discontinuously, prove that every point $[x] \in X/G$ has an open neighbourhood V^* such that

$$\pi^{-1}(V^*) = \bigcup_{g \in G} g(V),$$

where V is a suitable open neighbourhood of $x \in X$, so that $g_1(V) \cap g_2(V) = \varnothing$, for $g_1 \neq g_2$ and $\pi|_V : V \to V^*$ is a homeomorphism.
(b) Let M be a smooth n-manifold and G be a group of smooth diffeomorphisms which acts properly discontinuously on M. If the quotient space M/G is Hausdorff, prove that it is a smooth n-manifold.
(c) Let M be a smooth n-manifold and G be a finite group of smooth diffeomorphisms of M. If $g(x) \neq x$ for every $x \in M$, $g \in G$, $g \neq id$, prove that G acts properly discontinuously on M, the quotient space M/G is Hausdorff and therefore a smooth n-manifold.
(d) On S^n the antipodal map $a : S^n \to S^n$ with $a(x) = -x$ is a smooth diffeomorphism. If $G = \{id, a\}$, determine the smooth n-manifold S^n/G.
(e) On the 2-torus $T^2 = S^1 \times S^1$ let $f : T^2 \to T^2$ be the map

$$f(e^{2\pi ix}, e^{2\pi iy}) = (e^{-2\pi ix}, -e^{2\pi iy}).$$

If $G = \{id, f\}$, Prove that $K^2 = T^2/G$ is a smooth 2-manifold. This manifold is called Klein bottle.
(f) Prove that the group of translations by vectors with integer coordinates, which is isomorphic to $\mathbb{Z}^n$, acts properly discontinuously on $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomorphic to the n-torus T^n.

7. Prove that the 1-dimensional real projective space $\mathbb{R}P^1$ is deffeomorphic to the circle S^1.

8. Let $f : M \to N$ be a bijective smooth map of smooth manifolds. If its derivative $f_{*p} : T_pM \to T_{f(p)}N$ is a linear isomorphism for every $p \in M$, prove that f is a smooth diffeomorphism.

9. Let $f : M \to Q$ be a smooth map of smooth manifolds and $q \in Q$ be a regular value of f with $N = f^{-1}(q) \neq \varnothing$. If $i_N : N \hookrightarrow M$ is the inclusion, show that $(i_N)_{*p}(T_pN) = \mathrm{Ker} f_{*p}$ for every $p \in N$.

10. Prove that $T_pS^n = \{[\gamma]_p \in T_p\mathbb{R}^{n+1} : \langle \gamma'(0), p\rangle = 0\}$ for every $p \in S^n$, where $\langle , \rangle$ is the euclidean inner product.

11. Let $n > 1$ and $p : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree $m \in \mathbb{N}$. Prove that $p^{-1}(c)$ is a $(n-1)$-dimensional smooth submanifold of $\mathbb{R}^n$ for every $c \neq 0$.

12. Let M be a smooth m-manifold, N be a smooth n-manifold and let $f : M \to N$ be a smooth map. If $q \in N$ is such that $f^{-1}(q) \neq \varnothing$ and f has constant rank k on some open neighbourhood of $f^{-1}(q)$, prove that the level set $f^{-1}(q)$ is a $(m-k)$-dimensional smooth submanifld of M.

13. Prove that the set $N = \{A \in \mathbb{R}^{2\times 2} : A \text{ has rank } 1\}$ is a 3-dimensional smooth submanifold of $\mathbb{R}^{2\times 2}$.

14. The set S of all real $n \times n$ symmetric matrices is a vector subspace of $\mathbb{R}^{n\times n}$ of dimension $n(n+1)/2$. Let $f : GL(n, \mathbb{R}) \to S$ be the map $f(A) = A \cdot A^t$.
(a) Prove that $f_{*A}(H) = AH^t + HA^t$ for every $H \in T_A GL(n, \mathbb{R})$, $A \in GL(n, \mathbb{R})$.
(b) Prove that the identity $I_n \in S$ is a regular value of f.
(c) Prove that the orthogonal group $O(n, \mathbb{R})$ is a $\frac{n(n-1)}{2}$-dimensional smooth submanifold of $GL(n, \mathbb{R})$.
(d) Prove that $T_{I_n} O(n, \mathbb{R}) = \{H \in \mathbb{R}^{n\times n} : H + H^t = 0\}$.

15. Prove that the map $g : T^2 \to \mathbb{R}^3$ with

$$g(e^{2\pi i\phi}, e^{2\pi i\theta}) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta)$$

is an embedding of the 2-torus T^2 into $\mathbb{R}^3$ and its image is

$$g(T^2) = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - 2)^2 + z^2 = 1\}.$$

16. Prove that the map $f : S^2 \to \mathbb{R}^6$ with

$$f(x, y, z) = (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy)$$

an immersion which induces an embedding of the real projective plane $\mathbb{R}P^2$ into $\mathbb{R}^6$.

17. Prove that the map $f : \mathbb{R}P^2 \to \mathbb{R}^3$ with $f([x, y, z]) = (yz, zx, xy)$ is an immersion and the map $g : \mathbb{R}P^2 \to \mathbb{R}^4$ with $g([x, y, z]) = (yz, zx, xy, x^2 + 2y^2 + 3z^2)$ is an embedding.

18. Let M, N be two smooth n-manifolds and let $f : M \to N$ be an immersion.
(a) Prove that f is an open map.
(b) If M is compact and N is connected, prove that $f(M) = N$.

19. Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the orthogonal transformation (complex structure of $\mathbb{R}^{2n}$) with $J(x, y) = (-y, x)$ for every $(x, y) \in \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.
(a) Prove that the set $S = \{A \in \mathbb{R}^{2n\times 2n} : A^t J A = J\}$ is a smooth submanifold of $\mathbb{R}^{2n\times 2n}$.
(b) Describe $T_{I_{2n}} S$ as a vector subspace of $\mathbb{R}^{2n\times 2n}$.
(c) Find the dimension of S.
(Hint : Prove that $J \in \mathbb{R}^{2n\times 2n}$ is a regular value of the smooth map $f : GL(2n, \mathbb{R}) \to \{H \in \mathbb{R}^{2n\times 2n} : H + H^t = 0\}$ with $f(A) = A^t J A$.)

20. Let $d \in \mathbb{N}$, $n \geq 2$ and denote by V_d^{2n} the set of points $(z_0, z_1, ..., z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ which are solutions of the equation

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0.$$

(a) Prove that V_d^{2n} is a smooth $2n$-manifold.
(b) Prove that the set $W_d^{2n-1} = V_d^{2n} \cap S^{2n+1}$ is a smooth $(2n-1)$-manifold. W_d^{2n-1} is called Brieskorn manifold.

21. The unit tangent bundle of the 2-sphere S^2 is the subset

$$T^1S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|p\| = 1, \|v\| = 1, \langle p, v\rangle = 0\}$$

of $\mathbb{R}^6$, where $\langle , \rangle$ is the euclidean inner product on $\mathbb{R}^3$.
(a) Prove that T^1S^2 is a 3-dimensional smooth submanifold of $\mathbb{R}^6$.
(b) Prove that $F : SO(3, \mathbb{R}) \to T^1S^2$ with $F(A) = (Ae_3, Ae_1)$ is a smooth diffeomorphism.
(c) Let $D^3 = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ and let $g : D^3 \to SO(3, \mathbb{R})$ be the map with $g(0) = I_3$ and such that if $x \in D^3 \setminus \{0\}$ then $g(x)$ is the rotation with respect to the axis generated by x by the angle $\|x\| \cdot \pi$. Prove that g induces a smooth diffeomorphism from $\mathbb{R}P^3$ onto $SO(3, \mathbb{R})$.
(Hint : Observe that $T^1S^2 = f^{-1}(0)$, where $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the smooth map $f(p, v) = (\|p\|^2 - 1, \|v\|^2 - 1, \langle p, v\rangle)$.)

Chapter 2

Vector fields

2.1 The tangent bundle and vector fields

In this section we shall define the notion of vector field on a smooth manifold, which is a generalization and globalization of the notion o ordinary differential equation on an open subset of euclidean space. A continuous vector field is a map which to a point p assigns a tangent vector with point of application p and varies continuously with p. So, first we need to consider the set of all tangent vectors.

Let M be a smooth n-manifold and consider the disjoint union of all tangent spaces at points of M, that is the set

$$TM = \bigcup_{p \in M} \{p\} \times T_pM.$$

Let $\pi : TM \to M$ denote the natural projection $\pi(p, v) = p$, for $v \in T_pM$, $p \in M$. We shall endow TM with the structure of a smooth manifold, so that π becomes smooth and a submersion.

If $\mathcal{A}$ is a smooth atlas of M, we define the class

$$\tilde{\mathcal{A}} = \{(\pi^{-1}(U), \tilde{\phi}_U) : (U, \phi_U) \in \mathcal{A}\}$$

where $\tilde{\phi}_U : \pi^{-1}(U) \to \phi_U(U) \times \mathbb{R}^n$ is the bijection defined by

$$\tilde{\phi}_U(p, v) = (\phi_U(p), (\phi_U)_{*p}(v))$$

for every $p \in U$, $v \in T_pM$. In other words, if $\phi_U = (x^1, ..., x^n)$, then for $p \in M$ and

$$v = \sum_{k=1}^{n} v^k \left(\frac{\partial}{\partial x^k} \right)_p \in T_pM$$

we have $\tilde{\phi}_U(v, v) = (x^1(p), ..., x^n(p), v^1, ..., v^n)$.

If now (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ are such that $U \cap V \neq \varnothing$, then the transition map $\tilde{\phi}_U \circ \tilde{\phi}_V^{-1} : \phi_V(U \cap V) \times \mathbb{R}^n \to \phi_U(U \cap V) \times \mathbb{R}^n$ is given by the formula

$$(\tilde{\phi}_U \circ \tilde{\phi}_V^{-1})(x, y) = ((\phi_U \circ \phi_V^{-1})(x), D(\phi_U \circ \phi_V^{-1})(x)(y))$$

and is thus a smooth diffeomorphism. This means that $\tilde{\mathcal{A}}$ would be a smooth atlas of TM, if we had a topology on TM making it a topological $2n$-manifold in such a way the the sets $\pi^{-1}(U)$ were open and the maps $\tilde{\phi}_U$ homeomorphisms. This topology is provided by the following.

Lemma 2.1.1. *Let X be a non-empty set and $\mathcal{U}$ be a family of subsets of X which covers X. We assume that for every $U \in \mathcal{U}$ there exist a topological space X_U and a bijection $\psi_U : U \to X_U$ such that for U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$ the set $\psi_V(U \cap V)$ is open in X_V and the map $\psi_U \circ \psi_V^{-1} : \psi_V(U \cap V) \to X_U$ is continuous.*

Then there exists a unique topology on X with respect to which every element of $\mathcal{U}$ becomes an open set and every map ψ_U becomes a homeomorphism.

Proof. Our assumptions imply that $\psi_U \circ \psi_V^{-1} : \psi_V(U \cap V) \to \psi_U(U \cap V)$ is a homeomorphism for every U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$. The family

$$\mathcal{T} = \{A \subset X : \psi_U(U \cap A) \quad \text{is open in } X_U \text{ for every } U \in \mathcal{U}\}$$

is a topology on X which contains the family $\mathcal{U}$. By the definition of $\mathcal{T}$, each ψ_U is an open map. For the continuity of ψ_U let $W \subset X_U$ be an open set. Then,

$$(\psi_U \circ \psi_V^{-1})(\psi_V(\psi_U^{-1}(W) \cap V)) = W \cap \psi_U(U \cap V)$$

is open in X_U for every U, $V \in \mathcal{U}$ with $U \cap V \neq \varnothing$. Since $\psi_U \circ \psi_V^{-1}$ is a homeomorphism, $\psi_V(\psi_U^{-1}(W) \cap V))$ must be open in X_V. This shows that $\psi_U^{-1}(W) \in \mathcal{T}$ and that ψ_U is continuous. The uniqueness of the topology $\mathcal{T}$ is obvious. □

Applying now Lemma 2.1.1, we obtain a unique topology on TM with respect to which each set $\pi^{-1}(U)$ is open and each map $\tilde{\phi}_U$ is a homeomorphism for $(U, \phi_U) \in \mathcal{A}$. Since M and $\mathbb{R}^n$ are Hausdorff spaces and have countable basis for their topologies, the same is true for TM. Thus, TM becomes a smooth $2n$-manifold. For every $(U, \phi_U) \in \mathcal{A}$ the corresponding local representation $\phi_U \circ \pi \circ \tilde{\phi}_U^{-1} : \phi_U(U) \times \mathbb{R}^n \to \phi_U(U)$ of π is the projection $(\phi_U \circ \pi \circ \tilde{\phi}_U^{-1})(x, y) = x$. Hence π is a submersion.

The triple (TM, π, M) is the *tangent bundle* of M. The natural projection π is the bundle map and M is the base space of the bundle. The total space of the bundle is TM. Abusing terminology, we shall also use the term tangent bundle for TM itself.

Definition 2.1.2. A smooth *vector field* on a smooth n-manifold M is a smooth map $X : M \to TM$ which to every $p \in M$ assigns a tangent vector $X(p) \in T_pM$. Briefly, $X \circ \pi = id_M$ or in other words X is a smooth section of π.

The set $\mathcal{X}(M)$ of all smooth vector fields of a smooth manifold M is an infinite dimensional real vector space. It is also a module over the commutative ring $C^\infty(M)$ of all real valued smooth functions defined on M. Every smooth diffeomorphism $f : M \to M$ induces a linear isomorphism $f_* : \mathcal{X}(M) \to \mathcal{X}(M)$ defined by $(f_*X)(f(p)) = f_{*p}(X(p))$ for every $p \in M$. The smooth vector field X of M is called f-invariant if $f_*X = X$.

Let X be a smooth vector field on a smooth n-manifold M. If $\mathcal{A}$ is a smooth atlas of M anf $\tilde{\mathcal{A}}$ is the corresponding smooth atlas of TM, then $X(U) \subset \pi^{-1}(U)$ for every $(U, \phi_U) \in \mathcal{A}$. There exists a smooth map $F_U : \phi_U(U) \to \mathbb{R}^n$, which is called the principal part of X with respect to (U, ϕ_U), such that the corresponding local representation $\tilde{\phi}_U \circ X \circ \phi_U^{-1} : \phi_U(U) \to \phi_U(U) \times \mathbb{R}^n$ of X is

$$(\tilde{\phi}_U \circ X \circ \phi_U^{-1})(x) = (x, F_U(x)).$$

Thus, if $\phi_U = (x^1, ..., x^n)$ and $F_U = (F^1, ...F^n)$, then

$$X(p) = \sum_{k=1}^{n} F^k(\phi(p)) \left(\frac{\partial}{\partial x^k} \right)_p$$

for every $p \in U$ and the smoothness of X is equivalent to the smoothness of F_U. In particular, on U we have the basic smooth vector fields

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$$

defined by the smooth chart ϕ_U.

Apart for the notion of tangent vector field on a smooth manifold we need to have a notion of tangent vector field along a smooth curve.

Definition 2.1.3. A smooth *vector field along a smooth curve* $\gamma : I \to M$ on a smooth n-manifold M, for $I \subset \mathbb{R}$ an open interval, is a smooth map $X : I \to TM$ which to every $s \in I$ assigns a tangent vector $X(s) \in T_{\gamma(s)}M$.

If $\gamma : I \to M$ is a smooth curve on a smooth n-manifold M, then for every $s \in I$ the tangent vector

$$\dot{\gamma}(s) = \gamma_{*s}\left(\left(\frac{d}{dt} \right)_s \right)$$

is the *velocity of* γ *at* $\gamma(s)$, where $\dfrac{d}{dt}$ is the basic vector field on $\mathbb{R}$. Thus, $\dot{\gamma} : I \to TM$ is a smooth vector field along γ, which is called the *velocity field* of γ.

Recall that $\left(\dfrac{d}{dt} \right)_s$ is the usual derivation at s. Using the notation of section 1.4, note that $[\gamma]_p$ and $\dot{\gamma}(0)$ denote one and the same vector in T_pM for $p \in M$ and $\gamma \in S(M, p)$, namely the velocity of γ at $p = \gamma(0)$.

If $\gamma(I) \subset U$ for the smooth chart (U, ϕ_U) of M and $\phi_U \circ \gamma = (\gamma^1, ..., \gamma^n)$ is the corresponding local representation of γ, then

$$\dot{\gamma}(s) = \sum_{k=1}^{n} (\gamma^k)'(s) \left(\frac{\partial}{\partial x^k} \right)_{\gamma(s)}$$

for every $s \in I$.

2.2 Flows of smooth vector fields

Let M be a smooth n-manifold and let X be a smooth vector field on M. An *integral curve* of X is a smooth curve $\gamma : I \to M$, defined on an open interval $I \subset \mathbb{R}$, such that

$$\dot{\gamma}(s) = X(\gamma(s))$$

for every $s \in I$.

If (U, ϕ_U) is a smooth chart of M with $\phi_U = (x^1, ..., x^n)$ and $F_U = (F^1, ..., F^n)$ is the principal part of X on U with respect to ϕ_U, the discussion in the preceding section 2.1 shows that a smooth curve $\gamma : I \to U$ is an integral curve of X on U if and only if its local representation $\phi_U \circ \gamma = (\gamma^1, ..., \gamma^n)$ is a solution of the autonomous n-dimensional ordinary differential equation $x'(s) = F_U(x(s))$, which means that it satisfies the system of ordinary differential equations

$$(\gamma^k)'(s) = F_U^k((\gamma^1(s), ..., \gamma^n(s)), \quad s \in I, \quad k = 1, 2, ..., n.$$

Thus, locally on M the integral curves of smooth vector fields on M are the solutions of autonomous ordinary differential equations. The standard existence and uniqueness theorems combined with continuous and differentiable dependence on initial conditions imply that if X is a smooth vector field on M, then for every point $p \in M$ there exist an open neighbourhood V of p in M, some $\epsilon > 0$ and a smooth map $\Phi^V : (-\epsilon, \epsilon) \times V \to M$ such that $\Phi^V(0, q) = q$ for every $q \in V$ and

$$\frac{\partial \Phi^V}{\partial t}(s, q) = X(\Phi^V(s, q))$$

for every $(s, q) \in (-\epsilon, \epsilon) \times V$. Moreover, the map Φ^V is unique, in the sense that if W, $\delta > 0$ and $\Phi^W : (-\delta, \delta) \times W \to M$ is another triple like V, ϵ and Φ^V, then $\Phi^V = \Phi^W$ on $(-\epsilon, \epsilon) \times V \cap (-\delta, \delta) \times W$. Thus, for every $q \in V$ the smooth curve $\Phi^V(\cdot, q) : (-\epsilon, \epsilon) \to M$ is the unique integral curve of X defined on the interval $(-\epsilon, \epsilon)$ and satisfying the initial condition $\Phi^V(0, q) = q$. The map Φ^V is called the *local flow of X on the open set V.*

The existence of maximal integral curves globally on M can be established in the usual way.

Proposition 2.2.1. *If X is a smooth vector field on M, then for every $p \in M$ there exist $a_p < 0 < b_p$ and a maximal integral curve $\Phi^p : (a_p, b_p) \to M$ of X with $\Phi^p(00 = p$ in the sense that if $\gamma : I \to M$ is any other integral curve of X defined on an open interval $I \subset \mathbb{R}$ which contains 0 such that $\gamma(0) = p$ then $I \subset (a_p, b_p)$ and $\gamma = \Phi^p|_I$.*

Proof. Let $\gamma_j : I_j \to M$, $j = 1, 2$, be integral curves of X defined on open intervals such that $0 \in I_1 \cap I_2$, with $\gamma_1(0) = \gamma_(0) = p$. Then, $I_1 \cap I_2$ is a non-empty open interval and the set $I^* = \{s \in I_1 \cap I_2 : \gamma_1(s) = \gamma_2(s)\}$ is non-empty and closed in $I_1 \cap I_2$, by continuity. If $s \in I^*$, there exists $\delta > 0$ such hat $(s - \delta, s + \delta) \subset I_1 \cap I_2$. The smooth curves $\beta_j : (-\delta, \delta) \to M$ defined by $\beta_j(t) = \gamma(t + s)$, $j = 1, 2$, are integral curves of X with $\beta_1(0) = \gamma_1(s) = \gamma_2(s) = \beta_2(0)$. By uniqueness of solutions, there exists some $0 < \eta \leq \delta$ such hat $\beta_1 = \beta_2$ on $(-\eta, \eta)$. Therefore, $(s-\eta, s+\eta) \subset I^*$, which shows that I^* is open in $I_1 \cap I_2$. By connectedness now we must have $I^* = I_1 \cap I_2$. This shows that the union of all open intervals I containing 0 on which there is an integral curve $\gamma : I \to M$ of X with $\gamma(0) = p$, is an open interval (a_p, b_p) on which a maximal integral curve $\Phi^p : (a_p, b_p) \to M$ of X with $\Phi^p(00 = p$ is well defined. □

Recall that the open interval on which a maximal integral curve is defined is not necessarily the whole real line $\mathbb{R}$. For instance, the maximal solution of the autonomous ordinary differential equation $x'(s) = (x(s))^2$ on $\mathbb{R}$ with initial condition $x(0) = 1$ is $\Phi : (-\infty, 1) \to \mathbb{R}$ given by the formula

$$\Phi(s) = \frac{1}{1-s}.$$

Lemma 2.2.2. *Let $p \in M$ and $\Phi^p : (a_p, b_p) \to M$ be a maximal integral curve of a smooth vector field X om M with $\Phi^p(0) = p$. If $t \in (a_p, b_p)$ and $q \in \Phi^p(t)$, then the maximal integral curve Φ^q with $\Phi^q(0) = q$ is defined on the open interval $(a_p - t, b_p - t)$ and $\Phi^q(s) = \Phi^p(s+t)$.*

Proof. Since the smooth curve $\gamma : (a_p - t, b_p - t) \to M$ with $\gamma(s) = \Phi^p(s+t)$ is an integral curve of X with $\gamma(0) = q$, the maximal integral curve Φ^q with $\Phi^q(0) = q$ is defined at least on $(a_p - t, b_p - t)$. Conversely, if the interval of definition of Φ^q is the open interval (a_q, b_q), then $a_q \leq a_p - t$, $b_p - t \leq b_q$ and $\delta : (a_q + t, b_q + t) \to M$ defined by $\delta(s) = \Phi^q(s-t)$ is an integral curve with $\delta(0) = p$. Hence $a_p \leq a_q + t$, $b_q + t \leq a_p$. □

Using the notation of Lemma 2.2.2 for a smooth vector field X on M, we define

$$D = \bigcup_{p \in M} (a_p, b_p) \times \{p\}$$

and $\Phi : D \to M$ by $\Phi(s,p) = \Phi^p(s)$, which has the following properties:
(i) $\Phi(0,p) = p$ for every $p \in M$ and
(ii) $\Phi(t, \Phi(s,p)) = \Phi(t+s, p)$ for every $p \in M$ and s, $t \in \mathbb{R}$ such that at least one side of this equality is defined.

Theorem 2.2.3. *The set D is open in $\mathbb{R} \times M$ and $\Phi : D \to M$ is smooth.*

Proof. For $p \in M$ we consider the set I^* consisting of all $a_p < t < b_p$ for which there exist $\delta > 0$ and an open neighbourhood U of p in M such that $(t-\delta, t+\delta) \times U \subset D$ and Φ is smooth on $(t-\delta, t+\delta) \times U$. Then, $0 \in I^*$ and I^* is an open set. Thus, it suffices to prove that I^* is closed in the interval (a_p, b_p), by connectedness. Suppose that $a_p < s < b_p$ lies in the closure of I^*. There exist an open neighbourhood V of $\Phi(s.p)$ in M, some $\epsilon > 0$ and a local flow $\Phi^V : (-\epsilon, \epsilon) \times V \to M$, so that $\Phi^V = \Phi|_{(-\epsilon,\epsilon)\times V}$. By continuity, there exists some $t \in I^*$ with $|t-s| < \dfrac{\epsilon}{3}$ and $\Phi(t,p) \in V$. Since $t \in I^*$, there exist $0 < \delta < \dfrac{\epsilon}{3}$ and an open neighbourhood U of

p in M such that $(t-\delta, t+\delta) \times U \subset D$ and Φ is smooth on $(t-\delta, t+\delta) \times U$. By continuity of $\Phi(t, .) : U \to M$ and the fact that $\Phi(t, p) \in V$, shrinking U if necessary, we may take U so that $\Phi(\{t\} \times U) \subset V$. So, from Lemma 2.2.2 we have

$$(-\epsilon, \epsilon) \subset (a_{\Phi(t,q)}, b_{\Phi(t,q)}) = (a_q - t, b_q - t)$$

for every $q \in U$, which implies that $(t-\epsilon, t+\epsilon) \times U \subset D$, and Φ is smooth on $(t-\epsilon, t+\epsilon) \times U$, because

$$\Phi(r, q) = \Phi^V(r - t, \Phi(t, q))$$

for every $(r, q) \in (t-\epsilon, t+\epsilon) \times U$. Now

$$(s, p) \in (s-\delta, s+\delta) \times U \subset (t-\epsilon, t+\epsilon) \times U \subset D,$$

which means that $s \in I^*$. $\square$

The fact that D is an open subset of $\mathbb{R} \times M$ is equivalent to saying that the function $a : M \to [-\infty, 0)$ is upper semicontinuous and $b : M \to (0, +\infty]$ is lower semicontinuous.

The smooth map $\Phi : D \to M$ is called the *flow* of the smooth vector field X. The vector field X can be reconstructed from its flow by setting

$$X(p) = \frac{\partial \Phi}{\partial t}(0, p)$$

for every $p \in M$. The image $\Phi((a_p, b_p) \times \{p\})$ of the maximal integral curve of X through the point $p \in M$ is called the *orbit* of p with respect to X.

A smooth vector field X on M is called *complete* if every maximal integral curve of X is defined on the whole real line $\mathbb{R}$ or $D = \mathbb{R} \times M$, using the above notation. In this case, the flow $\Phi : \mathbb{R} \times M \to M$ is a smooth action of the additive group of real numbers $\mathbb{R}$ on M. For every $t \in \mathbb{R}$ the map $\Phi_t = \Phi(t, .) : M \to M$ is a smooth diffeomorphism. Moreover, $\Phi_0 = id_M$ and $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for every t, $s \in \mathbb{R}$ and the family $(\Phi_t)_{t \in \mathbb{R}}$ is called the *one-parameter group of diffeomorphisms* defined by X. For every $t \in \mathbb{R}$ and $p \in M$ we have

$$(\Phi_t)_{*p}(X(p)) = (\Phi_t)_{*p}\left(\frac{\partial \Phi}{\partial t}(0, p)\right) = \frac{\partial(\Phi_t \circ \Phi^p)}{\partial t}(0).$$

However,

$$(\Phi_t \circ \Phi^p)(s) = \Phi(t, \Phi(s, p)) = \Phi(t + s, p) = \Phi(s, \Phi(t, p))$$

for every $s \in \mathbb{R}$ and therefore

$$(\Phi_t)_{*p}(X(p)) = X(\Phi_t(p)).$$

This means that X is Φ_t-invariant for every $t \in \mathbb{R}$.

In case the smooth vector field X is not complete, the smooth diffeomorphisms Φ_t are defined on suitable open subsets of M.

The integral curves of a smooth vector field X which are not defined on the whole real line must necessarily explode at infinity. This is made more precise in the following.

Lemma 2.2.4. *Let X be a smooth vector field with flow $\Phi : D \to M$ and $p \in M$. If $b_p < +\infty$, then for every compact set $K \subset M$ there exists $0 < T < b_p$ such that $\Phi(t,p) \in M \setminus K$ for every $T < t < b_p$.*

Proof. For every $q \in K$ there exist $\delta_q > 0$ and an open neighbourhood V_q of q such that $(-\delta_q, \delta_q) \times V_q \subset D$. By compactness of K, there exist q_1,..., $q_m \in K$, for some $m \in \mathbb{N}$, such that $K \subset V_{q_1} \cup \cdots \cup V_{q_m}$. If now $\delta = \min\{\delta_{q_1}, ..., \delta_{q_m}\}$, then $(-\delta, \delta) \times K \subset D$. Thus, if there exists a sequence $t_k \nearrow b_p$ such that $\Phi(t_k, p) \in K$ for every $k \in \mathbb{N}$, we arrive at the contradiction $0 < \delta < b_p - t_k$ for all $k \in \mathbb{N}$. □

This implies the following important fact.

Corollary 2.2.5. *Every smooth vector field on a compact smooth manifold is complete.* □

It is possible to find all integral curves of a given smooth vector field only in very rare cases. The aim of the qualitative (or geometric) theory of dynamical systems is to find the distribution of the time oriented orbits of vector fields studying their asymptotic behaviour. In this point of view, we may replace X with $f \cdot X$ where $f : M \to (0, +\infty)$ is a smooth function, because both vector fields have the same orbits. Indeed, if $\Phi : D \to M$ is the flow of X, for every $p \in M$ the smooth map $h : (a_p, b_p) \to \mathbb{R}$ defined by

$$h(s,p) = \int_0^s \frac{1}{f(\Phi(t,p))} dt$$

is strictly increasing and $h((a_p, b_p))$ is an open interval. Also, $(h^{-1})'(s) = f(\Phi(h^{-1}(s)), p)$. It follows now that the maximal integral curve of $f \cdot X$ through p is just $\Phi^p \circ h^{-1} : h((a_p, b_p)) \to M$. In other words, the maximal integral curves of $f \cdot X$ are reparametrizations of the maximal integral curves of X.

The following can be obtained as a consequence of the existence of smooth partitions of unity.

Theorem 2.2.6. *If X is a smooth vector field of a smooth manifold M, then there exists a smooth function $f : M \to (0, 1]$ such that the smooth vector field $f \cdot X$ is complete.*

Proof. Let $\Phi : D \to M$ be the flow of X as above. Since D is an open subset of $\mathbb{R} \times M$, the function $g : M \to (0, 1]$ defined by

$$g(p) = \min\{1, -a_p, b_p\}$$

is lower semicontinuous. Thus, every $p \in M$ has an open neighbourhood W_p such that $g(q) > \frac{1}{2} g(p)$ for every $q \in W_p$. By Theorem 1.4.4, there exists a smooth partition of unity $\{f_p : p \in M\}$ subordinated to the open cover $\{W_p : p \in M\}$. The function $f : M \to (0,1]$ defined by

$$f(q) = \frac{1}{2}\sum_{p\in M} g(p) f_p(q)$$

is smooth and for every $q \in M$ there exist p_1,..., $p_k \in M$, for some $k \in \mathbb{N}$, such that $q \in \operatorname{supp} f_{p_1} \cap \cdots \cap \operatorname{supp} f_{p_k}$ and $f_p(q) = 0$ for $p \neq p_1, ..., p_k$. It follows that

$$f(q) = \frac{1}{2}\sum_{j=1}^{k} g(p_j) f_{p_j}(q) < \sum_{j=1}^{k} g(q) f_{p_j}(q) = g(q)) = \min\{1, -a_q, b_q\}$$

for every $q \in M$.

Let now $\psi : D \to \mathbb{R}$ be the smooth function defined by

$$\psi(s,p) = \int_0^s \frac{1}{f(\Phi(t,p))} dt.$$

The smooth map $h : D \to \mathbb{R} \times M$ with $h(s,p) = (\psi(s,p), p)$ is obviously injective, since

$$\frac{\partial \psi}{\partial t}(s,p) = \frac{1}{f(\Phi(s,p))} \geq 1.$$

Moreover, $\psi(s,p) \geq s$ for $0 \leq s < b_p$ and $\psi(s,p) \leq s$ for $a_p < s \leq 0$. Thus, $\lim\limits_{s\to b_p} \psi(s,p) = +\infty$, if $b_p = +\infty$. In case $b_p < +\infty$, for every $0 < s < b_p$ we have

$$\psi(s,p) > \int_0^s \frac{1}{b_{\Phi(t,p)}} dt = \int_0^s \frac{1}{b_p - t} dt = -\log\left(1 - \frac{s}{b_p}\right)$$

and therefore again $\lim\limits_{s\to b_p} \psi(s,p) = +\infty$. Similarly, $\lim\limits_{s\to a_p} \psi(s,p) = -\infty$ for all $p \in M$. This shows that h is surjective.

Since h is a bijection and its derivative $h_{*(s,p)}$ is a linear isomorphism at every point $(s,p) \in D$, it follows from the Inverse Map Theorem that h is a smooth diffeomorphism.

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "\Phi"'] & & \mathbb{R} \times M \arrow[dl, "\Psi"] \\ & M & \end{tikzcd}$$

The proof is now concluded by the observation that $\Psi = \Phi \circ h^{-1} : \mathbb{R} \times M \to M$ is the flow of $f \cdot X$, because

$$\frac{\partial \Psi}{\partial t}(0,p) = f(\Phi(h^{-1}(0,p))) \cdot \frac{\partial \Phi}{\partial t}(h^{-1}(0,p)) = f(p) \cdot \frac{\partial \Phi}{\partial t}(0,p) = f(p) \cdot X(p)$$

for every $p \in M$. $\square$

2.3 The Lie bracket

Let M be a smooth n-manifold and let X be a smooth vector field on M. At every point $p \in M$ the value $X(p) \in T_pM$ of X is a derivation on the algebra of germs $\mathcal{G}_p(M)$ of smooth functions defined on neighbourhoods of p and

$$X(p)(f) = \lim_{t \to 0} \frac{f(\Phi(t,p)) - f(p)}{t}$$

for every smooth function f which is defined on some open neighbourhood of p in M, where Φ is the flow of X.

Apart from functions, it is possible to define a special kind of derivation of another smooth vector field Y with respect to X, by transporting Y along the integral curves of X by the flow of X. The result can be defined in a purely algebraic way as follows.

Let $p \in M$. If $f \in C^\infty(M,p)$, then $Yf(q) = Y(q)(f)$ is a smooth function $Yf \in C^\infty(M,p)$ for every $Y \in \mathcal{X}(M)$. We define

$$[X,Y](p)(f) = X(p)(Yf) - Y(p)(Xf)$$

for every $f \in C^\infty(M,p)$ and X, $Y \in \mathcal{X}(M)$. We observe that

$$[X,Y](p)(f \cdot g) = X(p)(f \cdot Yg + g \cdot Yf) - Y(p)(f \cdot Xf + g \cdot Xf)$$

$$= f(p)X(p)(Yg) + Y(p)(g)X(p)(f) + Y(p)(f)X(p)(g) + g(p)X(p)(Yf)$$

$$-f(p)Y(p)(Xg) - Y(p)(f)X(p)(g) - Y(p)(g)X(p)(f) - g(p)Y(p)(Xf)$$

$$= f(p) \cdot [X,Y](p)(g) + g(p) \cdot [X,Y](p)(f).$$

Therefore, $[X,Y](p)$ is a derivation of the algebra of germs $\mathcal{G}_p(M)$ and so is a tangent vector in T_pM.

Let (U,ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$. Then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^i}\right) = 0$$

on U for all i, $j = 1, 2, ..., n$. If now X, $Y \in \mathcal{X}(U)$ and

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j},$$

then for every $p \in U$ and $f \in C^\infty(M,p)$ we have

$$[X,Y](p)(f) = \sum_{i,j=1}^n X^i(p)\left(\frac{\partial}{\partial x^i}\right)_p \left(Y^j \frac{\partial f}{\partial x^j}\right) - \sum_{i,j=1}^n Y^j(p)(p)\left(\frac{\partial}{\partial x^j}\right)_p \left(X^i \frac{\partial f}{\partial x^i}\right)$$

$$= \sum_{i,j=1}^n X^i(p) \frac{\partial Y^j}{\partial x^i}(p) \frac{\partial f}{\partial x^j}(p) + \sum_{i,j=1}^n X^i(p) Y^j(p) \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)(p)$$

$$- \sum_{i,j=1}^{n} Y^j(p) \frac{\partial X^i}{\partial x^j}(p) \frac{\partial f}{\partial x^i}(p) - \sum_{i,j=1}^{n} Y^j(p) X^i(p) \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)(p)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} X^i(p) \frac{\partial Y^j}{\partial x^i}(p) - Y^i(p) \frac{\partial X^j}{\partial x^i}(p) \right) \frac{\partial f}{\partial x^j}(p).$$

This means that

$$[X,Y] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

on U.

The above show that $[X,Y] \in \mathcal{X}(M)$ for every X, $Y \in \mathcal{X}(M)$, and is called the *Lie derivative* of Y with respect to X. The so defined function

$$[.,.] : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

is called the *Lie bracket* and has the following rather obvious properties:
(i) It is bilinear and alternating.
(ii) It satisfies the Jacobi identity, that is

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for every X, Y, $Z \in \mathcal{X}(M)$.
(iii) $[X, fY] = f[X,Y] + Xf \cdot Y$ for every $f \in C^\infty(M)$ and X, $Y \in \mathcal{X}(M)$.
(iv) If $F : M \to M$ is a smooth diffeomorphism, then $[F_*X, F_*Y] = [X,Y]$ for every X, $Y \in \mathcal{X}(M)$. More generally, let M be a smooth n-manifold, L be a smooth k-manifold, $k \leq n$, and let $g : L \to M$ be an injective immersion. Let X, $Y \in \mathcal{X}(M)$ be such that $X(g(x))$, $Y(g(x)) \in g_{*x}(T_xL)$ for every $x \in L$. Then, there exist unique $\tilde{X}(x)$, $\tilde{Y}(x) \in T_xL$ such that $g_{*x}(\tilde{X}(x)) = X(g(x))$ and $g_{*x}(\tilde{Y}(x)) = Y(g(x))$ and it follows from the local presentation of immersions provided by the Constant Rank Theorem 1.3.2 that $\tilde{X}$, $\tilde{Y} \in \mathcal{X}(L)$. Now we have

$$g_{*x}([\tilde{X},\tilde{Y}](x)) = [X,Y](g(x))$$

for every $x \in L$. Indeed, let $x \in L$ and let f be a smooth function defined on some open neighbourhood of $g(x)$. Note first that the chain rule implies that

$$\tilde{Y}(f \circ g) = Yf \circ g.$$

From the definitions now we have

$$g_{*x}([\tilde{X},\tilde{Y}](x))f = [\tilde{X},\tilde{Y}](x)(f \circ g) = \tilde{X}(x)(\tilde{Y}(f \circ g)) - \tilde{Y}(x)(\tilde{X}(f \circ g))$$

$$= \tilde{X}(x)(Yf \circ g) - \tilde{Y}(x)(Xf \circ g) = X(g(x))(Yf) - Y(g(x))(Xf) = [X,Y](g(x))f.$$

The structure on a vector space E imposed by an alternating, bilinear map $[.,.] : E \times E \to E$, which satisfies the Jacobi identity is called a *Lie algebra*. The following formula reveals the true nature of the Lie bracket.

Theorem 2.3.1. *Let M be a smooth n-manifold and X, $Y \in \mathcal{X}(M)$. If $\Phi : D \to M$ is the flow of X, then*

$$[X,Y](p) = \lim_{t\to 0} \frac{1}{t}\big((\Phi_{-t})_{*\Phi(t,p)}(Y(\Phi(t,p)) - Y(p)\big)$$

for every $p \in M$.

For the proof we shall need the following technical lemma.

Lemma 2.3.2. *Let U, $V \subset M$ be two open neighbourhoods of the point $p \in M$ for which there exists $\epsilon > 0$ such that $\Phi((-\epsilon,\epsilon)\times V) \subset U$. Then, for every smooth function $f : U \to \mathbb{R}$ there exists a smooth function $g : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ with the following properties:*
(i) $f(\Phi(-t,q)) = f(q) - tg(t,q)$ for every $(t,q) \in (-\epsilon,\epsilon)\times V$.
(ii) $X(q)(f) = g(0,q)$ for every $q \in V$.

Proof. If $h : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ is the smooth function defined by $h(s,q) = f(\Phi(-s,q)) - f(q)$, and if we define $g : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ by

$$g(t,q) = -\int_0^1 \frac{\partial h}{\partial s}(ts,q)ds,$$

then

$$-tg(t,q) = \int_0^t \frac{\partial h}{\partial s}(s,q)ds = h(t,q).$$

By continuity, we also have

$$g(0,q) = \lim_{t\to 0} g(t,q) = \lim_{t\to 0}\frac{f(\Phi(-t,q)) - f(q)}{-t} = X(q)(f). \quad \square$$

Proof of Theorem 2.3.1. Let $f : U \to \mathbb{R}$ be a smooth function defined on an open neighbourhood U of the point $p \in M$. There exist an open neighbourhood V of p and $\epsilon > 0$ such that $\Phi((-\epsilon,\epsilon)\times V) \subset U$. Let g be the smooth function supplied by Lemma 2.3.2 and let $g_t = g(t,.)$. Then, $Xf = g_0$ and

$$\lim_{t\to 0} \frac{1}{t}\big((\Phi_{-t})_{*\Phi(t,p)}(Y(\Phi(t,p)) - Y(p)\big)(f)$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[f_{*p}\big((\Phi_{-t})_{*\Phi(t,p)}(Y(\Phi(t,p)))\big) - Y(p)(f)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Y(\Phi(t,p))(f\circ\Phi_{-t}) - Y(p)(f)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Y(\Phi(t,p))(f - tg_t) - Y(p)(f)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Y(\Phi(t,p))(f) - Y(p)(f)\Big] - \lim_{t\to 0} Y(\Phi(t,p))(g_t)$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Yf(\Phi(t,p)) - Yf(p)\Big] - Y(p)(Xf)$$
$$= X(p)(Yf) - Y(p)(Xf) = [X,Y](p)(f). \quad \square$$

Definition 2.3.3. Two complete smooth vector fields X, Y on a smooth manifold M *commute* if $[X,Y] = 0$.

This terminology is justified by the following.

Proposition 2.3.4. *Let X and Y be two smooth vector fields on a smooth manifold M. Let $(\Phi_t)_{t\in\mathbb{R}}$ be the one-parameter group of smooth diffeomorphisms of M defined by the flow of X and $(\Psi_t)_{t\in\mathbb{R}}$ be the one-parameter group of smooth diffeomorphisms defined by the flow of Y. Then $[X,Y]=0$ if and only if $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for every t, $s \in \mathbb{R}$.*

Proof. If $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for every t, $s \in \mathbb{R}$, differentiating with respect to s at 0, we get $(\Phi_t)_*Y = Y$ for every $t \in \mathbb{R}$. It follows now from Theorem 2.3.1 that $[X,Y] = 0$.

Conversely, let $[X,Y] = 0$ and let $p \in M$ and $s \in \mathbb{R}$. The velocity of the smooth curve $\gamma : \mathbb{R} \to T_{\Psi_s(p)}M$ defined by $\gamma(t) = (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}(Y((\Phi_t(\Psi_s(p))))$ is

$$\dot\gamma(t) = \lim_{h\to 0}\frac{1}{h}\Big[(\Phi_{-t+h})_{*\Phi_{t+h}(\Psi_s(p))}(Y(\Phi_{t+h}(\Psi_s(p))) - (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}(Y((\Phi_t(\Psi_s(p))))\Big]$$
$$= (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}\left(\lim_{h\to 0}\frac{1}{h}\Big[(\Phi_{-h})_{*\Phi_{t+h}(\Psi_s(p))}(Y(\Phi_h(\Phi_t(\Psi_s(p))))) - Y(\Phi_t(\Psi_s(p)))\Big]\right)$$
$$= (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}([X,Y](\Phi_t(\Psi_s(p)))) = 0.$$

Thus, γ is constant, which means that $(\Phi_{-t})_{*\Phi_t(\Psi_s(p))}(Y((\Phi_t(\Psi_s(p)))) = Y(\Psi_s(p))$ or equivalently
$$Y(\Phi_t(\Psi_s(p))) = (\Phi_t)_{*\Psi_s(p)}(Y(\Psi_s(p)))$$
for every $p \in M$ and t, $s \in \mathbb{R}$. In other words, Y is Φ_t-invariant for every $t \in \mathbb{R}$. This implies that $\Phi_t \circ \Psi^p$ is an integral curve of Y and since $(\Phi_t \circ \Psi^p)(0) = \Phi_t(p)$, we must necessarily have $\Phi_t \circ \Psi^p = \Psi^{\Phi_t(p)}$, hence $\Phi_t(\Psi_s(p)) = \Psi_s(\Phi_t(p))$. $\square$

If X and Y are two commuting complete smooth vector fields on a smooth manifold M with corresponding one-parameter groups of smooth diffeomorphisms $(\Phi_t)_{t\in\mathbb{R}}$ and $(\Psi_t)_{t\in\mathbb{R}}$, respectively, then $F : \mathbb{R}^2 \times M \to M$ defined by
$$F(t,s,p) = (\Phi_t \circ \Psi_s)(p)$$
is a smooth action of the abelian group $(\mathbb{R}^2,+)$ on M. More generally, a finite family of mutually commuting complete smooth vector fields X_1,..., X_k with corresponding one-parameter groups of smooth diffeomorphisms $(\Phi_t^1)_{t\in\mathbb{R}}$,..., $(\Phi_t^k)_{t\in\mathbb{R}}$, respectively, defines a smooth action $F : \mathbb{R}^k \times M \to M$ of the abelian group $(\mathbb{R}^k,+)$ by the formula
$$F(t_1,...,t_k,p) = (\Phi^1_{t_1} \circ \cdots \Phi^k_{t_k})(p).$$

2.4 Geometric distributions

Let M be a smooth n-manifold and let $\mathcal{D} \subset TM$ be such that $\mathcal{D}_p = \mathcal{D} \cap T_pM \neq \varnothing$ for every $p \in M$. We denote by $\mathcal{X}^{\mathcal{D}}(M)$ the vector space of all smooth vector fields of M with values in $\mathcal{D}$ and by $\mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$ the set of all smooth vector fields defined on open subsets of M with values in $\mathcal{D}$. We shall call $\mathcal{D}$ a *geometric distribution* on M if it has the following two properties:
(i) $\mathcal{D}_p$ is a vector subspace of T_pM for every $p \in M$.
(ii) For every $p \in M$ and $v \in \mathcal{D}_p$ there exists $X \in \mathcal{X}^{\mathcal{D}}(M)$ such that $X(p) = v$.

The non-negative integer $k(p) = \dim \mathcal{D}_p$ is called the *rank* of $\mathcal{D}$ at p. Note that k is a lower semicontinuous function of p, because condition (ii) implies that every $p \in M$ has an open neighbourhood V such that $k(q) \geq k(p)$ for every $q \in V$.

An *integral manifold* of $\mathcal{D}$ is a pair (L, g) where L is a connected smooth manifold and $g : L \to M$ is an injective immersion such that $g_{*x}(T_xL) = \mathcal{D}_{g(x)}$ for every $x \in L$. In particular the rank of $\mathcal{D}$ is constant along an integral manifold. The geometric distribution $\mathcal{D}$ is called *integrable* if for every $p \in M$ there exists an integral manifold (L, g) of $\mathcal{D}$ with $p \in g(L)$.

Examples 2.4.1. (a) Every $X \in \mathcal{X}(M)$ generates a geometric distribution $\mathcal{D}$ so that $\mathcal{D}_p = \mathbb{R} \cdot X(p)$ for every $p \in M$. The maximal integral curves of X give integral manifolds of $\mathcal{D}$ which fill out M and so M is integrable. More precisely, let $\Phi : D \to M$ be the flow of X. If $X(p) = 0$, then the integral manifold through p is $(\{0\}, \Phi^p)$ and the rank at p is 0. If $X(p) \neq 0$ and the maximal integral curve $\Phi^p : (a_p, b_p) \to M$ is not injective, it is not hard to see that $(a_p, b_p) = \mathbb{R}$ and Φ^p is periodic of period $T = \min\{t > 0 : \Phi^p(t) = p\} > 0$. In this case Φ^p induces the embedding $\tilde{\Phi}^p : S^1 \to M$ well defined by $\tilde{\Phi}^p(e^{2\pi i t}) = \Phi^p(tT)$ and $(S^1, \tilde{\Phi}^p)$ is the integral manifold through p. In any other case the maximal integral curve $\Phi^p : (a_p, b_p) \to M$ is an injective immersion and $((a_p, b_p), \Phi^p)$ is the integral manifold through p.

(b) Let M be a smooth n-manifold and P be a smooth k-manifold with $n \geq k$. If $f : M \to P$ is a smooth submersion then $\mathcal{D} = \text{Ker} f_*$ is a geometric distribution of constant rank $n - k$, which is integrable. From Corollary 1.3.5, the connected components of the level sets of f are the integral manifolds of $\mathcal{D}$.

(c) On $\mathbb{R}^2$ let $\mathcal{D}$ be the geometric distribution globally defined by the smooth vector fields
$$\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y}.$$
The rank of $\mathcal{D}$ at points of the horizontal axis is 1 and it is 2 everywhere else. Obviously, $\mathcal{D}$ is integrable and has only three integral manifolds, These are the horizontal axis, the open upper half plane and the open lower half plane.

(d) Let $\mathcal{D}$ be the geometric distribution globally defined by the smooth vector fields
$$\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}.$$

The rank at points of the vertical axis is 1 and everywhere else it is 2. This time $\mathcal{D}$ is not integrable, because the only possible integral manifold through $(0,0)$ must be an open interval in the vertical axis, since the rank remains constant along integral manifolds. This contradicts the fact that $\mathcal{D}_{(0,0)}$ is not tangent to the vertical axis.

Let $\mathcal{D}$ be an integrable geometric distribution and let (L,g) be an integral manifold. We recall that if X, $Y \in \mathcal{X}^{\mathcal{D}}(M)$, there are unique $\tilde{X}$, $\tilde{Y} \in \mathcal{X}(L)$ such that $g_{*x}(\tilde{X}(x)) = X(g(x))$, $g_{*x}(\tilde{Y}(x)) = Y(g(x))$ and

$$[X,Y](g(x)) = g_{*x}([\tilde{X},\tilde{Y}](x)) \in g_{*x}(T_xL) = \mathcal{D}_{g(x)}$$

for every $x \in L$. This leads to the following.

Definition 2.4.2. A geometric distribution $\mathcal{D}$ on a smooth manifold M is called *involutive* if $\mathcal{X}^{\mathcal{D}}(M)$ is a Lie subalgebra of $\mathcal{X}(M)$, that is $[X,Y] \in \mathcal{X}^{\mathcal{D}}(M)$ for every X, $Y \in \mathcal{X}(M)$.

According to the above, every integrable geometric distribution is involutive.

Examples 2.4.3. (a) The geometric distribution defined by a smooth vector field is involutive.

(b) The geometric distribution on $\mathbb{R}^2$ of Example 2.4.1(c) is involutive, since

$$\left[\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right] = 0$$

but the one of Example 2.4.1(d) is not, because

$$\left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] = \frac{\partial}{\partial y}.$$

(c) The Heisenberg distribution on $\mathbb{R}^3$ is the constant rank 2 geometric distribution which is globally generated by the smooth vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}$$

that is not invilutive since $[X,Y] = \dfrac{\partial}{\partial z}$.

The question arises whether an involutive geometric distribution is integrable. In order to study this, we shall need the following two notions. First, a geometric distribution $\mathcal{D}$ on a smooth manifold M is said to be homogeneous if it is invariant by the flow of every $X \in \mathcal{X}^{\mathcal{D}}_{\mathrm{loc}}(M)$. The second notion is given in the following.

Definition 2.4.4. Let $\mathcal{D}$ be a geometric distribution on a smooth n-manifold M. Let $p \in M$ and $k = \dim \mathcal{D}_p$. A smooth chart (U,ϕ) of M where $\phi = (x^1, ..., x^n)$ is said to be $\mathcal{D}$-adapted at the point p if the following conditions are satisfied.

(i) $\phi(U) = \mathbb{R}^n$ and $\phi(p) = 0$.
(ii) $\dfrac{\partial}{\partial x^1}, ..., \dfrac{\partial}{\partial x^k} \in \mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$.
(iii) The rank of $\mathcal{D}$ is constant along the slices $\phi^{-1}(\mathbb{R}^k \times \{c\})$, $c \in \mathbb{R}^{n-k}$.

In the particular case of a constant rank k geometric distribution $\mathcal{D}$ condition (iii) is trivial and $\mathcal{D}|_U$ is integrable with integral manifolds the slices $\phi^{-1}(\mathbb{R}^k \times \{c\})$, $c \in \mathbb{R}^{n-k}$.

The adapted charts are the higher dimensional analogues of flow boxes in the theory of dynamical systems.

Proposition 2.4.5. *Let X be a smooth vector field on a smooth n-manifold M. If $p \in M$ is such that $X(p) \neq 0$, there exists a smooth chart (U, ϕ) of M with $\phi = (x^1, ..., x^n)$ such that $p \in U$ and $X|_U = \dfrac{\partial}{\partial x^1}$.*

Proof. Let $\Phi : D \to M$ be the flow of X. There exists a smooth chart (W, ψ) of M with $\psi(p) = 0$ and $X(p) = \left(\dfrac{\partial}{\partial y^1}\right)_p$, where $\psi = (y^1, ..., y^n)$. There exists an open neighbourhood $V \subset W$ of p and $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times V \subset D$ and $\Phi((-\epsilon, \epsilon) \times V) \subset W$. The set $S = \psi(V) \cap (\{0\} \times \mathbb{R}^{n-1})$ is an open neighbourhood of 0 in $\mathbb{R}^{n-1}$. If $F : (-\epsilon, \epsilon) \times S \to M$ is the smooth map defined by

$$F(t, x) = \Phi(t, \psi^{-1}(x))$$

we have $F_{*(0,0)}(e_1) = X(p)$ and $F_{*(0,0)}(e_j) = \left(\dfrac{\partial}{\partial y^j}\right)_p$ for $2 \leq j \leq n$. This means that $F_{*(0,0)}$ is a linear isomorphism and from the Inverse Map Theorem there exists an open neighbourhood $A \subset (-\epsilon, \epsilon) \times S$ of $(0, 0)$ such that $U = F(A)$ is an open neighbourhood of $p = F(0, 0)$ and $F|_A : A \to U$ is a diffeomorphism. Therefore, if $\phi = (F|_A)^{-1}$, then (U, ϕ) is a smooth chart of M with $\phi(p) = 0 \in \mathbb{R}^n$ and $X|_U = \dfrac{\partial}{\partial x^1}$, where $\phi = (x^1, ..., x^n)$. □

The following characterization of integrable geometric distributions is due to P. Stefan and H.J. Sussman.

Theorem 2.4.6. *For a geometric distribution $\mathcal{D}$ on a smooth n-manifold M the following statements are equivalent:*
(a) $\mathcal{D}$ is integrable.
(b) $\mathcal{D}$ is involutive and has constant rank along the maximal integral curves of every $X \in \mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$.
(c) $\mathcal{D}$ is homogeneous.
(d) At every point of M there exists some $\mathcal{D}$-adapted chart.

Proof. We have already shown above that (a) implies (b). In order to prove that (b) implies (c), we show first that every point $p \in M$ has an open neighbourhood U such that if $X \in \mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$ is defined on U with flow Φ, then $(\Phi_t)_{*p}(\mathcal{D}_p) = \mathcal{D}_{\Phi_t(p)}$ for all t for which $\Phi_t(p)$ is defined.

If $k = \dim \mathcal{D}_p$, there is an open neighbourhood U of p and Y_1,..., $Y_k \in \mathcal{X}(U)$ such that $\{Y_1(p), ..., Y_k(p)\}$ is a basis of $\mathcal{D}_p$ and $\{Y_1(q), ..., Y_k(q)\}$ is a linearly independent subset of $\mathcal{D}_q$ for every $q \in U$. Let $X \in \mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$ be defined on U with flow $\Phi : D \to U$. As in the proof of Proposition 3.3.4 we consider the smooth parametrized curves $\gamma_i : (a_p, b_p) \to T_pM$, $1 \leq i \leq k$, defined by

$$\gamma_i(t) = (\Phi_{-t})_{*\Phi_t(p)}(Y_i(\Phi_t(p))$$

where (a_p, b_p) is the interval of definition of the maximal integral curve of X through p. If we show that the linearly independent set $\{\gamma_1(t), ..., \gamma_k(t)\} \subset T_pM$ is contained in $\mathcal{D}_p$, we will have $(\Phi_t)_{*p}(\mathcal{D}_p) \subset \mathcal{D}_{\Phi_t(p)}$ and hence $(\Phi_t)_{*p}(\mathcal{D}_p) = \mathcal{D}_{\Phi_t(p)}$, by our assumption that the rank of $\mathcal{D}$ remains constant along the integral curves of the elements of $\mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$. From Theorem 2.3.1, the velocity field of γ_i is

$$\dot{\gamma}_i(t) = (\Phi_{-t})_{*\Phi_t(p)}([X, Y_i](\Phi_t(p))), \quad a_p < t < b_p.$$

Since by assumption the rank of $\mathcal{D}$ is constant along the integral curves of X, the set $\{Y_1(\Phi_t(p)), ..., Y_k(\Phi_t(p))\}$ is a basis of $\mathcal{D}_{\Phi_t(p)}$ and since $\mathcal{D}$ is involutive, there exist unique smooth functions $\lambda_{ji} : (a_p, b_p) \to \mathbb{R}$, $1 \leq i, j \leq k$ such that

$$[X, Y_i](\Phi_t(p)) = \sum_{j=1}^{k} \lambda_{ji}(t) Y_j(\Phi_t(p))$$

for every $a_p < t < b_p$ and $1 \leq i \leq k$. Thus, γ_1,..., γ_k satisfy the system of linear ordinary differential equations

$$\dot{\gamma}_i(t) = \sum_{j=1}^{k} \lambda_{ji}(t)\gamma_j(t), \quad a_p < t < b_p, \quad 1 \leq i \leq k.$$

From the existence and uniqueness of solutions and since $\gamma_i(0) \in \mathcal{D}_p$, $1 \leq i \leq k$, we conclude that $\gamma_i(t) \in \mathcal{D}_p$ for every $t \in (a_p, b_p)$ and $1 \leq i \leq k$.

Let now $X \in \mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$ be defined on an arbitrary open set $A \subset M$ with flow $\Phi : D \to A$ and let $(t, p) \in D$. By compactness of $\Phi([0, t] \times \{p\})$ and the above, there exists a partition $\{0 = t_0 < \cdots < t_m = t\}$ of $[0, t]$, for some $m \in \mathbb{N}$, such that $(\Phi_s)_{*\Phi_{t_i}(p)}(\mathcal{D}_{\Phi_{t_i}(p)}) = \mathcal{D}_{\Phi_{t_i+s}(p)}$ for every $0 \leq s \leq t_{i+1} - t_i$, $0 \leq i < m$. Therefore,

$$(\Phi_t)_{*p}(\mathcal{D}_p) = (\Phi_{t-t_{m-1}} \circ \cdots \circ \Phi_{t_1})_{*p}(\mathcal{D}_p) = \mathcal{D}_{\Phi_t(p)}.$$

In order to prove that (c) implies (d) we generalize the proof of Proposition 2.4.5. on the existence of flow boxes for smooth vector fields. Let $p \in M$ and suppose that $k = \dim \mathcal{D}_p$. As before, there is an open neighbourhood U of p and Y_1,..., $Y_k \in \mathcal{X}(U)$ such that $\{Y_1(p), ..., Y_k(p)\}$ is a basis of $\mathcal{D}_p$ and $\{Y_1(q), ..., Y_k(q)\}$ is a linearly independent subset of $\mathcal{D}_q$ for every $q \in U$. There are Y_{k+1},..., $Y_n \in \mathcal{X}(U)$ such that $\{Y_1(q), ..., Y_k(q)\}$ is a basis of T_qM for every $q \in U$. There exists $\epsilon > 0$ such that the smooth mao $\Psi : (-\epsilon, \epsilon)^n \to M$ with

$$\Psi(t_1, ..., t_n) = (\Psi^{Y_1}_{t_1} \circ \cdots \circ \Psi^{Y_n}_{t_n})(p)$$

is defined, where Ψ^{Y_i} denotes the flow of Y_i, $1 \leq i \leq n$. Since $\Psi_{*0}(e_i) = Y_i(p)$, $1 \leq i \leq n$, by the Inverse Map Theorem, we can choose $\epsilon > 0$ so that Ψ is a smooth diffeomorphism onto an open subset V of M. If $\psi = \Psi^{-1}$, then (V, ψ) is a smooth chart of M with $\psi(p) = 0$. Suppose that $\psi = (x^1, ..., x^n)$. If $q \in V$ and $\psi(q) = (t_1, ..., t_n)$, then

$$\left(\frac{\partial}{\partial x^i}\right)_q = (\Psi^{Y_1}_{t_1} \circ \cdots \circ \Psi^{Y_i}_{t_i})_* Y_i((\Psi^{Y_{i+1}}_{t_i+1} \circ \cdots \circ \Psi^{Y_n}_{t_n})(p))$$

belongs to $\mathcal{D}_q$ for $1 \leq i \leq k$, by our assumption that $\mathcal{D}$ is homogeneous. Finally, $\mathcal{D}$ has constant rank on each slice $\psi^{-1}((-\epsilon, \epsilon)^k \times \{c\})$, because $\mathcal{D}$ is homogeneous and every point $q \in \psi^{-1}((-\epsilon, \epsilon)^k \times \{c\})$ can be joined to $\Psi(0, c)$ with the concatenation of paths of integral curves of Y_1,... Y_k.

Obviously, (d) implies integrability. $\square$.

In the particular case of a geometric distribution of constant rank the preceding integrability criterion is known as the Frobenius' Theorem, although it had been originally proven by A. Clebsch in the context of partial differential equations.

Corollary 2.4.7. *A geometric distribution of constant rank on a smooth manifold is integrable if and only if it is involutive.*

In the rest of this section we shall restrict ourselves to the case of integrable geometric distributions of constant rank and be concerned with the existence and uniqueness of maximal integral manifolds. Two integral manifolds (L, g) and (K, h) of an integrable geometric distribution $\mathcal{D}$ of constant rank are called equivalent if there exists a diffeomorphism $f : K \to L$ such that $h = g \circ f$. In other words, equivalent integral manifolds are "reparametrizations" to each other. An integral manifold (L, g) is called *maximal* if there does not exist an integral manifold (K, h) such that $g(L)$ is a proper subset of $h(K)$.

Lemma 2.4.8. *Let $\mathcal{D}$ be an integrable geometric distribution of constant rank k on a smooth n-manifold M and let (L, g) be an integral manifold. If $p \in L$ and (U, ϕ) is a $\mathcal{D}$-adapted chart at p, then the connected components of $g(L) \cap U$ are countably many and each one of them is contained in a slice $\phi^{-1}(\mathbb{R}^k \times \{c\})$ for some $c \in \mathbb{R}^{n-k}$.*

Proof. Let C be a connected component of $g(L) \cap U$ and let $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ denote the projection. Since the topology of L has a countable basis and $g(L) \cap U$ is a union of slices, $(\pi \circ \phi)(g(L) \cap U)$ is a countable set. Thus, $(\pi \circ \phi)(C)$ is a connected subset of a countable subset of $\mathbb{R}^{n-k}$, hence a singleton. $\square$

Proposition 2.4.9. *Let $\mathcal{D}$ be an integrable geometric distribution of constant rank k on a smooth n-manifold M and let (L, g) be an integral manifold. If N is a smooth manifold and $f : N \to M$ is a smooth map such that $f(N) \subset g(L)$, there is a unique smooth map $\tilde{f} : N \to L$ such that $g \circ \tilde{f} = f$.*

$$\begin{tikzcd}
 & L \arrow[d, "g"] \\
N \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & M
\end{tikzcd}$$

Proof. Since g is an injective immersion, there is a unique map $\tilde{f} : N \to L$ such that $g \circ \tilde{f} = f$ and it suffices to show that f is continuous. Let $V \subset L$ be an open set, $x \in V$ and $y \in \tilde{f}^{-1}(x)$. Since $\mathcal{D}$ is integrable, there exists a $\mathcal{D}$-adapted chart (U, ϕ) at $g(x)$, so that $g^{-1}(\phi^{-1}(\mathbb{R}^k \times \{0\}))$ is an open neighbourhood of x contained in V. Since N is a manifold, hence locally connected, the connected component W of $f^{-1}(V)$ which contains y is open in N. To prove that $\tilde{f}$ is continuous, it suffices to show that $\tilde{f}(W) \subset g^{-1}(\phi^{-1}(\mathbb{R}^k \times \{0\}))$ or equivalently $f(W) \subset \phi^{-1}(\mathbb{R}^k \times \{0\})$. Indeed, since $f(W)$ is connected, it is contained in a connected component of $g(L) \cap U$. It follows form Lemma 2.4.8 that $f(W) \subset \phi^{-1}(\mathbb{R}^k \times \{0\})$, because $f(y) \in \phi^{-1}(\mathbb{R}^k \times \{0\})$. □

Theorem 2.4.10. *If $\mathcal{D}$ is an integrable geometric distribution of constant rank k on a smooth n-manifold M, then for every $p \in M$ there exists a unique maximal integral manifold (L, g) of $\mathcal{D}$ such that $p \in g(L)$ and for any other integral manifold (K, h) such that $p \in h(K)$ we have $h(K) \subset g(L)$.*

Proof. First we shall show the existence of maximal integral manifolds through the points of M. Let $p \in M$ and let L be the set of all points in M which can be joined to p by a concatenation of smooth paths on integral curves of elements of $\mathcal{X}^{\mathcal{D}}_{\mathrm{loc}}(M)$. Since the topology of M has a countable basis and $\mathcal{D}$ is integrable, there exists a countable smooth atlas $\mathcal{A}$ of M consisting of $\mathcal{D}$-adapted charts. Thus, for every $q \in L$ there exists a $\mathcal{D}$-adapted chart $(U, \phi) \in \mathcal{A}$ such that $q \in \phi^{-1}(\mathbb{R}^k \times \{c\}) \subset L$, for some $c \in \mathbb{R}^{n-k}$. Applying Lemma 2.1.1, there is a unique topology on L with respect to which all such slices become open subsets of L and is therefore finer than the subspace topology. It is clear that with this topology L will become a smooth k-manifold as soon as we show that it has a countable basis. For this it suffices to show that given $(U, \phi) \in \mathcal{A}$ only a countable number of the slices $\phi^{-1}(\mathbb{R}^k \times \{c\}) \subset L$, $c \in \mathbb{R}^{n-k}$ can be contained in L. Each point of $U \cap L$ can be joined to p with a piecewise smooth path which is a concatenation of smooth paths on integral curves of elements of $\mathcal{X}^{\mathcal{D}}_{\mathrm{loc}}(M)$ and so can be covered (not uniquely) by a finite sequence of $\mathcal{D}$-adapted charts in $\mathcal{A}$. Since there are only countably many such finite sequences, it suffices to show that only countably many of the slices $\phi^{-1}(\mathbb{R}^k \times \{c\}) \subset L$, $c \in \mathbb{R}^{n-k}$, are reachable in this way. This is true because such a slice can intersect at most countably many analogous slices in another $\mathcal{D}$-adapted chart in $\mathcal{A}$. Indeed, if $S = \phi^{-1}(\mathbb{R}^k \times \{c\})$ and $(V, \psi) \in \mathcal{A}$, then $S \cap V$ is open in S and so consists of countably many connected components each of which is an integral manifold of $\mathcal{D}$ in V and hence contained in a slice of (V, ψ).

If now $g : L \to M$ is the inclusion, then g is an injective immersion and (L, g) is an integral manifold of $\mathcal{D}$ by construction. In order to prove that is is maximal, let (K, h) be another integral manifold of $\mathcal{D}$ such that $p \in h(K)$. For every $q \in h(K)$ there exists a piecewise smooth path $\gamma : [0, 1] \to K$ from $h^{-1}(p)$ to $h^{-1}(q)$ and

$h \circ \gamma : [0,1] \to h(K)$ is a piecewise smooth path from p to q which is a concatenation of paths on integral curves of elements of $\mathcal{X}^{\mathcal{D}}_{\text{loc}}(M)$. Hence $q \in L$.

The uniqueness of (L, g) is a consequence of the preceding Proposition 2.4.9. If (K, h) is another maximal integral manifold of $\mathcal{D}$ and $p \in h(K)$, then $h(K) \subset L$, as we showed above, and actually $h(K) = L$, by maximality. From Proposition 2.4.9, there exists a unique smooth map $\tilde{h} : K \to L$ such that $g \circ \tilde{h} = h$.

$$\begin{tikzcd} & L \arrow[d, "g"] \\ K \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "h"'] & M \end{tikzcd}$$

Since $\tilde{h}$ is a bijective immersion between smooth manifolds of the same dimension k, it is a diffeomorphism. Hence (K, h) is equivalent to (L, g). $\square$

2.5 Exercises

1. Let M be a smooth n-manifold, $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ be a smooth atlas of M and $\bar{\mathcal{A}} = \{(\pi^{-1}(U_i), \bar{\phi}_i) : i \in I\}$ be the corresponding smooth atlas of TM, where $\pi : TM \to M$ is the tangent bundle projection. Prove that

$$\det D(\bar{\phi}_i \circ \bar{\phi}_j^{-1})(x, v) > 0$$

for every i, $j \in I$ with $U_i \cap U_j \neq \varnothing$ and $(x, v) \in \phi_j(U_i \cap U_j) \times \mathbb{R}^n$.

2. Let M be a smooth manifold and G be a group of diffeomorphisms of M which acts properly discontinuously on M. If $X \in \mathcal{X}(M)$ and $g_* X = X$ for every $g \in G$, prove that there exists a unique $\tilde{X} \in \mathcal{X}(M/G)$ such that $p_{*p}(X(p)) = \tilde{X}(\pi(p)))$ for every $p \in M$, where $\pi : M \to M/G$ is the quotient map. Construct a smooth vector field on the real projective plane $\mathbb{R}P^2$, which vanishes at exactly one point and every other maximal integral curve is periodic.

3. A smooth n-manifold M is called parallelizable if there are X_1,X_2,...,$X_n \in \mathcal{X}(M)$ such that $\{X_1(p), X_2(p), ..., X_n(p)\}$ is a basis of T_pM for every $p \in M$. Prove that M is parellelizable if and only if its tangent bundle is trivial, which means that there exists a smooth diffeomorphism $f : TM \to M \times \mathbb{R}^n$ such that the following diagram commutes

$$\begin{tikzcd} TM \arrow[rr, "f"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^n \arrow[dl, "\text{projection}"] \\ & M & \end{tikzcd}$$

and f maps linearly T_pM onto $\{p\} \times \mathbb{R}^n$ for every $p \in M$. Prove that the circle S^1 and the n-torus T^n are parallelizable.

4. On $\mathbb{R}^{2n}$ the nowhere vanishing smooth vector field

$$X = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + ... + x^{2n}\frac{\partial}{\partial x^{2n-1}} - x^{2n-1}\frac{\partial}{\partial x^{2n}}$$

is tangent to S^{2n-1}. In case $n = 2$, complete this vector field with two other vector fields to prove that the 3-sphere S^3 are parallelizable.

5. Let M be a smooth manifold and $f : M \to M$ be a diffeomorphism. If $X \in \mathcal{X}(M)$ has flow $\Phi : D \to M$, prove that the flow Ψ of f_*X is given by the formula $\Psi(t, f(p)) = f(\Phi(t, p))$.

6. Let $h : [0, 1] \to [0, \pi]$ be a smooth function with $h^{-1}(0) = [0, 1/5] \cup [4/5, 1]$ and $h^{-1}(\pi/2) = [2/5, 3/5]$. We extend h on $\mathbb{R}$ periodically by $h(x + 1) = h(x)$. Prove that the smooth vector fields

$$X(t) = t^2 \cos^2 h(t)\frac{d}{dt} \text{ and } Y(t) = t^2 \sin^2 h(t)\frac{d}{dt}$$

on $\mathbb{R}$ are complete, but $X + Y$ is not complete.

7. Let M be a smooth manifold, $X \in \mathcal{X}(M)$ with flow $\phi : D \to M$, where

$$D = \bigcup_{p\in M} (a_p, b_p) \times \{p\}.$$

If $f : M \to (0, 1]$ is a smooth function such that $f(p) < \min\{-a_p, b_p\}$ for every $p \in M$, prove that the smooth vector field $f \cdot X$ is complete.

8. On $\mathbb{R}^3$ we consider the smooth vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

(a) Prove that the map $g : \mathbb{R}^3 \to \mathcal{X}(\mathbb{R}^3)$ with

$$g(a, b, c) = aX + bY + cZ$$

is a linear monomorphism which has the additional property $g(A\times B) = [g(A), g(B)]$ for every A, $B \in \mathbb{R}^3$, where $\times$ is the usual exterior product on $\mathbb{R}^3$.
(b) Prove that the vector fields X, Y and Z generate a geometric distribution of constant rank 2 on $\mathbb{R}^3 \setminus \{0\}$ which is integrable. What are its maximal integral manifolds?

9. Let M be a smooth manifold and $X, Y \in \mathcal{X}(M)$ be complete with flows Φ and Ψ, respectively. If there exists a smooth function $h : M \to \mathbb{R}$ such that $[X, Y] = hX$, prove

$$(\Psi_t \circ \Phi_s)(p) = (\Phi_{T_p(t,s)} \circ \Psi_t)(p)$$

for every $p \in M$, t, $s \in \mathbb{R}$, where $T_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the smooth function

$$T_p(t, s) = \int_0^s \Big(\exp\Big(\int_0^t h(\psi_\tau(\phi_\sigma(p)))d\tau\Big)\Big)d\sigma.$$

Chapter 3

Riemannian manifolds

3.1 Connections

A straight line segment in euclidean n-space $\mathbb{R}^n$ is the unique piecewise smooth curve of minimum length between its endpoints. Equivalently, straight lines in $\mathbb{R}^n$ are the smooth curves whose acceleration vanishes identically. One way to define a notion of "straight line" on a smooth manifold is by defining first the notion of acceleration. The difficulty now lies in the fact that if M is a smooth manifold, $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth curve, the velocity vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(s)$ belong to different vector spaces for $t \neq s$ and their difference has no meaning. This difference can become meaningful if we have a way to connect the tangent spaces of M at the points $\gamma(t)$, $t \in I$. This requires the endowment of M with an extra structure. This structure can be described elegantly in an algebraic way.

Definition 3.1.1. A (linear) *connection* on a smooth n-manifold M is a map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

with the following properties, writing $\nabla_X Y$ instead of $\nabla(X, Y)$:
(i)$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$, for every f_1, $f_2 \in C^\infty(M)$ and X_1, X_2, $Y \in \mathcal{X}(M)$.
(ii) $\nabla_X(a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2$ for every a_1, $a_2 \in \mathbb{R}$ and X, Y_1, $Y_2 \in \mathcal{X}(M)$.
(iii) $\nabla_X(fY) = f \nabla_X Y + Xf \cdot Y$ for every $f \in C^\infty(M)$ and X, $Y \in \mathcal{X}(M)$.

The smooth vector field $\nabla_X Y$ is called the *covariant derivative of Y in the direction of X*. Some immediate consequences of the above definition are given in the following lemmas.

Lemma 3.1.2. *If ∇ is a connection on a smooth n-manifold M and $p \in M$, then for every X, $Y \in \mathcal{X}(M)$ the vector $(\nabla_X Y)(p) \in T_p M$ depends only on the values of X and Y in arbitrarily small open neighbourhoods of p.*

Proof. By bilinearity, it suffices to prove that $(\nabla_X Y)(p) = 0$ in case there exists an open neighbourhood V of p such that $X|_V = 0$ or $Y|_V = 0$. By Corollary 1.4.5, there exists a smooth function $f : M \to [0, 1]$ such that $f(p) = 1$ and $\mathrm{supp} f \subset V$.

If $Y|_V = 0$, then $fY = 0$ on M and so

$$0 = \nabla_X(fY)(p) = f(p)(\nabla_X Y)(p) + (Xf)(p) \cdot Y(p) = (\nabla_X Y)(p).$$

If $X|_V = 0$, we have $fX = 0$ on M, and

$$0 = (\nabla_{fX} Y)(p) = f(p)(\nabla_X Y)(p) = (\nabla_X Y)(p). \quad \square$$

Lemma 3.1.3. *If ∇ is a connection on a smooth n-manifold M and $p \in M$, then for every X, $Y \in \mathcal{X}(M)$ the vector $(\nabla_X Y)(p) \in T_pM$ depends only on the tangent vector $X(p)$ and the values of Y in arbitrarily small open neighbourhoods of p.*

Proof. It suffices to prove that $(\nabla_X Y)(p) = 0$ if $X(p) = 0$. In view of the preceding Lemma 3.1.2, we can work locally in the domain of a smooth chart (U, ϕ) of M with $p \in U$. If $\phi = (x^1, ..., x^n)$, there exist X^1,..., $X^n \in C^\infty(U)$ such that

$$X|_U = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}.$$

If $X(p) = 0$, we have $X^k(p) = 0$ for $1 \le k \le n$ and

$$(\nabla_X Y)(p) = \sum_{k=1}^{n} X^k(p)\big(\nabla_{\frac{\partial}{\partial x^k}} Y\big)(p) = 0. \quad \square$$

According to the above Lemma 3.1.3, it is legitimate to write henceforth $\nabla_{X(p)} Y$ instead of $(\nabla_X Y)(p)$. The same argument of the proof shows that if

$$S : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{X}(M)$$

is a $C^\infty(M)$-m-multilinear map, then for every X_1,..., $X_m \in \mathcal{X}(M)$ and $p \in M$ the value $S(X_1, ..., X_m)(p)$ depends only on the values $X_1(p)$,..., $X_m(p)$ and so we can write $S(X_1(p), ..., X_m(p))$ instead.

Lemma 3.1.4. *If ∇ is a connection on a smooth n-manifold M and $p \in M$, then for every X, $Y \in \mathcal{X}(M)$ the vector $(\nabla_X Y)(p) \in T_pM$ depends only on the tangent vector $X(p)$ and the values $Y(\gamma(t))$ for any smooth curve $\gamma : (-\epsilon, \epsilon) \to M$, $\epsilon > 0$, such that $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$.*

Proof. According to the preceding Lemmas 3.1.2 and 3.1.3, we may assume that $\gamma((-\epsilon, \epsilon)) \subset U$ for some smooth chart (U, ϕ) of M with $p \in U$. Let $\phi = (x^1, ..., x^n)$. There exist Y^1,..., $Y^n \in C^\infty(U)$ such that

$$Y|_U = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x^k}$$

and

$$\nabla_{X(p)}Y = \sum_{k=1}^{n} Y^k(p)\nabla_{X(p)}\frac{\partial}{\partial x^k} + \sum_{k=1}^{n}(Y^k \circ \gamma)'(0)\left(\frac{\partial}{\partial x^k}\right)_p.$$

If $Y(\gamma(t)) = 0$ for all $|t| < \epsilon$, then obviously $\nabla_{X(p)}Y = 0$. $\square$

We can now find a local formula for a given connection ∇ in the domain of a smooth chart (U, ϕ) of M with $\phi = (x^1, ..., x^n)$. There exist unique $\Gamma_{ij}^k \in C^\infty(U)$, $1 \leq i, j, k \leq n$, such that

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \sum_{k=1}^{n}\Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

for every $1 \leq i, j \leq n$. The smooth functions Γ_{ij}^k are called *the Christoffel symbols* of ∇ with respect to the smooth chart (U, ϕ). If now

$$X = \sum_{k=1}^{n} X^k\frac{\partial}{\partial x^k} \quad \text{and} \quad Y = \sum_{k=1}^{n} Y^k\frac{\partial}{\partial x^k},$$

a routine computation shows that on U we have

$$\nabla_X Y = \sum_{k=1}^{n}\left(X(Y^k) + \sum_{i,j=1}^{n}\Gamma_{ij}^k X^i Y^j\right)\frac{\partial}{\partial x^k}.$$

Conversely, given smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$, $1 \leq i, j, k \leq n$, the above formula defines a connection on U, because for every $f \in C^\infty(U)$ we have

$$\nabla_X(fY) = \sum_{k=1}^{n}\left(X(fY^k) + \sum_{i,j=1}^{n}\Gamma_{ij}^k X^i fY^j\right)\frac{\partial}{\partial x^k}$$

$$= \sum_{k=1}^{n}\left(Xf \cdot Y^k + fX(Y^k) + f\sum_{i,j=1}^{n}\Gamma_{ij}^k X^i Y^j\right)\frac{\partial}{\partial x^k} = Xf \cdot Y + f\nabla_X Y.$$

The connection on $\mathbb{R}^n$ with Christoffel symbols identically equal to zero is called the *euclidean connection* and is given by the formula

$$\nabla_X Y = \sum_{k=1}^{n} X(Y^k)\frac{\partial}{\partial x^k}.$$

In other words, the covariant derivative of Y in the direction of X with respect to the euclidean connection is the directional derivative of Y in the direction of X.

Example 3.1.5. A $(n-1)$-dimensional smooth submanifold M of $\mathbb{R}^n$ is usually called *hypersurface*. We identify the tangent space T_pM at a point $p \in M$ with its image under the derivative of the inclusion and consider it a vector subspace of $T_p\mathbb{R}^n$. The euclidean connection ∇ on $\mathbb{R}^n$ induces a connection on any hypersurface M in $\mathbb{R}^n$. We observe first that if $p \in M$ and (U, ϕ) is a M-straightening chart of $\mathbb{R}^n$ with $\phi(U \cap M) \subset \mathbb{R}^{n-1} \times \{0\}$ and $p \in U \cap M$, then for every $X \in \mathcal{X}(M)$ there exists an extension $\tilde{X} \in \mathcal{X}(U)$, that is $\tilde{X}|_{U\cap M} = X|_{U\cap M}$. For every X, $Y \in \mathcal{X}(M)$ we put now

$$\overline{\nabla}_{X(p)}Y = \pi_p(\nabla_{X(p)}\tilde{Y})$$

where $\pi_p : T_p\mathbb{R}^n \to T_pM$ is the projection with respect to the orthogonal splitting $T_p\mathbb{R}^n = T_pM \oplus (T_pM)^\perp$. By Lemma 5.1.4, this definition does not depend on the choice of the extension $\tilde{Y}$. Obviously, $\overline{\nabla}$ is a connection on M and is called the *euclidean connection* of the hypersurface M.

Proposition 3.1.6. *On every smooth manifold M there are connections.*

Proof. From the above, there are connections locally on M. Let $\mathcal{A}$ be a smooth atlas of M. For every $(U, \phi_U) \in \mathcal{A}$ there is a connection ∇^U on U. Let $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ be a smooth partition of unity subordinated to the open cover $\{U : (U, \phi_U) \in \mathcal{A}\}$ of M. Then, the formula

$$\nabla_X Y = \sum_{(U,\phi_U)\in\mathcal{A}} f_U \nabla^U_X Y$$

for X, $Y \in \mathcal{X}(M)$, defines a connection on M because if $f \in C^\infty(M)$, we have

$$\nabla_X(fY) = \sum_{(U,\phi_U)\in\mathcal{A}} f_U \nabla^U_X(fY) = \sum_{(U,\phi_U)\in\mathcal{A}} f_U(Xf \cdot Y + f\nabla^U_X Y)$$

$$= \Bigg(\sum_{(U,\phi_U)\in\mathcal{A}} f_U\Bigg) Xf \cdot Y + f \sum_{(U,\phi_U)\in\mathcal{A}} f_U \nabla^U_X Y = Xf \cdot Y + f\nabla_X Y. \quad \square$$

In view of Lemma 3.1.4, given a connection it is possible to define a covariant differentiation of smooth vector fields along a smooth curve. Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. The set $\mathcal{X}(\gamma)$ of smooth vector fields along γ is a vector space.

Proposition 3.1.7. *Let ∇ be a connection on a smooth n-manifold M. For every smooth curve $\gamma : I \to M$ there exists a unique linear operator*

$$\frac{D}{dt} : \mathcal{X}(\gamma) \to \mathcal{X}(\gamma)$$

with the following properties:
(i) $\dfrac{D}{dt}(fX) = f'X + f\dfrac{DX}{dt}$ *for every $X \in \mathcal{X}(\gamma)$ and smooth function $f : I \to \mathbb{R}$.*
(ii) If $X \in \mathcal{X}(\gamma)$ has a smooth extension $\tilde{X} \in \mathcal{X}(V)$ on an open set V which contains $\gamma(I)$, then

$$\frac{DX}{dt}(t) = \nabla_{\dot\gamma(t)}\tilde{X}, \quad t \in I.$$

The vector field $\dfrac{DX}{dt}$ *along γ is called the covariant derivative of X along γ.*

Proof. We shall prove uniqueness first. As in the proof of Lemma 3.1.2 we see that for every $t_0 \in I$ the value $\dfrac{DX}{dt}(t_0)$ depends only on the values of X on an arbitrarily small open interval with center t_0. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and $\gamma(t_0) \in U$. There exist $\epsilon > 0$ such that $\gamma((t_0-\epsilon, t_0+\epsilon)) \subset U$ and smooth functions X^1,..., $X^n : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}$ such that

$$X(t) = \sum_{k=1}^{n} X^k(t)\left(\frac{\partial}{\partial x^k}\right)_{\gamma(t)}$$

for $|t - t_0| < \epsilon$. By linearity and properties (i), (ii) we compute

$$\frac{DX}{dt}(t) = \sum_{k=1}^{n} (X^k)'(t)\left(\frac{\partial}{\partial x^k}\right)_{\gamma(t)} + \sum_{k=1}^{n} X^k(t)\nabla_{\dot\gamma(t)}\frac{\partial}{\partial x^k}$$

$$= \sum_{k=1}^{n}\left((X^k)'(t) + \sum_{i,j=1}^{n} \Gamma^k_{ij}(\gamma(t))(\gamma^i)'(t)X^j(t)\right)\left(\frac{\partial}{\partial x^k}\right)_{\gamma(t)},$$

where $(\phi\circ\gamma)(t) = (\gamma^1(t), ..., \gamma^n(t))$ for every $|t-t_0| < \epsilon$. This proves the uniqueness.

The existence follows covering $\gamma(I)$ by the domains of smooth charts of M and defining $\dfrac{D}{dt}$ locally by the above formula. By uniqueness, the local definitions coincide on overlapping intervals. □

In the rest of the section we shall see that the algebraic definition of a connection indeed gives a mechanism of "connecting" tangent spaces at various points of a smooth manifold. Let ∇ be a connection on a smooth n-manifold M.

Definition 3.1.8. If $\gamma : I \to M$ is a smooth curve defined on an open interval $I \subset \mathbb{R}$, a smooth vector field $X \in \mathcal{X}(\gamma)$ is said to be *parallel along* γ, if $\dfrac{DX}{dt} = 0$ on I. A smooth vector field $X \in \mathcal{X}(M)$ is called *parallel* if $\nabla_Y X = 0$ on M for every $Y \in \mathcal{X}(M)$.

Example 3.1.9. The parallel vector fields on $\mathbb{R}^n$ with respect to the euclidean connection are the constant ones, that is the vector fields

$$\sum_{k=1}^{n} a^k \frac{\partial}{\partial x^k}$$

for a^1,..., $a^n \in \mathbb{R}$.

Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and let $\gamma : I \to U$ be a smooth curve with local representation $\phi\circ\gamma = (\gamma^1, ..., \gamma^n)$. From the formula of the covariant differentiation along γ derived in the proof of Proposition 3.1.7 follows that a smooth vector field

$$X(t) = \sum_{k=1}^{n} X^k(t)\left(\frac{\partial}{\partial x^k}\right)_{\gamma(t)}, \quad t \in I$$

along γ is parallel if and only if the smooth functions X^1,..., X^n satisfy the system of linear ordinary differential equations

$$(X^k)'(t) = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)X^j(t), \quad t \in I, \quad 1 \le k \le n.$$

From the existence and uniqueness of solutions for linear ordinary differential equations we have that for every $t_0 \in I$ and every $v \in T_{\gamma(t_0)}M$ there exists a unique parallel vector field X along γ satisfying the initial condition $X(t_0) = v$.

Proposition 3.1.10. *Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. For every $t_0 \in I$ and every $v \in T_{\gamma(t_0)}M$ there exists a unique parallel vector field X along γ such that $X(t_0) = v$.*

Proof. From the above there exists $b > t_0$ such that there exists a unique parallel vector field along $\gamma|_{[t_0,b]}$ with $X(t_0) = v$. It suffices to prove that the supremum T of all such b does not belong to I. Suppose that it does. Choosing a smooth chart (V, ψ) of M with $\gamma(T) \in V$, there exists $\delta > 0$ such that $\gamma((T-\delta, T+\delta)) \subset V$. From the above, there exists a unique parallel vector field $\tilde{X}$ along $\gamma|_{(T-\delta,T+\delta)}$ satisfying the initial condition $\tilde{X}(T - \frac{\delta}{2}) = X(T - \frac{\delta}{2})$. From the uniqueness of solutions we get $\tilde{X} = X$ on $(T - \delta, T)$ and so X has a smooth extension on $[t_0, T + \delta)$. This contradicts the definition of T. □

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. The preceding Proposition 5.1.9 implies that for every a, $b \in I$ with $a < b$ there is a well defined map $\tau_{b,a} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ where $\tau_{b,a}(u)$ is the value $X(b)$ of the unique parallel vector field X along γ with $X(a) = u$. Since the parallel vector fields along γ are the solutions of a system of linear ordinary differential equations, $\tau_{b,a}$ is a linear isomorphism and it is called the *parallel translation along γ form $\gamma(a)$ to $\gamma(b)$*.

Theorem 3.1.11. *If $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ is a smooth curve, then for every $X \in \mathcal{X}(\gamma)$ and $s \in I$ we have*

$$\frac{DX}{dt}(s) = \lim_{h \to 0} \frac{1}{h}[\tau_{s,s+h}(X(s+h)) - X(s)].$$

Proof. It suffices to prove the assertion in case there exists a smooth chart (U, ϕ) and $\gamma(I) \subset U$. Since the parallel vector fields along γ are the solutions of a system of linear ordinary differential equations, there are parallel vector fields E_1,..., E_n along γ such that $\{E_1(t), ..., E_n(t)\}$ is a basis of $T_{\gamma(t)}M$ for every $t \in I$. Now there are unique smooth functions f_1,..., $f_n : I \to \mathbb{R}$ such that

$$X(t) = \sum_{k=1}^{n} f_k(t)E_k(t), \quad t \in I.$$

Therefore,

$$\frac{DX}{dt} = \sum_{k=1}^{n} f_k' \cdot E_k.$$

On the other hand, $\tau_{s,s+h}(E_k(s+h)) = E_k(s)$, because E_k is parallel along γ, $1 \leq k \leq n$, and hence

$$\tau_{s,s+h}(X(s+h)) - X(s) = \sum_{k=1}^{n} f_k(s+h)\tau_{s,s+h}(E_k(s+h)) - \sum_{k=1}^{n} f_k(s)E_k(s)$$
$$= \sum_{k=1}^{n} (f_k(s+h) - f_k(s))E_k(s).$$

It follows that

$$\lim_{h\to 0} \frac{1}{h}[\tau_{s,s+h}(X(s+h)) - X(s)] = \lim_{h\to 0} \sum_{k=1}^{n} \frac{f_k(s+h) - f_k(s)}{h} \cdot E_k(s) = \sum_{k=1}^{n} f_k'(s) \cdot E_k(s). \quad \square$$

3.2 Geodesics and exponential map

Let M be a smooth n-manifold and ∇ a connection on M. The *acceleration* of a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, is the smooth vector field $\dfrac{D\dot{\gamma}}{dt}$ along γ.

Definition 3.2.1. A smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, is called *geodesic* of the connection ∇ if $\dfrac{D\dot{\gamma}}{dt} = 0$.

Note that the differential equation of geodesics is independent of local coordinates of M. Its expression in the local coordinates of a smooth chart (U, ϕ) of M with $\phi = (x^1, ..., x^n)$, where $\phi \circ \gamma = (\gamma^1, ..., \gamma^n)$, is

$$(\gamma^k)''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)(\gamma^j)'(t) = 0, \quad 1 \leq k \leq n.$$

In the particular case of the euclidean connection on $\mathbb{R}^n$, where the Christoffel symbols vanish, it follows that the geodesics are the euclidean straight lines.

The geodesics in U are the projections under the tangent bundle projection $\pi : TM \to M$ of the integral curves of the smooth vector field

$$\sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} + \sum_{k=1}^{n} \Big(-\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j \Big) \frac{\partial}{\partial v^k}$$

on $\pi^{-1}(U)$, where $\tilde{\phi} = (x^1, ..., x^n, v^1, ..., v^n)$ is the smooth chart of TM corresponding to (U, ϕ). Since the differential equation of geodesics does not depend on smooth charts, we conclude that this is the local representation in the smooth chart $(\pi^{-1}(U), \tilde{\phi})$ of a smooth vector field G which is globally defined on TM and is called the *geodesic vector field of the connection* ∇. Its flow is called the *geodesic flow of* ∇.

The homogeneity of the differential equation of geodesics implies the following property.

Lemma 3.2.2. *If $\gamma : I \to M$ is the geodesic of the connection ∇ defined on the open interval I and satisfying the initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, then for every $\lambda \in \mathbb{R} \setminus \{0\}$ the maximal geodesic γ_λ satisfying the initial conditions $\gamma_\lambda(0) = p$ and $\dot{\gamma}_\lambda(0) = \lambda v$ is defined on the open interval $\frac{1}{\lambda} I$ and is given by $\gamma_\lambda(t) = \gamma(\lambda t)$.*

Proof. Indeed $\dot{\gamma}_\lambda = \lambda \dot{\gamma}$ and therefore $\frac{D\dot{\gamma}_\lambda}{dt} = \lambda^2 \frac{D\dot{\gamma}}{dt}$. Hence γ_λ is a geodesic if and only if γ is. $\square$

In the rest of the section we fix a connection ∇ on a smooth n-manifold M. Let $E \subset TM$ denote the set of all points $(p, v) \in TM$ such that the geodesic $\gamma_{(p,v)}$ from p with initial velocity v is defined on the unit interval $[0, 1]$. Let $\exp : E \to M$ be the smooth map $\exp(p, v) = \gamma_{(p,v)}(1)$. From Lemma 3.2.2, for every $p \in M$ the set $E_p = E \cap T_pM$ is an open neighbourhood of $0 \in T_pM$ and the map $\exp_p(v) = \exp(p, v)$ is smooth.

Lemma 3.2.3. *For every $p \in M$ the set E_p is star-shaped with respect to $0 \in T_pM$ and the geodesic $\gamma_{(p,v)}$ from p with initial velocity v is given by the formula*

$$\gamma_{(p,v)}(t) = \exp_p(tv)$$

for all $t \in \mathbb{R}$ for which at least one of the two sides is defined.

Proof. From Lemma 3.2.2. we have $\gamma_{(p,v)}(t) = \gamma_{(p,v)}(t \cdot 1) = \exp_p(tv)$ for every $t \in \mathbb{R}$ such that at least one of the two sides is defined. Moreover, if $v \in E_p$, then $\gamma_{(p,v)}$ is defined at least on $[0, 1]$ and hence $tv \in E_p$ for all $0 \le t \le 1$. This means that E_p is star-shaped with respect to $0 \in T_pM$. $\square$

Proposition 3.2.4. *For every point $p \in M$ there exist an open neighbourhood V of $0 \in T_pM$ and an open neighbourhood U of p in M such that $\exp_p(V) = U$ and $\exp_p : V \to U$ is a smooth diffeomorphism.*

Proof. According to the Inverse Map Theorem it suffices to prove that the derivative $(\exp_p)_{*0} : T_0(T_pM) \cong T_pM \to T_pM$ is a linear isomorphism. If $v \in T_pM$ and $\sigma : \mathbb{R} \to T_pM$ is the straight line $\sigma(t) = tv$, and $\gamma_{(p,v)}$ is the geodesic from p with initial velocity v, we have

$$(\exp_p)_{*0}(v) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(\sigma(t)) = \dot{\gamma}_{(p,v)}(0) = v.$$

Hence $(\exp_p)_{*0} = id_{T_pM}$. $\square$

Choosing a basis of T_pM, that is a linear isomorphism $h : T_pM \to \mathbb{R}^n$, the pair $(U, h \circ (\exp_p|_V)^{-1})$ is a smooth chart of M and is called a *normal chart* of M at p (with respect to the connection ∇). The neighbourhood U of p in Proposition 5.2.4 is called normal. Observe that the local representations of geodesics emanating from p with respect to a normal chart at p are straight lines through 0. Thus, if $(\gamma^1, ..., \gamma^n)$

is the local representation of any geodesic γ emanating from p with respect to a normal chart at p, then

$$\sum_{i,j=1}^{n} \Gamma_{ij}^{k}(p)(\gamma^i)'(0)\gamma^j)'(0) = 0, \quad 1 \le k \le n.$$

This means that the polynomial

$$\sum_{i,j=1}^{n} \Gamma_{ij}^{k}(p)v^i v^j$$

vanishes identically on some open neighbourhood of $0 \in \mathbb{R}^n$. Therefore,

$$\Gamma_{ij}^{k}(p) + \Gamma_{ji}^{k}(p) = 0$$

for every $1 \le i, j, k \le n$.

Given a connection ∇ on a smooth n-manifold M, we define its *torsion* to be the $C^\infty(M)$-bilinear map $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ with

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Thus the value of $T(X, Y)$ at a point $p \in M$ depends only on the values $X(p)$ and $Y(p)$.

The connection ∇ is said to be *symmetric* if its torsion vanishes. This terminology is justified as follows. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$. If X, $Y \in \mathcal{X}(M)$ and

$$X|_U = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k} \quad \text{and} \quad Y|_U = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x^k},$$

we have

$$T(X, Y)|_U = \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} (\Gamma_{ij}^{k} - \Gamma_{ji}^{k}) X^i Y^j \right) \frac{\partial}{\partial x^k}.$$

Hence ∇ is symmetric if and only if the Christoffel symbols with respect to any smooth chart are symmetric with respect to the lower indices, that is $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$ for every $1 \le i, j, k \le n$.

It follows from the above that if ∇ is a symmetric connection and $p \in M$, then the Christoffel symbols with respect to a normal chart at p vanish at the point p.

Proposition 3.3.5. *For every connection ∇ on a smooth n-manifold M there exists a unique symmetric connection $\overline{\nabla}$ on M which has the same geodesics as ∇.*

Proof. If T is the torsion of ∇, we define the connection $\overline{\nabla}$ by

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T(X, Y).$$

Since $T(X,X)=0$ for every $X \in \mathcal{X}(M)$, it follows that $\overline{\nabla}$ and ∇ have the same geodesics. The uniqueness is the fact that two symmetric connections with the same geodesics coincide. Indeed, if ∇^1 and ∇^2 are two symmetric connections, then

$$S = \nabla^1 - \nabla^2 : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

is a symmetric $C^\infty(M)$-bilinear map. If ∇^1 and ∇^2 have the same geodesics, $S(X,X)=0$ for every $X \in \mathcal{X}(M)$ and therefore

$$2S(X,Y) = S(X+Y, X+Y) = 0$$

for every X, $Y \in \mathcal{X}(M)$. □

3.3 Riemannian metrics

A *Riemannian metric* on a smooth n-manifold M is a family $g = (g_p)_{p\in M}$ of inner products

$$g_p : T_pM \times T_pM \to T_pM$$

which depend smoothly on p in the sense that if $U \subset M$ is an open set and X, $Y \in \mathcal{X}(U)$, then the function $f : U \to \mathbb{R}$ with $f(p) = g_p(X(p), Y(p))$ is smooth. A *Riemannian manifold* is a smooth manifold endowed with a Riemannian metric.

Let (M,g) and (N,h) be two Riemannian manifolds. A smooth map $f : M \to N$ is called *(Riemannian) isometry* if it is a smooth diffeomorphism and its derivative at each point preserves the Riemannian metrics, that is

$$h_{f(p)}(f_{*p}(v), f_{*p}(w)) = g_p(v,w)$$

for every v, $w \in T_pM$ and $p \in M$. The isometries are the isomorphisms of the category with objects the Riemannian manifolds and the aim of Riemannian Geometry is the classification of Riemannian manifolds up to isometry.

In the sequel we shall use in any case the symbol $\langle .,. \rangle$ to denote the Riemannian metric and the symbol $\|.\|$ for its corresponding norm on tangent spaces, if there is no danger of confusion.

If M is a Riemannian manifold, the set $I(M)$ of all isometries of M onto itself is a subgroup of its group of diffeomorphisms and is called the *isometry group* of M. If the action of $I(M)$ on M by evaluation is transitive, M is called *homogeneous*. Recall that the *isotropy group (or stabilizer)* at a point p is the subgroup

$$I_p(M) = \{f | f \in I(M) \quad \text{and } f(p) = p\}$$

of $I(M)$. The derivative of an element $f \in I_p(M)$ is an orthogonal transformation, that is linear isometry, $f_{*p} : T_pM \to T_pM$. It follows from the chain rule, that the assignment of f_{*p} to $f \in I_p(M)$ is a homomorphism of $I_p(M)$ into the group of the orthogonal transformations of T_pM which is usually called the *isotropic representation at* p. The point p is called *isotropic* if the action of $I_p(M)$ on the unit sphere in T_pM via the isotropic representation at p is transitive. Thus $p \in M$ is isotropic if for every v, $w \in T_pM$ with $\|v\| = \|w\| = 1$ there exists $f \in I_p(M)$ such that $f_{*p}(v) = w$. A Riamannian manifold M is called *isotropic* if every point of M is isotropic.

Example 3.3.1. On every open set $M \subset \mathbb{R}^n$, $n \geq 1$ the euclidean inner product of $\mathbb{R}^n$ defines a Riemannian metric in the obvious way which is called the *euclidean Riamannian metric.* Evidently, the euclidean n-space $\mathbb{R}^n$ is a homogeneous and isotropic Riemannian manifold.

Proposition 3.3.2. *On every smooth n-manifold there are Riemannian metrics.*

Proof. Let M be a smooth n-manifold and let $\mathcal{A}$ be a smooth atlas of M. For every $(U, \phi_U) \in \mathcal{A}$ there is a Riemannian metric g^U on U defined by

$$g_p^U(v, w) = \langle (\phi_U)_{*p}(v), (\phi_U)_{*p}(w) \rangle$$

for v, $w \in T_pM$, $p \in U$, where $\langle ., . \rangle$ is the euclidean inner product in $\mathbb{R}^n$. Let $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ be a smooth partition of unity subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. For every $p \in M$ and v, $w \in T_pM$ we define

$$g_p(v, w) = \sum_{(U,\phi_U)\in\mathcal{A}} f_U(p) g_p^U(v, w).$$

Since g is locally a convex combination of Riemannan metrics, it is a Riemannian metric itself. □

In the rest of the section we shall give in some detail several examples of Riemannian manifolds.

Example 3.3.3. Let (M, g) be a Riemannian manifold and let $i : N \to M$ be an immersion of the smooth manifold N into M. There is an induced by i Riemannian metric g^N on N defined by

$$g_p^N(v, w) = g_{i(p)}(i_{*p}(v), i_{*p}(w))$$

for every v, $w \in T_pN$ and $p \in N$. In particular, every smooth submanifold of M inherits a Riemannian metric.

The n-sphere $S_R^n = \{p \in \mathbb{R}^{n+1} : \|p\| = R\}$ of radius $R > 0$ inherits a Riamannian metric from the euclidean Riemannian metric $\langle ., . \rangle$ of $\mathbb{R}^{n+1}$. Obviously, the orthogonal group $O(n+1, \mathbb{R})$ is contained in the isometry group of $I(S_R^n)$. Actually, it can be proved that $O(n+1, \mathbb{R})$ coincides with $I(S_R^n)$, but we will not need this for the time being. We shall show that S_R^n is homogeneous and isotropic with one strike. Let $p \in S_R^n$ and let $\{E_1, ..., E_n\}$ be an orthonormal basis of $T_pS_R^n$. Then,

$$\{E_1, ..., E_n, \frac{1}{R}p\}$$

is an orthonormal basis of $T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ and there exists $f \in O(n+1, \mathbb{R})$ such that

$$f(e_k) = E_k, \quad 1 \leq k \leq n, \quad f(Re_{n+1}) = p.$$

This implies that S^n_R is homogeneous and isotropic, since every point p is the image of the north pole Re_{n+1} and $I_{Re_{n+1}}(S^n_R)$ acts transitively on the set of orthonormal basis of $T_{Re_{n+1}}S^n_R$.

Example 3.3.4. The hyperbolic metric on the upper half plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$$

is defined by

$$g_z(v, w) = \frac{1}{(\mathrm{Im} z)^2}\langle v, w\rangle = \frac{1}{(\mathrm{Im} z)^2}\mathrm{Re}(v\overline{w})$$

for v, $w \in T_z\mathbb{H}^2$, $z \in \mathbb{H}^2$, where $\langle v, w\rangle = \mathrm{Re}(v\overline{w})$ is the euclidean inner product in complex notation.

The reflection with respect to the imaginary semi-axis $\ell = \{it : t > 0\}$ is the map $\tau : \mathbb{H}^2 \to \mathbb{H}^2$ with $\tau(z) = -\overline{z}$ and is an orientation reversing isometry of $\mathbb{H}^2$.

If a, b, c, $d \in \mathbb{R}$ and $ad - bc = 1$, for the Möbius transformation $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with

$$T(z) = \frac{az + b}{cz + d}$$

we have

$$\mathrm{Im}(T(z)) = \frac{\mathrm{Im} z}{|cz + d|^2}$$

and

$$T'(z) = \frac{1}{(cz + d)^2}.$$

Therefore, $T(\mathbb{H}^2) = \mathbb{H}^2$ and

$$g_{T(z)}(T_{*z}(v), T_{*z}(w)) = g_{T(z)}(T'(z)v, T'(z)w) = \frac{1}{(\mathrm{Im} T(z))^2}\mathrm{Re}(|T'(z)|^2 v\overline{w})$$

$$= \frac{1}{(\mathrm{Im} z)^2}\mathrm{Re}(v\overline{w}) = g_z(v, w)$$

for every v, $w \in T_z\mathbb{H}^2$ and $z \in \mathbb{H}^2$. Therefore the group of Möbius transformations with real coefficients, which is isomorphic to $PSL(2, \mathbb{R})$, is a subgroup of the isometry group $I(\mathbb{H}^2)$. It can be proved that this is the group of orientation preserving isometries of $\mathbb{H}^2$ and it has index 2 in $I(\mathbb{H}^2)$, but we will not need this now.

The action of $PSL(2, \mathbb{R})$ on $\mathbb{H}^2$ by Möbius transformations is transitive because if $z_0 = a + ib$, $a \in \mathbb{R}$, $b > 0$, then $z_0 = T(i)$, where T is the Möbius transformation

$$T(z) = \frac{\sqrt{b}z + \frac{a}{\sqrt{b}}}{0z + \frac{1}{\sqrt{b}}} = bz + a.$$

Thus, $\mathbb{H}^2$ is homogeneous. It is isotropic as well. Indeed, if $v \in T_i\mathbb{H}^2$ and $g_i(v, v) = 1$, there exists $0 \le \theta < 2\pi$ such that $v = e^{-2i\theta}$. If

$$T(z) = \frac{\cos\theta \cdot z - \sin\theta}{\sin\theta \cdot z + \cos\theta},$$

then $T(i) = i$ and $T'(i) = e^{-2i\theta}$. Hence $v = T_{*i}(1)$.

The Riemannian manifold $\mathbb{H}^2$ is the Poincaré upper half-plane model of the *hyperbolic plane*.

Example 3.3.5. We shall describe two models of the higher dimensional version of the hyperbolic plane. The first one resembles the case of the sphere. Let $n \geq 2$, $R > 0$ and

$$\mathbb{H}^n_R = \{(x_1,,,,.x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -R^2, \quad x_{n+1} > 0\}$$

be the upper connected component of the two-sheeted hyperboloid in $\mathbb{R}^{n+1}$. On $\mathbb{H}^n_R$ we consider the Riemannian metric which on each tangent space is the restriction of the Minkowski non-degenerate symmetric bilinear form

$$\langle x, y \rangle = -x_{n+1}y_{n+1} + \sum_{k=1}^{n} x_k y_k$$

where $x = (x_1, ..., x_{n+1})$, $y = (y_1, ..., y_{n+1})$. Although the Minkowski form is not positive definite, its restriction on each tangent space $T_p\mathbb{H}^n_R$, $p \in \mathbb{H}^n_R$, is. To see this, suppose that $p = (p_1, ..., p_{n+1})$. If $v = (v_1, ..., v_{n+1}) \in T_p\mathbb{H}^n_R$, then

$$p_1v_1 + \cdots + x_nv_n - p_{n+1}v_{n+1} = 0$$

and

$$\langle v, v \rangle = \sum_{k=1}^{n} v_k^2 - \frac{1}{p_{n+1}^2}\left(\sum_{k=1}^{n} p_k v_k\right)^2 \geq \left(1 - \frac{p_{n+1}^2 - R^2}{p_{n+1}^2}\right)\sum_{k=1}^{n} v_k^2 \geq 0$$

from the Cauchy-Schwarz inequality, and $\langle v, v \rangle = 0$ if and only if $v_1 = \cdots = v_n = 0$ and therefore $v_{n+1} = 0$ as well, since $p_{n+1} > 0$.

The Riamannian manifold $\mathbb{H}^n_R$ is called the *hyperbolic n-space of radius $R > 0$*. An alternative model is the upper half n-space, which we denote temporarily by $\mathbb{U}^n_R = \{(p_1, ..., p_n) \in \mathbb{R}^n : p_n > 0\}$, endowed with the Riemannian metric

$$g_p(v, w) = \frac{R^2}{p_n^2}\sum_{k=1}^{n} v_k w_k$$

where $p = (p_1, ..., p_n) \in \mathbb{U}^n_R$ and $v = (v_1, ..., v_n)$, $w = (w_1, ..., w_n) \in T_p\mathbb{U}^n_R$. A tedious calculation shows that the map $F : \mathbb{H}^n_R \to \mathbb{U}^n_R$ defined by

$$F(x_1, ..., x_n, x_{n+1}) = \left(\frac{x_1(R + x_{n+1})}{x_{n+1} - x_n}, ..., \frac{x_{n-1}(R + x_{n+1})}{x_{n+1} - x_n}, \frac{R^2}{x_{n+1} - x_n}\right)$$

is an isometry. So we use henceforth the notation $\mathbb{H}^n_R$ for both models.

The group $O_+(n, 1)$ of linear automorphisms of $\mathbb{R}^{n+1}$ which preserve the Minkowski form and send $\mathbb{H}^n_R$ onto itself is contained in the isometry group $I(\mathbb{H}^n_R)$. In this case too, it can be proved that this is the entire isometry group, but we will not need this fact now. In a similar way as in the case of the n-sphere S^n_R we can prove that $\mathbb{H}^n_R$ is homogeneous and isotropic. Let $p = (p_1, ..., p_n) \in \mathbb{H}^n_R$, so $\langle p,p\rangle = -R^2$, $p_{n+1} > 0$. and let $\{E_1,...,E_n\}$ be an orthonormal basis of $T_p\mathbb{H}^n_R$. Then, $\langle E_k, p\rangle = 0$, $1 \le k \le n$ and so

$$\{E_1,...,E_n,\frac{1}{R}p\}$$

is a basis of $\mathbb{R}^{n+1}$. If now $A \in O_+(n,1)$ is the matrix with columns E_1,..., E_n, $\frac{1}{R}p$, then $A(Re_{n+1}) = p$, which shows that $O_+(n,1)$ acts transitively on $\mathbb{H}^n_R$, and $Ae_k = E_k$, $1 \le k \le n$, which shows that $\mathbb{H}^n_R$ is isotropic, since $\{e_1,...,e_n\}$ is an orthonormal basis of $T_{Re_{n+1}}\mathbb{H}^n_R$.

Example 3.3.6. Let $n \ge 1$ and $\pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}P^n$ be the quotient map. Recall that in the canonical atlas $\{(V_j,\phi_j) : 0 \le j \le n\}$ of $\mathbb{C}P^n$ we have

$$V_j = \{[z_0,...,z_n] \in \mathbb{C}P^n : z_j \neq 0\}$$

and

$$\phi_j[z_0,...,z_n] = (\frac{z_0}{z_j},...,\frac{z_{j-1}}{z_j},\frac{z_{j+1}}{z_j},...,\frac{z_n}{z_j}).$$

The quotient map π is a submersion. To see this note first that its local representation $\phi_0 \circ \pi : \pi^{-1}(V_0) \to \mathbb{C}^n$ with respect to the smooth chart (V_0,ϕ_0) is given by the formula

$$(\phi_0 \circ \pi)(z_0,...,z_n) = (\frac{z_1}{z_0},...,\frac{z_n}{z_0}).$$

Let $z = (z_0,...,z_n) \in \pi^{-1}(V_0)$ and $v = (v_0,...,v_n) \in T_z\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ be non-zero. Then $v = \dot\gamma(0)$, where $\gamma(t) = z + tv$, and

$$(\phi_0 \circ \pi \circ \gamma)(t) = \left(\frac{z_1 + tv_1}{z_0 + tv_0},...,\frac{z_n + tv_n}{z_0 + tv_0}\right)$$

so that

$$(\phi_0 \circ \pi \circ \gamma)'(0) = \left(\frac{v_1}{z_0} - \frac{z_1v_0}{z_0^2},...,\frac{v_n}{z_0} - \frac{z_nv_0}{z_0^2}\right).$$

This implies that $v \in \text{Ker }\pi_{*z}$ if and only if $[v_0,...,v_n] = [z_0,...,z_n]$. In other words $\text{Ker }\pi_{*z} = \{\lambda z : \lambda \in \mathbb{C}\}$. Obviously, for every $(\zeta_0,...,\zeta_n) \in \mathbb{C}^n$ there exists $v = (v_0,...,v_n) \in \mathbb{C}^{n+1}$ such that

$$\zeta_j = \frac{v_j}{z_0} - \frac{z_jv_0}{z_0^2}.$$

Since the same holds for any other chart (V_j,ϕ_j) instead of (V_0,ϕ_0), this shows that π is a submersion.

The inclusion $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}\setminus\{0\}$ is an embedding and so its derivative at every point of S^{2n+1} is a linear monomorphism. For every $z \in S^{2n+1}$ we have

$$\text{Ker}(\pi|_{S^{2n+1}})_{*z} = \text{Ker}\pi_{*z} \cap T_zS^{2n+1} = \{\lambda z : \lambda \in \mathbb{C} \text{ and } \text{Re}\lambda = 0\}$$

which is a real line. On the other hand, $\pi^{-1}(\pi(z)) \cap S^{2n+1}$ is the trace of the smooth curve $\sigma : \mathbb{R} \to S^{2n+1}$ with $\sigma(t) = e^{it}z$ for which $\sigma(0) = z$ and $\dot\sigma(0) = iz$. Therefore $\text{Ker}(\pi|_{S^{2n+1}})_{*z}$ is generated by $\dot\sigma(0)$.

Let h be the usual hermitian product on $\mathbb{C}^{n+1}$. If

$$W_z = \{\eta \in T_z\mathbb{C}^{n+1} : h(\eta, z) = 0\},$$

then $\pi_{*z}|_{W_z} : W_z \to T_{[z]}\mathbb{C}P^n$ is a linear isomorphism for every $z \in \mathbb{C}^{n+1}\setminus\{0\}$. Indeed, for every $v \in T_z\mathbb{C}^{n+1}$ there are unique $\lambda \in \mathbb{C}$ and $\eta \in W_z$ such that $v = \lambda z + \eta$. Obviously,

$$\lambda = \frac{h(v,z)}{h(z,z)}, \qquad \eta = v - \frac{h(v,z)}{h(z,z)} \cdot z.$$

The restricted hermitian product on W_z can be transfered isomorphically by π_{*z} on $T_{[z]}\mathbb{C}P^n$. If now

$$g_{[z]}(v,w) = \mathrm{Re}\; h((\pi_{*z}|_{W_z})^{-1}(v), (\pi_{*z}|_{W_z})^{-1}(w))$$

for v, $w \in T_{[z]}\mathbb{C}P^n$, then g is Riemannian metric on $\mathbb{C}P^n$ called the *Fubini-Study metric*. If $z \in S^{2n+1}$, then $W_z = \{v \in T_zS^{2n+1} : \langle v, \dot\sigma(0)\rangle = 0\}$.

Each element $A \in U(n+1)$ induces a diffeomorphism $\tilde{A} : \mathbb{C}P^n \to \mathbb{C}P^n$. Moreover, $A(W_z) = W_{A(z)}$ for every $z \in \mathbb{C}^{n+1} \setminus \{0\}$ and therefore $\tilde{A}$ is an isometry of the Fubini-Study metric. In this way, $U(n+1)$ acts on $\mathbb{C}P^n$ by isometries. The action is transitive and so $\mathbb{C}P^n$ is a homogeneous Riemannian manifold with respect to the Fubibi-Study metric. Indeed, $U(n+1)$ acts transitively on S^{2n+1}, because if $z \in S^{2n+1}$, there exist $E_1, \dots E_n \in \mathbb{C}^{n+1}$ such that $\{E_1, \dots E_n, z\}$ is an h-orthonormal basis of $\mathbb{C}^{n+1}$. The matrix U with columns $E_1, \dots, E_n$, z is an element of $U(n+1)$ such that $U(e_j) = E_j$ for $1 \le j \le n$ and $U(e_{n+1}) = z$. This last equality shows that $U(n+1)$ acts transitively on $\mathbb{C}P^n$.

The isotropy group of $[e_{n+1}] = [0, \dots, 0, 1]$ consists of all $A \in U(n+1)$ such that $\lambda A(e_{n+1}) = e_{n+1}$ for some $\lambda \in S^1$. This means that

$$\lambda A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $B \in U(n)$. Since $\tilde{A} = \widetilde{\lambda A}$, this implies that the isotropy group of $[e_{n+1}]$ is $U(n)$, considered as a subgroup of $U(n+1)$ as above, and therefore $\mathbb{C}P^n$ is diffeomorphic to the homogeneous space $U(n+1)/U(n)$.

If $A \in U(n+1)$, then $\det A \in S^1$ and taking $a \in S^1$ such that $a^n = \det A$ we have $a^{-1}A \in SU(n+1)$ and $\tilde{A} = \widetilde{a^{-1}A}$. Hence $SU(n+1)$ acts also transitively on $\mathbb{C}P^n$ and $\mathbb{C}P^n$ is diffeomorphic to $SU(n+1)/U(n)$, if we identify $U(n)$ with the subgroup of $SU(n+1)$ consisting of matrices of the form

$$\begin{pmatrix} B & 0 \\ 0 & \frac{1}{\det B} \end{pmatrix}$$

for $B \in U(n)$. If $A \in SU(n+1)$ belongs to the isotropy group of $[e_{n+1}]$ and λA has the above form, then $\det B = \lambda^{n+1}$ and putting $B' = \dfrac{1}{\lambda}B$, we have now

$$A = \begin{pmatrix} B' & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

where $\det B' = \lambda$. Therefore $A \in U(n)$, as a subgroup of $SU(n+1)$.

Example 3.3.7. If (M, g) and (N, h) are two Riemannian manifolds, on the product manifold $M \times N$ there is a Riemannian metric $\langle ., . \rangle$ defined by

$$\langle v, w \rangle_p = g_{p_1}(v_1, w_1) + h_{p_2}(v_2, w_2)$$

for $v = (v_1, v_2)$, $w = (w_1, w_2) \in T_p(M \times N) = T_{p_1}M \oplus T_{p_2}N$, $p = (p_1, p_2) \in M \times N$, which is called the product Riemannian metric.

Example 3.3.8. Let M be a Riemannian manifold and let G be a subgroup of its isometry group $I(M)$ which acts properly discontinuously on M, that is every point $p \in M$ has an open neighbourhood U in M such that $g(U) \cap U = \varnothing$ for all $g \in G$, $g \neq id_M$. If the orbit space M/G is Hausdorff, it is a smooth manifold and the quotient map $\pi : M \to M/G$ is a smooth covering map, in particular a local diffeomorphism as it maps each open neighbourhood like U above diffeomorphically onto $\pi(U)$.

Let $p \in M$, $g \in G$ and $q = g(p)$. Since $\pi \circ g = \pi$, from the chain rule we have $\pi_{*q} \circ g_{*p} = \pi_{*p}$, and since g is an isometry, it follows that

$$\langle \pi_{*q}^{-1}(v), \pi_{*q}^{-1}(w) \rangle_q = \langle g_{*p}^{-1}(\pi_{*q}^{-1}(v)), g_{*p}^{-1}(\pi_{*q}^{-1}(w)) \rangle_p = \langle \pi_{*p}^{-1}(v), \pi_{*p}^{-1}(w) \rangle_p$$

for every v, $w \in T_{\pi(p)}(M/G)$. This means that there is a unique well defined Riemannian metric $\tilde{g}$ on M/G with respect to which π becomes a local isometry, as it maps each open neighbourhood U as above isometrically onto $\pi(U)$.

In the special case $M = S^n$ and $G = \{id_{S^n}, a\} \cong \mathbb{Z}_2$, where $a(x) = -x$ is the antipodal map, we obtain a Riemannian metric on the real projective n-space $\mathbb{R}P^n$ which is locally isometric to the euclidean Riemannian metric on S^n. Similarly, the group of translations of $\mathbb{R}^n$ by a vector in $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^n$ and acts properly discontinuously on $\mathbb{R}^n$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomotphic to the n-torus $T^n = S^1 \times \cdots \times S^1$, n-times. Since translations are euclidean isometries, we obtain a Riemannian metric on T^n such that the quotient map $\pi : \mathbb{R}^n \to T^n$ which is given by

$$\pi(t_1, ..., t_n) = (e^{it_1}, ..., e^{it_n})$$

becomes a local isometry. The n-torus T^n equipped with this Riemannian metric is usually called *flat n-torus*.

3.4 The Levi-Civita connection

In this section we shall prove that on a Riemannian manifold there exists a unique symmetric connection which is compatible with the Riemannian metric in the sense that parallel translation along smooth curves with respect to this connection is a linear isometry of inner product vector spaces. This result is sometimes called the Fundamental Theorem of Riemannian Geometry. Connections on a Riamannian manifold which are compatible with the Riemannian metric are characterized as follows.

Proposition 3.4.1. *Let M be a Riemannian smooth n-manifold. For a connection ∇ on M the following statements are equivalent.*
(i) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ *for every* X, Y, $Z \in \mathcal{X}(M)$.
(ii) If $I \subset \mathbb{R}$ *is an open interval and* $\gamma : I \to M$ *is a smooth curve, then*

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$$

for every V, $W \in \mathcal{X}(\gamma)$.
(iii) If a, $b \in \mathbb{R}$, $a < b$, *and* $\gamma : [a, b] \to M$ *is a smooth curve, then the parallel translation* $\tau_{b,a} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ *from* $\gamma(a)$ *to* $\gamma(b)$ *along* γ *with respect to* ∇ *is a linear isometry of inner product vector spaces.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of Lemma 5.1.4 and Proposition 3.1.7. If (ii) holds and V, W are parallel along γ then

$$\frac{d}{dt}\langle V, W\rangle = 0$$

and so $\langle V, W\rangle$ is constant on $[a, b]$. This implies (iii). Conversely, there are parallel E_1,..., $E_n \in \mathcal{X}(\gamma)$ such that $\{E_1(t_0), ..., E_n(t_0)\}$ is n orthonormal basis of $T_{\gamma(t_0)}M$ for some $t_0 \in I$. If (iii) holds, $\{E_1(t), ..., E_n(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M$ for every $t \in I$. If V, $W \in \mathcal{X}(\gamma)$, there are unique smooth functions f_k, $g_k : I \to \mathbb{R}$, $1 \leq k \leq n$, such that

$$V = \sum_{k=1}^{n} f_k E_k \quad \text{and} \quad \sum_{k=1}^{n} g_k E_k.$$

Then, $\langle V, W\rangle = f_1 g_1 + \cdots + f_n g_n$ and

$$\frac{d}{dt}\langle V, W\rangle = \sum_{k=1}^{n} f_k' g_k + \sum_{k=1}^{n} f_k g_k' = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle. \quad \square$$

Corollary 3.4.2. *Let M be a Riemannian smooth n-manifold and ∇ be a connection on M. If ∇ is compatible with the Riemannian metric, then the velocity field of each geodesic of ∇ has constant length.*

Proof. Indeed, if γ is a geodesic of ∇ and the latter is compatible with the Riemannian metric, we have

$$\frac{d}{dt}\|\dot{\gamma}\|^2 = \langle \frac{D\dot{\gamma}}{dt}, \dot{\gamma}\rangle + \langle \dot{\gamma}, \frac{D\dot{\gamma}}{dt}\rangle = 0. \quad \square$$

For every $c > 0$ the set

$$T^c M = \{(p, v) \in TM : p \in M, v \in T_p M, \|v\| = c\}$$

is a $(2n-1)$-dimensional smooth submanifold of TM, by Corollary 1.3.5, because $T^c M = f^{-1}(\frac{1}{2}c^2)$ and $\frac{1}{2}c^2$ is a regular value of the kinetic energy $f : TM \to \mathbb{R}$ defined by

$$f(p, v) = \frac{1}{2}\|v\|^2.$$

Indeed, if (U, ϕ) is a smooth chart of M and $(\pi^{-1}(U), \tilde{\phi})$ is the corresponding chart of TM, then the local representation of f is

$$(f \circ \tilde{\phi}^{-1})(x^1, ..., x^n, v^1, ..., v^n) = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(\phi^{-1}(x^1, ..., x^n)) v^i v^j$$

and differentiating

$$\frac{\partial (f \circ \tilde{\phi}^{-1})}{\partial v^i}(x^1, ..., x^n, v^1, ..., v^n) = \sum_{j=1}^{n} g_{ij}(\phi^{-1}(x^1, ..., x^n)) v^j$$

because the matrix $(g_{ij})_{1 \le i,j \le n}$ of the Riemannian metric is symmetric. Since it is invertible at every point as well,

$$\frac{\partial (f \circ \tilde{\phi}^{-1})}{\partial v^i}(x^1, ..., x^n, v^1, ..., v^n) = 0$$

for all $1 \le i \le n$ if and only if $v^1 = \cdots = v^n = 0$.

The tangent space $T_{(p,v)}T^cM$ is the $\mathrm{Ker} f_{*(p,v)}$ for every $(p, v) \in T^cM$. Now γ is a geodesic of a connection ∇ on M if and only if $(\gamma, \dot{\gamma})$ is an integral curve of the geodesic vector field G of ∇. If ∇ is compatible with the Riemannian metric, Corollary 3.4.2 says that $\|\dot{\gamma}\|$ takes on a constant value c. If γ is not constant, $c > 0$ and $(\gamma, \dot{\gamma})$ lies entirely on the constant kinetic energy level set T^cM. Thus, the geodesic vector field is tangent to constant kinetic energy level sets. In particular, T^1M is called the *unit tangent bundle* of M and from Lemma 3.2.2 every geodesic is a reparametrization of a geodesic whose velocities lie in T^1M.

Theorem 3.4.3. *On every Riemannian smooth n-manifold M there exists a unique symmetric connection which is compatible with the Riemannian metric.*

Proof. We shall prove first the uniqueness by finding an explicit formula for such a connection ∇. For every X, Y, $Z \in \mathcal{X}(M)$ we have

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle$$

since ∇ is symmetric and compatible with the Riemannian metric. From these we get

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle + \langle Z, [Y, X]\rangle - \langle X, [Z, Y]\rangle.$$

This equality uniquely determines ∇ because the Riemannian metric on each tangent space is a non-degenerate symmetric bilinear form.

The existence of ∇ will be proved locally by providing the Christoffel symbols from which it is determined. Due to uniqueness the local definitions will coincide on the overlapping domains. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and let

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad 1 \leq i, j \leq n.$$

By the above formula, a symmetric connection ∇ which is compatible with the Riemannian metric must satisfy

$$\sum_{k=1}^{n} \Gamma_{ij}^k g_{km} = \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m} \right\rangle = \frac{1}{2} \left[\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right]$$

on U, for every $1 \leq i, j, m \leq n$. The Christoffel symbols are uniquely determined from the above linear systems, because the Riemannian metric on each tangent space is a non-degenerate symmetric bilinear form and therefore the symmetric matrix $(g_{ij})_{1 \leq i,j \leq n}$ is invertible at each point of U. If we denote by g^{ij} the entries of the inverse matrix of the Riemannian metric $(g_{ij})_{1 \leq i,j \leq n}^{-1}$, the the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \quad 1 \leq i, j, k \leq n.$$

It remains to show that the connection on ∇ on U whose Christoffel symbols are the solutions of the above linear systems is symmetric and compatible with Riemannian metric. The first is obvious, because the matrix $(g_{ij})_{1 \leq i,j \leq n}$ is symmetric and so the (i, j) linear system is the same as the (j, i) one. To prove compatibility, we let

$$X = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}, \quad Y = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x^k}, \quad Z = \sum_{k=1}^{n} Z^k \frac{\partial}{\partial x^k},$$

and then we have

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

$$= \sum_{k,l=1}^{n} \left[g_{kl} \big(Z^l X(Y^k) + Y^k X(Z^l) \big) + \sum_{i,j=1}^{n} X^i Y^j \Gamma_{ij}^k g_{kl} Z^l + \sum_{i,j=1}^{n} X^i Z^j \Gamma_{ij}^l g_{kl} Y^k \right].$$

Since the matrix $(g_{ij})_{1 \leq i,j \leq n}$ is symmetric, substituting we compute

$$\sum_{j,k,l=1}^{n} (Y^j Z^l \Gamma_{ij}^k g_{kl} + Z^j Y^k \Gamma_{ij}^l g_{kl}) = \sum_{j,k,l=1}^{n} Y^j Z^l \Gamma_{ij}^k g_{kl} + \sum_{j,k,l=1}^{n} Y^k Z^j \Gamma_{ij}^k g_{kl}$$

$$= \sum_{j,l=1}^{n} (Z^l Y^j + Y^l Z^j) \left(\sum_{k=1}^{n} \Gamma_{ij}^k g_{kl} \right)$$

$$= \frac{1}{2} \sum_{j.l=1}^{n} Z^l Y^j \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) + \frac{1}{2} \sum_{j,l=1}^{n} Z^j Y^l \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$$= \sum_{j,l=1}^{n} Z^l Y^j \frac{\partial g_{jl}}{\partial x^i}.$$

Therefore,

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \sum_{k,l=1}^{n} g_{kl}\big(Z^l X(Y^k) + Y^k X(Z^l)\big) + \sum_{i,j,l=1}^{n} X^i Z^l Y^j \frac{\partial g_{jl}}{\partial x^i}$$

$$= X\left(\sum_{k,l=1}^{n} g_{kl} Y^k Z^l\right) = X\langle Y, Z\rangle. \quad \square$$

The unique connection of a Riemannian manifold M which is symmetric and compatible with the Riemannian metric is called the *Levi-Civita connection* of M. The geodesics of the Levi-Civita sonnection of M will be simply called geodesics of M. It easy to see that if ∇ is a connection on M and $f : M \to M$ is a smooth diffeomorphism, then the formula

$$\overline{\nabla}_X Y = f_*^{-1}\big(\nabla_{f_*X} f_*Y\big)$$

for X, $Y \in \mathcal{X}(M)$ defines a new connection on M. If ∇ is symmetric, so is $\overline{\nabla}$. If ∇ is compatible with the Riemannian metric of M and f is an isometry, then $\overline{\nabla}$ is also compatible with the Riemannian metric. By uniqueness, if ∇ is the Levi-Civita connection of M, it is preserved by isometries, that is

$$f_*(\nabla_X Y) = \nabla_{f_*X} f_*Y$$

for every X, $Y \in \mathcal{X}(M)$ and $f \in I(M)$. In particular, every isometry sends geodesics to geodesics. This observation is crucial for the determination of the geodesics of a Riemennian manifold with sufficiently large isometry group.

Example 3.4.4. The Levi-Civita connection of the euclidean n-space $\mathbb{R}^n$ is the euclidean connection with vanishing Christoffel symbols. If $M \subset \mathbb{R}^n$ is a hypersurface, the induced euclidean connection on M defined in Example 5.1.5 is the Levi-Civita connection of M for the restricted euclidean Riemannian metric, as it is easily seen.

Example 3.4.5. We shall describe the geodesics on a n-sphere S^n_R of radius $R > 0$. Let $\gamma : I \to S^n_R$ be the geodesic satisfying the initial conditions $\gamma(0) = Re_{n+1}$ and $\dot\gamma(0) = e_1$, defined on some open interval $I \subset \mathbb{R}$ containing zero. Suppose that $\gamma(t) = (\gamma^1(t), ..., \gamma^{n+1}(t))$ for $t \in I$. For $2 \le j \le n$, the reflection $a_j : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with

$$a_j(x^1, ..., x^{n+1}) = (x^1, ..., x^{j-1}, -x^j, x^{j+1}, ..., x^{n+1})$$

is an isometry of S^n_R such that $a_j(Re_{n+1}) = Re_{n+1}$ and

$$(a_j)_{*Re_{n+1}}(\dot\gamma(0)) = a_j(e_1) = e_1 = \dot\gamma(0).$$

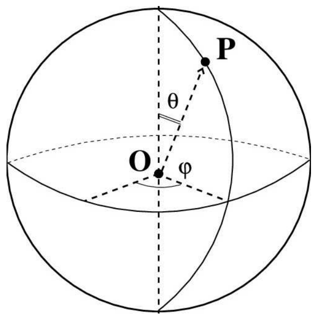
From the invariance of geodesics under isometries and uniqueness follows now that $a_j \circ \gamma = \gamma$ and hence $\gamma^j(y) = -\gamma^j(t)$, that is $\gamma^j(t) = 0$ for every $t \in I$ and $2 \le j \le n$. This means that $\gamma(I)$ is an arc on the great circle which is the intersection of S^n_R with the plane generated by $\{e_1, e_{n+1}\}$. Since S^n_R is homogeneous and isotropic, again the existence and uniqueness of geodesics implies that all geodesics are great circles. In particular, the geodesic vector field on TS^n_R is complete.

As an illustration we shall write down the system of differential equations of geodesics on S^2 with respect to the spherical coordinates (θ, ϕ), where the point $(x, y, z) \in S^2$ is written

$$x = \cos\phi \cdot \sin\theta, \quad y = \sin\phi \cdot \sin\theta, \quad z = \cos\theta.$$

The basic vector fields are

$$\frac{\partial}{\partial\theta} = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}, \quad \frac{\partial}{\partial\phi} = \begin{pmatrix} -\sin\phi\sin\theta \\ \cos\phi\sin\theta \\ 0 \end{pmatrix}$$



and so the matrix of the Riemannian metric is

$$(g_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

It follows that almost all Christoffel symbols vanish except

$$\Gamma^1_{22} = -\frac{1}{2}\sin 2\theta, \quad \Gamma^2_{12} = \cot\theta.$$

Therefore, the system of differential equations of geodesics in spherical coordinates is

$$\theta'' - \frac{1}{2}\sin 2\theta \cdot (\phi')^2 = 0,$$
$$\phi'' + 2\cot\theta \cdot \phi'\theta' = 0.$$

The meridians are obvious solutions of this system.

Example 3.4.6. The matrix of the hyperbolic Riemannian metric on the upper half plane $\mathbb{H}^2$ is

$$(g_{ij})_{1\le i,j\le 2} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

and so the Christoffel symbols are

$$\Gamma^1_{12} = -\frac{1}{y}, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^2_{22} = -\frac{1}{y},$$

and the rest are zero, at the point $z = x + iy \in \mathbb{H}^2$. So the system of differential equations of geodesics is

$$x'' - \frac{2}{y}x'y' = 0,$$

$$y'' - \frac{1}{y}[(x')^2 - (y')^2] = 0.$$

An obvious solution is $\ell(t) = ie^t$, $t \in \mathbb{R}$, whose image is the imaginary semi-axis. Since $\mathbb{H}^2$ is homogeneous and isotropic with respect to the subgroup $PSL(2, \mathbb{R})$ of its isometry group which acts by Möbius transformations, the geodesics are euclidean semi-circles with center on $\partial\mathbb{H}^2$ (the boundary taken in the Riemann sphere $\hat{\mathbb{C}}$), because the Möbius transformations send circles onto circles on $\hat{\mathbb{C}}$ and preserve angles.

Let M be a Riemannian smooth n-manifold. On M we shall always consider the Levi-Civita connection and all the related notions associated with it such as parallel translation, geodesics and exponential map. Let $p \in M$ and U be a normal neighbourhood of p, that is there exists an open neighbourhood V of $0 \in T_pM$ in T_pM such that $\exp : V \to U$ is a smooth diffeomorphism. We denote by $B_p(0, \epsilon)$ the open ball in T_pM of radius $\epsilon > 0$ and center $0 \in T_pM$. There exists $\epsilon_0 > 0$ such that $\overline{B_p(0, \epsilon_0)} \subset V$. The set $\exp_p(\overline{B_p(0, \epsilon)})$ will be called the *closed geodesic ball* of radius $0 < \epsilon \leq \epsilon_0$ and center p and its interior $\exp(B_p().\epsilon))$ *open geodesic ball.* Its boundary $\exp_p(\partial B_p(0, \epsilon))$ will be called *geodesic sphere.* Fixing an orthonormal basis $\{E_1, ..., E_n\}$ of T_pM we have a linear isometry of inner product spaces $\sigma : \mathbb{R}^n \to T_pM$ with $\sigma(e_k) = E_k$, $1 \leq k \leq n$, and a normal chart (U, ϕ) where $\phi = \sigma^{-1} \circ (\exp_p|_V)^{-1}$. Let $\phi = (x^1, ..., x^n)$ and

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad 1 \leq i, j \leq n.$$

Then $g_{ij}(p) = \delta_{ij}$, $1 \leq i, j \leq n$, Since the Levi-Civita connection is symmetric, the Christoffel symbols with respect to this normal chart vanish at p. From the formula in the proof of Theorem 3.4.3 giving the Christoffel symbols we compute

$$\sum_{k=1}^{n} \Gamma_{ij}^k g_{kl} + \sum_{k=1}^{n} \Gamma_{il}^k g_{kj} = \frac{\partial g_{jl}}{\partial x^i}$$

and in particular $\dfrac{\partial g_{jl}}{\partial x^i}(p) = 0$ for every $1 \leq i, j, l \leq n$.

In order a normal neighbourhood of p, in particular a geodesic ball, to be useful for local calculations near p, it is desirable to be a normal neighbourhood of nearby points also. An open set $W \subset M$ will be called *uniformly normal* if it is a normal neighbourhood of all its points. More precisely, W is uniformly normal if there exists some $\delta > 0$ such that $W \subset \exp_p(B_p(0, \delta))$ and $\exp_p : B_p(0, \delta) \to \exp_p(B_p(0, \delta))$ is a smooth diffeomorphism onto the open set $\exp_p(B_p(0, \delta)) \subset M$ for every $p \in W$. In order to prove the existence of uniformly normal neighbourhoods we shall need the following technical remark which is a parametrized version of the equivalence of norms in finite dimensional real vector spaces.

Lemma 3.4.7. *If M is a Riemannian smooth n-manifold and $p \in M$, for every open neighbourhood $A \subset TM$ of $(p, 0)$ there exists an open neighbourhood U of p in M and some $\delta > 0$ such that*

$$U_\delta = \{(q, v) \in TM : q \in U, v \in B_q(0, \delta)\} \subset A.$$

Proof. Let (W, ψ) be a smooth chart of M with $p \in W$ and $\psi(p) = 0$. Let $\psi = (x^1, ..., x^n)$. We denote by r the euclidean norm on $\mathbb{R}^n$. If $(\pi^{-1}(W), \tilde{\psi})$ is the corresponding smooth chart of TM, where $\pi : TM \to M$ is the tangent bundle projection, we have $\tilde{\psi}(p, 0) = 0$ and we may assume that $A \subset \pi^{-1}(W)$. Since $\tilde{\psi}(A) \subset \mathbb{R}^n \times \mathbb{R}^n$ is open, there exists $\epsilon > 0$ such that $B(0, 2\epsilon) \times B(0, 2\epsilon) \subset \tilde{\psi}(A)$. The set

$$K = \big\{ \big(q, \sum_{k=1}^{n} v_k \Big(\frac{\partial}{\partial x^k}\Big)_q \big) \in \pi^{-1}(W) : r(\psi(q)) \le \epsilon, \quad \sum_{k=1}^{n} v_k^2 = \epsilon^2 \big\}$$

is compact and so there exist $0 < \delta \le c$ such that

$$0 < \delta^2 \le \sum_{i,j=1}^{n} g_{ij}(q) v_i v_j \le c^2$$

for $\big(q, \sum_{k=1}^{n} v_k \Big(\frac{\partial}{\partial x^k}\Big)_q \big) \in K$. If now $r(\psi(q)) \le \epsilon$, then

$$\big(q, \frac{\epsilon}{(\sum_{k=1}^{n} v^2)^{1/2}} \cdot \sum_{k=1}^{n} v_k \Big(\frac{\partial}{\partial x^k}\Big)_q \big) \in K$$

and thus

$$\frac{\delta}{\epsilon}\Big(\sum_{k=1}^{n} v_k^2\Big)^{1/2} \le \Big\| \sum_{k=1}^{n} v_k \Big(\frac{\partial}{\partial x^k}\Big)_q \Big\| \le \frac{c}{\epsilon}\Big(\sum_{k=1}^{n} v_k^2\Big)^{1/2}$$

for every v_1,..., $v_n \in \mathbb{R}$. If we take $U = \psi^{-1}(B(0, \epsilon))$, we have

$$U_\delta \subset \tilde{\psi}^{-1}(B(0, \epsilon) \times B(0, \epsilon)) \subset A. \quad \square$$

Proposition 3.4.8. *If M is a Riemannian smooth n-manifold and $p \in M$, then every open neighbourhood of p contains a uniformly normal open neighbourhood of p.*

Proof. Let $E \subset TM$ be the domain of definition of the exponential map and let $F : E \to M \times M$ be the smooth map

$$F(p, v) = (p, \exp_p(v)).$$

For every $p \in M$, the derivative $F_{*(p,0)}$ is a linear isomorphism and from the Inverse Map Theorem there exists an open neighbourhood $A \subset E \subset TM$ of $(p, 0)$ such that $F(A) \subset M \times M$ is open and $F|_A : A \to F(A)$ is a smooth diffeomorphism. From the preceding Lemma 3.4.7 there exists an open neighbourhood U of p and some $\delta > 0$ such that $U_\delta \subset A$. Since $F(p, 0) = (p.p)$, there exists an open neighbourhood

$W \subset U$ of p such that $W \times W \subset F(U_\delta)$. We shall show that W uniformly normal. We observe first that $\exp_q$ is defined on $B_q(0,\delta) \subset T_qM$ for all $q \in W$. Moreover, $(\exp_q|_{B_q(0,\delta)})^{-1} = (F|_{\{0\}\times B_q(0,\delta)})^{-1}$ is smooth for $q \in W$. Finally, if $(q,y) \in W \times W$, there exists $v \in B_q(0,\delta)$ such that $(q,y) = F(q,v)$, that is $y = \exp_q(v)$. This shows that $W \subset \exp_q(B_q(0,\delta))$ for every $q \in W$. □

Note that if U is a (closed or open) geodesic ball with center $p \in M$, for every $q \in U$ there exists a unique geodesic path in U from p to q, but if p, q are two points in a uniformly normal open set W, there exists a geodesic path from p to q, which however may not lie entirely in W.

3.5 The Riemannian distance

On a Riemannian manifold M it is possible to define the length of curves as follows. Let a, $b \in \mathbb{R}$, $a < b$, and $\gamma : [a,b] \to M$ be a piecewise smooth parametrized curve. The non-negative real number

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\| dt$$

is defined to be the *length* of γ with respect to the Riemennian metric. By the change of variables formula, it is invariant by piecewise smooth reparametrizations.

If $\gamma : I \to M$ is a smooth parametrized curve defined on an open interval $I \subset \mathbb{R}$ such that $\dot\gamma(t) \neq 0$ for every $t \in I$, then taking any $t_0 \in I$ and putting

$$h(t) = \int_{t_0}^t \|\dot\gamma(s)\| ds$$

the smooth function $h : I \to \mathbb{R}$ is strictly increasing and maps I diffeomorphically onto an open interval $h(I) \subset \mathbb{R}$. The smooth parametrized curve

$$\sigma = \gamma \circ h^{-1} : h(I) \to M$$

is a reparametriztion of γ such that $\|\dot\sigma\| = 1$.

A smooth parametrized curve γ with $\|\dot\gamma\| = 1$ is said to be parametrized by arclength or unit speed. By Corollary 3.4.2, every non-constant geodesic is parametrized proportionally to arclength and from Lemma 3.2.2 every such geodesic can be reparametrized to a unit speed geodesic.

If M is connected, for every p, $q \in M$ the non-negative real number

$$d(p,q) = \inf\{L(\gamma) | \gamma : [a,b] \to M \quad \text{is a piecewise smooth parametrized curve}$$
$$\text{with } \gamma(a) = p \text{ and } \gamma(b) = q \text{ for some } a,\ b \in \mathbb{R},\ a < b\}$$

is called the *(Riemannian) distance* of p and q. The function $d : M \times M \to \mathbb{R}$ has the following obvious properties:
(i) $d(p,q) \geq 0$ and $d(p,p) = 0$,
(ii) $d(p,q) = d(q,p)$ and
(ii) $d(p,q) \leq d(p,z) + d(z,q)$

for every p, q, $z \in M$. In other words, d is a pseudo-distance on M. It can be proved directly that the topology defined by d coincides with the topology of M and hence d is actually a distance. However, we shall derive this from considerations showing the strong connection of d with geodesics, at least locally. We shall need a couple of lemmas, which are of independent interest.

Lemma 3.5.1. *let M be a smooth n-manifold endowed with a symmetric connection ∇ and let $A \subset \mathbb{R}^2$ be an open set. If $\sigma : A \to M$ is a smooth map then*

$$\frac{D}{dt}\left(\frac{\partial \sigma}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial \sigma}{\partial t}\right).$$

Proof. It suffices to prove the formula in case there is a smooth chart $U, \phi)$ of M such that $\sigma(A) \subset U$. If $\phi = (x^1, ..., x^n)$ and $\phi \circ \sigma = (\sigma_1, ..., \sigma_n)$, we have

$$\frac{\partial \sigma}{\partial s} = \sum_{k=1}^{n} \frac{\partial \sigma_k}{\partial s} \cdot \frac{\partial}{\partial x^k}$$

and

$$\frac{D}{dt}\left(\frac{\partial \sigma}{\partial s}\right) = \sum_{k=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial \sigma_k}{\partial s}\right) + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{\partial \sigma_i}{\partial t} \cdot \frac{\partial \sigma_j}{\partial s}\right] \frac{\partial}{\partial x^k}$$

and similarly

$$\frac{D}{ds}\left(\frac{\partial \sigma}{\partial t}\right) = \sum_{k=1}^{n} \left[\frac{d}{ds}\left(\frac{\partial \sigma_k}{\partial t}\right) + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{\partial \sigma_i}{\partial s} \cdot \frac{\partial \sigma_j}{\partial t}\right] \frac{\partial}{\partial x^k}.$$

Since ∇ is symmetric, $\Gamma_{ij}^k = \Gamma_{ji}^k$, $1 \leq i, j, k \leq n$, and the result follows from Schwartz's theorem. $\Box$.

The next lemma is due to C.F. Gauss.

Lemma 3.5.2. *Let M be a Riemannian smooth n-manifold, $p \in M$ and let $V = \exp_p(B_p(0, \epsilon))$ be an open geodesic ball of radius $\epsilon > 0$ with center p. Then every geodesic emanating from p intersects orthogonally the geodesic spheres $\exp_p(\partial B_p(0, \delta))$, $0 < \delta < \epsilon$.*

Proof. Let $I \subset \mathbb{R}$ be an open interval and let $u : I \to T_pM$ be a smooth curve with $\|u(t)\| = 1$ for every $t \in I$. If $\sigma : I \times (-\epsilon, \epsilon) \to M$ is the smooth map

$$\sigma(t, s) = \exp_p(su(t)),$$

it suffices to prove that $\left\langle \frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s} \right\rangle = 0.$

We compute

$$\frac{\partial}{\partial s}\left\langle \frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s} \right\rangle = \left\langle \frac{D}{ds}\left(\frac{\partial \sigma}{\partial t}\right), \frac{\partial \sigma}{\partial s} \right\rangle + \left\langle \frac{\partial \sigma}{\partial t}, \frac{D}{ds}\left(\frac{\partial \sigma}{\partial s}\right) \right\rangle = \left\langle \frac{D}{dt}\left(\frac{\partial \sigma}{\partial s}\right), \frac{\partial \sigma}{\partial s} \right\rangle + 0$$

by Lemma 5.5.1 and since $\sigma(t,.) : (-\epsilon, \epsilon) \to M$ is a geodesic for every $t \in I$. For the same reason,

$$\left\| \frac{\partial \sigma}{\partial s} \right\|^2 = 1$$

by Corollary 3.4.2, and differentiating

$$2\left\langle \frac{D}{dt}\left(\frac{\partial \sigma}{\partial s}\right), \frac{\partial \sigma}{\partial s} \right\rangle = 0.$$

Thus,

$$\frac{\partial}{\partial s}\left\langle \frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s} \right\rangle = 0$$

and $\left\langle \frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s} \right\rangle$ is independent of s. However $\sigma(t,0) = p$ for all $t \in I$ and so $\frac{\partial \sigma}{\partial t}(.,0) = 0$. Therefore,

$$\left\langle \frac{\partial \sigma}{\partial t}(t,s), \frac{\partial \sigma}{\partial s}(t,s) \right\rangle = \left\langle \frac{\partial \sigma}{\partial t}(t,0), \frac{\partial \sigma}{\partial s}(t,0) \right\rangle = 0. \quad \square$$

As in the situation of the preceding Lemma 3.5.2, let M be a Riemannian smooth n-manifold, $p \in M$ and $V = \exp_p(B_p(0,\epsilon))$ be an open geodesic ball of radius $\epsilon > 0$ with center p. A piecewise smooth parametrized curve $\gamma : [a,b] \to V \setminus \{p\}$, where a, $b \in \mathbb{R}$, $a < b$, is a the form

$$\gamma(t) = \exp_p(r(t)u(t))$$

where $r : [a,b] \to (0,\epsilon)$ is a unique piecewise smooth function and $u : [a,b] \to T_pM$ is a unique piecewise smooth parametrised curve with $\|u(t)\| = 1$ for $t \in [a,b]$. Using the notation of the proof of Lemma 3.5.2 we have $\gamma(t) = \sigma(t, r(t))$ and

$$\dot{\gamma}(t) = \frac{\partial \sigma}{\partial t} + r'(t)\frac{\partial \sigma}{\partial s}.$$

From Lemma 3.5.2 we have

$$\|\dot{\gamma}(t)\|^2 = \left\| \frac{\partial \sigma}{\partial t} \right\|^2 + (r'(t))^2 \left\| \frac{\partial \sigma}{\partial s} \right\|^2 \geq (r'(t))^2$$

and the equality holds if and only if u is constant. This implies that

$$L(\gamma) \geq \int_a^b |r'(t)| dt \geq \left| \int_a^b r'(t) dt \right| = |r(b) - r(a)|$$

and the equality holds if and only if u is constant and r is monotone.

Proposition 3.5.3. *Let M be a Riemannian smooth n-manifold, $p \in M$ and let $V = \exp_p(B_p(0,\epsilon))$ be an open geodesic ball of radius $\epsilon > 0$ with center p. Let $\gamma : [0,\ell] \to V$ be a geodesic from $\gamma(0) = p$ to a point $q = \gamma(\ell) \in V$. If a, $b \in \mathbb{R}$, $a < b$, and $\sigma : [a,b] \to M$ is any piecewise smooth curve from $\sigma(a) = p$ to $\sigma(b) = q$,* *then* $L(\gamma) \leq L(\sigma)$. *Moreover, if* $L(\gamma) = L(\sigma)$, *then* $\sigma([a, b]) = \gamma([0, \ell])$.

Proof. We may assume that γ is parametrized by arclength, so that $\ell = L(\gamma)$ and γ is given by $\gamma(t) = \exp_p(tv)$, where $v = \dot{\gamma}(0)$ and $\|v\| = 1$. Obviously, $\ell < \epsilon$. We shall prove first that $L(\sigma) \geq \ell$. Let $0 < \delta < \ell$. By continuity and connectedness, there exist $a < c < d \leq b$ such that $\sigma(c) \in \exp_p(\partial B_p(0, \delta))$, $\sigma(d) \in \exp_p(\partial B_p(0, \ell))$ and $\sigma((c, d)) \subset \exp_p(B_p(0, \ell)) \setminus \exp_p(\overline{B_p(0, \delta)})$. Then,

$$L(\sigma) \geq L(\sigma|_{[c,d]}) \geq \ell - \delta$$

from the above considerations and letting δ go to zero this implies that $L(\sigma) \geq \ell$. This proves the first part.

Suppose now that $L(\sigma) = \ell$. Applying what we have already proved to $\sigma|_{[a,c]}$ we have $L(\sigma|_{[a,c]}) \geq \delta$ and therefore

$$L(\sigma|_{[c,d]}) \leq L(\sigma|_{[c,d]}) + L(\sigma|_{[d,b]}) = \ell - L(\sigma|_{[a,c]}) \leq \ell - \delta.$$

Hence $L(\sigma|_{[c,d]}) = \ell - \delta$ and from the above the trace $\sigma([c, d])$ is the same as the trace of a geodesic path $\exp_p(tv)$, $\delta \leq t \leq \ell$, for some $v \in T_pM$ with $\|v\| = 1$. Letting again δ go to zero we get a geodesic $\exp_p(tv)$, $0 \leq t \leq \ell$ whose trace is the same as $\sigma(|_{[a,d]}$. Thus, necessarily $L(\sigma|_{[d,b]}) = 0$ and $\gamma(l) = q = \exp_p(lv)$. It follows that $\gamma(t) = \exp_p(tv)$ for all $0 \leq t \leq \ell$. $\square$

Corollary 3.5.4. *Let* M *be a Riemannian smooth* n*-manifold with Riemannian distance* d. *For every* $p \in M$ *there exists* $\epsilon > 0$ *such hat*

$$\exp_p(B_p(0, \delta)) = \{q \in M : d(p, q) < \delta\}$$

for every $0 < \delta < \epsilon$.

Proof. By Proposition 3.2.4, there exists $\epsilon > 0$ such that $\exp_p$ maps $B_p(0, \epsilon) \subset T_pM$ diffeomorphocally onto the open neighbourhood $\exp_p(B_p(0, \epsilon))$ of p. Obviously then

$$\exp_p(B_p(0, \delta)) \subset \{q \in M : d(p, q) < \delta\}$$

for every $0 < \delta < \epsilon$, since each geodesic path in the open geodesic ball $\exp_p(B_p(0, \delta))$ emanating from p has length $< \delta$.

Conversely, if $q \notin \exp_p(B_p(0, \delta))$, then every piecewise smooth parametrized curve σ from p to q intersects the geodesic sphere $\exp_p(\partial B_p(0, \rho))$ for all $0 < \rho < \delta$, and so $L(\sigma) \geq \rho$, by Proposition 5.5.3. Consequently, $L(\sigma) \geq \delta$. This shows that $d(p, q) \geq \delta$. $\square$

Corollary 3.5.5. *On a Riemannian smooth manifold* M *the Riamannian distance* d *induces the original manifold topology and the pair* (M, d) *is a metric space.* $\square$

In the sequel we shall denote by $B(p, \delta)$ the open d-ball in M with radius δ and center p.. According to Proposition 3.5.3, for every $p \in M$ there exists some $\epsilon > 0$ such that $B(p, \delta)$ is the geodesic open ball of radius δ and center p and for each

$q \in B(p, \delta)$ the distance $d(p, q)$ is the length of the unique geodesic path in $B(p, \epsilon)$ from p to q for all $0 < \delta < \epsilon$. It follows from this that geodesics locally minimize length.

Proposition 3.5.6. *Let M be a Riamannian smooth manifold and $\gamma : [a, b] \to M$, where a, $b \in \mathbb{R}$, $a < b$, be a piecewise smooth parametrized curve from $\gamma(a) = p$ to $\gamma(b) = q$. If $L(\gamma) = d(p, q)$, then $\gamma([a, b])$ is the trace of a geodesic path. In particular, if γ is parametrized by arclength, it is a geodesic path and in particular smooth.*

Proof. Since being a geodesic is a local property, it suffices to show that the trace of γ is locally the same as that of a geodesic. Let $a < t_0 < b$. By Proposition 3.4.8, there exists a uniformly normal neighbourhood W of $\gamma(t_0)$. So there exists $\epsilon > 0$ such that $W \subset \exp_y(B_y(0, \epsilon))$ and $\exp_y|_{B_y(0,\epsilon)}$ is a diffeomorphism for every $y \in W$. There exists $\eta > 0$ such that $\gamma([[t_0 - \eta, t_0 + \eta]) \subset \exp_{\gamma(t_0)}(B_{\gamma(t_0)}(0, \epsilon))$. Our assumption implies that $L(\gamma|_{[t_0-\eta,t_0+\eta]}) = d(\gamma(t_0 - \eta), \gamma(t_0 + \eta))$ and thus, by Proposition 3.5.3, $\gamma([t_0 - \eta, t_0 + \eta])$ is the trace of a geodesic path. □

Definition 3.5.7. A geodesic path $\gamma : [a, b] \to M$, a, $b \in \mathbb{R}$, $a < b$, on a Riemennian smooth manifold M with Riemannian distance d is called *minimizing* if $L(\gamma) = d(\gamma(a), \gamma(b))$.

Note that if γ is a minimizing geodesic path as in the above definition, then $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s))$, that is $\gamma|_{[t,s]}$ is minimizing, for every $a \leq t < s \leq b$. According to Proposition 3.5.3, every geodesic of a Riemannian manifold is locally minimizing. However, the example of the sphere shows that on a Riemennian manifold there may exist non-minimizing geodesic paths. The question now arises whether any two points on a connected Riemennian manifold can be joined by a minimizing geodesic path. This is answered by the following theorem which is due to H, Hopf and his student W. Rinow. The proof we present here is due G. de Rham.

Theorem 3.5.8. *Lei M be a connected Riemannian smooth n-manifold. If the geodesic vector field of M is complete, then any two given points of M can be joined by a minimizing geodesic path.*

Proof. Let p, $q \in M$ and $r = d(p, q) > 0$. According to Corollary 3.5.4, there exists $0 < \epsilon < r$ such that $\exp_p(B_p(0, \delta)) = B(p, \delta)$ is a normal neighbourhood of p for every $0 < \delta < \epsilon$. Fixing such a δ, by compactness, there exists $p_0 \in \exp_p(\partial B_p(0, \delta))$ such that

$$d(p_0, q) = \inf\{d(z, q) : z \in \exp_p(\partial B_p(0, \delta))\}.$$

Then, $p_0 = \exp_p(\delta v)$ for some $v \in T_pM$ with $\|v\| = 1$ and the unit speed geodesic

$$\gamma(t) = \exp_p(tv)$$

is defined on the entire real line $\mathbb{R}$, because we assume the the geodesic vector field is complete. It suffices to prove now that $d(\gamma(t), q) = r - t$ for every $\delta \leq t \leq r$, because then for $t = r$ we will get $\gamma(r) = q$ and $\gamma|_{[0,r]}$ will be minimizing.

Firstly, we have

$$r = d(p,q) \leq d(p,\gamma(t)) + d(\gamma(t),q) \leq t + d(\gamma(t),q)$$

and hence $d(\gamma(t),q) \geq r - t$ for every $0 \leq t \leq r$.

On the other hand we have

$$r \geq \inf\{d(p,z) + d(z,q) : z \in \exp_p(\partial B_p(0,\delta))\} = \delta + d(p_0,q)$$

and so $d(p_0,q) \leq r - \delta$. Hence $d(\gamma(\delta),q) = d(p_0,q) = r - \delta$. Let

$$T = \sup\{t \in [\delta,r] : d(\gamma(t),q) = r - t\}.$$

By continuity, $d(\gamma(T),q) = r - T$. Moreover, $d(\gamma(t),q) = r - t$ for all $\delta \leq t \leq T$, because

$$r - t \leq d(\gamma(t),q) \leq d(\gamma(t),\gamma(T)) + d(\gamma(T),q) \leq T - t + r - t = r - t.$$

It remains to prove that $T = r$. Suppose that $T < r$. We apply what we have already proved for p to $\gamma(T)$. Thus, there are some $\eta > 0$ and $p_0' \in \exp_{\gamma(T)}(\partial B_{\gamma(T)}(0,\eta))$ with

$$d(p_0',q) = \inf\{d(z,q) : z \in \exp_{\gamma(T)}(\partial B_{\gamma(T)}(0,\eta))\}$$

and $d(p_0',q) = d(\gamma(T),q) - \eta = r - T - \eta$. Therefore,

$$d(p,p_0') \geq d(p,q) - d(p_0',q) = r - (r - T - \eta) = T + \eta.$$

However the piecewise smooth parametrized curve, which is the concatenation of $\gamma|_{0,T]}$ and the unique geodesic in $\exp_{\gamma(T)}(\overline{B_{\gamma(T)}(0,\eta)})$ from $\gamma(T)$ to p_0' has length $T + \eta$, and from Proposition 5.5.6 its trace must be the trace of a geodesic path. Since part of this path coincides with $\gamma|_{0,T]}$, it follows from uniqueness of geodesics that this geodesic path is $\gamma|_{[0,T+\eta]}$. Hence $p_0' = \gamma(T + \eta)$ and $d(\gamma(T+\eta),q) = r - (T + \eta)$. This contradicts the definition of T. □

A topological characterization of the completeness of the geodesic vector field is given by the following theorem also due to H. Hopf and W. Rinow.

Theorem 3.5.9. *For a connected Riemannian smooth manifold M with Riemannian distance d the following statements are equivalent:*
(i) The geodesic vector field of M is complete.
(ii) The metric space (M,d) is complete.

Proof. Suppose that the geodesic vector field of M is complete. In order to prove that (M,d) is a complete metric space, it suffices to show that every d-bounded set $C \subset M$ is contained in a compact set. Let $p \in M$. Since C is bounded, there exists $c > 0$ such that $d(p,q) < c$ for every $q \in C$. From Theorem 3.5.8, there exists some $v \in T_pM$ such that $q = \exp_p(v)$ and $\|v\| = d(p,q)$. This shows that $C \subset \exp_p(\overline{B_p(0,c)})$, and $\exp_p(\overline{B_p(0,c)})$ is compact, because $\exp_p$ is continuous.

Conversely, suppose that there exists a geodesic parametrized by arclength γ whose maximal interval of definition is an open interval (a,b) for some $a < b < +\infty$.

Then, $d(\gamma(t), \gamma(s)) \leq |t - s|$ for every t, $s \in (a, b)$. If (M, d) is complete, then $p = \lim_{t \to b^-} \gamma(t)$ exists in M. From Proposition 3.4.8 there exists a uniformly normal open neighbourhood W of p, for which there exists some $\delta > 0$ such that $W \subset \exp_q(B_q(0, \delta))$ for every $q \in W$. There exists $b - \delta < T < b$ such that $\gamma(T) \in W$ and then the geodesic form $\gamma(T)$ with initial velocity $\dot{\gamma}(T)$ is defined at least on the interval $[0, \delta)$. By uniqueness of geodesics, this implies that γ is defined at least on $(a, T+\delta)$ and since $T+\delta > b$ this contradicts our assumption the $b < +\infty$. $\square$

If any of the two equivalent conditions of the preceding theorem is satisfied, we shall call the Riemannian manifold M *complete*. As the proof shows, the following also holds.

Corollary 3.5.10. *A connected Riemannian smooth manifold M is complete if and only if there exists a point $p \in M$ such that $\exp_p$ is defined on the entire tangent space T_pM.* $\square$

Corollary 3.5.11. *The geodesic vector field of a compact Riemmannian smooth manifold is complete.* $\square$

The fact that homogeneous Riemannian manifolds are complete is a consequence of the following.

Proposition 3.5.12. *Let (M, d) be a locally compact metric space. If it is homogeneous in the sense that for every x, $y \in M$ there exists a d-isometry $f : M \to M$ such that $f(x) = y$, then it is complete.*

Proof. Let $p \in M$. Since M is assumed to be locally compact, there exists some $r > 0$ such that $\overline{B(p, r)}$ is compact. The homogeneity implies now that $\overline{B(x, r)}$ is compact for every $x \in M$. If $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in M, there exists some $k_0 \in \mathbb{N}$ such that $d(x_{k_0}, x_k) < r$ for every $k \geq k_0$. Hence the sequence has a convergent subsequence, by compactness of $\overline{B(x_{k_0}, r)}$, which implies that it converges in M. $\square$

Corollary 3.5.13. *A homogeneous connected Riemannian smooth manifold is complete.* $\square$

The euclidean space, the spheres and the hyperbolic spaces are all complete Riemannian manifolds.

3.6 Geodesic convexity

Let M be a Riemannian smooth n-manifold and $p \in M$. By Proposition 3.4.8 and Proposition 3.5.3, there exists a uniformly normal open neighbourhood W of p for which there exists some $\delta > 0$ such that $W \subset \exp_q(B_q(0, \delta))$, for every $q \in W$, and for every q_1, $q_2 \in W$ there exists a unique minimizing geodesic path from q_1 to q_2

of length $< \delta$. However this geodesic path may not lie entirely in W.

Definition 3.6.1. A subset C of a Riemannian smooth manifold is said to be *strongly (geodesically) convex* if for every $x,\ y \in \overline{C}$ there exists a unique and minimizing geodesic path $\gamma : [a,b] \to \overline{C}$, for some $a,\ b \in \mathbb{R}$, $a < b$, from $x = \gamma(a)$ to $y = \gamma(b)$ such that $\gamma(t) \in C$ for $a < t < b$.

In this section we shall prove that sufficiently small geodesic balls with center any given point on a Riemennian smooth manifold are strongly convex (and of course uniformly normal). This result on the existence of strongly convex open neighbourhoods is due to J.H.C. Whitehead and is based on the following.

Lemma 3.6.2. *Let M be a Riemannian smooth n-manifold. For every $p \in M$ there exists some $\epsilon_0 > 0$ such that for $0 < \delta < \epsilon_0$ if $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a geodesic which is tangent to the geodesic sphere $\exp_p(\partial B_p(0,\delta))$ at the point $\gamma(t_0)$, for some $t_0 \in I$, then there exists some $\eta > 0$ such that*

$$\gamma((t_0 - \eta, t_0 + \eta) \setminus \{t_0\}) \subset M \setminus \exp_p(\overline{B_p(0,\delta)}).$$

Proof. There exists some $\epsilon > 0$ such that $\exp_p$ maps $B_p(0,\epsilon)$ diffeomorphically onto $U = \exp_p(B_p(0,\delta))$. Let $0 < \delta < \epsilon$. We choose an orthonormal basis $\{E_1, ..., E_n\}$ of T_pM and consider the normal chart (U, ϕ) at p, where $\phi = h \circ (\exp_p |_{B_p(0,\epsilon)})^{-1}$ and $h : T_pM \to \mathbb{R}^n$ is the linear isommetry with $h(E_i) = e_i$, $1 \le i \le n$. Let $\gamma : I \to U$ be a geodesic which is tangent to the geodesic sphere $\exp_p(\partial B_p(0,\delta))$ at the point $\gamma(t_0)$. Suppose that $\phi = (x^1, ..., x^n)$ and $\phi \circ \gamma = (\gamma^1, ..., \gamma^n)$. We consider the smooth function $f : I \to \mathbb{R}$ with

$$f(t) = \sum_{k=1}^{n} (\gamma^k(t))^2.$$

Since γ is tangent to $\exp_p(\partial B_p(0,\delta))$ at $\gamma(t_0)$, we have

$$f'(t_0) = 2\sum_{k=1}^{n} \gamma^k(t_0)(\gamma^k)'(t_0) = 0.$$

Since γ is a geodesic,

$$(\gamma^k)''(t) = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)(\gamma^j)'(t)$$

and substituting

$$f''(t) = 2\sum_{k=1}^{n} [((\gamma^k)'(t))^2 + (\gamma^k)(t)(\gamma^k)''(t)]$$

$$= \sum_{i,j=1}^{n} \left(2\delta_{ij} - 2\sum_{k=1}^{n} \Gamma_{ij}^k(\gamma(t))\gamma^k(t)\right)(\gamma^i)'(t)(\gamma^j)'(t)$$

for every $t \in I$. Since $\Gamma_{ij}^k(p) = 0$, $1 \le i, j, k \le n$, there exists some $0 < \epsilon_0 < \epsilon$ such that the quadratic form

$$\sum_{i,j=1}^{n} \left(\delta_{ij} - \sum_{k=1}^{n} \Gamma_{ij}^k(q) x^k(q) \right) v^i v^j$$

is positive definite for every $q \in \exp_p(B_p(0, \epsilon_0))$. Thus, if $0 < \delta < \epsilon_0$, then $f''(t_0) > 0$ and f has has a strict local minimum at t_0, which means that there exists $\eta > 0$ such that $f(t) > \delta^2$ for $t \in (t_0 - \eta, t_0 + \eta) \setminus \{t_0\}$. This proves the assertion. $\square$

We shall also use the following remark. If $p \in M$, for every open neighbourhood U of p there exists an open neighbourhood V of $(p, 0)$ in TM such that $\exp_q(tv) \in U$ for every $0 \le t \le 1$ and $(q, v) \in V$. To see this, it suffices to consider the smooth map $g : [0, 1] \times E \to M$ with $g(t, q, v) = \exp_q(tv)$, where $E \subset TM$ is the domain of definition of the exponential map and note that $g(t, p, 0) = p$ for all $0 \le t \le 1$. By continuity, for every $t \in [0, 1]$ there exists an open neighbourhood $V_t \subset E$ of $(p, 0)$ and $\delta_t > 0$ such that $g((t - \delta_t, t + \delta_t) \times V_t) \subset U$. By compactness of $[0, 1]$, there exist t_1,..., $t_m \in [0, 1]$, for some $m \in \mathbb{N}$, such that

$$[0, 1] = \bigcup_{k=1}^{m} (t_k - \delta_{t_k}, t_k + \delta_{t_k}).$$

It suffices now to take $V = V_{t_1} \cap \cdots \cap V_{t_m}$.

Theorem 3.6.3. *If M is a Riemannian smooth n-manifold, then for every $p \in M$ there exists some $\epsilon > 0$ such that for every $0 < \delta < \epsilon$ the geodesic ball $\exp_p(B_p(0, \delta))$ is strongly convex.*

Proof. Let $\epsilon_0 > 0$ be as in the preceding Lemma 3.6.2 and let $F : E \to M \times M$ be the smooth map $F(q, v) = (q \exp_q(v))$, where $E \subset TM$ is the domain of definition of the exponential map. As in the proof of Proposition 3.4.8, there exists an open neighbourhood $V \subset TM$ of $(p, 0)$ and some $0 < \epsilon < \epsilon_0$ such that F maps V diffeomorphically onto $\exp_p(B_p(0, \epsilon)) \times \exp_p(B_p(0, \epsilon))$ and $\exp_q(tv) \in \exp_p(B_p(0, \epsilon_0))$ for every $(q, v) \in V$ and $0 \le t \le 1$, form the above remark. Moreover, there exists some $\eta > 0$ such that $\exp_p(B_p(0, \epsilon)) \subset \exp_q(B_q(0, \eta))$ for every $q \in \exp_p(B_p(0, \epsilon))$.

We shall prove that $\exp_p(B_p(0, \delta))$ is strongly convex for every $0 < \delta < \epsilon$. Let q_1, $q_2 \in \overline{\exp_p(B_p(0, \delta))} = \exp_p(\overline{B_p(0, \delta)})$, Since $(q_1, q_2) \in F(V)$ there exists $v \in T_{q_1}M$ such that $q_1 = \exp_{q_1}(v)$ and $\gamma(t) = \exp_{q_1}(tv) \in \exp_p(B_p(0, \epsilon_0))$ for every $0 \le t \le 1$. By Proposition 3.5.3, γ is the unique and minimizing geodesic path from q_1 to q_2 in $\exp_{q_1}(B_{q_1}(0, \eta))$, hence in $\exp_p(B_p(0, \epsilon_0))$, and it suffices to show that $\gamma(t) \in \exp_p(B_p(0, \delta))$ for $0 < t < 1$. Let $(\gamma^1, ..., \gamma^n)$ be its local representation with respect to the normal chart on $\exp_p(B_p(0, \epsilon_0))$ and let again $f : [0, 1] \to \mathbb{R}$ be the smooth function

$$f(t) = \sum_{k=1}^{n} (\gamma^k(t))^2$$

as in the beginning of the proof of Lemma 3.6.2. If $\gamma((0,1))$ has points outside $\exp_p(B_p(0,\delta))$, then f takes its maximal value on $[0,1]$ at some $0 < t_0 < 1$ and

$$\delta^2 \leq f(t_0) < \epsilon_0^2$$

or equivalently $\gamma([0,1]) \in \exp_p(\overline{B_p(0,\sqrt{f(t_0)})})$. On the other hand, we must have

$$0 = f'(t_0) = 2\sum_{k=1}^{n}(\gamma^k)(t_0)(\gamma^k)'(t_0)$$

which means that the geodesic path $\gamma((0,1))$ is tangent to the geodesic sphere $\exp_p(\partial B_p(0,\sqrt{f(t_0)}))$. This contradicts Lemma 3.6.2. $\square$

Corollary 3.6.4. *If M is a Riemennian smooth manifold with Riemannian distance d, then for every $p \in M$ there exists some $\epsilon > 0$ such that for every $0 < \delta < \epsilon$ the open d-ball $B(p,\delta)$ is the geodesic ball with center p and radius δ and is uniformly normal and strongly convex.* $\square$

The existence of strongly convex geodesic balls can be applied to facilitate algebraic calculations on smooth manifolds involving de Rham and Čech cohomology, as we shall see in chapters 5 and 6.

3.7 Exercises

1. Prove that the euclidean connection on $\mathbb{R}^n$ is the unique connection for which $\nabla_X Y = 0$ for every $X \in \mathcal{X}(\mathbb{R}^n)$ and every constant $Y \in \mathcal{X}(\mathbb{R}^n)$.

2. Let ∇ be a connection on a smooth n-manifold M. A smooth diffeomorphism $f : M \to M$ is called *affine*, if it preserves ∇, that is $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$, for every X, $Y \in \mathcal{X}(M)$. The set of all affine diffeomorphisms of ∇ is a group. Prove that in case $M = \mathbb{R}^n$ and ∇ is the euclidean connection, for every affine diffeomorphism f there exist $A \in GL(n,\mathbb{R})$ and $b \in \mathbb{R}^n$ such that $f(x) = Ax + b$ for every $x \in \mathbb{R}^n$.

3. A smooth n-manifold M is said to be *affinely flat*, if there exists a smooth atlas $\mathcal{A} = \{(U_i,\phi_i) : i \in I\}$ of M such that for every i, $j \in I$ with $U_i \cap U_j \neq \varnothing$ there exist $A_{ij} \in GL(n,\mathbb{R})$ and $b_{ij} \in \mathbb{R}^n$ such that

$$\phi_i \circ \phi_j^{-1}(x) = A_{ij}x + b_{ij}$$

for every $x \in \phi_j(U_i \cap U_j)$. Prove that then there exists a natural connection ∇ on M such that every $\phi_i : U_i \to \phi_i(U_i)$ transfers $\nabla|_U$ to the euclidean connection on $\phi_i(U_i) \subset \mathbb{R}^n$.

4. Let $A \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix and

$$M = \{x \in \mathbb{R}^n : \langle A^{-1}x, x\rangle = 1\}$$

be the $(n-1)$-dimensional ellipsoid with semi-axis the eigenvalues of A. Prove that a smooth parametrized curve $\gamma : \mathbb{R} \to M$ is a geodesic of M (with respect to the euclidean connection) if and only if

$$\gamma'' + \frac{\langle A^{-1}\gamma', \gamma' \rangle}{\|A^{-1}\gamma\|^2} A^{-1}\gamma = 0.$$

5. On $\mathbb{R}^2$ we consider the connection whose Christoffel symbols are $\Gamma^1_{11} = x$, $\Gamma^1_{12} = 1$, $\Gamma^2_{22} = 2y$ and the rest vanish.
(a) Write down the system of differential equations of its geodesics.
(b) Let $\gamma : [0,1] \to \mathbb{R}^2$ be the smooth parametrized curve $\gamma(t) = (t, 0)$. Find the parallel translation of the vector $\left(\frac{\partial}{\partial y}\right)_{(0,0)}$ along γ on $(1,0)$ with respect to this connection.

6. Let M be a smooth manifold endowed with a connection ∇ and let $\rho : M \to \mathbb{R}$ be a smooth function. For every X, $Y \in \mathcal{X}(M)$ we put

$$\nabla^\rho_X Y = \nabla_X Y - Y(\rho)X - X(\rho)Y.$$

(a) Prove that ∇^ρ is a connection on M.
(b) Let $\epsilon > 0$ and $\gamma : (-\epsilon, \epsilon) \to M$ be a geodesic of ∇^ρ. If $h : (-\epsilon, \epsilon) \to \mathbb{R}$ is the smooth function with

$$h(t) = \int_0^t e^{2\rho(\gamma(s))} ds,$$

prove that $\gamma \circ h^{-1}$ is a geodesic of ∇. Thus, the two connections ∇ and ∇^ρ have the same non-parametrized geodesics.

7. On $\mathbb{R}^3$ we define $\nabla : \mathcal{X}(\mathbb{R}^3) \times \mathcal{X}(\mathbb{R}^3) \to \mathcal{X}(\mathbb{R}^3)$ by

$$\nabla_X Y = D_X Y + \frac{1}{2} X \times Y,$$

where $D_X Y$ is the directional derivetive of Y with respect to X and $X \times Y$ is the usual exterior product on $\mathbb{R}^3$.
(a) Prove that ∇ is a connection.
(b) Is ∇ symmetric?
(c) Is ∇ compatible with the euclidean Riemannian metric?

8. Let M, N be two connected Riemannian manifolds and let $f : M \to N$ be a smooth diffeomorphism. Assume that there exists some point $p \in M$ such that $f_{*p} : T_pM \to T_{f(p)}N$ is a linear isometry. Prove that f is an isometry if and only if it preserves the corresponding Levi-Civita connections.

9. Let M be a Riemannian smooth n-manifold and let $f : M \to \mathbb{R}$ be a smooth function. The gradient of f is the unique smooth vector field $\mathrm{grad} f$ such that

$$f_{*p}(v) = \langle \mathrm{grad} f(p), v \rangle$$

for every $v \in T_pM$, $p \in M$.
(a) Prove that in the local coordinates $(x^1, ..., x^n)$ of a smooth chart of M the gradient of f is given by the formula

$$\mathrm{grad} f = (g_{ij})^{-1}_{1\leq i,j\leq n} \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial x^n} \end{pmatrix}.$$

(b) If $\|\mathrm{grad} f\| = 1$ everywhere on M, prove that the integral curves of $\mathrm{grad} f$ are geodesics.

10. On $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ we consider the Riemannian metric

$$\langle v, w\rangle = \frac{4}{(1-|z|^2)^2} \cdot \mathrm{Re}(v\bar{w}), \qquad v, w \in T_z\mathbb{D}^2, \quad z \in \mathbb{D}^2.$$

(a) Prove that the map $C : \mathbb{D}^2 \to \mathbb{H}^2$ defined by

$$C(z) = -i\frac{z+i}{z-i}$$

is an isometry. C is called the Cayley transformation.
(b) Prove that if a, $b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$, then

$$h(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

is an isometry of $\mathbb{D}^2$.
(c) Describe the geodesics of $\mathbb{D}^2$.

11. Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be the smooth parametrized curve $\gamma(t) = (t, 1)$. Find the parallel vector field X along γ with $X(0) = \left(\frac{\partial}{\partial y}\right)_{\gamma(0)}$ and draw X on the interval $[-\frac{\pi}{2}, \pi]$.

12. Let M and N be two connected Riemannian manifolds.
(a) Let $p \in M$, $q \in N$ and $T : T_pM \to T_qN$ be a linear isometry. If there exists an isometry $h : M \to N$ such that $h(p) = q$ and $h_{*p} = T$, prove that there exist normal open neighbourhoods V of p and W of q such that $h(V) = W$ and

$$h|V = \exp_q \circ T \circ \exp_p^{-1}.$$

(b) Prove that if g, $h : M \to N$ are two isometries for which there exists some $p \in M$ such that $g(p) = h(p)$ and $g_{*p} = h_{*p}$, then $g = h$.

13. Let M ne a Riemannian smooth n-manifold and let G be a non-empty set of isometries of M. If $F = \{p \in M : g(p) = p \text{ for every } g \in G\}$, prove that F is a smooth submanifold of M.

(Hint: Consider for every $p \in F$ the vector subspace

$$V = \{v \in T_pM : g_{*p}(v) = v \text{ for every } g \in G\}$$

of T_pM and show that $\exp_p(U \cap V) = F \cap \exp_p(U)$ for a suitable open neighbourhood U of $0 \in T_pM$.)

14. Let M be a Riemannian smooth manifold with group of isometries $I(M)$. For a properly discontinuous subgroup G of $I(M)$, the orbit space M/G inherits a Riemannian metric, if it is a Hausdorff space, and the quotient map $p : M \to M/G$ is a local isometry. If M is complete, prove that M/G is complete as well. Describe the geodesics of the flat 2-torus T^2 and the geodesics of $\mathbb{R}P^2$ with respect to the induced Riemannian metric from S^2.

15. Prove that a connected isotropic and complete Riemannian manifold is homogeneous.

16. Let M be a connected, non-compact, complete Riemannian manifold with Riemannian distance d. Prove that for every $p \in M$ there exists a geodesic $\gamma : [0, +\infty) \to M$ with $\gamma(0) = p$ and $d(p, \gamma(t)) = t$ for every $t \geq 0$.

17. Let M and N be two Riemannian smooth manifolds and let $h : M \to N$ be a smooth diffeomorphism for which there exists $c > 0$ such hat $c\|h_{*p}(v)\| \leq \|v\|$ for every $v \in T_pM$ and $p \in M$. If N is complete, prove that M is also complete.

18. Let M be a Riemannian smooth manifold with Riemannian distance d. For every piecewise smooth parametrized curve $\gamma : [a, b] \to M$, where a, $b \in \mathbb{R}$, $a < b$, the non-negative real number

$$J(\gamma) = \frac{1}{2}\int_a^b \|\dot{\gamma}(t)\|^2 dt$$

is called the energy of γ and is not invariant under reparametrizations.
(a) Prove that $(L(\gamma))^2 \leq 2(b-a)J(\gamma)$ and the equality holds if and only if $\|\dot{\gamma}\|$ is constant.

For every p, $q \in M$ we define

$$e(p, q) = \inf\{2J(\gamma) | \gamma : [0, 1] \to M \quad \text{piecewise smooth with} \quad \gamma(0) = p, \gamma(1) = q\}.$$

(b) Prove that $(d(p, q))^2 = e(p, q)$ for every p, $q \in M$.
(c) If p, $q \in M$ and γ is a piecewise smooth parametrized curve from p to q, prove that γ minimizes the energy, that is $2J(\gamma) = e(p, q)$, if and only if γ is a minimizing geodesic.

Chapter 4

Differential forms

4.1 The cotangent bundle

Let M be a smooth n-manifold. The disjoint union of the algebraic duals of tangent spaces at points of M, that is the set

$$T^*M = \bigcup_{p\in M} \{p\} \times (T_pM)^*$$

can be endowed with a smooth structure in a similar way as the tangent bundle can, so that the natural projection $\pi : TM \to M$ with $\pi(p, a) = p$, for $a \in (T_pM)^*$, $p \in M$, becomes smooth and a submersion.

Let (U, ϕ) be a smooth chart of M, where $\phi = (x^1, ..., x^n)$ and let $\left\{ \frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n} \right\}$ be the corresponding set of basic vector fields on U. For every $p \in U$, we have a dual basis $\{(dx^1)_p, ..., (dx^n)_p\}$ of $(T_pM)^*$, so that

$$(dx^i)_p \left(\frac{\partial}{\partial x^j} \right)_p = \delta_{ij}$$

for all i, $j = 1, 2, ..., n$. Let $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ be defined by

$$\tilde{\phi}(p, a) = (x^1(p), ..., x^n(p), a_1, ...a_n)$$

for $a = a_1(dx^1)_p + \cdots + a_n(dx^n)_p \in (T_pM)^*$ and $p \in U$.

If (V, ψ) is another smooth chart of M with $U \cap V \neq \varnothing$, then

$$(\tilde{\psi} \circ \tilde{\phi}^{-1})(x, y) = ((\psi \circ \phi^{-1})(x), (D(\psi \circ \phi^{-1})(x)^{-1})^t(y)).$$

Applying Lemma 2.1.1, precisely as in the case of the tangent bundle, T^*M can be made a smooth $2n$-manifold with respect to which each $(\pi^{-1}(U), \tilde{\phi})$ is a smooth chart and $\pi : T^*M \to M$ is a submersion. The triple (T^*M, π, M) is the *cotangent bundle* of M. As in the case of the tangent bundle, the natural projection π is the bundle map, M is the base space of the bundle and T^*M is the total space of the bundle. We shall also use the term cotangent bundle for T^*M itself.

Definition 4.1.1. A *differential* 1*-form* on a smooth n-manifold M is a smooth map $\omega : M \to T^*M$ which to every $p \in M$ assigns a cotangent vector $\omega_p \in (T_pM)^*$. Briefly, $\omega \circ \pi = id_M$ or in other words ω is a smooth section of π.

The set $A^1(M)$ of all differential 1-forms of a smooth manifold M is an infinite dimensional real vector space and a $C^\infty(M)$-module. As for vector fields, if (U, ϕ) is a smooth chart of M, where $\phi = (x^1, ..., x^n)$, then for every $\omega \in A^1(U)$ there is a unique smooth function $F = (F_1, ..., F_n) : \phi(U) \to \mathbb{R}^n$ such that ω has a local representation

$$(\tilde{\phi} \circ \omega \circ \phi^{-1})(x) = (x, F(x)).$$

If we put $f_j = F_j \circ \phi$, $j = 1, ..., n$, then

$$\omega_p = \sum_{j=1}^{n} f_j(p)(dx^j)_p$$

for every $p \in U$. In particular, dx^j is a differential 1-form on U, $j = 1, ..., n$ and in analogy with the basic vector fields on U defined by the chart ϕ, we call dx^1,.., dx^n the *basic differential* 1*-forms* on U with respect to the smooth chart (U, ϕ).

Example 4.1.2. Let M be a smooth n-manifold and let $f : M \to \mathbb{R}$ be any smooth function. At every point $p \in M$, the derivative $f_{*p} : T_pM \to T_{f(p)}\mathbb{R}$ of f at p, can be considered an element of $(T_pM)^*$, identifying $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ via the linear isomorphism which sends $\left(\frac{d}{dt}\right)_{f(p)}$ to 1. So we obtain a map $df : M \to T^*M$, that is $(df)_p = f_{*p}$. If (U, ϕ) is a smooth chart of M and $\phi = (x^1, ..., x^n)$, the corresponding local representation of df on U is given by the formula

$$df|_U = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \cdot dx^j.$$

Therefore, df is a differential 1-form and is called *the differential of* f. Note that in particular the basic differential 1-form dx^j is the differential of the j-th coordinate $x^j : U \to \mathbb{R}$ of the smooth chart ϕ.

The differential is a linear map $d : C^\infty(M) \to A^1(M)$ which has the additional property

$$d(fg) = gdf + fdg$$

for every f, $g \in C^\infty(M)$. Indeed, if $p \in M$, a tangent vector $v \in T_pM$ is a derivation of the algebra $\mathcal{G}_p(M)$ of germs smooth real valued functions defined on neighbourhoods of p and so

$$(d(fg))_p(v) = v(fg) = g(p)v(f) + f(p)v(g) = g(p)(df)_p(v) + f(p)(dg)_p(v).$$

A smooth map $f : M \to N$ of smooth manifolds induces *transpose* linear maps $f^* : C^\infty(N) \to C^\infty(M)$ and $f^* : A^1(N) \to A^1(M)$ by $f^*h = h \circ f$ for $h \in C^\infty(M)$ and

$$(f^*\omega)_p(v) = \omega_{f(p)}(f_{*p}(v))$$

for every $v \in T_pM$, $p \in M$. The differential 1-form $f^*\omega$ is called the *pull-back of* ω with respect to f. If $g : N \to P$ is a second smooth map of smooth manifolds, it follows immediately from the chain rule that $(g \circ f)^* = f^* \circ g^*$.

Another consequence of the chain rule is the fact that the differential is natural. This means that if $f : M \to N$ is any smooth map of smooth manifolds, then the following diagram commutes.

$$\begin{tikzcd} C^\infty(N) \arrow[r, "d"] \arrow[d, "f^*"'] & A^1(N) \arrow[d, "f^*"] \\ C^\infty(M) \arrow[r, "d"] & A^1(M) \end{tikzcd}$$

4.2 Alternating multilinear forms

Let V be a real vector space of finite dimension n. Recall that a k-multilinear form on V, for $k \in \mathbb{N}$, is any function $\phi : V^k \to \mathbb{R}$ which is linear with respect to each variable separately. The set $\mathcal{J}^k(V)$ of all k-multilinear forms of V carries an obvious vector space structure. Note that $\mathcal{J}^1(V) = V^*$ is the algebraic dual space of V. We also put $\mathcal{J}^0(V) = \mathbb{R}$.

The graded vector space $\mathcal{J}(V) = \bigoplus_{k=0}^{\infty} \mathcal{J}^k(V)$ of all multilinear forms on V can be endowed with the *tensor product* $\otimes$ defined by

$$(\phi \otimes \psi)(v_1, ..., v_k, u_1, ..., u_l) = \phi(v_1, ..., v_k) \cdot \psi(u_1, ..., u_l)$$

for $\phi \in \mathcal{J}^k(V)$, $\psi \in \mathcal{J}^l(V)$ and $v_1, ..., v_k, u_1, ..., u_l \in V$ with respect to which it becomes a graded associative (non-commutative) algebra.

If $\{v_1, ..., v_n\}$ is a basis of V and $\{v_1^*, ..., v_n^*\}$ is its dual basis of V^*, then

$$\{v_{i_1}^* \otimes \cdots \otimes v_{i_k}^* : 1 \leq i_1, ..., i_k \leq n\}$$

is a basis of $\mathcal{J}^k(V)$. Note that

$$(v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*)(v_{j_1}, ..., v_{j_k}) = \begin{cases} 0, & \text{if} \quad (i_1, ..., i_k) \neq (j_1, ..., j_k), \\ 1, & \text{if} \quad (i_1, ..., i_k) = (j_1, ..., j_k) \end{cases}$$

and

$$\phi = \sum_{i_1, ..., i_k = 1}^{n} \phi(v_{i_1}, ..., v_{i_k}) \cdot v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*$$

for every $\phi \in \mathcal{J}^k(V)$.

Every linear map $f : V \to W$ of finite dimensional real vector spaces induces a linear map $f^* : \mathcal{J}(W) \to \mathcal{J}(V)$ which is defined by

$$(f^*\phi)(u_1, ..., u_k) = \phi(f(u_1), ..., f(u_k))$$

for every $u_1, ..., u_k \in V$ and $\phi \in \mathcal{J}^k(V)$ and which is called the *transpose* of f. It is immediate from the definitions that f^* preserves the tensor product and is thus an algebra homomorphism.

The determinant is an example of an n-multilinear form which has the additional property that is alternating.

Definition 4.2.1. A k-multilinear form $\omega \in \mathcal{J}^k(V)$ is called *alternating* if

$$\omega(u_1, ..., u_k) = (\mathrm{sgn}\sigma) \cdot \omega(u_{\sigma(1)}, ..., u_{\sigma(k)})$$

for every $u_1, ..., u_k \in V$ and every permutation $\sigma \in S_n$.

The set $\Lambda^k(V)$ of alternating k-multilinear forms of V is a vector subspace of $\mathcal{J}^k(V)$. If $\{v_1, ..., v_n\}$ is a basis of V and $\omega \in \Lambda^k(V)$ for $k > n$, then $\omega(v_{i_1}, ..., v_{i_k}) = 0$ for every $1 \leq i_1, ..., i_k \leq n$, because at least two of $v_{i_1}, ..., v_{i_k}$ must coincide. Therefore, $\omega = 0$. This means that $\Lambda^k(V) = 0$ for $k > n$.

The tensor product of two alternating k-multilinear forms need not be alternating. In order to define an algebra structure on the vector space $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ of all alternating forms we consider the linear map $A : \mathcal{J}(V) \to \mathcal{J}(V)$ defined by

$$A(\phi)(u_1, ..., u_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma) \cdot \omega(u_{\sigma(1)}, ..., u_{\sigma(k)})$$

and we observe that $A(\phi) \in \Lambda^k(V)$ for every $\phi \in \mathcal{J}^k(V)$. Indeed, if $\tau = (i \quad j)$ is the transposition which permutes the symbols i and j, we have

$$A(\phi)(u_{\tau(1)}, ..., u_{\tau(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma) \cdot \omega(u_{\sigma(\tau(1))}, ..., u_{\sigma(\tau(k))})$$

$$= -\frac{1}{k!} \sum_{\sigma \circ \tau \in S_n} \mathrm{sgn}(\sigma \circ \tau) \cdot \omega(u_{\sigma(\tau(1))}, ..., u_{\sigma(\tau(k))}) = -A(\phi)(u_1, ..., u_k).$$

Moreover, $A(\omega) = \omega$, if $\omega \in \Lambda(V)$.

If now $\omega \in \Lambda^k(V)$ and $\theta \in \Lambda^l(V)$, the element

$$\omega \wedge \theta = \frac{(k+l)!}{k! \cdot l!} A(\omega \otimes \theta) \in \Lambda^{k+l}(V)$$

is called the *wedge product* of ω with θ. It follows from the linearity of A that the wedge product

$$\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$$

is bilinear.

If $f : V \to W$ is a linear map of finite dimensional real vector spaces, then $f^*(\Lambda^k(W)) \subset \Lambda^k(V)$ and $f^*(\omega \wedge \theta) = f^*(\omega) \wedge f^*(\theta)$ for every $\omega \in \Lambda^k(W)$, $\theta \in \Lambda^l(W)$.

Lemma 4.2.2. If $\omega \in \Lambda^k(V)$ and $\theta \in \Lambda^l(V)$, then $\omega \wedge \theta = (-1)^{kl} \theta \wedge \omega$.

Proof. If $\tau = (1 \quad 2 \cdots k+l)^k = ((1 \quad k+l) \cdots (1 \quad 2))^k$, that is

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & \cdots & k \end{pmatrix},$$

then $\operatorname{sgn}\tau = (-1)^{(k+l-1)k} = (-1)^{kl}$ and we have

$$A(\omega\otimes\theta)(u_1,...,u_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\cdot\omega(u_{\sigma(1)},...,u_{\sigma(k)})\cdot\theta(u_{\sigma(k+1)},...,u_{\sigma(k+l)})$$

$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\cdot\omega(u_{\sigma(\tau(l+1))},...,u_{\sigma(\tau(k+l))})\cdot\theta(u_{\sigma(\tau(1))}),...,u_{\sigma(\tau(l))})$$

$$= (-1)^{kl}\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}(\sigma\tau))\cdot\omega(u_{\sigma(\tau(l+1))},...,u_{\sigma(\tau(k+l))})\cdot\theta(u_{\sigma(\tau(1))}),...,u_{\sigma(\tau(l))})$$

$$= (-1)^{kl}A(\theta\otimes\omega). \quad \square$$

As a consequence, if k is odd, then $\omega\wedge\omega = 0$ for every $\omega\in\Lambda^k(V)$. For the proof of the associativity of the wedge product we shall need the following.

Lemma 4.2.3. *Let $\phi\in\mathcal{J}^k(V)$ and $\psi\in\mathcal{J}^l(V)$. If $A(\phi)=0$, then*

$$A(\phi\otimes\psi) = A(\psi\otimes\phi) = 0.$$

Proof. For every u_1,..., $u_{k+l}\in V$ we have by definition

$$A(\phi\otimes\psi)(u_1,...,u_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\cdot\phi(u_{\sigma(1)},...,u_{\sigma(k)})\cdot\psi(u_{\sigma(k+1)},...,u_{\sigma(k+l)})$$

The set $G = \{\sigma\in S_{k+l} : \sigma(k+1) = k+1, ..., \sigma(k+l) = k+l\}$ is a subgroup of S_{k+l} isomorphic to S_k and S_{k+l} is the disjoint union of the right cosets of G in S_{k+l}. Now we have

$$\sum_{\sigma\in G}(\operatorname{sgn}\sigma)\cdot\phi(u_{\sigma(1)},...,u_{\sigma(k)})\cdot\psi(u_{\sigma(k+1)},...,u_{\sigma(k+l)})$$

$$= k!A(\phi)(u_1,...,u_k)\cdot\psi(u_{k+1},...,u_{k+l}) = 0$$

and

$$\sum_{\sigma\in G\tau}(\operatorname{sgn}\sigma)\cdot\phi(u_{\sigma(1)},...,u_{\sigma(k)})\cdot\psi(u_{\sigma(k+1)},...,u_{\sigma(k+l)})$$

$$= (\operatorname{sgn}\tau)\sum_{\sigma\tau^{-1}\in G}(\operatorname{sgn}(\sigma\tau^{-1}))\cdot\phi(u_{\sigma\tau^{-1}\tau(1)},...,u_{\sigma\tau^{-1}\tau(k)})\cdot\psi(u_{\sigma\tau^{-1}\tau(k+1)},...,u_{\sigma\tau^{-1}\tau(k+l)})$$

$$= (\operatorname{sgn}\tau)k!A(\phi)(u_{\tau(1)},...,u_{\tau(k)})\cdot\psi(u_{\tau(k+1)},...,u_{\tau(k+l)}) = 0$$

for every $\tau\in S_{k+l}$. This proves that $A(\phi\otimes\psi)=0$ and similarly one can prove that $A(\psi\otimes\phi)=0$. $\square$

Corollary 4.2.4. *If $\omega\in\Lambda^k(V)$, $\theta\in\Lambda^l(V)$ and $\eta\in\Lambda^m(V)$, then*

$$A(A(\omega\otimes\theta)\otimes\eta) = A(\omega\otimes A(\theta\otimes\eta)) = A(\omega\otimes\theta\otimes\eta).$$

Proof. Since $A(A(\omega\otimes\theta)-\omega\otimes\theta)=0$, it follows from Lemma 4.2.3 that

$$0 = A((A(\omega\otimes\theta)-\omega\otimes\theta)\otimes\eta) = A(A(\omega\otimes\theta)\otimes\eta) - A(\omega\otimes\theta\otimes\eta). \quad \square$$

Proposition 4.2.5. *If $\omega \in \Lambda^k(V)$, $\theta \in \Lambda^l(V)$ and $\eta \in \Lambda^m(V)$, then*

$$(\omega \wedge \theta) \wedge \eta = \omega \wedge (\theta \wedge \eta) = \frac{(k+l+m)!}{k! \cdot l! \cdot m!} \cdot A(\omega \otimes \theta \otimes \eta).$$

Proof. Using Corollary 4.2.4 we compute

$$(\omega \wedge \theta) \wedge \eta = \frac{(k+l+m)!}{(k+l)! \cdot m!} \cdot A((\omega \wedge \theta) \otimes \eta)$$

$$= \frac{(k+l+m)!}{(k+l)! \cdot m!} \cdot \frac{(k+l)!}{k! \cdot l!} \cdot A(A(\omega \otimes \theta) \otimes \eta = \frac{(k+l+m)!}{k! \cdot l! \cdot m!} \cdot A(\omega \otimes \theta \otimes \eta). \quad \square$$

The above show that $\Lambda(V)$ endowed with the wedge product is a graded commutative associative algebra with unity. If now $\{v_1, ..., v_n\}$ is a basis of V and $\{v_1^*, ..., v_n^*\}$ is its dual basis of V^*, then

$$\{v_{i_1}^* \wedge \cdots \wedge v_{i_k}^* : 1 \leq i_< \cdots < i_k \leq n\}$$

generates $\Lambda^k(V)$, since $A(\mathcal{J}^k(V)) = \Lambda^k(V)$. Actually, it is a basis, because if $a_{i_1 \cdots i_k} \in \mathbb{R}$, $1 \leq i_1 < \cdots < i_k \leq n$ are such that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \cdot v_{i_1}^* \wedge \cdots \wedge v_{i_k}^* = 0,$$

then

$$0 = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \cdot (v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*)(v_{j_1}, ..., v_{j_k})$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \cdot k! \cdot A(v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*)(v_{j_1}, ..., v_{j_k})$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \cdot \sum_{\sigma \in S_k} (\text{sgn}\sigma) \cdot v_{i_1}^*(v_{\sigma(j_1)}) \cdots v_{i_k}^*(v_{\sigma(j_k)}) = a_{j_1 \cdots j_k}.$$

Therefore, the dimension of $\Lambda^k(V)$ is $\binom{n}{k}$. In particular, $\dim \Lambda^n(V) = 1$ and $\Lambda^n(V)$ is generated by the determinant. If $w_j = \sum_{i=1}^{n} a_{ij} v_i$, $j = 1, ..., n$, then

$$\omega(w_1, ..., w_n) = \omega(v_1, ..., v_n) \cdot \det(a_{ij})_{1 \leq i,j \leq n}.$$

4.3 The exterior algebra of a smooth manifold

In analogy to the tangent and the cotangent bundle of a smooth n-manifold M, the disjoint union of the spaces of alternating k-multilinear forms, $1 \leq k \leq n$, of the tangent spaces

$$\Lambda^k(M) = \bigcup_{p \in M} \{p\} \times \Lambda^k(T_pM)$$

can be endowed with a smooth structure so that the projection $\pi : \Lambda^k(M) \to M$ with $\pi(p,a) = p$ for $p \in M$, $a \in \Lambda^k(T_pM)$ becomes a submersion. Note that $\Lambda^1(M) = T^*M$.

Let (U,ϕ) be a smooth chart of M, where $\phi = (x^1,...,x^n)$. Let $\left\{\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n}\right\}$ be the corresponding set of basic vector fields and $\{dx^1,...,dx^n\}$ the corresponding set of basic differential 1-forms on U. For each $p \in U$ the set

$$\{(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p : 1 \le i_1 < \cdots < i_k \le n\}$$

is a basis of $\Lambda^k(T_pM)$.

Let $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^{\binom{n}{k}}$ be defined by

$$\tilde{\phi}(p,a) = (\phi(p),(a_{i_1\cdots i_k})_{1\le i_1<\cdots<i_k\le n})$$

for $a = \sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\cdots i_k}(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p$ and $p \in U$. If (V,ψ) is another smooth chart of M, then $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a smooth diffeomorphism, since $(D(\psi \circ \phi^{-1})(x)^{-1})^t$ depends smoothly on $x \in \phi(U \cap V)$. Thus, applying Lemma 2.1.1 we obtain a topology and a smooth structure on $\Lambda^k(M)$ turning it into a smooth manifold and the natural projection $\pi : \Lambda^k(M) \to M$ a submersion. The triple $(\Lambda^k(M),\pi,M)$ is called the *exterior k-bundle* of M. As usual, we shall also use the same term for its total space $\Lambda^k(M)$.

Definition 4.3.1. A *differential k-form* on a smooth n-manifold M, $1 \le k \le n$, is a smooth map $\omega : M \to \Lambda^k(M)$ which to every $p \in M$ assigns an element $\omega_p \in \Lambda^k(T_pM)$. So, $\omega \circ \pi = id_M$, which means that ω is a smooth section of π. The non-negative integer k is the *degree* of ω.

The set $A^k(M)$ of all differential k-forms of a smooth manifold M is an infinite dimensional real vector space and a $C^\infty(M)$-module. We also put $A^0(M) = C^\infty(M)$.

If (U,ϕ) is a smooth chart of M, where $\phi = (x^1,...,x^n)$, then for every $\omega \in A^k(U)$ there is a unique smooth function $F = (F_{i_1i_2\cdots i_k})_{1\le i_1<\cdots<i_k\le n} : \phi(U) \to \mathbb{R}^{\binom{n}{k}}$ such that ω has a local representation

$$(\tilde{\phi} \circ \omega \circ \phi^{-1})(x) = (x,F(x)).$$

If we put $f_{i_1i_2\cdots i_k} = F_{i_1i_2\cdots i_k} \circ \phi$, then

$$\omega_p = \sum_{1\le i_1<\cdots<i_k\le n} f_{i_1i_2\cdots i_k}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p$$

for every $p \in U$. In particular, every $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ is a differential k-form on U, which we call a *basic differential k-form* on U with respect to the smooth chart (U,ϕ).

On the graded vector space $A^*(M) = \bigoplus_{k=0}^{n} A^k(M)$ we have a wedge product

$$\wedge : A^k(M) \times A^l(M) \to A^{k+l}(M)$$

which is defined by $(\omega \wedge \theta)_p = \omega_p \wedge \theta_p$ for every $p \in M$. Therefore, all the properties that the wedge product of alternating multilinear forms of vector spaces have transfer verbatim to differential forms making thus $A^*(M)$ a graded commutative associative algebra with unity, which is called *the exterior algebra* of the smooth manifold M.

Every smooth map $f : M \to N$ of smooth manifolds induces a *transpose map* $f^* : A^*(N) \to A^*(M)$ defined by

$$(f^*\omega)_p(v_1, ..., v_k) = \omega_{f(p)}(f_{*p}(v_1), ..., f_{*p}(v_k))$$

for v_1,..., $v_k \in T_pM$, $p \in M$ and every $\omega \in A^k(N)$. The differential form $f^*\omega$ is called the *pull-back* of ω with respect to f. The transpose f^* is a homomorphism of graded algebras, since it preserves the wedge product. If $g : N \to P$ is another smooth map of smooth manifolds, then $(g \circ f)^* = f^* \circ g^*$, by the chain rule, and evidently $(id_M)^* = id_{A^*(M)}$. It follows that if f is a smooth diffeomorphism, then $f^* : A^*(N) \to A^*(M)$ is an isomorphism of graded algebras.

On the exterior algebra of a smooth manifold there exists a natural linear endomorphism, which is not an algebra homomorphism, but satisfies a graded Leibniz formula. This unifies and extends the classical operators of vector analysis in $\mathbb{R}^3$. We shall construct it starting locally from open subsets of $\mathbb{R}^n$.

For every differential k-form ω on an open subset $S \subset \mathbb{R}^n$ there exist unique smooth functions $f_{i_1 i_2 \cdots i_k} : S \to \mathbb{R}$, $1 \leq i_1 < \cdots < i_k \leq n$, such that

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 i_2 \cdots i_k} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The *differential* of ω is the differential $(k+1)$-form defined by the formula

$$d\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} df_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

So we get a linear endomorphism $d : A^*(S) \to A^*(S)$ which is called the *exterior differential*.

Proposition 4.3.2. *The exterior differential $d : A^*(S) \to A^*(S)$ for an open set $S \subset \mathbb{R}^n$ has the following properties:*
(i) If $B \subset S$ is an open subset of S, then $d\omega|_B = d(\omega|_B)$ for every $\omega \in A^*(S)$.
(ii) d has degree 1, which means that $d(A^k(S)) \subset A^{k+1}(S)$, $0 \leq k \leq n$.
(iii) If $f \in A^0(M) = C^\infty(M)$, then df is the usual differential of f which was defined in Example 3.1.2.
(iv) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$ for every $\omega \in A^k(S)$ and $\theta \in A^l(S)$, $0 \leq k, l \leq n$ (graded Leibniz formula).
(v) $d \circ d = 0$, that is $d(d\omega) = 0$ for every $\omega \in A^*(S)$.

Proof. The properties (i), (ii) and (iii) are immediate from the definitions. For (iv) we suppose that

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 i_2 \cdots i_k} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \theta = \sum_{1 \leq j_1 < \cdots < j_l \leq n} g_{j_1 j_2 \cdots j_l} \cdot dx^{j_1} \wedge \cdots \wedge dx^{l_l}$$

and compute

$$
\begin{aligned}
d(\omega \wedge \theta) &= \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} d(f_{i_1 i_2 \cdots i_k} g_{j_1 j_2 \cdots j_l}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{l_l} \\
&= \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} g_{j_1 j_2 \cdots j_l} df_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{l_l} \\
&+ \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} f_{i_1 i_2 \cdots i_k} dg_{j_1 j_2 \cdots j_l} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{l_l} \\
&= \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} (df_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge g_{j_1 j_2 \cdots j_l} dx^{j_1} \wedge \cdots \wedge dx^{l_l} \\
&+ \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} (f_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge (-1)^k dg_{j_1 j_2 \cdots j_l} \wedge dx^{j_1} \wedge \cdots \wedge dx^{l_l} \\
&= d\omega \wedge \theta + (-1)^k \omega \wedge d\theta.
\end{aligned}
$$

To prove (v), we start with a $f \in A^0(M)$. Then, by definition,

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \cdot dx^j$$

and

$$d(df) = \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial x^j}\right) \wedge dx^j = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j} \cdot dx^i \wedge dx^j = 0.$$

In particular, it follows inductively form this and (iv) that $d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = 0$. If now

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 i_2 \cdots i_k} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then

$$d(d\omega) = d\Bigg(\sum_{1 \le i_1 < \cdots < i_k \le n} df_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\Bigg)$$

$$= \sum_{1 \le i_1 < \cdots < i_k \le n} d(df_{i_1 i_2 \cdots i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} - \sum_{1 \le i_1 < \cdots < i_k \le n} df_{i_1 i_2 \cdots i_k} \wedge d(dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

$$= 0 - 0 = 0. \quad \square$$

An additional important property of the exterior differential is that it is natural.

Proposition 4.3.3. *Let $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ be open sets and let $f : S \to T$ be a smooth map. Then $f^* \circ d = d \circ f^*$, that is the following diagram commutes.*

$$
\begin{array}{ccc}
A^*(T) & \xrightarrow{\ d\ } & A^*(T) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
A^*(S) & \xrightarrow{\ d\ } & A^*(S)
\end{array}
$$

Proof. We already now from the chain rule that $f^*(dg) = d(g \circ f) = d(f^*g)$ for $g \in A^0(T) = C^\infty(T)$. If $\omega \in A^1(T)$ and

$$\omega = \sum_{j=1}^m g_j dx^j,$$

we have

$$d(f^*\omega) = \sum_{j=1}^m d((g_j \circ f) \cdot f^*(dx^j)) = \sum_{j=1}^m d(g_j \circ f) \wedge f^*(dx^j) + \sum_{j=1}^m (g_j \circ f) \cdot d(f^*(dx^j))$$

$$= \sum_{j=1}^m f^*(dg_j) \wedge f^*(dx^j) + \sum_{j=1}^m f^* g_j \cdot f^*(d(dx^j)) = f^*\left(\sum_{j=1}^m g_j dx^j\right) = f^*(d\omega).$$

The proof now can be concluded by induction on the degree. If the conclusion is true for differential forms of degree smaller than k and $\omega = g dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, then

$$d(f^*\omega) = d(f^*(g dx^{i_1}) \wedge f^*(dx^{i_2} \wedge \cdots \wedge dx^{i_k}))$$

$$= d(f^*(g dx^{i_1})) \wedge f^*(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) - f^*(g dx^{i_1}) \wedge d(f^*(dx^{i_2} \wedge \cdots \wedge dx^{i_k}))$$

$$= f^*(d(g dx^{i_1})) \wedge f^*(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) - f^*(g dx^{i_1}) \wedge f^*(d(dx^{i_2} \wedge \cdots \wedge dx^{i_k}))$$

$$= f^*(dg \wedge dx^{i_1} \wedge f^*(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) - 0 = f^*(d\omega).$$

By linearity of the exterior differential this proves the assertion. □

We are now in a position to extend the definition of the exterior differential from open subsets of euclidean spaces to smooth manifolds. The crucial fact that we shall need is that the definition we gave for open sets of euclidean spaces is invariant under smooth diffeomorphisms. This is provided by Proposition 4.3.3.

Definition 4.3.4. An *exterior differential* is a linear endomorphism

$$d : A^*(M) \to A^*(M)$$

of degree 1 which is defined for every smooth manifold M and has the following properties:
(i) If $f \in A^0 M)$, then df is the usual differential of f.
(ii) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$ for every $\omega \in A^k(M)$ and $\theta \in A^l(M)$, $0 \le k, l \le n$.
(iii) $d \circ d = 0$.
(iv) If $f : M \to N$ is a smooth map of smooth manifolds, then $f^* \circ d = d \circ f^*$.

In particular, if $U \subset M$ is an open set and $i : U \hookrightarrow M$ is the inclusion, then $d\omega|_U = i^*(d\omega) = d(i^*\omega) = d(\omega|_U)$, by (iv).

Theorem 4.3.5. *There exists a unique exterior differential.*

Proof. For the uniqueness it suffices to prove that for every smooth chart (U, ϕ) of a smooth n-manifold M the differential $(k+1)$-form $d(\omega|_U)$ on U is uniquely determined for every $\omega \in A^k(M)$. Since $\phi : U \to \phi(U)$ is a smooth diffeomorphism, its transpose $\phi^* : A^*(\phi(U)) \to A^*(U)$ is an isomorphism of graded algebras. This implies that it suffices to prove uniqueness for open subsets of $\mathbb{R}^n$. Indeed, if $S \subset \mathbb{R}^n$ is an open set and

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 i_2 \cdots i_k} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in A^k(S),$$

it follows from properties (i)-(iv) of Definition 3.3.4 that necessarily

$$d\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} df_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

This proves uniqueness because the smooth functions $f_{i_1 i_2 \cdots i_k}$ are uniquely determined by ω.

The existence of the exterior differential has already been proved on open subsets of euclidean spaces in Proposition 3.3.2. Let M be a smooth n-manifold and let $\mathcal{A}$ be a smooth atlas of M. If (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ are such that $U \cap V \neq \varnothing$, then $\phi_{UV} = \phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \to \phi_U(U \cap V)$ is a smooth diffeomorphism and we have a commutative diagram

$$\begin{array}{ccc} A^*(\phi_U(U \cap V)) & \xrightarrow{\;d\;} & A^*(\phi_U(U \cap V)) \\ {\scriptstyle \phi_{UV}^* = (\phi_U \circ \phi_V^{-1})^*} \Big\downarrow & & \Big\downarrow {\scriptstyle \phi_{UV}^* = (\phi_U \circ \phi_V^{-1})^*} \\ A^*(\phi_V(U \cap V)) & \xrightarrow{\;d\;} & A^*(\phi_V(U \cap V)) \end{array}$$

from Proposition 4.3.3. So, $d = ((\phi_U \circ \phi_V^{-1})^*)^{-1} \circ d \circ (\phi_U \circ \phi_V^{-1})^*$. For every $\omega \in A^*(M)$ we define

$$(d\omega)|_U = \phi_U^*(d((\phi_U^{-1})^*(\omega|_U))).$$

From the above commutative diagram we have

$$\begin{aligned} \phi_U^*(d((\phi_U^{-1})^*(\omega|_{U \cap V}))) &= ((\phi_{UV}^{-1} \circ \phi_U)^* \circ d \circ (\phi_U^{-1} \circ \phi_{UV})^*)(\omega|_{U \cap V}) \\ &= \phi_V^*(d((\phi_V^{-1})^*(\omega|_{U \cap V}))). \end{aligned}$$

Since $(d\omega)|_U$ and $(d\omega)|_V$ coincide on $U \cap V$ for every (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ such that $U \cap V \neq \varnothing$, we get a globally well defined differential $(k+1)$-form $d\omega$ on M. This concludes the proof. □.

Thus, the exterior algebra $A^*(M)$ of a smooth manifold M becomes a differential graded algebra, which is invariant under smooth diffeomorphisms, and is called *the de Rham covhain complex of* M.

$$C^\infty(M) = A^0(M) \xrightarrow{d} A^1(M) \xrightarrow{d} \cdots \xrightarrow{d} A^k(M) \xrightarrow{d} A^{k+1}(M) \xrightarrow{d} \cdots$$

This is infinite dimensional and impossible to compute. Its cohomology is also invariant under smooth diffeomorphisms and we can use traditional homological methods to compute it.

We call $\omega \in A^k(M)$ a *closed differential k-form (or k-cocycle)* if $d\omega = 0$ and an *exact differential k-form (or k-coboundary)* if there exists some $\eta \in A^{k-1}(M)$ such that $d\eta = \omega$. Since $d \circ d = 0$, an exact differential form is always closed. The converse however is not true.

Example 4.3.6. Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Then ω is a closed differential 1-form, because

$$d\omega = -\frac{x^2-y^2}{(x^2+y^2)^2}dy \wedge dx + \frac{-x^2+y^2}{(x^2+y^2)^2}dx \wedge dy = 0.$$

However ω is not exact. Indeed, suppose that there exists a smooth function (the potential) $f : M \to \mathbb{R}$ such that $\omega = df$. Let $\gamma : \mathbb{R} \to M$ be the standard parametrization of the unit circle, that is $\gamma(t) = (\cos t, \sin t)$. Then $\omega_{\gamma(t)}(\dot{\gamma}(t)) = 1$ and from the Fundamental Theorem of Calculus we arrive at the contradiction

$$2\pi = \int_0^{2\pi} \omega_{\gamma(t)}(\dot{\gamma}(t))dt = \int_0^{2\pi} (f \circ \gamma)'(t)dt = f(\gamma(2\pi)) - f(\gamma(0)) = 0.$$

The set of closed differential k-forms on a smooth manifold M is the vector subspace $Z^k(M) = A^k(M) \cap \mathrm{Ker}d$ and the set of exact differential k-forms is the vector subspace $B^k(M) = A^k(M) \cap \mathrm{Im}d$ of $Z^k(M)$. The quotient vector space

$$H^k(M) = \frac{Z^k(M)}{B^k(M)}$$

is called the *de Rham cohomology of M at degree k* or the *k-th de Rham cohomology of M*. The total de Rham cohomology of a smooth n-manifold M is the graded vector space $H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$ and it can be given the structure of a graded commutative associative algebra with unity. Indeed, the wedge product on $A^*(M)$ induces a product $\smile: H^k(M) \times H^l(M) \to H^{k+l}(M)$ well defined by

$$[\omega] \smile [\theta] = [\omega \wedge \theta]$$

for $\omega \in Z^k(M)$, $\theta \in Z^l(M)$, which is called the *cup product* on $H^*(M)$. To see this, note first that $\omega \wedge \theta$ is closed , by the Leibniz formula. If now η, $\zeta \in A^{k-1}(M)$, then for the cohomologous closed differential k-forms $\omega + d\eta$ to ω and $\theta + d\zeta$ to θ we have

$$(\omega+d\eta)\wedge(\theta+d\zeta) = \omega\wedge\theta+d\eta\wedge\theta+\omega\wedge d\zeta+d\eta\wedge d\zeta = \omega\wedge\theta+d(\eta\wedge\theta\pm\omega\wedge d\zeta+\eta\wedge d\zeta)$$

and therefore $[\omega \wedge \theta] = [(\omega + d\eta) \wedge (\theta + d\zeta)]$. Evidently, the cup product on $H^*(M)$ inherits the properties of the wedge product on $A^*(M)$. The graded algebra $H^*(M)$ is called the *de Rham cohomology algebra of M*.

If $f : M \to N$ is now a smooth map of smooth manifolds, then the transpose $f^* : A^*(N) \to A^*(M)$ maps closed differential forms on N to closed differential forms on M and exact differential forms to exact differential forms, because it commutes with the exterior differential. Thus it induces a homomorphism of graded algebras (denoted again by) $f^* : H^*(N) \to H^*(M)$. If $g : N \to P$ is another smooth map of smooth manifolds, then $(g \circ f)^* = f^* \circ g^*$ and $(id_M)^* = id_{H^*(M)}$. It follows that if f is a smooth diffeomorphism, then $f^* : H^*(N) \to H^*(M)$ is an algebra isomorphism. Thus, the de Rham cohomology at every degree is a diffeomorphism invariant, as well as the total de Rham cohomolody algebra.

In Chapter 4 we shall use powerful algebraic methods for the computation of the de Rham cohomology. For the time being, we can compute the de Rham cohomology of every smooth manifold at degree 0.

Theorem 4.3.7. *If M is a connected smooth n-manifold, then $H^0(M) \cong \mathbb{R}$.*

Proof. Note first that $B^0(M) = 0$ and $Z^0(M) = \{f \in C^\infty(M) : df = 0\}$. Since every point of M has an open neighbourhood which is diffeomorphic to $\mathbb{R}^n$, every $f \in Z^0(M)$ is locally constant on M. The connectedness of M implies now that f is constant on M. Therefore, $H^0(M) = Z^0(M) \cong \mathbb{R}$. □

4.4 Orientable smooth manifolds

Let V be a real n-dimensional vector space, $n \geq 1$. We say that two ordered basis $[v_1, ..., v_n]$ and $[w_1, ..., w_n]$ define the same orientation of V if the change of basis matrix has positive determinant. This is an equivalence relation on the set of all ordered basis of V with exactly two equivalence classes, which are called *orientations* of V. The choice of an orientation of V turns it into an *oriented vector space.*

Recall that if $w_j = \sum_{i=1}^{n} a_{ij} v_i$, $j = 1, ..., n$, then

$$\omega(w_1, ..., w_n) = \omega(v_1, ..., v_n) \cdot \det(a_{ij})_{1 \leq i,j \leq n}.$$

for every $\omega \in \Lambda^n(V) \cong \mathbb{R}$. This implies that two ordered basis $[v_1, ..., v_n]$ and $[w_1, ..., w_n]$ define the same orientation of V if and only if

$$(v_1^* \wedge \cdots \wedge v_n^*)(w_1, ..., w_n) > 0$$

or equivalently

$$\omega(v_1, ..., v_n) \cdot \omega(w_1, ..., w_n) > 0$$

for every non-zero $\omega \in \Lambda^n(V)$. Thus the choice of an orientation on V can be determined by the choice of a non-zero element of $\Lambda^n(V)$. More precisely, having chosen chosen a non-zero $\omega \in \Lambda^n(V)$, we usually say that the ordered basis $[v_1, ..., v_n]$ is *positively oriented with respect to* ω if $\omega(v_1, ..., v_n) > 0$. Two non-zero elements ω, $\theta \in \Lambda^n(V)$ determine the same orientation if and only if $\theta = \lambda\omega$ for some $\lambda > 0$. This is again an equivalence relation with two equivalence classes on the set of non-zero elements of $\Lambda^n(V)$. So, we could have equally well defined an orientation of V to be one of these two equivalence classes.

An orientation of a smooth n-manifold is now the choice of an orientation on each tangent space coherently, so that they vary smoothly. However, this choice my not be always possible.

Defibition 4.4.1. A smooth n-manifold M, $n \geq 1$, is called *orientable* if there exists a nowhere vanishing differential n-form ω on M. Any such form is called a *volume element* of M.

We say that two volume elements ω, $\theta \in A^n(M)$ define the same orientation on M if there exists a smooth function $f : M \to (0, +\infty)$ such that $\theta = f\omega$. This is an equivalence relation on the set of volume elements of $A^n(M)$, an equivalence class of which is called an *orientation* of M. The choice of an orientation on M makes it an *oriented manifold.*

On a connected orientable smooth n-manifold M there are exactly two orientations. Indeed, let ω be a nowhere vanishing differential n-form on M. If θ is any other nowhere vanishing differential n-form on M, there exists a smooth function $f : M \to \mathbb{R} \setminus \{0\}$ such that $\theta = f\omega$. Since M is connected, we must have $f > 0$ everywhere of M or $f < 0$. In the first case θ and ω define the same orientation, and in the second θ and $-\omega$ define the same orientation.

Examples 4.4.2. (a) Any open subset M of $\mathbb{R}^n$, $n \geq 1$, is orientable. An orientation is defined by the volume element $dx^1 \wedge \cdots \wedge dx^n$ restricted on M. Note that at each tangent space $T_p\mathbb{R}^n \cong \mathbb{R}^n$ its value is the determinant. This is usually called the positive orientation of $\mathbb{R}^n$.

(b) The n-sphere S^n is an orientable smooth n-manifold. We shall prove that if

$$\omega = \sum_{j=1}^{n+1} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^{n+1} \in A^n(\mathbb{R}^{n+1}),$$

and $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion, then $i^*\omega$ is nowhere vanishing. This is the standard volume element of S^n.

Let $p = (x^1, ..., x^{n+1}) \in S^n$. The tangent space T_pS^n is the hyperplane in $\mathbb{R}^{n+1}$ which is orthogonal to the vector p. The subgroup $G = \{\sigma \in S_{n+1} : \sigma(1) = 1\}$ of the symmetric group S_{n+1} is isomorphic to S_n. Let also $\sigma_j = (1 \quad j)$, $1 \leq j \leq n+1$. The right cosets of G in S_{n+1} are $G\sigma_j$, $1 \leq j \leq n+1$. Putting $v_1 = p$, for every v_2,..., $v_{n+1} \in T_pS^n$ we compute

$$\begin{aligned}
&(dx^1 \wedge \cdots \wedge dx^{n+1})_p(v_1, v_2, ..., v_{n+1}) \\
&= \sum_{\sigma \in S_{n+1}} (\text{sgn}\sigma)(dx^1)_p(v_{\sigma(1)}) \cdots (dx^{n+1})_p(v_{\sigma(n+1)}) \\
&= \sum_{j=1}^{n+1} \sum_{\sigma\sigma_j \in G} (\text{sgn}\sigma)(dx^1)_p(v_{\sigma(1)}) \cdots (dx^{n+1})_p(v_{\sigma(n+1)}) \\
&= \sum_{j=1}^{n+1} \sum_{\tau \in G} (-\text{sgn}\tau)(dx^1)_p(v_{\tau\sigma_j(1)}) \cdots (dx^{n+1})_p(v_{\tau\sigma_j(n+1)})
\end{aligned}$$

$$= \sum_{j=1}^{n+1} \sum_{\tau \in G} (-\mathrm{sgn}\tau)(dx^1)_p(v_{\tau(j)}) \cdots (dx^j)_p(v_{\tau(1)}) \cdots (dx^{n+1})_p(v_{\tau(n+1)})$$

$$= \sum_{j=1}^{n+1} \sum_{\tau \in G} (-\mathrm{sgn}\tau) x^j (dx^1)_p(v_{\tau(j)}) \cdots (dx^{j-1})_p(v_{\tau(j-1)})$$

$$(dx^{j+1})_p(v_{\tau(j+1)}) \cdots (dx^{n+1})_p(v_{\tau(n+1)})$$

$$= \sum_{j=1}^{n+1} \sum_{\rho \in G} (-1)^{j-1} x^j (\mathrm{sgn}\rho)(dx^1)_p(v_{\rho(2)}) \cdots (dx^{j-1})_p(v_{\rho(j)})$$

$$(dx^{j+1})_p(v_{\rho(j+1)}) \cdots (dx^{n+1})_p(v_{\rho(n+1)})$$

(putting $\rho = \tau(2 \quad 3 \cdots j)^{-1}$)

$$= \left(\sum_{j=1}^{n+1} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^{n+1} \right)(v_2, ..., v_{n+1}).$$

If now $\{v_2, ..., v_{n+1}\}$ is a basis of T_pS^n, then $\{v_1, v_2, ..., v_{n+1}\}$ is a basis of $\mathbb{R}^{n+1}$ and therefore $(dx^1 \wedge \cdots \wedge dx^{n+1})_p(v_1, v_2, ..., v_{n+1}) \neq 0$. It follows that $i^*\omega$ nowhere vanishes on S^n.

(c) If $a : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denotes the antipodal map $a(x) = -x$ and ω is the differential n-form of (b), then

$$a^*\omega = \sum_{j=1}^{n+1} (-1)^{j-1}(-x^j) d(-x^1) \wedge \cdots \wedge d(-x^{j-1}) \wedge d(-x^{j+1}) \wedge \cdots \wedge d(-x^{n+1})$$

$$= (-1)^{n+1}\omega.$$

Thus, ω is a-invariant, that is $a^*\omega = \omega$, if n is odd. In this case, if $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion as before, then $i^*\omega$ induces a unique well defined differential n-form Ω on $\mathbb{R}P^n$ such that $\pi^*\Omega = i^*\omega$, where $\pi : S^n \to \mathbb{R}P^n$ is the quotient map. Since π a local smooth diffeomorphism, that is its derivative at each point of S^n is a linear isomorphism, and $i^*\omega$ nowhere vanishes, it follows that Ω vanishes nowhere on $\mathbb{R}P^n$. This shows that the odd dimensional real projective spaces are orientable smooth manifolds.

Suppose now that n is even. If there exists a nowhere vanishing $\Omega \in A^n(\mathbb{R}P^n)$, then $\pi^*\Omega \in A^n(S^n)$ nowhere vanishes and is a-invariant. There exists a smooth function $f : S^n \to \mathbb{R} \setminus \{0\}$ such that $\pi^*\Omega = f \cdot i^*\omega$. Since S^n is connected, $f > 0$ everywhere on S^n or $f < 0$. Now we have

$$f\omega = \pi^*\Omega = a^*(\pi^*\Omega) = a^*(f\omega) = (f \circ a)a^*\omega = -(f \circ a)\omega,$$

because n is even. It follows that $f = -(f \circ a)$, contradiction. Thus, $\mathbb{R}P^n$ is non-orientable in case n is even.

Theorem 4.4.3. *A smooth n-manifold M, $n \geq 1$, is orientable if and only if there exists a smooth atlas $\mathcal{A}$ of M such that*

$$\det D(\phi_V \circ \phi_U^{-1})(x) > 0$$

for every $x \in \phi_U(U \cap V)$ and (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ with $U \cap V \neq \varnothing$.

Proof. Suppose first that M is orientable and let $\omega \in A^n(M)$ be nowhere vanishing on M. There exists a smooth atlas $\mathcal{B}$ of M such that $\psi_U(U) = \mathbb{R}^n$ for every $(U, \psi_U) \in \mathcal{B}$. There exist smooth functions $f_U : \mathbb{R}^n \to \mathbb{R} \setminus \{0\}$ such that

$$(\psi_U^{-1})^*(\omega|_U) = f_U dx^1 \wedge \cdots \wedge dx^n.$$

If $f_U > 0$, we put $\phi_U = \psi_U$, but if $f_U < 0$, we put $\phi_U = g \circ \psi_U$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is the linear isomorphism

$$g(x^1, x^2, ..., x^n) = (-x^1, x^2, ..., x^n)$$

which has negative determinant. In this second case, where $f_U < 0$, we have

$$(\phi_U^{-1})^*(\omega|_U) = (f_U \circ g^{-1}) \cdot (g^{-1})^*(dx^1 \wedge \cdots \wedge dx^n)$$

$$= (f_U \circ g^{-1}) \cdot \det g^{-1} \cdot dx^1 \wedge \cdots \wedge dx^n = -(f_U \circ g^{-1}) \cdot dx^1 \wedge \cdots \wedge dx^n.$$

Thus, putting $g_U = -f_U \circ g^{-1}$, in case $f_U < 0$, and $g_U = f_U$, in case $f_U > 0$, we have

$$(\phi_U^{-1})^*(\omega|_U) = g_U dx^1 \wedge \cdots \wedge dx^n$$

in any case and $g_U > 0$. The class $\mathcal{A} = \{(U, \phi_U) : (U, \psi_U) \in \mathcal{B}\}$ is a smooth atlas of M and if (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ are such hat $U \cap V \neq \varnothing$, we have

$$g_U dx^1 \wedge \cdots \wedge dx^n = (\phi_U^{-1})^*(\omega|_{U \cap V}) = (\phi_V \circ \phi_U^{-1})^*((\phi_V^{-1})^*(\omega|_{U \cap V}))$$

$$\det D(\phi_V \circ \phi_U^{-1}) \cdot (g_V \circ (\phi_V \circ \phi_U^{-1})) \cdot dx^1 \wedge \cdots \wedge dx^n$$

on $U \cap V$ and therefore $\det D(\phi_V \circ \phi_U^{-1}) > 0$

Conversely, suppose that there exists a smooth atlas $\mathcal{A}$ such that

$$\det D(\phi_V \circ \phi_U^{-1})(x) > 0$$

for every $x \in \phi_U(U \cap V)$ and (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ with $U \cap V \neq \varnothing$. There exists a smooth partition of unity $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ which is subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M, by Theorem 1.4.4. We shall show that the differential n-form

$$\omega = \sum_{(U,\phi_U) \in \mathcal{A}} f_U \cdot \phi_U^*(dx^1 \wedge \cdots \wedge dx^n)$$

vanishes nowhere on M.

Let $p \in M$. There exists an open neighbourhood W of p contained in in some $U_0 \in \mathcal{U}$, which intersects only a finite number $\text{supp} f_{U_1}$,..., $\text{supp} f_{U_k}$, for some $k \in \mathbb{N}$, of elements of the class $\{\text{supp} f_U : (U, \phi_U) \in \mathcal{A}\}$. Thus,

$$\omega_q = \sum_{j=1}^{k} f_{U_j}(q) \cdot \phi_{U_j}^*(dx^1 \wedge \cdots \wedge dx^n)_q$$

$$= \phi_{U_0}^*\left(\sum_{j=1}^{k}(f_{U_j}\circ\phi_{U_0}^{-1})\cdot\det D(\phi_{U_j}\circ\phi_{U_0}^{-1})\circ\phi_{U_0}\cdot dx^1\wedge\cdots\wedge dx^n\right)_q$$

$$= \left(\sum_{j=1}^{k} f_{U_j}(q)\cdot\det D(\phi_{U_j}\circ\phi_{U_0}^{-1})(\phi_{U_0}(q))\right)\phi_{U_0}^*(dx^1\wedge\cdots\wedge dx^n)_q$$

for every $q\in W$. Since $f_{U_1}(p)+\cdots+f_{U_k}(p)=1$, at least one of $f_{U_1}(p)$,..., $f_{U_k}(p)$ must be positive. This together with our assumption imply that $\omega_p\neq 0$. □

Example 4.4.4. The transition maps of the smooth charts of the canonical atlas of the complex projective n-space $\mathbb{C}P^n$ described in Example 1.1.4(d) are biholomorphic maps of open subsets of $\mathbb{C}^n$. Hence its Jacobian matrix at every point in its domain of definition has positive determinant. From the above Theorem 4.4.3 follows that $\mathbb{C}P^n$ is orientable for every $n\in\mathbb{Z}^+$.

Let M be an oriented smooth n-manifold by a volume element ω. A smooth chart (U,ϕ) of M will be called *positively oriented* if there exists some smooth function $g:\phi(U)\to(0,+\infty)$ such that $(\phi^{-1})^*(\omega|_U)=g dx^1\wedge\cdots\wedge dx^n$. A smooth diffeomorphism $f:M\to M$ is called *orientation preserving* if $f^*\omega$ and ω define the same orientation. If $f^*\omega$ and $-\omega$ define the same orientation, we say that f reverses orientation. In particular, a smooth diffeomorphism $f:\mathbb{R}^n\to\mathbb{R}^n$ is orientation preserving if and only if $\det Df(x)>0$ for every $x\in\mathbb{R}^n$, because

$$f^*(dx^1\wedge\cdots\wedge dx^n)=(\det Df)\cdot dx^1\wedge\cdots\wedge dx^n.$$

If $\det Df<0$, then f is reverses orientation.

4.5 Integration on oriented manifolds

A differential k-form ω on a smooth n-manifold M has compact support if there exists a compact set $K\subset M$ such that $\omega_p=0$ for every $p\in M\setminus K$. The closed set $\mathrm{supp}\omega=\overline{\{p\in M:\omega_p\neq 0\}}$ is the support of ω. The set $A_c^k(M)$ of all differential k-forms with compact supports on M is a vector subspace of $A^k(M)$, $0\leq k\leq n$. Of course $A^0(M)$ is just the set $C_c^\infty(M)$ of all smooth real valued functions on M with compact supports.

If $\omega\in A_c^n(\mathbb{R}^n)$, there exists a unique $g\in C_c^\infty(\mathbb{R}^n)$ such that $\omega=g dx^1\wedge\cdots\wedge dx^n$. We define

$$\int_{\mathbb{R}^n}\omega=\int_{\mathbb{R}^n}g.$$

If $f:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth diffeomorphism (or more generally f is a smooth diffeomorphism of open subsets of $\mathbb{R}^n$), then

$$f^*\omega=(f\circ g)\cdot\det Df\cdot dx^1\wedge\cdots\wedge dx^n.$$

On the other hand, from the change of variables formula we have

$$\int_{\mathbb{R}^n}g=\int_{\mathbb{R}^n}(f\circ g)\cdot|\det Df|.$$

It follows that

$$\int_{\mathbb{R}^n} f^*\omega = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \omega, & \text{if } f \text{ is orientation preserving,} \\ \displaystyle -\int_{\mathbb{R}^n} \omega, & \text{if } f \text{ is orientation reversing.} \end{cases}$$

Let now M be a oriented smooth n-manifold and let $\mathcal{A}$ be a smooth atlas of M consisting of positively oriented smooth charts of M. Thus,

$$\det D(\phi_V \circ \phi_U^{-1})(x) > 0$$

for every $x \in \phi_U(U \cap V)$ and (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ with $U \cap V \neq \varnothing$, as the proof of Theorem 4.4.3 shows. There exists a smooth partition of unity $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. For every $\omega \in A_c^n(M)$ the differential n-form $f_U\omega$ has compact support and vanishes outside U. We define

$$\int_M \omega = \sum_{(U,\phi_U)\in\mathcal{A}} \int_{\mathbb{R}^n} (\phi_U^{-1})^*(f_U\omega).$$

In order this definition to be sound, we must show that it does not depend on the choice of the smooth atlas $\mathcal{A}$ and the subordinated smooth partition of unity. Let $\mathcal{B}$ be another smooth atlas of M consisting of positively oriented charts of M and let $\{h_W : (W, \psi_W) \in \mathcal{B}\}$ be smooth partition of unity subordinated to the open cover $\mathcal{W} = \{W : (W, \psi_W) \in \mathcal{B}\}$ of M. The transition maps $\phi_U \circ \psi_W^{-1}$ for $(U, \phi_U) \in \mathcal{A}$, $(W, \psi_W) \in \mathcal{B}$ with $U \cap W \neq \varnothing$, are orientation preserving smooth diffeomorphisms between open subsets of $\mathbb{R}^n$. We compute

$$\sum_{(U,\phi_U)\in\mathcal{A}} \int_{\mathbb{R}^n} (\phi_U^{-1})^*(f_U\omega) = \sum_{\substack{(U,\phi_U)\in\mathcal{A} \\ (W,\psi_W)\in\mathcal{B}}} \int_{\mathbb{R}^n} (\phi_U^{-1})^*(f_U h_W \omega)$$

$$= \sum_{\substack{(U,\phi_U)\in\mathcal{A} \\ (W,\psi_W)\in\mathcal{B}}} \int_{\mathbb{R}^n} (\phi_U \circ \psi_W^{-1})^*((\phi_U^{-1})^*(f_U h_W \omega)) = \sum_{\substack{(U,\phi_U)\in\mathcal{A} \\ (W,\psi_W)\in\mathcal{B}}} \int_{\mathbb{R}^n} (\psi_W^{-1})^*(f_U h_W \omega)$$

$$= \sum_{(W,\psi_W)\in\mathcal{B}} \int_{\mathbb{R}^n} (\psi_W^{-1})^*(h_W\omega).$$

In this way we get a linear map $\displaystyle\int_M : A_c^n(M) \to \mathbb{R}$ which is called the *(oriented Riemann) integral* on the oriented smooth n-manifold M.

If $f : M \to M$ is a smooth diffeomorphism of a connected, smooth n-manifold M, then

$$\int_M f^*\omega = \begin{cases} \displaystyle\int_M \omega, & \text{if } f \text{ is orientation preserving,} \\ \displaystyle -\int_M \omega, & \text{if } f \text{ is orientation reversing.} \end{cases}$$

for every $\omega \in A_c^n(M)$.

Theorem 4.5.1. *If M is an oriented smooth n-manifold then*

$$\int_M d\omega = 0$$

for every $\omega \in A_c^{n-1}(M)$.

Proof. Suppose first that there exists a positively oriented smooth chart (U, ϕ) of M such that $\text{supp}\omega \subset U$. There exist $g_1,..., g_n \in C_c^\infty(\phi(U))$ such that

$$(\phi^{-1})^*\omega = \sum_{j=1}^{n} (-1)^{j-1} g_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

and differentiating

$$(\phi^{-1})^*(d\omega) = d((\phi^{-1})^*\omega) = \sum_{j=1}^{n} (-1)^{j-1} dg_j \wedge dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

$$= \sum_{i,j=1}^{n} (-1)^{j-1} \frac{\partial g_j}{\partial x^i} \cdot dx^i \wedge dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

$$= \left(\sum_{j=1}^{n} \frac{\partial g_j}{\partial x^j} \right) \cdot dx^1 \wedge \cdots \wedge dx^n.$$

Therefore,

$$\int_M d\omega = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial g_j}{\partial x^j} dx^1 \cdots dx^n$$

$$= \sum_{j=1}^{n} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial g_j}{\partial x^j} dx^j \right) dx^1 \cdots dx^{j-1} dx^{j+1} \cdots dx^n = 0$$

by Fubini's theorem and the Fundamental Theorem of Calculus.

In the general case we consider a smooth atlas $\mathcal{A}$ of M consisting of positively oriented charts and a smooth partition of unity $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. Then, $\text{supp}(f_U\omega) \subset U$ and from the above we get

$$\int_M d\omega = \sum_{(U,\phi_U) \in \mathcal{A}} \int_M d(f_U\omega) = 0. \quad \square$$

Corollary 4.5.2. *Let M be an oriented smooth n-manifold.*
(a) If M is compact, then

$$\int_M d\omega = 0$$

for every $\omega \in A^{n-1}(M)$.
(b) $\int_M : A_c^n(M) \to \mathbb{R}$ is a linear epimorphism.

Proof. Only the second assertion requires proof. For this it suffices to construct a differential n-form with compact support on M with non-zero integral. Let (U, ϕ) be a positively oriented smooth chart of M and let $p \in U$ be any point. There exists a smooth function $f : M \to [0, 1]$ such that $f(p) = 1$ and $\mathrm{supp} f$ is a compact subset of U, by Corollary 1.4.5. If we take

$$\omega = \begin{cases} \phi^*((f \circ \phi^{-1}) \cdot dx^1 \wedge \cdots \wedge dx^n), & \text{on } U \\ 0, & \text{on } M \setminus U \end{cases}$$

then $\omega \in A^n_c(M)$ and

$$\int_M \omega = \int_{\mathbb{R}^n} f \circ \phi^{-1} > 0. \quad \square$$

The kernel of the linear epimorphism $\displaystyle\int_M : A^n_c(M) \to \mathbb{R}$ contains $d(A^{n-1}_c(M))$, by Theorem 4.5.1. It is a non-trivial fact that this is precisely the kernel in case M is connected. The proof can be divided into a series of steps, the most crucial of which is the first one.

Lemma 4.5.3. *The kernel of* $\displaystyle\int_{\mathbb{R}^n} : A^n_c(\mathbb{R}^n) \to \mathbb{R}$ *is* $d(A^{n-1}_c(\mathbb{R}^n))$.

Proof. Let $\omega \in A^n_c(\mathbb{R}^n)$ be such that $\displaystyle\int_{\mathbb{R}^n} \omega = 0$. There exists a unique $f \in C^\infty_c(\mathbb{R}^n)$ such that $\omega = f dx^1 \wedge \cdots \wedge dx^n$. If $\theta \in A^{n-1}_c(\mathbb{R}^n)$, there exist $f_1,..., f_n \in C^\infty_c(\mathbb{R}^n)$ such that

$$\theta = \sum_{j=1}^n (-1)^{j-1} f_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

and

$$d\theta = \left(\sum_{j=1}^n \frac{\partial f_j}{\partial x^j} \right) \cdot dx^1 \wedge \cdots \wedge dx^n.$$

Thus it suffices to prove that given $f \in C^\infty_c(\mathbb{R}^n)$ such that $\displaystyle\int_{\mathbb{R}^n} f = 0$, there exist $f_1,..., f_n \in C^\infty_c(\mathbb{R}^n)$ such that

$$f = \sum_{j=1}^n \frac{\partial f_j}{\partial x^j}.$$

We proceed by induction. For $n = 1$, it suffices to take $g_1(t) = \displaystyle\int_{-\infty}^t f$. Suppose that the problem can be solved in dimension $n - 1$. There exists $R > 0$ such that $\mathrm{supp} f \subset (-R, R)^n$. Let $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be defined by

$$g(x^1, ..., x^{n-1}) = \int_{\mathbb{R}} f(x^1, ..., x^{n-1}, x^n) dx^n.$$

Then, $g \in C^\infty_c(\mathbb{R}^{n-1})$ and

$$\int_{\mathbb{R}^{n-1}} g = \int_{\mathbb{R}^n} f = 0,$$

by Fubini's theorem. So there exist g_1,..., $g_{n-1} \in C_c^\infty(\mathbb{R}^{n-1})$ such that

$$g = \sum_{j=1}^{n-1} \frac{\partial g_j}{\partial x^j}.$$

Let now $\rho \in C_c^\infty(\mathbb{R})$ be such that supp$\rho \subset (-R, R)$ and $\int_{\mathbb{R}} \rho = 1$. We define $f_j \in C_c^\infty(\mathbb{R}^n)$ by

$$f_j(x^1, ..., x^n) = g_j(x^1, ..., x^{n-1}) \cdot \rho(x^n)$$

for $j = 1, ..., n-1$. Let $h \in C_c^\infty(\mathbb{R}^n)$ be the function with

$$h(x^1, ..., x^{n-1}, x^n) = f(x^1, ..., x^n) - g(x^1, ..., x^{n-1})\rho(x^n)$$

and let $f_n : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f_n(x^1, ..., x^n) = \int_{-\infty}^{x^n} h(x^1, ..., x^{n-1}, t)dt.$$

Then, $f = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x^j}$, by construction. Finally, f_n has compact support because $h(x^1, ..., x^n) = 0$ for $x^n < -R$ and for $x^n > R$ we have

$$\begin{aligned} f_n(x^1, ..., x^n) &= \int_{-\infty}^{x^n} f(x^1, ..., x^{n-1}, t)dt - g(x^1, ..., x^{n-1}) \int_{-\infty}^{x^n} \rho(t)dt \\ &= \int_{\mathbb{R}} f(x^1, ..., x^{n-1}, t)dt - g(x^1, ..., x^{n-1}) \int_{\mathbb{R}} \rho(t)dt \\ &= \int_{\mathbb{R}} f(x^1, ..., x^{n-1}, t)dt - g(x^1, ..., x^{n-1}) = 0. \quad \square \end{aligned}$$

Lemma 4.5.4. *For every non-empty open set $W \subset \mathbb{R}^n$ and every $\omega \in A_c^n(\mathbb{R}^n)$ there exists some $\theta \in A_c^{n-1}(\mathbb{R}^n)$ such that* supp$(\omega - d\theta) \subset W$.

Proof. There exists some $\omega_1 \in A_c^n(\mathbb{R}^n)$ with supp$\omega_1 \subset W$ and $\int_{\mathbb{R}^n} \omega_1 = 1$. Then,

$$\int_{\mathbb{R}^n} \left(\omega - \left(\int_{\mathbb{R}^m} \omega \right) \omega_1 \right) = 0$$

and by Lemma 4.5.3 there exists some $\theta \in A_c^{n-1}(\mathbb{R}^n)$ such that

$$d\theta = \omega - \left(\int_{\mathbb{R}^m} \omega \right) \omega_1.$$

Therefore, supp$(\omega - d\theta) \subset$ supp$\omega_1 \subset W$. $\square$

Lemma 4.5.5. *If M is a connected smooth n-manifold, then for every non-empty open set $W \subset \mathbb{R}^n$ and every $\omega \in A_c^n(M)$ there exists some $\theta \in A_c^{n-1}(M)$ such that*

$\mathrm{supp}(\omega - d\theta) \subset W$.

Proof. First suppose that $\omega \in A_c^n(M)$ is such that its support is contained in an open subset U of M which is diffeomorphic to $\mathbb{R}^n$. Since M is connected, there is a finite chain of open subsets $U_1, ..., U_k$ of M, for some $k \in \mathbb{N}$, which are all diffeomorphic to $\mathbb{R}^n$ and are such that $U_1 \subset U$, $U_k \subset W$ and $U_j \cap U_{j+1} \neq \varnothing$, for $j = 1, ..., n-1$. From Lemma 4.5.4, there exist $\theta_1, ..., \theta_{k-1} \in A_c^{n-1}(M)$ such that

$$\mathrm{supp}\left(\omega - \sum_{i=1}^{j} d\theta_i\right) \subset U_j \cap U_{j+1}$$

for every $j = 1, ..., n-1$. Thus, it suffices to take $\theta = \sum_{i=1}^{k-1} d\theta_i$.

In the general case using a smooth partition of unity it is possible to write

$$\omega = \sum_{j=1}^{m} \omega_j$$

for some $m \in \mathbb{N}$, where each $\omega_j \in A_c^n(M)$ has support which is contained in some open subset of M which is diffeomorphic to $\mathbb{R}^n$. According to the above, there exists $\eta_j \in A_c^{n-1}(M)$ such that $\mathrm{supp}(\omega_j - d\eta_j) \subset W$, $j = 1, ..., m$. If now $\theta = \sum_{j=1}^{m} \eta_j$, then

$$\mathrm{supp}(\omega - d\theta) \subset \bigcup_{j=1}^{m} \mathrm{supp}(\omega_j - d\eta_j) \subset W. \quad \square$$

Theorem 4.5.6. *If M is a connected smooth n-manifold, the kernel of $\int_M : A_c^n(M) \to \mathbb{R}$ is $d(A_c^{n-1}(M))$.*

Proof. Let $\omega \in A_c^n(M)$ be such that $\int_M \omega = 0$. Let $W \subset M$ be an open set which is diffeomorphic to $\mathbb{R}^n$. From Lemma 4.5.5 there exists some $\theta \in A_c^{n-1}(M)$ such that $\mathrm{supp}(\omega - d\theta) \subset W$. From Theorem 4.5.1 we have

$$\int_M (\omega - d\theta) = 0$$

and from Lemma 4.5.3 there exists some $\eta \in A^{n-1}(M)$ such that $\mathrm{supp}\eta \subset W$ and $\omega - d\theta = d\eta$. Thus $\omega = d(\theta + \eta)$ and $\theta + \eta \in A_c^{n-1}(M)$. $\square$

It follows immediately from Theorem 4.5.1 and its Corollary 4.5.2 that integration on a compact oriented smooth n manifold M induces a linear epimorphism

$$\int_M : H^n(M) \longrightarrow \mathbb{R}.$$

In particular, $H^n(M)$ is non-trivial. In case M is connected and compact, Theorem 4.5.6 gives the following.

Corollary 4.5.7. *If M is a connected compact oriented smooth n-manifold, then the integration of differential n-forms on M induces a linear isomorphism*

$$\int_M : H^n(M) \xrightarrow{\cong} \mathbb{R}. \quad \square$$

4.6 Stokes' formula

Let M be a smooth n-manifold. An open set $D \subset M$ is called a *domain with smooth boundary* if for every $p \in \partial D$ there exists a smooth chart (U, ϕ) of M such that $p \in U$ and

$$\phi(U \cap D) = \phi(U) \cap \{(x^1, ..., x^{n-1}, x^n) \in \mathbb{R}^n : x^n > 0\},$$

$$\phi(U \cap \partial D) = \phi(U) \cap (\mathbb{R}^{n-1} \times \{0\}).$$

In particular, ∂D is a $(n-1)$-dimensional smooth submanifold of M. A smooth chart (U, ϕ) as above will be called $\overline{D}$-half space smooth chart. Each such smooth chart is ∂D-straightening. Let (V, ψ) be another $\overline{D}$-half space smooth chart such that $\partial D \cap U \cap V \neq \varnothing$. If $\phi \circ \psi^{-1} = (g_1, ..., g_n)$ is the transition smooth diffeomorphism, then $g_n(x^1, ..., x^{n-1}, 0) = 0$ and $g_n(x^1, ..., x^n) > 0$ for $x^n > 0$. So,

$$D(\phi \circ \psi^{-1})(x^1, ..., x^{n-1}, 0) = \begin{pmatrix} \frac{\partial g_1}{\partial x^1} & \cdots & \frac{\partial g_1}{\partial x^{n-1}} & \frac{\partial g_1}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_{n-1}}{\partial x^1} & \cdots & \frac{\partial g_{n-1}}{\partial x^{n-1}} & \frac{\partial g_{n-1}}{\partial x^n} \\ 0 & \cdots & 0 & \frac{\partial g_n}{\partial x^n} \end{pmatrix}$$

and

$$\frac{\partial g_n}{\partial x^n}(x^1, ..., x^{n-1}, 0) = \lim_{t \to 0^+} \frac{g_n(x^1, ..., x^{n-1}, t)}{t} > 0.$$

If $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection onto the first $n-1$ coordinates and $i : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ is the inclusion, then

$$(\pi \circ (\phi \circ \psi^{-1}) \circ i)(x^1, ..., x^{n-1}) = (g_1(x^1, ..., x^{n-1}, 0), ..., g_{n-1}(x^1, ..., x^{n-1}, 0))$$

and

$$D(\pi \circ (\phi \circ \psi^{-1}) \circ i)(x^1, ..., x^{n-1}, 0) = \begin{pmatrix} \frac{\partial g_1}{\partial x^1} & \cdots & \frac{\partial g_1}{\partial x^{n-1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial g_{n-1}}{\partial x^1} & \cdots & \frac{\partial g_{n-1}}{\partial x^{n-1}} \end{pmatrix}.$$

If now M is orientable and we have chosen a specific orientation, we can cover ∂D by positively oriented smooth charts of M, which are ∂D-straightening as above. It follows that $\det D(\pi \circ (\phi \circ \psi^{-1}) \circ i)(x^1, ..., x^{n-1}, 0) > 0$. This means that ∂D is orientable and has an orientation induced by the orientation of M.

Let now M be oriented and let $\mathcal{A}$ be a smooth atlas of M which consists of positively oriented smooth charts of M, so that every element of $\mathcal{A}$ whose domain of definition intersects ∂D is a $\overline{D}$-half space smooth chart as above. We choose a smooth partition of unity $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. For every $\omega \in A^n_c(M)$ we define its integral over $\overline{D}$ by

$$\int_{\overline{D}} \omega = \sum_{(U,\phi_U)\in\mathcal{A}} \int_{\phi_U(U\cap\overline{D})} (\phi_U^{-1})^*(f_U\omega).$$

The definition does not depend on the choice of the smooth atlas $\mathcal{A}$ consisting of positively oriented $\overline{D}$-half space smooth charts, as above, and the choice of the subordinated smooth partition of unity. The following is a generalization of Theorem 4.5.1, as well as its proof.

Theorem 4.6.1. *Let M be an oriented smooth n-manifold and let $D \subset M$ be a domain with smooth boundary. Let $i : \partial D \hookrightarrow M$ denote the inclusion. Then*

$$(-1)^n \int_{\partial D} i^*\omega = \int_{\overline{D}} d\omega$$

for every $\omega \in A^n_c(M)$.

Proof. We assume first that there exists a positively oriented $\overline{D}$-half space smooth chart (U, ϕ) as above such that $U \cap \partial D \neq \varnothing$ and $\phi(\overline{D} \cap \mathrm{supp}\omega) \subset (0,1)^n \subset \phi(U)$. As in the proof of Theorem 4.5.1, there exist g_1,..., $g_n \in C^\infty_c(\phi(U))$ such that

$$(\phi^{-1})^*\omega = \sum_{j=1}^n (-1)^{j-1} g_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

and

$$(\phi^{-1})^*(d\omega) = \left(\sum_{j=1}^n \frac{\partial g_j}{\partial x^j}\right) \cdot dx^1 \wedge \cdots \wedge dx^n.$$

Therefore,

$$\int_{\overline{D}} d\omega = \sum_{j=1}^n \int_{[0,1]^n} \frac{\partial g_j}{\partial x^j} dx^1 \cdots dx^n = -\int_{[0,1]^{n-1}} g_n(x^1, ..., x^{n-1}, 0) dx^1 \cdots dx^{n-1}$$

by Fubini's theorem and the Fundamental Theorem of Calculus.

On the other hand $\phi(\mathrm{supp} i^*\omega) \subset (0,1)^{n-1}$ and so

$$\int_{\partial D} i^*\omega = \int_{[0,1]^{n-1}} ((\pi \circ \phi)^{-1})^*(i^*\omega)$$

$$= \int_{[0,1]^{n-1}} (-1)^{n-1} g_n(x^1, ..., x^{n-1}, 0) dx^1 \cdots dx^{n-1} = (-1)^n \int_{\overline{D}} d\omega.$$

In case (U, ϕ) is a positively oriented chart of M such that $\mathrm{supp}\omega \subset U \subset D$, we have

$$\int_{\partial D} i^*\omega = (-1)^n \int_{\overline{D}} d\omega = 0$$

from Theorem 4.5.1.

In the general case, we take a smooth atlas $\mathcal{A}$ of M which consists of positively oriented smooth charts, so that every element of $\mathcal{A}$ whose domain of definition intersects ∂D is a $\overline{D}$-half space smooth chart as in the beginning, and a smooth partition of unity $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ which is subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. Since $\mathrm{supp}(f_U\omega) \subset U$ we have

$$\int_{\overline{D}} d\omega = \sum_{(U,\phi_U)\in\mathcal{A}} \int_{\overline{D}} d(f_U\omega) = \sum_{(U,\phi_U)\in\mathcal{A}} (-1)^n \int_{\partial D} i^*(f_U\omega) = (-1)^n \int_{\partial D} i^*\omega. \quad \square$$

It is worth to describe the induced orientation used to define integration over ∂D. We shall need the notion of tangent vector which is directed inward or outward of D. Let $p \in \partial D$. A tangent vector $v \in T_pM \setminus T_p\partial D$ is directed inward of D if it is the velocity of a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$, that is $\gamma(0) = p$ and $v = \dot{\gamma}(0)$, such that $\gamma(t) \in D$ for all $0 < t < \epsilon$. If (U, ϕ) is any $\overline{D}$-half space smooth chart with $\phi = (x^1, ..., x^{n-1}, x^n)$ and $p \in U$, then $x^n(p) = 0$ and $x^n(\gamma(t)) > 0$ for every $0 < t < \epsilon$. Therefore

$$(dx^n)_p(v) = \lim_{t\to 0^+} \frac{x^n(\gamma(t))}{t} > 0.$$

The converse is evidently also true, that is v is directed inward of D if and only if $(dx^n)_p(v) > 0$ for any $\overline{D}$-half space smooth chart $\phi = (x^1, ..., x^{n-1}, x^n)$. Similarly, v is directed outward of D if there is a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$, $v = \dot{\gamma}(0)$ and $\gamma(t) \in D$ for all $-\epsilon < t < 0$ or equivalently $(dx^n)_p(v) < 0$ for any $\overline{D}$-half space smooth chart $\phi = (x^1, ..., x^{n-1}, x^n)$. Obviously, v is directed outward of D if and only if $-v$ is directed inward of D.

Let $\mathcal{A}$ be a smooth atlas of M such that each $(U, \phi) \in \mathcal{A}$ with $U \cap \partial D \neq \varnothing$ is a $\overline{D}$-half space smooth chart and let $\mathcal{A}_{\partial D} = \{(U, \phi) \in \mathcal{A} : U \cap \partial D \neq \varnothing\}$. Let $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ be a smooth partition of unity subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. The smooth map $Y : \partial D \to TM$ defined by

$$Y(p) = \sum_{(U,\phi)\in\mathcal{A}_{\partial D}} f_U(p) \left(\frac{\partial}{\partial x^n} \right)_p$$

where in the sum $\phi = (x^1, ..., x^{n-1}, x^n)$, satisfies $Y(p) \in T_pM$ for every $p \in \partial D$. In other words Y is a smooth vector field along the smooth submanifold ∂D. If $(V, \psi) \in \mathcal{A}_{\partial D}$ and $\psi = (y^1, ..., y^{n-1}, y^n)$ with $p \in V$, then

$$(dy^n)_p(Y(p)) = \sum_{(U,\phi)\in\mathcal{A}_{\partial D}} f_U(p)(dy^n)_p \left(\frac{\partial}{\partial x^n} \right)_p > 0,$$

because $f_U(p) \geq 0$ and there exists at least one $(U, \phi) \in \mathcal{A}_{\partial D}$ such that $f_U(p) > 0$, while

$$(dy^n)_p \left(\frac{\partial}{\partial x^n} \right)_p > 0$$

for all $(U, \phi) \in \mathcal{A}_{\partial D}$. Hence $Y(p)$ is directed inward of D for every $p \in \partial D$. Also $X = -Y$ is a smooth vector field along ∂D which is directed outward of D.

Let now M be oriented and let the smooth atlas $\mathcal{A}$ as above consist of positively oriented smooth charts. As the proof of Theorem 4.4.3 shows, the orientation of M is defined by the volume element

$$\Omega = \sum_{(U,\phi)\in\mathcal{A}} f_U \cdot \phi^*(e_1^* \wedge \cdots \wedge e_n^*).$$

For every v_1,..., $v_{n-1} \in T_p\partial D$ we have

$$\Omega_p(v_1, ..., v_{n-1}, Y(p)) = \sum_{(U,\phi)\in\mathcal{A}} f_U(p) \cdot (dx^n)_p(Y(p)) \cdot \phi^*(e_1^* \wedge \cdots \wedge e_{n-1}^*)(v_1, ..., v_{n-1}).$$

This implies that an ordered basis $[v_1, ..., v_{n-1}]$ of $T_p\partial D$ is positively oriented with respect to the induced orientation from M if and only if $\Omega_p(v_1, ..., v_{n-1}, Y(p)) > 0$. Thus, the induced orientation on ∂D is given by $\Omega_{\partial D} \in A^{n-1}(\partial D)$ which is defined by

$$(\Omega_{\partial D})_p(v_1, ..., v_{n-1}) = \Omega_p(v_1, ..., v_{n-1}, Y(p))$$

for v_1,..., $v_{n-1} \in T_p\partial D$, $p \in \partial D$, where $Y : \partial D \to TM$ is any smooth vector field along ∂D which is directed inward of D.

The left hand side of the asserted formula in Theorem 3.6.1 is however the integral of $i^*\omega$ with respect to the orientation of ∂D given by $(-1)^n\Omega_{\partial D}$. In odd dimensions this orientation is given by a $\tilde{\Omega}_{\partial D} \in A^{n-1}(\partial D)$ which is defined by

$$(\tilde{\Omega}_{\partial D})_p(v_1, ..., v_{n-1}) = \tilde{\Omega}_p(v_1, ..., v_{n-1}, X(p))$$

for v_1,..., $v_{n-1} \in T_p\partial D$, $p \in \partial D$, where $\tilde{\Omega} \in A^n(M)$ gives the orientation of M and $X : \partial D \to TM$ is any smooth vector field along ∂D which is directed outward of D.

Theorem 4.6.1 can now be rephrased as follows.

Theorem 4.6.2. *Let M be an oriented smooth n-manifold whose orientation is given by a volume element Ω. Let $D \subset M$ be a domain with smooth boundary which is considered oriented by $(-1)^n\Omega_{\partial D}$ and let $i : \partial D \hookrightarrow M$ denote the inclusion. Then*

$$\int_{\partial D} i^*\omega = \int_{\overline{D}} d\omega$$

for every $\omega \in A_c^n(M)$. □

This is known as the (generalized) Stokes' formula and is a generalization of the Fundamental Theorem of Calculus.

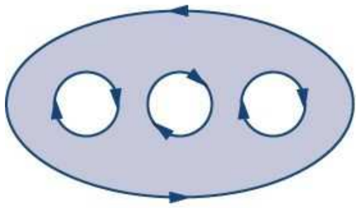
Examples 4.6.3. (a) The boundary ∂D of a domain with smooth boundary D and with compact closure in $\mathbb{R}^2$ is a compact 1-dimensional smooth submanifold of $\mathbb{R}^2$. A differential 1-form ω defined on some open neighbourhood of $\overline{D}$ is given by $\omega = Pdx + Qdy$, for a pair of smooth functions P, Q. Then

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

and according to Stokes' formula

$$\int_{\overline{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} Pdx + Qdy.$$

This is Green's theorem.



(b) Let $D \subset \mathbb{R}^3$ be a domain with smooth boundary and compact closure. A differential 2-form ω on an open neighbourhood of $\overline{D}$ can be written

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.$$

Then,

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz.$$

From Stokes' formula we get Gauss' Divergence Formula

$$\int_{\partial D} Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = \int_{\overline{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz.$$

Recall that in this case ∂D is considered oriented so that an order basis $[v_1, v_2]$ of $T_p\partial D$, $p \in \partial D$, is positively oriented if and only if it can be completed with a third vector v_3 which is directed outward of D such that $[v_1, v_2, v_3]$ is a positively oriented ordered basis of $\mathbb{R}^3$.

(c) Let $\gamma = \gamma_1 + i\gamma_2$ be a parametrised smooth simple closed curve in the complex plane $\mathbb{C}$ whose image is the boundary of a domain with smooth boundary D. Let f be a holomorphic complex function defined on some open neighbourhood of $\overline{D}$. Then, the smooth real valued functions $u = \mathrm{Re} f$, $v = \mathrm{Im} f$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

The complex line integral of f along γ can be written

$$\int_\gamma f(z)dz = \int_\gamma \omega_1 + i \int_\gamma \omega_2$$

where $\omega_1 = udx - vdy$ and $\omega_2 = vdx + udy$. The Cauchy-Riemann equations are equivalent to $d\omega_1 = 0$ and $d\omega_2 = 0$. It follows from Stokes' formula that

$$\int_\gamma f(z)dz = \int_{\overline{D}} d\omega_1 + i \int_{\overline{D}} d\omega_2 = 0.$$

This is known as Cauchy's Theorem in Complex Analysis.

4.7 Vector fields and differential forms

Let V be a real n-dimensional vector space. For each $0 \leq k \leq n$ we define the bilinear map $i : V \times \Lambda^k(V) \to \Lambda^{k-1}(V)$ by

$$(i_X\omega)(u_1, ..., u_{k-1}) = \omega(X, u_1, ..., u_{k-1})$$

for every $X \in V$, $\omega \in \Lambda^k(V)$ and u_1,..., $u_{k-1} \in V$. In case $k = 0$ we define $i = 0$. We call $i_X\omega$ the *contraction* of ω by X. Fixing the vector $X \in V$ we get thus a linear map $i_X : \Lambda(V) \to \Lambda(V)$ of degree -1, which has the following important property.

Proposition 4.7.1. *If $X \in V$, $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$, then*

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^k\omega \wedge i_X\eta.$$

Proof. Let v_1,..., $v_{k+l-1} \in V$ and put $v_0 = X$. Then,
$i_X\omega \wedge \eta(v_0, v_1, ..., v_{k+l-1})$

$$= \frac{1}{(k-1)!l!} \sum_{\sigma \in S_{k+l-1}} (\text{sgn}\sigma)\omega(X, v_{\sigma(1)}, ..., v_{\sigma(k-1)})\eta(v_{\sigma(k)}, ..., v_{\sigma(k+l-1)})$$

and
$(-1)^k\omega \wedge i_X\eta(v_0, v_1, ..., v_{k+l-1})$

$$= \frac{(-1)^k}{k!(l-1)!} \sum_{\sigma \in S_{k+l-1}} (\text{sgn}\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma k)})\eta(X, v_{\sigma(k+1)}, ..., v_{\sigma(k+l-1)}).$$

The symmetric group S_{k+l} on the set of symbols $\{0, 1, ..., k + l - 1\}$ is the disjoint union of the two sets

$$A = \{\pi \in S_{k+l} : \pi(0) \in \{0, ..., k - 1\}\},$$

$$B = \{\pi \in S_{k+l} : \pi(0) \in \{k, ..., k + l - 1\}\}.$$

Now we have
$i_X(\omega \wedge \eta)(v_0, v_1, ..., v_{k+l-1})$

$$= \frac{1}{k!l!} \sum_{\pi \in A} (\text{sgn}\pi)\omega(v_{\pi(0)}, ..., v_{\pi(k-1)})\eta(v_{\pi(k)}, ..., v_{\pi(k+l-1)})$$

$$+ \frac{1}{k!l!} \sum_{\pi \in B} (\text{sgn}\pi)\omega(v_{\pi(0)}, ..., v_{\pi(k-1)})\eta(v_{\pi(k)}, ..., v_{\pi(k+l-1)}).$$

If $\pi \in A$, we need to make $\pi^{-1}(0)$ transpositions in order to move v_0 to the first entry and so
$\omega(v_{\pi(0)}, ..., v_{\pi(k-1)})\eta(v_{\pi(k)}, ..., v_{\pi(k+l-1)})$

$$= (-1)^{\pi^{-1}(0)}\omega(X, v_{\sigma(1)}, ..., v_{\sigma(k-1)})\eta(v_{\sigma(k)}, ..., v_{\sigma(k+l-1)})$$

for some unique $\sigma \in S_{k+l-1}$ such that $\text{sgn}\pi = (-1)^{\pi^{-1}(0)}\text{sgn}\sigma$. Since $v_0 = X$ can be at any of the first k entries it follows that the first sum is equal to

$$\frac{k}{k!l!} \sum_{\sigma \in S_{k+l-1}} (\text{sgn}\sigma)\omega(X, v_{\sigma(1)}, ..., v_{\sigma(k-1)})\eta(v_{\sigma(k)}, ..., v_{\sigma(k+l-1)}).$$

Similarly, the second sum is equal to

$$\frac{(-1)^k l}{k!)!} \sum_{\sigma \in S_{k+l-1}} (\text{sgn}\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma k)})\eta(X, v_{\sigma(k+1)}, ..., v_{\sigma(k+l-1)}),$$

because in this case we need to perform k extra transpositions in order to move $v_0 = X$ to the first entry. □

Example 4.7.2. A particularly interesting case of contraction is the following. Let $\{v_1, ..., v_n\}$ be a basis of $V$and $\omega = v_1^* \wedge \cdots \wedge v_n^*$. If $X = X_1 v_1 + \cdots + X_n v_n$, then

$$i_X\omega = \sum_{j=1}^{n} (-1)^{j-1} X_j v_1^* \wedge \cdots \wedge v_{j-1}^* \wedge v_{j+1}^* \wedge \cdots \wedge v_n^*.$$

This can be seen by a computation which is similar to the computation of Example 3.4.2 and which we repeat for the sake of clarity. Let $G = \{\sigma \in S_n : \sigma(1) = 1\}$ and $\sigma_j = (1 \quad j)$. For u_2,..., $u_n \in V$ and putting $v_1 = X$ we compute

$$i_X\omega(u_2, ..., u_n) = \sum_{\sigma \in S_n} (\text{sgn}\sigma) v_1^*(u_{\sigma(1)} \cdots v_n^*(u_{\sigma(n)})$$

$$= \sum_{j=1}^{n} \sum_{\sigma\sigma_j \in G} (\text{sgn}\sigma) v_1^*(u_{\sigma(1)} \cdots v_n^*(u_{\sigma(n)})$$

$$= \sum_{j=1}^{n} \sum_{\tau \in G} (\text{sgn}\tau) v_1^*(u_{\tau\sigma_j(1)} \cdots v_n^*(u_{\tau\sigma_j(n)})$$

$$\sum_{j=1}^{n} \sum_{\tau \in G} -(\text{sgn}\tau) v_1^*(u_{\tau(j)}) \cdots v_{j-1}^*(u_{\tau(j-1)}) X_j v_{j+1}^*(u_{\tau(j+1)}) \cdots v_n^*(u_{\tau(n)})$$

$$= \sum_{j=1}^{n} \sum_{\rho \in G} (-1)^{j-1} X_j (\text{sgn}\rho) v_1^*(u_{\rho(2)}) \cdots v_{j-1}^*(u_{\rho(j-1)}) v_{j+1}^*(u_{\rho(j+1)}) \cdots v_n^*(u_{\rho(n)})$$

$$\sum_{j=1}^{n} (-1)^{j-1} X_j v_1^* \wedge \cdots \wedge v_{j-1}^* \wedge v_{j+1}^* \wedge \cdots \wedge v_n^*(u_2, ..., u_n).$$

It follows immediately from this that the linear map $F : V \to \Lambda^{n-1}(V)$ defined by $F(X) = i_X\omega$ is a monomorphism and hence an isomorphism since $\dim \Lambda^{n-1}(V) = \dim V = n$.

Let now M be a smooth n-manifold. For every $X \in \mathcal{X}(M)$ and $\omega \in A^k(M)$, the differential $(k-1)$-form $i_X\omega$ defined by

$$(i_X\omega)_p(u_1, ..., u_{k-1}) = \omega_p(X(p), u_1, ..., u_{k-1})$$

for every u_1,..., $u_{k-1} \in T_pM$, $p \in M$, is called the *contraction* of ω by the vector field X.

From Proposition 4.7.1 follows that the linear map $i_X : A^*(M) \to A^*(M)$ of degree -1 satisfies the graded Leibliz formula

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^k\omega \wedge i_X\eta$$

for $\omega \in A^k(M)$, $\eta \in A^*(M)$.

Proposition 4.7.3. *If M is an oriented smooth n-manifold by a volume element $\omega \in A^n(M)$ then the linear map $F : \mathcal{X}(M) \to A^{n-1}(M)$ defined by $F(X) = i_X\omega$ is an isomorphism.*

Proof. Let (U, ϕ) be a positively oriented smooth chart of M and $\phi = (x^1, ..., x^n)$. There exists a unique smooth function $f : U \to (0, +\infty)$ such that

$$\omega|_U = f \cdot dx^1 \wedge \cdots \wedge dx^n.$$

For every $X \in (M)$ there exist unique smooth functions X_1,..., $X_n : U \to \mathbb{R}$ such that

$$X|_U = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x^j}.$$

As in Example 4.7.2 we have then

$$i_x\omega = \sum_{j=1}^{n} (-1)^{j-1} f X_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n.$$

This implies that F is injective. Unlike Example 4.7.2 we need an extra globalization argument in order to show that F is surjective, since this time we deal with infinite dimensional vector spaces. Let $\theta \in A^{n-1}(M)$. There are unique smooth functions X_1,..., $X_n : U \to \mathbb{R}$ such that

$$\theta|_U = \sum_{j=1}^{n} (-1)^{j-1} f X_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n.$$

Let (V, ψ) be another positively oriented smooth chart of M and $\psi = (y^1, ..., y^n)$. There exists a unique smooth function $g : V \to (0, +\infty)$ such that

$$\omega|_V = g \cdot dy^1 \wedge \cdots \wedge dy^n$$

and unique smooth functions Y_1,..., $Y_n : V \to \mathbb{R}$ such that

$$\theta|_V = \sum_{j=1}^{n} (-1)^{j-1} g Y_j dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^n.$$

If $U \cap V \neq \varnothing$, then

$$\sum_{j=1}^{n} (-1)^{j-1} f X_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

$$= \sum_{j=1}^{n} (-1)^{j-1} g Y_j dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^n$$

on $U \cap V$, because θ is globally defined. Since ω is also globally defined, on $U \cap V$ for each $1 \leq i \leq n$ we have

$$Y_i \omega|_{U\cap V} = dy^i \wedge \theta|_{U\cap V} = \sum_{j=1}^{n} (-1)^{j-1} f X_j dy^i \wedge dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

$$= \sum_{j=1}^{n} (-1)^{j-1} f X_j \left(\sum_{k=1}^{n} \frac{\partial y^i}{\partial x^k} dx^k \right) dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

$$= \left(\sum_{j=1}^{n} X_j \frac{\partial y^i}{\partial x^j} \right) \omega|_{U\cap V}.$$

Hence

$$Y_i = \sum_{j=1}^{n} X_j \frac{\partial y^i}{\partial x^j}$$

which implies that

$$\sum_{j=1}^{n} X_j \frac{\partial}{\partial x^j} = \sum_{j=1}^{n} Y_j \frac{\partial}{\partial y^j}$$

on $U \cap V$. Thus, these local vector fiends piece together to a globally defined smooth vector field X such that $i_X \omega = \theta$. $\square$

If M is an oriented smooth n-manifold by a volume element $\omega \in A^n(M)$, the differential $(n-1)$-form $i_X \omega$ is called the *flux form* of the smooth vector field X.

There is a useful formula for the exterior differential in terms of vector fields considered as derivations and the Lie bracket.

Theorem 4.7.4. *Let M be a smooth n-manifold, $\omega \in A^k(M)$, $0 \leq k \leq n$, and let $X_0, \dots, X_k \in \mathcal{X}(M)$. Then*

$$d\omega(X_0, ..., X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, ..., X_{i-1}, X_{i+1}, ..., X_k)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., X_{i-1}, X_{i+1}, ..., X_{j-1}, X_{j+1}, ..., X_k).$$

Proof. A first observation is that since both sides involve derivations, it suffices to prove the formula locally. A second observation is that both sides are $C^\infty(M)$-multilinear on the $C^\infty(M)$-module $\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)$. This is trivial for the left hand side. In order to confirm it for the right hand side, we put

$$S(X_0, ..., X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, ..., X_{i-1}, X_{i+1}, ..., X_k)$$

and

$$T(X_0, ..., X_k) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., X_{i-1}, X_{i+1}, ..., X_{j-1}, X_{j+1}, ..., X_k).$$

For every $f \in C^\infty(M)$ and $1 \leq m \leq k$ we have

$$S(X_0, ..., fX_m, ..., X_k) = fS(X_0, ..., X_k) + \sum_{i \neq m} (-1)^i X_i f \cdot \omega(X_0, ..., X_{i-1}, X_{i+1}, ..., X_k).$$

On the other hand,
$T(X_0, ..., fX_m, ..., X_k)$

$$\begin{aligned}
&= \sum_{\substack{i<j \\ i,j\neq m}} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., fXm, ..., X_{i-1}, X_{i+1}, ..., X_{j-1}, X_{j+1}, ..., X_k) \\
&\quad + \sum_{i<m} (-1)^{i+m} \omega([X_i, fX_m], X_0, ..., X_{i-1}, X_{i+1}, ..., X_{m-1}, X_{m+1}, ..., X_k) \\
&\quad + \sum_{m<j} (-1)^{m+j} \omega([fX_m, X_j], X_0, ..., X_{m-1}, X_{m+1}, ..., X_{j-1}, X_{j+1}, ..., X_k) \\
&= fT(X_), ..., X_k) + \sum_{i<m} (-1)^{i+m} X_i f \cdot \omega(X_m, X_0, ..., X_{i-1}, X_{i+1}, ..., X_{m-1}, X_{m+1}, ..., X_k) \\
&\quad - \sum_{m<j} (-1)^{m+j} X_j f \cdot \omega(X_m, X_0, ..., X_{m-1}, X_{m+1}, ..., X_{j-1}, X_{j+1}, ..., X_k) \\
&= fT(X_), ..., X_k) + \sum_{i<m} (-1)^{i+m+m-1} X_i f \cdot \omega(X_0, ..., X_{i-1}, X_{i+1}, ..., X_k) \\
&\quad - \sum_{m<j} (-1)^{m+j+m} X_j f \cdot \omega(X_0, ..., X_{j-1}, X_{j+1}, ..., X_k) \\
&= fT(X_), ..., X_k) - \sum_{i \neq m} (-1)^i X_i f \cdot \omega(X_0, ..., X_{i-1}, X_{i+1}, ..., X_k).
\end{aligned}$$

Hence the right hand side $S+T$ is $C^\infty(M)$-multilinear. From these two observations we see that it is sufficient to prove the formula on the domain U of a smooth chart (U, ϕ), where $\phi = (x^1, ..., x^n)$, for any set of k basic vector fields. There are unique smooth functions $\omega_{i_0 i_1 \cdots i_{k-1}} : U \to \mathbb{R}$, $1 \leq i_0 < \cdots < i_{k-1} \leq n$, such that

$$\omega = \sum_{1 \leq i_0 < \cdots < i_{k-1} \leq n} \omega_{i_0 \cdots i_{k-1}} dx^{i_0} \wedge \cdots \wedge dx^{i_{k-1}}.$$

For any $1 \leq j_0 < \cdots < j_k \leq n$ we have

$$d\omega\left(\frac{\partial}{\partial x^{j_0}}, ..., \frac{\partial}{\partial x^{j_k}}\right)$$

$$= \sum_{m=1}^{n} \sum_{1\leq i_0<\cdots<i_{k-1}\leq n} \frac{\partial \omega_{i_0\cdots i_{k-1}}}{\partial x^m} dx^m \wedge dx^{i_0} \wedge \cdots \wedge dx^{i_{k-1}}\left(\frac{\partial}{\partial x^{j_0}}, ..., \frac{\partial}{\partial x^{j_k}}\right)$$

$$= \sum_{i=1}^{k} (-1)^i \frac{\partial \omega_{j_0\cdots j_{i-1}j_{i+1}\cdots j_k}}{\partial x^{j_i}}$$

and

$$(S+T)\left(\frac{\partial}{\partial x^{j_0}}, ..., \frac{\partial}{\partial x^{j_k}}\right) = \sum_{i=0}^{k} (-1)^i \frac{\partial}{\partial x^{j_i}} \omega\left(\frac{\partial}{\partial x^{j_0}}, ..., \frac{\partial}{\partial x^{j_k}}\right)$$

$$= \sum_{i=1}^{k} (-1)^i \frac{\partial \omega_{j_0\cdots j_{i-1}j_{i+1}\cdots j_k}}{\partial x^{j_i}}. \quad \square$$

Let now $\Phi : D \to M$ be the flow of a smooth vector field X on M. For $\omega \in A^*(M)$ the differential form

$$L_X\omega = \frac{d}{dt}\bigg|_{t=0} \Phi_t^*\omega = \lim_{t\to 0} \frac{1}{t}(\Phi_t^*\omega - \omega)$$

is called the *Lie derivative of ω with respect to X*. Note that $L_X f = Xf$ for $f \in A^0(M) = C^\infty(M)$. It is obvious that the Lie derivative operator

$$L_X : A^*(M) \to A^*(M)$$

commutes with the exterior differentiation d, that is $d \circ L_X = L_X \circ d$. Finally, the Lie derivative L_X with respect to X is a derivation of the exterior algebra $A^*(M)$ since it satisfies a (non-graded) Leibliz formula

$$L_X(\omega \wedge \eta) = L_X\omega \wedge \eta + \omega \wedge L_X\eta$$

for every ω, $\eta \in A^*(M)$. Actually, these properties characterize L_X.

Proposition 4.7.5. *Let M be a smooth n-manifold and let $X \in \mathcal{X}(M)$. Let $D : A^*(M) \to A^*(M)$ be a linear map with the following properties:*
(a) $D(A^k(M)) \subset A^k(M)$ for all $0 \leq k \leq n$.
(b) $D(\omega \wedge \eta) = D\omega \wedge \eta + \omega \wedge D\eta$ for every ω, $\eta \in A^*(M)$.
(c) D commutes with the exterior differentiation d, that is $D \circ d = d \circ D$.
(d) $Df = Xf$ for every $f \in C^\infty(M)$.
Then, $D = L_X$.

Proof. It suffices to prove the assertion locally in the domain U of a smooth chart (U, ϕ) with $\phi = (x^1, ..., x^n)$. If $\omega \in A^k(M)$, there exist unique smooth functions $\omega_{i_1\cdots i_k} : U \to \mathbb{R}$, $1 \leq i_1 < \cdots < i_k \leq n$ such that

$$\omega|_U = \sum_{1\leq i_1<\cdots<i_k\leq n} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Then,

$$D\omega|_U = \sum_{1\le i_1<\cdots<i_k\le n} D\omega_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$+\sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\cdots i_k}\sum_{m=1}^{k} dx^{i_1}\wedge\cdots\wedge D(dx^{i_m})\wedge\cdots dx^{i_k}$$

$$=\sum_{1\le i_1<\cdots<i_k\le n} L_X\omega_{i_1\cdots i_k}\cdot dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$+\sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\cdots i_k}\sum_{m=1}^{k} dx^{i_1}\wedge\cdots\wedge L_X(dx^{i_m})\wedge\cdots dx^{i_k} = L_X\omega|_U. \quad \square$$

The Lie derivative is closely related with the contraction and the exterior differentiation through a formula which is due to E. Cartan.

Theorem 4.7.6. *If X is a smooth vector field of a smooth n-manifold M, then*

$$L_X = i_X\circ d + d\circ i_X.$$

Proof. It suffices to check that $D = i_X\circ d + d\circ i_X$ has the properties (a)-(d) in the statement of Proposition 4.7.5. Obviously, D is linear of degree 0. Also, D is a derivation, because if $\omega\in A^k(M)$ and $\eta\in A^l(M)$ we have

$$D(\omega\wedge\eta) = d(i_X\omega\wedge\eta + (-1)^k\omega\wedge i_X\eta) + i_X(d\omega\wedge\eta + (-1)^k\omega\wedge d\eta)$$

$$= di_X\omega\wedge\eta + (-1)^{k-1}i_X\omega\wedge d\eta + (-1)^k d\omega\wedge i_X\eta + \omega\wedge di_X\eta$$

$$+ i_X d\omega\wedge\eta + (-1)^{k+1}d\omega\wedge i_X\eta + (-1)^k i_X\omega\wedge d\eta + \omega\wedge i_X d\eta$$

$$= (di_X\omega + i_X d\omega)\wedge\eta + \omega\wedge(di_X\eta + i_X d\eta) = D\omega\wedge\eta + \omega\wedge D\eta.$$

Finally, $D\circ d = d\circ i_X\circ d = d\circ D$ and $Df = i_X(df) = df(X) = Xf$ for every $f\in C^\infty(M)$. $\square$

Corollary 4.7.7. *If $\omega\in A^k(M)$, $1\le k\le n$, and X, X_1,..., $X_k\in\mathcal{X}(M)$, then*

$$L_X\omega(X_1,...,X_k) = X\omega(X_1,...,X_k) - \sum_{j=1}^{k}\omega(X_1,...,X_{j-1},[X,X_j],X_{j+1},...,X_k).$$

Proof. Applying Theorem 3.7.6 we have

$$i_X d\omega(X_1,...,X_k) = X\omega(X_1,...,X_k) + \sum_{i=1}^{k} X_i\omega(X,X_1,...,X_{i-1},X_{i+1},...,X_k)$$

$$+\sum_{j=1}^{k}(-1)^j\omega([X,X_j],X_1,...,X_{j-1},X_{j+1},...,X_k)$$

$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X,X_1,...,X_{i-1},X_{i+1},...,X_{j-1},X_{j+1},...,X_k)$$

and

$$di_X\omega(X_1,...,X_k) = \sum_{i=1}^{k}(-1)^{k-1}X_i\omega(X,X_1,...,X_{i-1},X_{i+1},...,X_k)$$

$$+\sum_{i<j}(-1)^{i+j}\omega(X,[X_i,X_j],...,X_{i-1},X_{i+1},...,X_{j-1},X_{j+1},...,X_k).$$

Therefore,

$$L_X\omega(X_1,...,X_k) = di_X\omega(X_1,...,X_k) + i_Xd\omega(X_1,...,X_k)$$

$$= X\omega(X_1,...,X_k) - \sum_{j=1}^{k}\omega(X_1,...,X_{j-1},[X,X_j],X_{j+1},...,X_k). \quad \square$$

Corollary 4.7.8. *If M is a smooth n-manifold and X, $Y \in \mathcal{X}(M)$, then*

$$i_{[X,Y]} = L_X \circ i_Y - i_Y \circ L_X.$$

Proof. Applying the formula for the Lie derivative proved in the preceding Corollary 4.7.7, for any $\omega \in A^k(M)$ and X_1,...,$K_{k-1} \in \mathcal{X}(M)$ we have
$L_X(i_Y\omega)(X_1,...,X_{k-1})$

$$= X\omega(Y,X_1,...,X_{k-1}) - \sum_{j=1}^{k-1}\omega(Y,X_1,...,X_{j-1},[X,X_j],X_{j+1},...,X_{k-1})$$

and

$$i_Y(L_X\omega)(X_1,...,X_{k-1}) = X\omega(Y,X_1,...,X_{k-1}) - \omega([X,Y],X_1,...,X_{k-1})$$

$$-\sum_{j=1}^{k-1}\omega(Y,X_1,...,X_{j-1},[X,X_j],X_{j+1},...,X_{k-1}).$$

Therefore,

$$(L_Xi_Y\omega - i_YL_X\omega)(X_1,...,X_{k-1}) = \omega([X,Y],X_1,...,X_{k-1}) = i_{[X,Y]}\omega(X_1,...,X_k). \quad \square$$

4.8 Integration on Riemannian manifolds

Let V be a n-dimensional real vector space equipped with an inner product $\langle .,. \rangle$ which we assume that it is oriented by a non-zero element of $\Lambda^n(V)$. There exists a unique $\Omega \in \Lambda^n(V)$ such that $\Omega(v_1,...,v_n) = 1$ for every positively oriented ordered orthonormal basis $[v_1,...,v_n]$ of V or equivalently $\Omega = v_1^* \wedge \cdots \wedge v_n^*$, where $[v_1^*,...,v_n^*]$ is the dual basis. Indeed, if $[w_1,...,w_n]$ is another such basis of V, then

$$w_j = \sum_{i=1}^{n} a_{ij}v_i, \quad 1 \le j \le n$$

for some $a_{ij} \in \mathbb{R}$, $1 \le i,j \le n$. The matrix $A = (a_{ij})_{1 \le i,j \le n}$ is orthogonal and has $\det A = 1$. Since

$$\omega(w_1,...,w_n) = (\det A)\omega(v_1,...,v_n) = \omega(v_1,...,v_n)$$

for every $\omega \in \Lambda^n(V)$, it follows that $v_1^* \wedge \cdots \wedge v_n^* = w_1^* \wedge \cdots \wedge w_n^*$.

Let now M be an oriented Riemannian smooth n-manifold. According to the above, on each tangent space T_pM, $p \in M$, there exists a unique element $\Omega_p \in \Lambda^n(T_pM)$ such that $\Omega_p(v_1, ..., v_n) = 1$ for every positively oriented ordered orthonormal basis $[v_1, ..., v_n]$ of T_pM. This defines a volume element of M which gives its orientation and is called *the Riemannian volume element of* M. We need only show that Ω is indeed smooth. To see this, let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$. Let $p \in U$ and let $[v_1, ..., v_n]$ be a positively oriented ordered orthonormal basis of T_pM. Then,

$$\left(\frac{\partial}{\partial x^j}\right)_p = \sum_{i=1}^n a_{ij} v_i, \quad 1 \leq j \leq n$$

for some $a_{ij} \in \mathbb{R}$, $1 \leq i, j \leq n$. Let $A = (a_{ij})_{1 \leq i,j \leq n}$. The matrix of the Riemannian metric at p with respect to the chosen smooth chart has entries

$$g_{ij}(p) = \left\langle \left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial x^j}\right)_p \right\rangle = \left\langle \sum_{k=1}^n a_{ki} v_k. \sum_{l=1}^n a_{lj} v_l \right\rangle = \sum_{k=1}^n a_{ki} a_{kj}.$$

Thus, $(g_{ij}(p))_{1 \leq i,j \leq n} = A^t A$ and since

$$\Omega_p\left(\left(\frac{\partial}{\partial x^1}\right)_p, ..., \left(\frac{\partial}{\partial x^n}\right)_p\right) = (\det A)\omega(v_1, ..., v_n) = \det A$$

we have

$$\Omega_p = \sqrt{\det(g_{ij}(p))_{1 \leq i,j \leq n}} \cdot (dx^1)_p \wedge \cdots \wedge (dx^n)_p.$$

Since this holds for every $p \in U$, we conclude that Ω is smooth.

Let now ∇ be the Levi-Civita connection of M. If $X \in \mathcal{X}(M)$, the smooth function

$$\mathrm{div} X = \mathrm{Tr}(\nabla_. X)$$

is called the *divergence* of X with respect to the Riemannian metric and can be alternatively characterized as follows.

Proposition 4.8.1. *Let M be an oriented Riemannian smooth n-manifold with Riemannian volume element Ω. The divergence* $\mathrm{div} X$ *of $X \in \mathcal{X}(M)$ is the unique smooth function such that*

$$d(i_X \Omega) = (\mathrm{div} X) \cdot \Omega.$$

Proof. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and suppose that

$$X|_U = \sum_{k=1}^n X^k \frac{\partial}{\partial x^k}.$$

Using Example 4.7.2 and the above local formula for Ω, we compute

$$d(i_X \Omega)|_U = d(i_X(\sqrt{\det(g_{ij})_{1 \leq i,j \leq n}} \cdot dx^1 \wedge \cdots \wedge dx^n))$$

$$= d\Big(\sum_{k=1}^{n}(-1)^{k-1}\sqrt{\det(g_{ij})_{1\le i,j\le n}}X^k \cdot dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge dx^n\Big)$$

$$= \Big(\sum_{k=1}^{n}\frac{\partial}{\partial x^k}(\sqrt{\det(g_{ij})_{1\le i,j\le n}}X^k)\Big) \cdot dx^1 \wedge \cdots \wedge dx^n$$

$$= \Big(\frac{1}{\sqrt{\det(g_{ij})_{1\le i,j\le n}}}\sum_{k=1}^{n}\frac{\partial}{\partial x^k}(\sqrt{\det(g_{ij})_{1\le i,j\le n}}X^k)\Big) \cdot \Omega.$$

On the other hand, for every $1 \le i \le n$ we have

$$\nabla_{\frac{\partial}{\partial x^i}} X = \sum_{k=1}^{n}\left(\frac{\partial X^k}{\partial x^i} + \sum_{j=1}^{n}\Gamma_{ij}^k X^j\right)\frac{\partial}{\partial x^k}$$

and so

$$\mathrm{div}X = \sum_{k=1}^{n}\left(\frac{\partial X^k}{\partial x^k} + \sum_{j=1}^{n}\Gamma_{kj}^k X^j\right) = \sum_{k=1}^{n}\frac{\partial X^k}{\partial x^k} + \sum_{j=1}^{n}\left(\sum_{k=1}^{n}\Gamma_{kj}^k\right)X^j.$$

Using the formula for the Christoffel symbols derived in the proof of Theorem 5.4.3 we have

$$\sum_{k=1}^{n}\Gamma_{kj}^k = \frac{1}{2}\sum_{k,l=1}^{n}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^l}\right) = \frac{1}{2}\sum_{k,l=1}^{n}g^{kl}\frac{\partial g_{lk}}{\partial x^j} + \frac{1}{2}\sum_{k,l=1}^{n}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l}\right)$$

$$= \frac{1}{2}\sum_{k,l=1}^{n}g^{kl}\frac{\partial g_{lk}}{\partial x^j} + 0 = \frac{1}{2}\cdot\frac{1}{\det(g_{kl})_{1\le k,l\le n}}\cdot\frac{\partial}{\partial x^j}\det(g_{kl})_{1\le k,l\le n}$$

$$= \frac{1}{\sqrt{\det(g_{kl})_{1\le k,l\le n}}}\cdot\frac{\partial}{\partial x^j}\sqrt{\det(g_{kl})_{1\le k,l\le n}}.$$

Substituting we arrive at

$$\mathrm{div}X = \sum_{j=1}^{n}\left(\frac{\partial X^j}{\partial x^j} + \frac{X^j}{\sqrt{\det(g_{kl})_{1\le k,l\le n}}}\cdot\frac{\partial}{\partial x^j}\sqrt{\det(g_{kl})_{1\le k,l\le n}}\right)$$

$$= \frac{1}{\sqrt{\det(g_{klj})_{1\le k,l\le n}}}\sum_{j=1}^{n}\frac{\partial}{\partial x^j}\left(\sqrt{\det(g_{klj})_{1\le kl,j\le n}}X^j\right). \quad \square$$

In the end of the proof of the preceding Proposition 4.8.1 we have used the following fact. Let $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a smooth map. If $p \in \mathbb{R}^n$ and $\det A(p) > 0$, then

$$\frac{1}{\det A(p)}\cdot\frac{\partial \det A}{\partial x^k}(p) = \mathrm{Tr}\Big(\frac{\partial A}{\partial x^k}(p)\cdot(A(p))^{-1}\Big), \quad 1 \le k \le n.$$

Indeed, if $G : (-\epsilon, \epsilon) \to \mathbb{R}^{n\times n}$ is a smooth curve for some $\epsilon > 0$ with $G(0) = I_n$, then from Taylor's formula we have

$$G(t) = I_n + tG'(0) + O(t^2)$$

and therefore

$$\det G(t) = 1 + t\mathrm{Tr}G'(0) + O(t^2).$$

This implies $(\det G)'(0) = \mathrm{Tr}G'(0)$. Applying this to $G(t) = B(t)B(0)^{-1}$ we obtain

$$\frac{(\det B)'(0)}{\det B(0)} = \mathrm{Tr}(B'(0)B(0)^{-1})$$

for any smooth $B : (-\epsilon, \epsilon) \to \mathbb{R}^{n\times n}$.

Let now D be a domain with smooth boundary in an oriented Riemannian smooth n-manifold M. There exists a unique smooth vector field $\nu : \partial D \to TM$ along ∂D which is directed outward of D and is orthogonal to ∂D and has unit length. We shall call ν the *unit outer normal* to ∂D. As we saw in section 3.6, the orientation of ∂D with respect to which Stokes' formula holds is represented by $i_\nu\Omega$, where Ω is the Riemannian volume element of M. Let $p \in \partial D$ and let v_2,..., $v_n \in T_p\partial D$ be such that $[\nu(p), v_2, ..., v_n]$ is a positively oriented ordered orthonormal basis of $T_p\partial D$. Then,

$$i_\nu\Omega(p) = v_2^* \wedge \cdots \wedge v_n^*.$$

If $X \in \mathcal{X}(M)$, from Example 4.7.2 we have

$$\begin{aligned} i_X\Omega(p) &= \langle X(p), \nu(p)\rangle + \sum_{k=2}^{n} (-1)^{k-1}\langle X(p), v_k\rangle \nu(p)^* \wedge v_2^* \wedge \cdots \wedge v_{k-1}^* \wedge v_{k+1}^* \wedge \cdots \wedge v_n^* \\ &= \langle X(p), \nu(p)\rangle i_\nu\Omega(p) + 0. \end{aligned}$$

Thus, $i_X\Omega|_{\partial D} = \langle X, \nu\rangle i_\nu\Omega$. Stokes' formula has the following version on Riemannian manifolds, which is known also as the Divergence Theorem.

Theorem 4.8.2. *Let M be an oriented Riemannian smooth n-manifold with Riemannian volume element Ω and let $D \subset M$ be a domain with smooth boundary. Let ν be the unit outer normal to ∂D. If $X \in \mathcal{X}(M)$ has compact support in M, then*

$$\int_{\overline{D}} \mathrm{div}X \cdot \Omega = \int_{\partial D} \langle X, \nu\rangle i_\nu\Omega.$$

Proof. From the above considerations, Theorem 4.6.2 and Proposition 4.8.1 we have

$$\int_{\overline{D}} \mathrm{div}X \cdot \Omega = \int_{\overline{D}} d(i_X\Omega) = \int_{\partial D} i_X\Omega|_{\partial D} = \int_{\partial D} \langle X, \nu\rangle i_\nu\Omega. \quad \square$$

4.9 Differential ideals

Let M be a smooth n-manifold and let $\mathcal{D}$ be a geometric distribution of constant rank k on M. A differential r-form ω on M is said to *annihilate* $\mathcal{D}$ if $\omega_p(v_1, ..., v_r) = 0$ for every v_1,..., $v_r \in \mathcal{D}_p$ and $p \in M$. An element of the exterior algebra $A^*(M)$ annihilates $\mathcal{D}$ if all its components annihilate $\mathcal{D}$. The set $\mathcal{E}(\mathcal{D})$ of all elements of $A^*(M)$ which annihilate $\mathcal{D}$ is an ideal in $A^*(M)$, by the definition of the wedge product. We shall analyse further the structure of the annihilating ideal $\mathcal{E}(\mathcal{D})$.

In general, an ideal $\mathcal{S}$ in $A^*(M)$ is said to be *locally generated by* $n-k$ *independent differential* 1*-forms* if there exists an open cover $\mathcal{U}$ of M such that for every $U \in \mathcal{U}$ there exist pointwise linearly independent $\theta_1,..., \theta_{n-k} \in A^1(U)$ such that a differential form ω on M belongs to $\mathcal{E}(\mathcal{D})$ if and only if $\omega|_U$ belongs to the ideal in $A^*(U)$ which is generated by $\theta_1,..., \theta_{n-k}$.

Proposition 4.9.1. *If $\mathcal{D}$ is a geometric distribution of constant rank k on a smooth n-manifold M, then its annihilator $\mathcal{E}(\mathcal{D})$ is an ideal locally generated by $n-k$ independent differential* 1*-forms.*

Proof. Let $p \in M$. There exists an open neighbourhood U of p and $Y_1,..., Y_k \in \mathcal{X}(U)$ such that $\{Y_1(q),..., Y_k(q)\}$ is a basis of $\mathcal{D}_q$ for every $q \in U$. There exist some $Y_{k+1},...,$ $Y_n \in \mathcal{X}(U)$ such that $\{Y_1(q),..., Y_k(q), Y_{k+1}(q),..., Y_n(q)\}$ is a basis of T_qM for every $q \in U$. There are unique dual differential 1-forms $\omega_1,..., \omega_n \in A^1(U)$, that is $\omega_i(Y_j) = \delta_{ij}$, $1 \leq i, j \leq n$. Then, $\omega_{k+1},..., \omega_n \in \mathcal{E}(\mathcal{D})$ and they are pointwise linearly independent. If now $\omega \in \mathcal{E}(\mathcal{D})$ is a differential r-form, there are $f_{i_1 \cdots i_r} \in C^\infty(U)$, $\{i_1,...,i_r\} \subset \{1,...,n\}$, such that

$$\omega = \sum_{\{i_1,...,i_r\} \subset \{1,...,n\}} f_{i_1 \cdots i_r} \omega_{i_1} \wedge \cdots \wedge \omega_{i_r}$$

where $f_{i_1 \cdots i_r} = 0$ in case $\{i_1,...,i_r\} \cap \{k+1,...,n\} = \varnothing$. Hence $\omega|_U$ belongs to the ideal in $A^*(U)$ which is generated by $\omega_{k+1},..., \omega_n$. Conversely, if $\omega \in A^*(M)$ is such hat $\omega|_U$ belongs to the ideal in $A^*(U)$ generated by $\omega_{k+1},..., \omega_n$, then evidently $\omega \in \mathcal{E}(\mathcal{D})$. $\square$

Proposition 4.9.2. *Let M be a smooth n-manifold and let $\mathcal{S}$ be an ideal in $A^*(M)$. If $\mathcal{S}$ is locally generated by $n-k$ independent differential* 1*-forms, there exists a unique geometric distribution $\mathcal{D}$ of constant rank k such that $\mathcal{S} = \mathcal{E}(\mathcal{D})$.*

Proof. Let $p \in M$ and let $\theta_1,..., \theta_{n-k}$ be pointwise linearly independent differential 1-forms defined on some open neighbourhood U of p which generated $\mathcal{S}$ on U. Then,

$$\mathcal{D}_p = \bigcap_{i=1}^{n-k} \mathrm{Ker}\theta_i(p)$$

is a k-dimensional vector subspace of T_p. It is obvious that $\mathcal{D} = \bigcup_{p \in M} \mathcal{D}_p$ is a geometric distribution of constant rank k and $\mathcal{S} = \mathcal{E}(\mathcal{D})$. The uniqueness is immediate from the fact that if $\mathcal{D}^1$ and $\mathcal{D}^2$ are two geometric distributions of the same constant rank and $\mathcal{D}^1 \neq \mathcal{D}^2$, then $\mathcal{E}(\mathcal{D}^1) \neq \mathcal{E}(\mathcal{D}^2)$. $\square$

Thus, there is a bijective correspondence between geometric distributions of constant rank k on a smooth n-manifold M and ideals in its exterior algebra $A^*(M)$ that are locally generated by $n-k$ independent differential 1-forms. In terms of annihilating ideals the Frobenius' theorem can be stated as follows.

Theorem 4.9.3. *A geometric distribution $\mathcal{D}$ of constant rank k on a smooth n-manifold M is integrable if and only if $d(\mathcal{E}(\mathcal{D})) \subset \mathcal{E}(\mathcal{D})$.*

Proof. If $\mathcal{D}$ is integrable, it is involutive and so if $\omega \in A^r(M)$ annihilates $\mathcal{D}$, from Theorem 4.7.4 we have

$$d\omega(X_1, ..., X_r) = 0$$

for every X_1,... $X_r \in \mathcal{X}^{\mathcal{D}}(M)$. Hence $d\omega$ annihilates $\mathcal{D}$ as well.

Conversely, suppose that $d(\mathcal{E}(\mathcal{D})) \subset \mathcal{E}(\mathcal{D})$ and let X, $Y \in \mathcal{X}^{\mathcal{D}}(M)$. By Proposition 3.8.1, every point $p \in M$ has an open neighbourhood U such that $\mathcal{E}(\mathcal{D})$ is generated on U by pointwise linearly independent differential 1-forms θ_1,..., $\theta_{n-k} \in A^1(U)$. By Corollary 1.4.5, we may assume that these are restrictions to U of globally defined differential 1-forms on M with support contained in U. From Theorem 4.7.4 we have

$$\theta_j([X, Y]) = -d\theta_j(X, Y) + X\theta_j(Y) - Y\theta_j(X) = 0$$

for all $1 \leq j \leq n - k$. Therefore,

$$[X, Y](p) \in \bigcap_{j=1}^{n-k} \mathrm{Ker}\theta_j(p) = \mathcal{D}_p.$$

This shows that $\mathcal{D}$ is involutive, hence integrable, by Corollary 2.4.7. □

Combined with Proposition 4.9.1, the preceding version of Frobenius' theorem can be restated in local terms as follows.

Corollary 4.9.4. *Let $\mathcal{D}$ be a geometric distribution of constant rank k on a smooth n-manifold M with annihilating ideal $\mathcal{E}(\mathcal{D})$. The following statements are equivalent.*
(a) $\mathcal{D}$ is integrable.
(b) There exists an open cover $\mathcal{U}$ of M such that for every $U \in \mathcal{U}$ the ideal $\mathcal{E}(\mathcal{D})$ on U is generated by $n - k$ independent differential 1-forms θ_1,..., θ_{n-k} for which there exist $a_{ij} \in A^1(U)$, $1 \leq i, j \leq n - k$, such that

$$d\theta_j = \sum_{i=1}^{n-k} \theta_i \wedge a_{ij}, \quad 1 \leq j \leq n - k.$$

(c) There exists an open cover $\mathcal{U}$ of M such that for every $U \in \mathcal{U}$ the ideal $\mathcal{E}(\mathcal{D})$ on U is generated by $n - k$ independent differential 1-forms θ_1,..., θ_{n-k} such that

$$d\theta_j \wedge \theta_1 \wedge \cdots \wedge \theta_{n-k} = 0, \quad 1 \leq j \leq n - k. \quad \square$$

Example 4.9.5. Let M be an open subset of $\mathbb{R}^3$ and $\theta \in A^1(M)$ be nowhere vanishing. Then $\mathrm{Ker}\theta$ is geometric distribution of constant rank 2 on M and $\mathcal{E}(\mathrm{Ker}\theta)$ is generated by θ. According to Theorem 4.9.3, $\mathrm{Ker}\theta$ is integrable if and only if $d\theta \wedge \theta = 0$. In particular, $\mathrm{Ker}\theta$ is integrable, if θ is closed. The euclidean inner product $\langle ., . \rangle$ gives a natural linear isomorphism $\phi : \mathcal{X}(M) \to A^1(M)$ defined by $\phi(X) = \langle ., X\rangle$. If $X = \phi^{-1}(\theta)$, by a routine computation we see that the integrability condition translates to $\langle X, \mathrm{curl}X\rangle = 0$. This observation is due to G. Reeb and is considered to have given birth to the theory of foliations.

4.10 Exercises

1. Let M be a smooth manifold and $\omega \in A^1(M)$. If there exists $f \in C^\infty(M)$, such that $f(p) \neq 0$ for every $p \in M$ and $f\omega$ is closed, prove that $\omega \wedge d\omega = 0$.

2. Let M and N be two smooth manifolds and $f : M \to N$ be a submersion onto N. Prove that the transpose $f^* : A^*(N) \to A^*(M)$ is injective.

3. Prove that $H^1(\mathbb{R}) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth periodic function of period 1, that is $f(x+1) = f(x)$ for every $x \in \mathbb{R}$. Prove that there exists $\lambda \in \mathbb{R}$ and a smooth periodic function $g : \mathbb{R} \to \mathbb{R}$ of period 1 such that $fdx = \lambda dx + dg$ on $\mathbb{R}$. Use this to prove that $H^1(S^1) \cong \mathbb{R}$.

5. On $\mathbb{R}^2 \setminus \{(0,0)\}$ we consider the differential 1-form

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Let $F : (0,+\infty) \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0,0)\}$ the local smooth diffeomorphism defined by

$$F(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta).$$

(a) Prove that $F^*\omega = d\theta$.
(b) Let η be a closed differential 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$. Prove that there exist $\lambda \in \mathbb{R}$, a smooth periodic function $g : \mathbb{R} \to \mathbb{R}$ of period 2π and a smooth function $h : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ such that $h(\rho,\theta+2\pi) = h(\rho,\theta)$ for every $\rho > 0$, $\theta \in \mathbb{R}$ and

$$F^*\eta = dh + \lambda d\theta + g'(\theta)d\theta$$

on $(0,+\infty) \times \mathbb{R}$.
(c) Use the above to prove that $H^1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \mathbb{R}$.

6. Let $M \subset \mathbb{R}^3$ be an open set. For every $\alpha \in A^1(M)$ there exist $\alpha_1, \alpha_2, \alpha_3 \in C^\infty(M)$ such that $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$. The map $\phi : \mathcal{X}(M) \to A^1(M)$ with

$$\phi(\alpha_1\frac{\partial}{\partial x^1} + \alpha_2\frac{\partial}{\partial x^2} + \alpha_3\frac{\partial}{\partial x^3}) = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$$

is a linear isomorphism. For every $\theta \in A^2(M)$ there exist $\beta_1, \beta_2, \beta_3 \in C^\infty(M)$ such that $\theta = \beta_1 dx^2 \wedge dx^3 + \beta_2 dx^3 \wedge dx^1 + \beta_3 dx^1 \wedge dx^2$ and $\psi : \mathcal{X}(M) \to A^2(M)$ with

$$\psi(\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} + \beta_3\frac{\partial}{\partial x^3}) = \theta$$

is a linear isomorphism. Finally, $\mu : C^\infty(M) \to A^3(M)$ with $\mu(f) = fdx^1 \wedge dx^2 \wedge dx^3$ is a linear isomorphism. Prove that $\phi(\xi) \wedge \phi(\zeta) = \psi(\xi \times \zeta)$ and $\phi(\xi) \wedge \psi(\zeta) = \mu(\langle \xi, \zeta \rangle)$

for every ξ, $\zeta \in \mathcal{X}(M)$, where $\times$ is the usual exterior product on $\mathbb{R}^3$ and $\langle , \rangle$ is the euclidean inner product, and the following diagram commutes.

$$\begin{array}{ccccccc} C^\infty(M) & \stackrel{\text{grad}}{\longrightarrow} & \mathcal{X}(M) & \stackrel{\text{curl}}{\longrightarrow} & \mathcal{X}(M) & \stackrel{\text{div}}{\longrightarrow} & C^\infty(M) \\ \Big\downarrow {\scriptstyle id} & & \Big\downarrow {\scriptstyle \phi} & & \Big\downarrow {\scriptstyle \psi} & & \Big\downarrow {\scriptstyle \mu} \\ C^\infty(M) & \stackrel{d}{\longrightarrow} & A^1(M) & \stackrel{d}{\longrightarrow} & A^2(M) & \stackrel{d}{\longrightarrow} & A^3(M) \end{array}$$

7. Let $M \subset \mathbb{R}^n$ be an open set and $\omega \in A^1(M)$ such that $\omega \wedge dx^1 \wedge \cdots \wedge dx^k = 0$, where $k < n$. Prove that there exist $f_1, ..., f_k \in C^\infty(M)$ such that $\omega = f_1 dx^1 + \cdots + f_k dx^k$.

8. Prove that the (total space of the) tangent bundle of a smooth manifold is always an orientable smooth manifold.

9. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a smooth function. If $c \in \mathbb{R}$ is a regular value of f and $M = f^{-1}(c) \neq \varnothing$, prove that M is an orientable $(n-1)$-dimensional smooth submanifold of $\mathbb{R}^n$.
(Hint : The pull-back of $\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial f}{\partial x^j} \cdot dx^1 \wedge ... \wedge dx^{j-1} \wedge dx^{j+1} \wedge ... \wedge dx^n$ on M vanishes nowhere on M.)

10. Prove that orientability is a property of smooth manifolds which remains invariant under smooth diffeomorphisms.

11. Let M be a smooth n-manifold and $\omega \in A^k(M)$, $0 \leq k \leq n$. Let G be a group of diffeomorphisms of M which acts properly discontinuously on M so that M/G is a Hausdorff space. If $g^*\omega = \omega$ for every $g \in G$, prove that there exists a unique $\tilde{\omega} \in A^k(M/G)$ such that $p^*\tilde{\omega} = \omega$, where $p : M \to M/G$ is the quotient map. Use this to prove that if M is orientable and ω is a volume element such that $g^*\omega = \omega$ for every $g \in G$, then M/G is orientable.

12. Let M be a smooth n-manifold and $\omega \in A^k(M)$, $0 \leq k \leq n$. Let G be a group of diffeomorphisms of M which acts properly discontinuously on M and let M/G be Hausdorff. If $\tilde{\omega} \in A^k(M/G)$, $0 \leq k \leq n$ and $\omega = p^*\tilde{\omega}$, where $p : M \to M/G$ is the quotient map, prove that $g^*\omega = \omega$ for every $g \in G$. Thus, if M/G is orientable, then M is necessarily orientable.

13. Let $G =< g, h >$, where $g, h : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $g(x, y) = (x + 1, y)$ and $h(x, y) = (1 - x, y + 1)$. In other words $G =< g, h : h^{-1}gh = g^{-1} >$. Prove that the quotient space, $K^2 = \mathbb{R}^2/G$, which is the Klein bottle, is a non-orientable connected compact smooth 2-manifold.

14. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and let ω be the standard volume element of S^{n-1}. Prove that

$$\int_{S^{n-1}} \langle Ax, x \rangle \omega = \frac{1}{n} \mathrm{Tr} A \cdot \mathrm{vol}(S^{n-1})$$

where $\langle , \rangle$ is the euclidean inner product.

(Hint: Use the Spectral Theorem.)

15. If $k \in \mathbb{Z}^+$, prove that the differential $(n-1)$-form

$$\omega_k = \sum_{j=1}^{n+1} (-1)^{j-1} \frac{x^j}{\|x\|^k} \cdot dx^1 \wedge ... \wedge dx^{j-1} \wedge dx^{j+1} \wedge ... \wedge dx^{n+1}$$

is not exact on $\mathbb{R}^{n+1} \setminus \{0\}$.

16. Let M be an oriented smooth n-manifold by a volume element $\omega \in A^n(M)$. For every $X \in \mathcal{X}(M)$ there exists a unique smooth function $\mathrm{div}_\omega X \in C^\infty(M)$, which is called the ω-divergence of X such that $d(i_X\omega) = (\mathrm{div}_\omega X)\omega$. If $M = \mathbb{R}^n$ and $\omega = f dx^1 \wedge ... \wedge dx^n$, where $f \in C^\infty(\mathbb{R}^n)$ with $f \neq 0$, prove that for

$$X = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}$$

we have

$$\mathrm{div}_\omega X = \frac{1}{f} \sum_{k=1}^{n} \frac{\partial (f X^k)}{\partial x^k}.$$

17. If M is a smooth manifold and X, $Y \in \mathcal{X}(M)$, prove that

$$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X.$$

18. Let M be a compact connected oriented smooth n-manifold by a volume element $\omega \in A^n(M)$. A smooth vector field $X \in \mathcal{X}(M)$ with corresponding one-parameter group of diffeomorphisms $(\Phi_t)_{t\in\mathbb{R}}$ is called *ω-volume preserving* if $\Phi_t^*\omega = \omega$ for every $t \in \mathbb{R}$.
(a) Prove that $X \in \mathcal{X}(M)$ is ω-volume preserving if and only if the flux form $i_X\omega$ is closed.
(b) Prove that that the vector space $\mathcal{X}_\omega(M)$ of all ω-volume preserving smooth vector fields of M is isomorphic to $A^{n-1}(M) \cap \mathrm{Ker}d$.

A ω-volume preserving smooth vector field $X \in \mathcal{X}(M)$ is called *ω-homologically trivial* if the flux form $i_X\omega$ is exact.
(c) Prove that for every X, $Y \in \mathcal{X}_\omega(M)$ the smooth vector field $[X,Y]$ is always ω-homologically trivial.

Let now M be 3-dimensional such that $H^1(M) = \{0\}$ and $H^2(M) = \{0\}$.
(d) If X, $Y \in \mathcal{X}_\omega(M)$ and $\eta \in A^1(M)$ is such that $d\eta = i_Y\omega$, prove that the integral

$$\ell(X,Y) = \int_M i_X\omega \wedge \eta$$

does not depend on the choice of the primitive η of the flux form $i_Y\omega$.
(e) Prove that $\ell : \mathcal{X}_\omega(M) \times \mathcal{X}_\omega(M) \to \mathbb{R}$ is a non-degenerate, symmetric, bilinear form.

19. Let M be an open subset of $\mathbb{R}^3$ and $\theta \in A^1(M)$ be nowhere vanishing. Prove that $\mathrm{Ker}\theta$ is integrable if and only if every $p \in M$ has an open neighbourhood U on

which there exists a nowhere vanishing $f \in C^\infty(U)$ such that $f\theta|_U$ is exact.

20. Let M be a Riemannian smooth n-manifold and let $f : M \to \mathbb{R}$ be a smooth function. We assume that M is oriented with Riemannian volume element Ω. The function

$$\triangle f = \mathrm{div}(\mathrm{grad} f)$$

is the (Riemannian) Laplacian of f.
(c) If $h : M \to \mathbb{R}$ is another smooth function, prove that
(i) $\mathrm{div}(f\mathrm{grad}h) = \langle \mathrm{grad}\ f, \mathrm{grad}h\rangle + f\triangle h$ and
(ii) $\triangle(fh) = 2\langle \mathrm{grad}\ f, \mathrm{grad}h\rangle + f\triangle h + h\triangle f$.
(d) Let $D \subset M$ be a domain with smooth boundary and ν be the unit outer normal on ∂D. If $f, h : M \to \mathbb{R}$ are two smooth functions at least one of which has compact support, prove Green's formulas

$$\int_{\overline{D}} (\langle \mathrm{grad}\ f, \mathrm{grad}h\rangle + f\triangle h)\Omega = \int_{\partial D} (f\langle \mathrm{grad} f, \nu\rangle i_\nu \Omega,$$

$$\int_{\overline{D}} (h\triangle f - f\triangle h)\Omega = \int_{\partial D} (h\langle \mathrm{grad} f, \nu\rangle - f\langle \mathrm{grad} h, \nu\rangle) i_\nu \Omega.$$

(e) The smooth function $f : M \to \mathbb{R}$ is called harmonic if $\triangle f = 0$. Prove that if M is connected, then every harmonic function on M with compact support is constant.

21. Let $n \geq 2$ be an integer and $g : (0, +\infty) \times (0, \pi)^{n-1} \times (0, 2\pi) \to \mathbb{R}^{n+1}$ be the smooth map with $g(\rho, \theta_1, ..., \theta_n) = (x^1, ..., x^n, x^{n+1})$ where

$x^1 = \rho \cos\theta_1$
$x^2 = \rho \sin\theta_1 \cos\theta_2$
....
$x^n = \rho \sin\theta_1 \cdots \sin\theta_{n-1} \cos\theta_n$
$x^{n+1} = \rho \sin\theta_1 \cdots \sin\theta_{n-1} \sin\theta_n$.

(a) Prove that $g^*\omega = \rho^{n+1} \sin^{n-1}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1} \cdot d\theta_1 \wedge \cdots \wedge d\theta_n$, where

$$\omega = \sum_{j=1}^{n+1} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^{n+1} \in A^n(\mathbb{R}^{n+1}).$$

(b) Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion. Prove that if $n = 2m - 1$, then

$$\mathrm{vol}(S^{2m-1}) = \int_{S^{2m-1}} i^*\omega = \frac{2\pi^m}{(m-1)!}$$

and if $n = 2m$, then

$$\mathrm{vol}(S^{2m}) = \int_{S^{2m}} i^*\omega = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m-1)}.$$

Chapter 5

De Rham cohomology

5.1 Homotopy invariance

This chapter is devoted to the development of methods of computation of the de Rham cohomology of smooth manifolds. The first important property of the de Rham cohomology is homotopy invariance. This will give the de Rham cohomology of $\mathbb{R}^n$, a result which is traditionally known as the Poincaré Lemma.

Let M be a smooth n-manifold. In order to compute the de Rham cohomology of the smooth $(n+1)$-manifold $\mathbb{R}\times M$ we consider the projection $\pi : \mathbb{R}\times M \to M$ and the inclusion $i : M \to \mathbb{R}\times M$ with $i(p) = (0, p)$. Since $\pi \circ i = id_M$, we immediately have that $i^* \circ \pi^* = id$. The greater part of this section is devoted to proving that $\pi^* \circ i^* = id$ also, and therefore $\pi^* : H^*(M) \to H^*(\mathbb{R}\times M)$ is an isomorphism of graded algebras with inverse i^*. We note that in place of the inclusion i we could very well use the inclusion $i_t : M \to \mathbb{R}\times M$ with $i_t(p) = (t, p)$ for any $t \in \mathbb{R}$.

Let $\mathcal{A}$ be a smooth atlas of M and let $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ be a smooth partition of unity subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. Then, $\tilde{\mathcal{A}} = \{(\mathbb{R}\times U, id\times\phi_U) : (U, \phi_U) \in \mathcal{A}\}$ is a smooth atlas of $\mathbb{R}\times M$ and $\{\tilde{f}_U : (U, \phi_U) \in \mathcal{A}\}$ is a smooth partition of unity subordinated to the open cover $\tilde{\mathcal{U}} = \{\mathbb{R}\times U : (U, \phi_U) \in \mathcal{A}\}$ of $\mathbb{R}\times M$, where $\tilde{f}_U = f_U \circ \pi$.

Let now $\omega \in A^k(\mathbb{R}\times M)$. If $\phi_U = (x^1, ..., x^n)$, there are smooth functions $f^U_{i_1\cdots i_{k-1}}$, $g^U_{j_1\cdots j_k}$ on $\mathbb{R}\times U$ such that

$$\omega|_{\mathbb{R}\times U} = \sum_{1\leq i_1<\cdots<i_{k-1}\leq n} f^U_{i_1\cdots i_{k-1}} dt\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}$$

$$+ \sum_{1\leq j_1<\cdots<j_k\leq n} g^U_{j_1\cdots_k} dx^{j_1}\wedge\cdots\wedge dx^{j_k}.$$

and globally

$$\omega = \sum_{(U,\phi_U)\in\mathcal{A}} \left(\sum_{1\leq i_1<\cdots<i_{k-1}\leq n} \tilde{f}_U f^U_{i_1\cdots i_{k-1}} dt\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}} \right)$$

$$+ \sum_{(U,\phi_U)\in\mathcal{A}} \left(\sum_{1\leq j_1<\cdots<j_k\leq n} \tilde{f}_U g^U_{j_1\cdots_k} dx^{j_1}\wedge\cdots\wedge dx^{j_k} \right).$$

From Corollary 1.4.5, for every $U \in \mathcal{U}$ there is a smooth function $h_U : M \to [0,1]$ such that $\text{supp} f_U \subset h_U^{-1}(1)$ and $\text{supp} h_U \subset U$. If $\tilde{h}_U = h_U \circ \pi$, then $\tilde{f}_U \tilde{h}_U = \tilde{f}_U$ and

$$\omega = \sum_{(U,\phi_U)\in\mathcal{A}} \Bigg(\sum_{1\leq i_1<\cdots<i_{k-1}\leq n} \tilde{f}_U f^U_{i_1\cdots i_{k-1}} dt \wedge (\tilde{h}_U dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}) \Bigg)$$

$$+ \sum_{(U,\phi_U)\in\mathcal{A}} \Bigg(\sum_{1\leq j_1<\cdots<j_k\leq n} \tilde{f}_U g^U_{j_1\cdots k} (\tilde{h}_U dx^{j_1} \wedge \cdots \wedge dx^{j_k}) \Bigg).$$

On each strip $\mathbb{R}\times U$ only a finite number of elements of $\mathcal{A}$ give non-zero terms of the above sum. Note that each differential form $\tilde{h}_U dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$ can be smoothly extended to all of $\mathbb{R} \times M$ by setting it zero outside $\mathbb{R} \times U$ so that

$$\tilde{h}_U dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} = \pi^* \eta^U_{i_1\cdots i_{k-1}}$$

where

$$\eta^U_{i_1\cdots i_{k-1}} = \begin{cases} h_U dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}, & \text{on } U \\ 0, & \text{on } M \setminus U. \end{cases}$$

Similarly, in the second sum we have $\tilde{h}_U dx^{j_1} \wedge \cdots \wedge dx^{j_k} = \pi^* \zeta^U_{j_1\cdots j_k}$, where

$$\zeta^U_{j_1\cdots j_k} = \begin{cases} h_U dx^{j_1} \wedge \cdots \wedge dx^{j_k}, & \text{on } U \\ 0, & \text{on } M \setminus U. \end{cases}$$

Thus, every $\omega \in A^k(\mathbb{R} \times M)$ is a locally finite sum of differential k-forms of (the compressed) type

$$f(t,x)dt \wedge \pi^*\eta + g(t,x)\pi^*\zeta$$

for suitable smooth functions f, g and $\eta \in A^{k-1}(M)$, $\zeta \in A^k(M)$.

Now set $A^k(M) = 0$ for every integer $k < 0$ and define $S : A^k(M) \to A^{k-1}(M)$ by

$$S\omega = \left(\int_0^t f(s,x)ds \right) \pi^*\eta$$

if $\omega = f(t,x)dt \wedge \pi^*\eta + g(t,x)\pi^*\zeta$ and extending using the above. Thus we obtain a linear map $S : A^*(M) \to A^*(M)$ of degree -1 of the graded vector space $A^*(M)$, which according to the following crucial lemma is a cochain homotopy between $\pi^* \circ i^*$ and the identity.

Lemma 5.1.1. *$d \circ S + S \circ d = id - \pi^* \circ i^*$.*

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & A^{k-1}(M) \arrow[r, "d"] \arrow[d, shift right, "\pi^*\circ i^*"'] \arrow[d, shift left, "id"] & A^k(M) \arrow[r, "d"] \arrow[dl, "S"'] \arrow[d, shift right, "\pi^*\circ i^*"'] \arrow[d, shift left, "id"] & A^{k+1}(M) \arrow[r, "d"] \arrow[dl, "S"'] \arrow[d, shift right, "\pi^*\circ i^*"'] \arrow[d, shift left, "id"] & \cdots \\
\cdots \arrow[r, "d"] & A^{k-1}(M) \arrow[r, "d"] & A^k(M) \arrow[r, "d"] & A^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Proof. If $\omega = g(t,x)\pi^*\zeta$, we have $S\omega = 0$, by definition, and so

$$d(S\omega) + S(d\omega) = S(dg \wedge \pi^*\zeta + g\pi^*(d\zeta)) = S(dg \wedge \pi^*\zeta) = S\big(\frac{\partial g}{\partial t}dt \wedge \pi^*\zeta\big)$$

$$= \Big(\int_0^t \frac{\partial g}{\partial s}(s,x)ds\Big)\pi^*\zeta = \big(g(t,x) - g(0,x)\big)\pi^*\zeta = \omega - (\pi^* \circ i^*)\omega.$$

If now $\omega = f(t,x)dt \wedge \pi^*\eta$, then $\omega - (\pi^* \circ i^*)\omega = \omega$, because $i^*(dt) = 0$. On the other hand, we have

$$d(S\omega) = d\bigg(\Big(\int_0^t f(s,x)ds\Big)\pi^*\eta\bigg) = d\Big(\int_0^t f(s,x)ds\Big)\pi^*\eta + \Big(\int_0^t f(s,x)ds\Big)d(\pi^*\eta)$$

$$= \bigg[\Big(\int_0^t \frac{\partial f}{\partial x}(s,x)ds\Big)dx + f(t,x)dt\bigg] \wedge \pi^*\eta + \Big(\int_0^t f(s,x)ds\Big)(\pi^*\eta)$$

and

$$S(d\omega) = S\big(\frac{\partial f}{\partial x}dx \wedge dt \wedge \pi^*\eta - f(t,x)dt \wedge d(\pi^*\eta)\big)$$

$$= -\Big(\int_0^t \frac{\partial f}{\partial x}(s,x)ds\Big)dx \wedge \pi^*\eta - \Big(\int_0^t f(s,x)ds\Big)d(\pi^*\eta).$$

therefore,

$$d(S\omega) + S(d\omega) = f(t,x)dt \wedge \pi^*\eta = \omega = \omega - (\pi^* \circ i^*)\omega.$$

This completes the proof. □

Corollary 5.1.2. *For every smooth manifold M the canonical projection $\pi : \mathbb{R} \times M \to M$ induces an isomorphism $\pi^* : H^*(M) \to H^*(\mathbb{R} \times M)$ in de Rham cohomology.*

Proof. Indeed, for every closed differential form $\omega \in A^*(\mathbb{R} \times M)$ we have

$$\omega - (\pi^* \circ i^*)\omega = d(S\omega)$$

from Lemma 5.1.1 and hence $id - \pi^* \circ i^* = 0$ in the level of cohomology. □

Since $\mathbb{R}^0$ is a singleton, from Theorem 4.3.7 and we get inductively the following.

Corollary 5.1.3. *The de Rham cohomology of $\mathbb{R}^n$, $n \in \mathbb{Z}^+$, is*

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & \text{for } k = 0, \\ \{0\}, & \text{for } k > 0. \end{cases} \qquad \Box$$

Definition 5.1.4. Let M and N be two smooth manifolds. Two smooth maps $f,\, g : M \to N$ are said to be *(smoothly) homotopic* if there exists a smooth map $F : \mathbb{R} \times M \to M$ such that $F(t,p) = f(p)$ for all $t \leq 0$, $p \in M$ and $F(t,p) = g(p)$ for all $t \geq 1$, $p \in M$ or equivalently $F \circ i_t = f$ for $t \leq 0$ and $F \circ i_t = g$ for $t \geq 1$. In this case we write $f \simeq g$ and call F a *(smooth) homotopy* from f to g.

It is obvious that (smooth) homotopy is an equivalence relation in the set of all smooth maps from M to N.

Theorem 5.1.5. *Let M and N be two smooth manifolds. If two smooth maps f, $g : M \to N$ are (smoothly) homotopic, then $f^* = g^* : H^*(N) \to H^*(M)$.*

Proof. If $F : \mathbb{R} \times M \to M$ is a smooth homotopy from f to g, then

$$f^* = (F \circ i_0)^* = i_0^* \circ F^* = (\pi^*)^{-1} \circ F = i_1^* \circ F^* = (F \circ i_1)^* = g^*. \quad \square$$

As we know, the de Rham cohomology is a diffeomorphism invariant. Actually, Theorem 5.1.5 implies a much more stronger statement.

Definition 5.1.6. Two smooth manifolds M and N are said to have the same *smooth homotopy type* if there are smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f \simeq id_M$ and $f \circ g \simeq id_N$. Such maps f and g are called *homotopy equivalences* and *homotopy inverses* to each other.

Corollary 5.1.7. *If two smooth manifolds have the same smooth homotopy type, they have isomorphic de Rham cohomology algebras.*

Two smooth manifolds with the same smooth homotopy type may be quite different, for instance they may not even have the same dimension.

Examples 5.1.8. (a) The n-dimensional euclidean space has the homotopy type of a singleton for every $n \in \mathbb{Z}^+$. Indeed, if $i : \{0\} \hookrightarrow \mathbb{R}^n$ is the inclusion and $r : \mathbb{R}^n \to \{0\}$ the unique obvious map, then $r \circ i = id_{\{0\}}$. On the other hand, if $h : \mathbb{R} \to [0, 1]$ is a smooth function such that $h^{-1}(0) = (-\infty, 0]$ and $h^{-1}(1) = [1, +\infty)$, then $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(t, x) = h(t)x$ is a smooth homotopy from $i \circ r$ to $id_{\mathbb{R}}$. A smooth manifold with the smooth homotopy type of a singleton is called *contractible.*

(b) The n-sphere S^n has the same smooth homotopy type with the punctured $(n+1)$-dimensional euclidean space $\mathbb{R}^{n+1} \setminus \{0\}$. To see this, let $i : S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ be the inclusion and let $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ be the smooth map

$$r(x) = \frac{1}{\|x\|} \cdot x.$$

Then, obviously $r \circ i = id_{S^n}$, and $i \circ r \simeq id_{\mathbb{R}^{n+1} \setminus \{0\}}$. Indeed, the smooth map $F : \mathbb{R} \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ defined by

$$F(t, x) = (1 - h(t)) \frac{1}{\|x\|} \cdot x + h(t) \cdot x,$$

where h is the smooth function of (a), is a smooth homotopy from $i \circ r$ to $id_{\mathbb{R}^{n+1} \setminus \{0\}}$.

In the terminology of Algebraic Topology, the map r is a retraction and S^n is a (strong) deformation retract of $\mathbb{R}^{n+1}$.

5.2 The degree of a smooth map

If M is a compact, connected, oriented smooth n-manifold, there exists a unique de Rham cohomology class $o_M \in H^n(M)$ whose integral over M is equal to 1, which is called the *cohomological fundamental class (or orientation class)* of M. By Theorem 4.5.7, the cohomology class of a differential n-form $\omega \in A^n(M)$ is then

$$[\omega] = \left(\int_M \omega\right) o_M.$$

Let N be another compact, connected, oriented smooth n-manifold and suppose that $f : M \to N$ is a smooth map. We call

$$\deg f = \int_M f^* o_N$$

the *degree of* f. Then, for every $\theta \in A^n(N)$ we have

$$\int_M f^*\theta = (\deg f) \cdot \left(\int_N \theta\right)$$

and so the transpose $f^* : H^n(N) \to H^n(M)$ is given by the formula

$$f^*\theta = (\deg f) \cdot \left(\int_N \theta\right) \cdot o_M.$$

The degree has the following properties.

Proposition 5.2.1. *Let M, N be two compact, connected, oriented smooth n-manifolds and $f : M \to N$ be a smooth map.*
(a) If f is a diffeomorphism, then $\deg f = 1$*, in case f preserves orientation, and* $\deg f = -1$*, if f reverses orientation.*
(b) If f is smoothly homotopic to a smooth map $g : M \to N$, then $\deg f = \deg g$.
(c) If P is compact, connected, oriented smooth n-manifold and $h : N \to P$ is a smooth map, then $\deg(h \circ f) = (\deg h) \cdot (\deg f)$.
(d) If $\deg f \neq 0$*, then f is onto N.*

Proof. Assertions (a) and (c) are obvious from the definition of the degree, and assertion (b) is an immediate consequence of Theorem 5.1.5. To prove (d), suppose that f is not onto N. Then $N \setminus f(M)$ is a non-empty open subset of N and there exists a smooth function $h : N \to [0.1]$ such that $\varnothing \neq \mathrm{supp} h \subset N \setminus f(M)$, by Corollary 1.4.5. Thus, $h \circ f = 0$ and therefore

$$\int_M f^*(h o_N) = \int_M (h \circ f) f^* o_N = 0.$$

This means that $\deg f = 0$. $\square$.

We shall give an important application of the notion of degree to tangent vector fields of even dimensional spheres which is known as the "Hairy Ball Theorem".

We observe first that the antipodal map $a : S^n \to S^n$, $n \geq 1$ with $a(x) = -x$ has degree $(-1)^{n+1}$. This follows immediately from Example 4.4.2(c).

Lemma 5.2.2. *If two smooth maps f, $g : S^n \to S^n$, $n \geq 1$, satisfy $f(x) \neq -g(x)$ for every $x \in S^n$, then they are smoothly homotopic and so $\deg f = \deg g$.*

Proof. If $h : \mathbb{R} \to [0,1]$ is a smooth function such that $h^{-1}(0) = (-\infty, 0]$ and $h^{-1}(1) = [1, +\infty)$, then $F : \mathbb{R} \times S^n \to S^n$ defined by

$$F(t,x) = \frac{1}{\|(1-h(t))f(x) + h(t)g(x)\|} \cdot [(1-h(t))f(x) + h(t)g(x)]$$

is a smooth homotopy from f to g. $\square$

Theorem 5.2.3. *Every smooth tangent vector field on an even dimensional sphere vanishes in at least one point.*

Proof. Let $X \in \mathcal{X}(S^n)$, $n \geq 1$, be nowhere vanishing. There exists a unique smooth map $F : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $X(p) = (p, F(p))$ and $\langle p, F(p) \rangle = 0$ for every $p \in S^n$. We consider the smooth map $f : S^n \to S^n$ defined by

$$f(p) = \frac{1}{\|F(p)\|} \cdot F(p).$$

Again $\langle p, f(p) \rangle = 0$, and so $f(p) \neq \pm p$ for every $p \in S^n$. From the preceding Lemma 5.2.2, f must be smoothly homotopic to the identity and to the antipodal map a. Therefore,

$$1 = \deg f = \deg a = (-1)^{n+1}$$

and n must be odd. $\square$.

In the sequel we shall give another more geometric description of the degree from which will follow that the degree is always an integer. As before, let M, N be two compact, connected, oriented smooth n-manifolds and let $f : M \to N$ be a smooth map. Let $y \in N$ be a regular value of f such that $f^{-1}(y) \neq \varnothing$. For each $p \in f^{-1}(y)$ the derivative $f_{*p} : T_pM \to T_yN$ is a linear isomorphism and so there exists an open neighbourhood $V \subset M$ of p such that $f(V) \subset N$ is open and $f|_V : V \to f(V)$ is a smooth diffeomorphism, by the Inverse Map Theorem. In particular, $f^{-1}(y) \cap V = \{p\}$. This means that $f^{-1}(y)$ is a closed discrete subset of M, hence finite, because M is compact. So, there are $p_1, ..., p_m \in M$ for some $m \in \mathbb{N}$ such that $f^{-1}(y) = \{p_1, ..., p_m\}$, and each p_k has an open neighbourhood $V_k \subset M$ such that $f|_{V_k} : V_k \to f(V_k)$ is a smooth diffeomorphism. Moreover, $V_k \cap V_l = \varnothing$ for $k \neq l$. The set $C = M \setminus V_1 \cup \cdots \cup V_k$ is compact and so is $f(A)$. The set

$$W = \bigcap_{k=1}^{m} f(V_k) \cap (N \setminus f(C))$$

is an open neighbourhood of y and $f^{-1}(W) \subset V_1 \cup \cdots \cup V_k$. If now $U_k = V_k \cap f^{-1}(W)$ for $1 \leq k \leq m$, we have

$$f^{-1}(W) = U_1 \cup \cdots \cup U_k$$

each U_k is an open neighbourhood of p_k and $f(U_k) = W$. Finally, U_1,..., U_m are mutually disjoint and $f|_{U_k} : U_k \to W$ is a smooth diffeomorphism. Shrinking W, if necessary, we may always pick it to be connected.

In the particular case where f is a local diffeomorphism onto N the above considerations show that f is a finite covering map.

For every $p \in M$ we set now

$$\epsilon(p) = \begin{cases} 0, & \text{if } f_{*p} \text{ is not a linear isomorphism,} \\ +1, & \text{if } f_{*p} \text{ is an orientation preserving linear isomorphism,} \\ -1, & \text{if } f_{*p} \text{ is an orientation reversing linear isomorphism.} \end{cases}$$

Theorem 5.2.4. *Let M, N be compact, connected, oriented smooth n-manifolds and let $f : M \to N$ be a smooth map. If $y \in N$ is a regular value of f such that $f^{-1}(y) \neq \varnothing$, then*

$$\deg f = \sum_{p \in f^{-1}(p)} \epsilon(p).$$

Proof. We continue to use the notations of the preceding considerations. The cohomological fundamental class o_N can be represented by a differential n-form $\omega \in A^n(N)$ such that $\mathrm{supp}\omega \subset W$. Then, $\mathrm{supp} f^*\omega \subset f^{-1}(W)$ and

$$\deg f = \int_M f^*\omega = \sum_{k=1}^m \int_{U_k} f^*\omega|_{U_k} = \sum_{k=1}^m \int_{U_k} (f|_{U_k})^*\omega.$$

If $\epsilon(p) = +1$, then $f|_{U_k}$ is orientation preserving, since U_k is connected, and for the same reason if $\epsilon(p) = -1$, then $f|_{U_k}$ is orientation reversing. It follows that

$$\deg f = \sum_{k=1}^m \int_{U_k} (f|_{U_k})^*\omega = \sum_{k=1}^m \epsilon(p_k) \int_W \omega|_W = \sum_{k=1}^m \epsilon(p_k). \quad \square$$

5.3 The Mayer-Vietoris exact sequence

In this section we shall develop the Mayer-Vietoris long exact sequence for de Rham cohomology, which is a powerful tool for computations. Let M be a smooth n-manifold and let U, $V \subset M$ be two open sets such that $M = U \cup V$. We denote by $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ the inclusions. We also consider the inclusions $i_U : U \hookrightarrow M$ and $i_V : V \hookrightarrow M$.

$$U \cap V \xrightarrow[j]{i} U \coprod V \xrightarrow{\text{inclusion}} M$$

Passing to the level of differential forms we get the following sequence of cochain maps

$$0 \longrightarrow A^*(M) \xrightarrow{(i_U^*, i_V^*)} A^*(U) \oplus A^*(V) \xrightarrow{\rho} A^*(U \cap V) \longrightarrow 0$$

where $\rho(\omega,\theta) = j^*\theta - i^*\omega$, which is exact and is called the Mayer-Vietoris exact sequence. Its exactness at $A^*(M)$ and $A^*(U)\oplus A^*(V)$ is obvious. In order to see that ρ is an epimorphism, let $\omega \in A^*(U\cap V)$ and $\{f_U, f_V\}$ be a smooth partition of unity subordinated to the open cover $\{U,V\}$ of M. At every point $p\in U\cap V$ we have

$$j^*(f_U\omega)_p - i^*(-f_V\omega)_p = f_U(p)\omega_p + f_V(p)\omega_p = \omega_p.$$

Therefore, $\rho(-f_V\omega, f_U\omega) = \omega$ and $-f_V\omega$ can be considered in $A^*(U)$, extended by zero on $U\setminus U\cap V$, and similarly $f_U\omega$ can be considered in $A^*(V)$.

From the fundamental theorem of homological algebra (also known as "the snake lemma") we get the Mayer-Vietoris long exact sequence for the de Rham cohomology.

$$\cdots \xrightarrow{d^*} H^k(M) \xrightarrow{(i_U^*,i_V^*)} H^k(U)\oplus H^k(V) \xrightarrow{\rho} H^k(U\cap V) \xrightarrow{d^*} H^{k+1}(M) \longrightarrow \cdots$$

We shall describe in detail the connecting homomorphism d^*. The following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{k-1}(M) & \xrightarrow{(i_U^*,i_V^*)} & A^{k-1}(U)\oplus A^{k-1}(V) & \xrightarrow{\rho} & A^{k-1}(U\cap V) & \longrightarrow & 0\\
 & & \big\downarrow d & & \big\downarrow d\oplus d & & \big\downarrow d & & \\
0 & \longrightarrow & A^{k}(M) & \xrightarrow{(i_U^*,i_V^*)} & A^{k}(U)\oplus A^{k}(V) & \xrightarrow{\rho} & A^{k}(U\cap V) & \longrightarrow & 0\\
 & & \big\downarrow d & & \big\downarrow d\oplus d & & \big\downarrow d & & \\
0 & \longrightarrow & A^{k+1}(M) & \xrightarrow{(i_U^*,i_V^*)} & A^{k+1}(U)\oplus A^{k+1}(V) & \xrightarrow{\rho} & A^{k+1}(U\cap V) & \longrightarrow & 0
\end{array}$$

has exact rows. Let $\omega\in A^k(U\cap V)$ be a closed differential k-form. From the above, $\rho(-f_V\omega, f_U\omega) = \omega$ and $\rho(-d(f_V\omega), d(f_U\omega)) = 0$, by exactness. Thus,

$$j^*(d(f_U\omega)) = i^*(-d(f_V\omega))$$

and we obtain a well defined closed differential $(k+1)$-form $\theta\in A^{k+1}(M)$ by

$$\theta = \begin{cases} -d(f_V\omega), & \text{on } U,\\ d(f_U\omega), & \text{on } V.\end{cases}$$

The cohomology class $[\theta]\in H^{k+1}(M)$ depends only on the cohomology class of ω and $d^*[\omega] = [\theta]$.

Example 5.3.1. Using a Mayer-Vietoris long exact sequence combined with the homotopy invariance we shall compute the de Rham cohomology of the spheres S^n, $n\geq 0$. We already know from Theorem 4.3.7 and Theorem 4.5.7 that $H^0(S^n)\cong\mathbb{R}$ and $H^n(S^n)\cong\mathbb{R}$ for $n\geq 1$. In particular,

$$H^k(S^1) = \begin{cases}\mathbb{R}, & \text{for } k = 0,1,\\ \{0\}, & \text{for } k>1.\end{cases}$$

Moreover, $H^0(S^0) \cong \mathbb{R} \oplus \mathbb{R}$ and $H^k(S^0) = \{0\}$ for $k > 0$. So we assume that $n \geq 2$ in the sequel.

To begin with, we note first that for every $0 < \epsilon < 1$ the set

$$A_\epsilon = \{x \in S^n : |\langle x, e_{n+1} \rangle| < \epsilon\}$$

where $\langle , \rangle$ is the euclidean inner product in $\mathbb{R}^{n+1}$, has the smooth homotopy type of S^{n-1}, which is identified with the set $\{x \in S^n : \langle x, e_{n+1} \rangle = 0\}$. Indeed, let $i : S^{n-1} \hookrightarrow S^n$ be the inclusion and let $r : A_\epsilon \to S^{n-1}$ be the smooth map defined by

$$r(x) = \frac{1}{\|x - \langle x, e_{n+1} \rangle\|} \cdot (x - \langle x, e_{n+1} \rangle).$$

Then, obviously $r \circ i = id_{S^{n-1}}$. On the other hand, let $h : \mathbb{R} \to [0, 1]$ be a smooth function such that $h^{-1}(0) = (-\infty, 0]$ and $h^{-1}(1) = [1, +\infty)$. The smooth map $F : \mathbb{R} \times A_\epsilon \to A_\epsilon$ defined by

$$F(t, x) = \frac{1}{\|x - h(t)\langle x, e_{n+1} \rangle\|} \cdot (x - h(t)\langle x, e_{n+1} \rangle)$$

is a smooth homotopy of id_{A_ϵ} with $i \circ r$. Hence $i \circ r \simeq id_{A_\epsilon}$ and the transpose of the inclusion on cohomology $i^* : H^*(A_\epsilon) \to H^*(S^{n-1})$ is an isomorphism of graded algebras.

Let now $U = \{x \in S^n : \langle x, e_{n+1} \rangle > -\epsilon\}$ and $V = \{x \in S^n : \langle x, e_{n+1} \rangle < \epsilon\}$. Then, $S^n = U \cup V$ and $U \cap V = A_\epsilon$. Moreover, the open subsets U, V are both contractible, because the smooth map $G : \mathbb{R} \times U \to U$ defined by

$$G(t, x) = \frac{1}{\|(1 - h(t))e_{n+1} + h(t)x\|} \cdot ((1 - h(t))e_{n+1} + h(t)x)$$

is a smooth homotopy of id_U with the constant map of U with value e_{n+1}. Therefore,

$$H^k(U) = \begin{cases} \mathbb{R}, & \text{for } k = 0, \\ \{0\}, & \text{for } k > 0. \end{cases} \qquad \square$$

and similarly for V. It follows that the corresponding Mayer-Vietoris long exact sequence splits in short exact sequences

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^0(S^{n-1}) \to H^1(S^n) \to 0$$

$$0 \to H^k(S^{n-1}) \xrightarrow{d^*} H^{k+1}(S^n) \to 0$$

for $k \geq 1$. The first short exact sequence gives $H^1(S^n) = \{0\}$ for every $n \geq 2$ and the second one gives inductively

$$H^k(S^n) \cong \cdots \cong H^1(S^{n-k+1}), \quad 2 \leq k \leq n.$$

It follows that

$$H^k(S^n) = \begin{cases} \mathbb{R}, & \text{for } k = 0, n, \\ \{0\}, & \text{for } k \neq 0, n. \end{cases}$$

Example 5.3.2. Let $\mathcal{A} = \{(U_k, \phi_k) : k = 0, 1, ...n\}$ be the canonical atlas of the complex projective n-space $\mathbb{C}P^n$, $n \geq 0$. Since $\mathbb{C}P^0$ is a singleton, $H^0(\mathbb{C}P^0) \cong \mathbb{R}$ and $H^k(\mathbb{C}P^0) = \{0\}$, for $k > 0$. So we assume that $n \geq 1$ in the sequel. We already know that $H^0(\mathbb{C}P^n) \cong \mathbb{R}$ and $H^{2n}(\mathbb{C}P^n) \cong \mathbb{R}$, since $\mathbb{C}P^n$ is a connected, compact orientable smooth $2n$-manifold.

If $E = \mathbb{C}P^n \setminus \{[0, ..., 0, 1]\}$, then $\mathbb{C}P^n = E \cup U_n$ and E has the smooth homotopy type of $\mathbb{C}P^{n-1}$. Indeed, let $i : \mathbb{C}P^{n-1} \to E$ be the smooth embedding $i[z_0, ..., z_{n-1}] = [z_0, ..., z_{n-1}, 0]$ and let $r : E \to \mathbb{C}P^{n-1}$ be the smooth submersion $r[z_0, ..., z_{n-1}, z_n] = [z_0, ..., z_{n-1}]$. Obviously, $r \circ i = id_{\mathbb{C}P^{n-1}}$. On the other hand, the smooth map $F : \mathbb{R} \times E \to E$ defined by

$$F(t, [z_0, ..., z_{n-1}, z_n]) = [z_0, ..., z_{n-1}, h(t)z_n],$$

where h is the smooth function of the previous Example 5.3.1, is a smooth homotopy of $i \circ r$ with id_E. Therefore, $i^* : H^*(E) \to H^*(\mathbb{C}P^{n-1})$ is an isomorphism of graded algebras.

Recall that the canonical smooth chart $\phi_n : U_n \to \mathbb{C}^n$ is given by

$$\phi_n[z_0, ..., z_{n-1}, z_n] = \Big(\frac{z_0}{z_n}, ..., \frac{z_{n-1}}{z_n}\Big)$$

and so

$$\phi_n(E \cap U_n) = \Big\{\Big(\frac{z_0}{z_n}, ..., \frac{z_{n-1}}{z_n}\Big) : (z_0, ..., z_{n-1}) \neq (0, ..., 0)\Big\} = \mathbb{C}^n \setminus \{0\}$$

has the homotopy type of S^{2n-1}, according to the Example 5.1.8(b). Hence from the previous Example 5.3.1 the de Rham cohomology of $E \cap U_n$ is

$$H^k(E \cap U_n) = \begin{cases} \mathbb{R}, & \text{for } k = 0, 2n-1, \\ \{0\}, & \text{for } k \neq 0, 2n-1. \end{cases}$$

From the corresponding Mayer-Vietoris long exact sequence

$$\cdots \longrightarrow H^{k-1}(E \cap U_n) \xrightarrow{d^*} H^k(\mathbb{C}P^n) \longrightarrow H^k(E) \oplus H^k(U_n) \xrightarrow{\rho} H^k(E \cap U_n) \xrightarrow{d^*} \cdots$$

follows that the inclusion $i_n : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ with $i_n[z_0, ..., z_{n-1}] = [z_0, ..., z_{n-1}, 0]$ induces a linear monomorphism $i_n^* : H^1(\mathbb{C}P^n) \to H^1(\mathbb{C}P^{n-1})$ and hence $H^1(\mathbb{C}P^n) = \{0\}$ for every $n \geq 1$. Also $H^{2n-1}(\mathbb{C}P^n) = \{0\}$, because $H^{2n-2}(S^{2n-1}) = \{0\}$ and $H^{2n-1}(E) \cong H^{2n-1}(\mathbb{C}P^{n-1}) = \{0\}$, since $\mathbb{C}P^{n-1}$ has dimension $2n-2$. For $1 < k < 2n-1$ the Mayer-Vietoris long exact sequence gives a linear isomorphism

$$i_n^* : H^k(\mathbb{C}P^n) \to H^k(\mathbb{C}P^{n-1}).$$

It follows now inductively that the de Rham cohomology of the complex projective n-space is

$$H^k(\mathbb{C}P^n) = \begin{cases} \mathbb{R}, & \text{for } k = 0, 2, 4, ..., 2n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Note that this computation only gives $H^*(\mathbb{C}P^n)$ as a graded vector space. It gives no information about the algebra structure. One way to obtain the de Rham cohomology algebra $H^*(\mathbb{C}P^n)$ is by applying the Poincaré Duality Theorem which will be proved in the next section.

In principle, using the Mayer-Vietoris long exact sequence we can compute the de Rham cohomology vector spaces of a smooth manifold M inductively from a finite open cover if we have control over the cohomologies of its elements as well as their intersections. This is possible if the open cover is admissible. An open cover $\mathcal{U}$ of M is called *admissible* if for every $m \in \mathbb{N}$ and any U_1,..., $U_m \in \mathcal{U}$ the set $U_1 \cap \cdots \cap U_m$ is contractible.

Theorem 5.3.3. *Let M be a smooth n-manifold. For every open cover $\mathcal{U}$ of M there exists a countable open cover $\mathcal{V}$ of M which is an admissible locally finite refinement of $\mathcal{U}$ consisting of relatively compact sets.*

Proof. From Lemma 1.4.3 there exists an open cover $\mathcal{B}$ which is a locally finite refinement of $\mathcal{U}$ and consists of relatively compact sets. We can choose any Riemannian metric on M, by Proposition 3.3.2. Each point $p \in M$ has a strongly convex uniformly normal open ball W_p contained in some element of $\mathcal{B}$, by Corollary 3.6.4. Then $\mathcal{W} = \{W_p : p \in M\}$ is an open cover of M and for each $B \in \mathcal{B}$ there exists a finite set $\mathcal{W}_B \subset \mathcal{W}$ which covers $\overline{B}$. Now

$$\mathcal{V} = \bigcup_{B \in \mathcal{B}} \mathcal{W}_{\mathcal{B}}$$

is an open cover of M which is a locally finite refinement of $\mathcal{U}$ consisting of relatively compact sets. For every $m \in \mathbb{N}$ and V_1,..., $V_m \in \mathcal{V}$ the open set $C = V_1 \cap \cdots \cap V_m$ is strongly convex and is contained in V_1 which is a uniformly normal strongly convex open ball. It follows that C is contractible, because fixing any point $p \in C$, and choosing a smooth function $h : \mathbb{R} \to [0, 1]$ such such that $h^{-1}(0) = [1, +\infty)$ and $h^{-1}(1) = (-\infty, 0]$, the smooth map $H : \mathbb{R} \times C \to C$ with $H(t, q) = \exp_p(h(t) \exp_p^{-1}(q))$ is a smooth homotopy from $H(0, .) = id_C$ to the constant $H(1, .) = p$. □

Thus the set of admissible covers of a smooth manifold constitutes a cofinal subset of the directed set of its open covers.

A smooth manifold M is said to be *of finite type* if it has a finite admissible cover. Obviously, every compact manifold is of finite type. More generally, if C is a compact subset of a smooth manifold, then every open neighbourhood of C in M contains an open neighbourhood of C which as a smooth manifold is of finite type. The terminology *of finite type* is justified by the following fact whose proof is an illustration of the inductive use of the Mayer-Vietoris long exact sequence in computing cohomologies.

Proposition 5.3.4. *If M is a smooth manifold of finite type, then $H^*(M)$ has finite dimension.*

The proof relies on the following elementary observation. Let V_1, V_2, V_3 be three real vector spaces and let

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$$

be a short exact sequence of linear maps. If V_1 and V_3 have finite dimension, then also V_2 has finite dimension. Indeed, there exist v_1,..., $v_k \in V_2$, for some $k \in \mathbb{N}$ such that $\{g(v_1), ..., g(v_k)\}$ is a basis of $g(V_2)$ and also v_{k+1},..., $v_m \in V_2$, for some $m \in \mathbb{N}$ such that $\{v_{k+1}, ..., v_m\}$ is a basis of $f(V_1) = \mathrm{Ker} g$. For every $v \in V_2$ there exist a_1,..., $a_k \in \mathbb{R}$ such that

$$g(v) = \sum_{i=1}^{k} a_i g(v_i) = g\Big(\sum_{i=1}^{k} a_i v_i\Big)$$

and so there exist a_{k+1},..., $a_m \in \mathbb{R}$ such that

$$v - \sum_{i=1}^{k} a_i v_i = \sum_{i=k+1}^{m} a_i v_i.$$

Thus, V_2 is finitely generated.

Proof of Proposition 5.3.4. We proceed by induction on the number m of the elements of the admissible finite cover. If $m = 1$, the conclusion is trivial, by Corollary 5.1.7. Suppose that the conclusion holds for smooth manifolds which have an admissible cover with $m - 1$ elements. Let M be a smooth manifold which has an admissible cover $\{U_1, U_2, ..., U_m\}$. Putting $V = U_2 \cup \cdots \cup U_m$, by the inductive hypothesis $H^*(V)$ has finite dimension. Since $M = U_1 \cup V$ from the corresponding Mayer-Vietoris long exact sequence we obtain short exact sequences

$$H^{k-1}(U_1 \cap V) \xrightarrow{d^*} H^k(M) \longrightarrow H^k(U_1) \oplus H^k(V).$$

Since $\{U_1 \cap U_2, ..., U_1 \cap U_m\}$ is an admissible cover of $U_1 \cap V$, by the inductive hypothesis $H^k(U_1 \cap V)$ has finite dimension. From the above elementary observation, $H^k(M)$ has finite dimension. □

Corollary 5.3.5. *The de Rham cohomology of a compact smooth manifold has finite dimension.* □

5.4 Poincaré Duality

Let M be a smooth n-manifold. Since $d(A_c^k(M)) \subset A_c^{k+1}(M)$ for every $k \in \mathbb{Z}^+$, the pair $(A_c^*(M), d)$ is a cochain complex. The quotient vector space

$$H^k(M) = \frac{Z^k(M) \cap A_c^k(M)}{B^k(M) \cap A_c^k(M)}$$

is called the *de Rham cohomology of M with compact supports* at degree k. Since the wedge product of two differential forms with compact supports also has compact support, the graded vector space $H_c^*(M) = \bigoplus_{k=0}^{n} H_c^k(M)$ endowed with the cup product becomes an associative commutative graded algebra which is a diffeomorphism invariant. In general however if $f : M \to N$ is a smooth map, $f^*(A_c^*(N))$ may not be a subset of $A_c^*(M)$.

According to Theorem 4.5.6, if M is a connected oriented smooth n-manifold, then integration over M induces a well defined linear isomorphism

$$\int_M : H_c^n(M) \xrightarrow{\cong} \mathbb{R}.$$

Also, the proof of Theorem 4.3.7 shows that if M is a connected, non-compact smooth manifold, then $H_c^0(M) = \{0\}$. The version of the Poincaré Lemma for the de Rham cohomology with compact supports can be stated as follows.

Proposition 5.4.1. *The de Rham cohomology with compact supports of* $\mathbb{R}^n$ *is*

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & \text{for } k = n, \\ \{0\}, & \text{for } k \neq n. \end{cases}$$

Proof. From the above, this is obviously true for $n = 0, 1$ and for $n > 1$, it suffices to prove that $H_c^k(\mathbb{R}^n) = \{0\}$ for all $0 < k < n$. Since $\mathbb{R}^n$ is diffeomorphic to $S^n \setminus \{e_{n+1}\}$, it suffices to prove that $H_c^k(S^n \setminus \{e_{n+1}\}) = \{0\}$ for $0 < k < n$. The elements of $A_c^k(S^n \setminus \{e_{n+1}\})$ are differential k-forms on S^n which vanish on an open neighbourhood of the north pole e_{n+1}. Let $\omega \in A_c^k(S^n \setminus \{e_{n+1}\})$ with $d\omega = 0$. Since $H^k(S^n) = \{0\}$, by Example 5.3.1, there exists $\theta \in A^{k-1}(S^n)$ such that $\omega = d\theta$. It remains to show that there exists such a θ that vanishes on an open neighbourhood of e_{n+1}.

There exists an open neighbourhood $V \subset S^n$ of e_{n+1} which is diffeomorphic to $\mathbb{R}^n$ such that $\omega|_V = 0$. If $k = 1$, then $\theta \in C^\infty(S^n) = A^0(S^n)$ is a smooth function such that $d\theta|_V = 0$ and therefore θ is constant on V. We denote this constant value by $\theta|_V$. Now $\tilde{\theta} = \theta - (\theta|_V) \in C^\infty(S^n)$ vanishes on V and $d\tilde{\theta} = \omega$. This proves the assertion for $k = 1$. Let $2 \leq k < n$. From Corollary 5.1.3, there exists $\eta \in A^{k-2}(V)$ such that $d\eta = \theta|_V$, because $d(\theta|_V) = d\theta|_V = \omega|_V = 0$. Let U be an open neighbourhood of e_{n+1} with $\overline{U} \subset V$. There exists a smooth function $f : S^n \to [0, 1]$ such that $\overline{U} \subset f^{-1}(1)$ and supp$f \subset V$, by Corollary 1.4.5. The differential $(k-2)$-form $f\eta \in A^{k-2}(V)$ can be extended to the differential $(k-2)$-form $\tilde{\eta} \in A^{k-2}(S^n)$ defined by

$$\tilde{\eta} = \begin{cases} f\eta, & \text{on } V, \\ 0, & \text{on } S^n \setminus V. \end{cases}$$

If $\tilde{\theta} = \theta - d\tilde{\eta}$, then $d\tilde{\theta} = d\theta = \omega$ and $\tilde{\theta}|_U = \theta|_U - d\eta|_U = 0$. This completes the proof. $\square$

There is a Mayer-Vietoris exact sequence for de Rham cohomology with compact supports. We observe first that if $W \subset U \subset M$ are open sets of a smooth n-manifold

M, the inclusion $i : W \hookrightarrow U$ induces a cochain map $i_* : A_c^*(W) \to A_c^*(U)$ defined by

$$(i_*\omega)_p = \begin{cases} \omega_p, & \text{for } p \in W, \\ 0, & \text{for } p \in U \setminus \text{supp}\omega. \end{cases}$$

Let now U, $V \subset M$ be two open sets such that $M = U \cup V$. Let $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ denote the inclusions and $i_U : U \hookrightarrow M$ and $i_V : V \hookrightarrow M$ be the inclusions in M.

$$U \cap V \ \xrightarrow[j]{i} U \coprod V \xrightarrow{\text{inclusion}} M$$

Passing to the level of differential forms with compact supports we get the following sequence of cochain maps

$$0 \longrightarrow A_c^*(U \cap V) \xrightarrow{\tau} A_c^*(U) \oplus A_c^*(V) \xrightarrow{\sigma} A_c^*(M) \longrightarrow 0$$

where $\tau(\omega) = (-i_*\omega, j_*\omega)$ and $\sigma(\omega_1, \omega_2) = (i_U)_*\omega_1 + (i_V)_*\omega_2$, which is exact. Its exactness at $A_c^*(U \cap V)$ and at $A_c^*(U) \oplus A_c^*(V)$ is obvious from the definitions of τ and σ. To see that σ is onto $A_c^*(M)$, let $\{f_U, f_V\}$ be a smooth partition of unity subordinated to the open cover $\{U, V\}$ of M. If $\omega \in A_c^*(M)$, then $\omega = \sigma(f_u\omega|_U, f_V\omega|_V)$.

Thus we get a Mayer-Vietoris long exact sequence for the de Rham cohomology with compact supports.

$$\cdots \xrightarrow{d_*} H_c^k(U \cap V) \xrightarrow{\tau} H_c^k(U) \oplus H_c^k(V) \xrightarrow{\sigma} H_c^k(M) \xrightarrow{d_*} H_c^{k+1}(U \cap V) \longrightarrow \cdots$$

The connecting homomorphism d_* can be described as follows. If $\omega \in A_c^k(M)$, there are $\omega_1 = f_U\omega$, $\omega_2 = f_V\omega \in A_c^k(M)$, so that $\text{supp}\omega_1 \subset U$, $\text{supp}\omega_2 \subset V$ and $\omega = (i_U)_*(\omega_1|_U) + (i_V)_*(\omega_2|_V)$. If moreover $d\omega = 0$, then $-d\omega_1|_{U\cap V} = d\omega_2|_{U\cap V} = \eta \in A_c^{k+1}(U \cap V)$ and $d\eta = 0$. We have now $d_*[\omega]_c = [\eta]_c$.

If $\omega \in A^k(M)$ and $\theta \in A_c^l(M)$, then $\omega \wedge \theta \in A_c^{k+l}(M)$. If ω and θ are closed and $\eta \in A^{k-1}(M)$, $\zeta \in A_c^{l-1}(M)$, we have

$$(\omega + d\eta) \wedge (\theta + d\zeta) - \omega \wedge \theta = \pm d(\omega \wedge \zeta) \pm d(\eta \wedge \theta) \pm d(\eta \wedge d\zeta)$$

and the differential forms $\omega \wedge \zeta$, $\eta \wedge \theta$, $\eta \wedge d\zeta$ have compact supports. This means that the wedge product induces a well defined cup product

$$\smile : H^k(M) \times H_c^l(M) \to H_c^{k+l}(M)$$

which inherits its properties.

Let now M be an oriented smooth n-manifold. From the above, we get a well defined bilinear map

$$H^k(M) \times H_c^{n-k}(M) \xrightarrow{\smile} H_c^n(M) \xrightarrow{\int_M} \mathbb{R}$$

and a linear map $D_M : H^k(M) \to H_c^{n-k}(M)^*$ with

$$D_M([\omega])([\theta]_c) = \int_M \omega \wedge \theta$$

which we call the Poincaré Duality map.

Theorem 5.4.2. *If M is an oriented smooth n-manifold, then the Poincaré Duality map $D_M : H^k(M) \to H_c^{n-k}(M)^*$ is a linear isomorphism for every $k = 0, 1, ..., n$.*

The proof will be given in several steps, starting locally and going to global using a Mayer-Vietoris argument.

Lemma 5.4.3. *The Poincaré Duality map $D_{\mathbb{R}^n} : H^k(\mathbb{R}^n) \to H_c^{n-k}(\mathbb{R}^n)^*$ is a linear isomorphism for all $0 \le k \le n$.*

Proof. By Corollary 5.1.3 and Proposition 5.4.1, we need only check that

$$D_{\mathbb{R}^n} : H^0(\mathbb{R}^n) \to H_c^n(\mathbb{R}^n)^*$$

is a linear isomorphism. Indeed, as the proof of Theorem 4.3.7 shows, $H^0(\mathbb{R}^n) \cong \mathbb{R}$ is generated by the constant function with value 1. This is sent from $D_{\mathbb{R}^n}$ to the integration

$$\int_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) \to \mathbb{R}$$

over M, which is a linear isomorphism, according to Theorem 4.5.6. □

Lemma 5.4.4. *If U, $V \subset M$ are two open subsets of an oriented smooth n-manifold M such that $M = U \cup V$, then the following diagram, with first row the Mayer-Vietoris long exact sequence in de Rham cohomology and second the dual Mayer-Vietoris long exact sequence in de Rham cohomology with compact supports, commutes.*

$$\begin{tikzcd}
\cdots \arrow[r, "d^*"] & H^k(M) \arrow[r, "(i_U^*, i_V^*)"] \arrow[d, "D_M"] & H^k(U) \oplus H^k(V) \arrow[r, "\rho"] \arrow[d, "D_U \oplus D_V"] & H^k(U \cap V) \arrow[r, "d^*"] \arrow[d, "D_{U \cap V}"] & H^{k+1}(M) \arrow[r] \arrow[d, "D_M"] & \cdots \\
\cdots \arrow[r, "\pm d_*^t"] & H_c^{n-k}(M)^* \arrow[r, "\sigma^t"] & H_c^{n-k}(U)^* \oplus H_c^{n-k}(V)^* \arrow[r, "\tau^t"] & H_c^{n-k}(U \cap V)^* \arrow[r, "\pm d_*^t"] & H_c^{n-k-1}(M)^* \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The left square commutes because if $\omega \in A^k(M)$, $\phi \in A_c^{n-k}(U)$, $\theta \in A_c^{n-k}(V)$ are closed, then

$$D_U([i_U^*\omega])([\phi]_c) + D_V([i_V^*\omega])([\theta]_c) = \int_U i_U^*\omega \wedge \phi + \int_V i_V^*\omega \wedge \theta$$

$$= \int_M \omega \wedge ((i_U)_*\phi + (i_V)_*\theta) = D_M([\omega])(\sigma([\phi]_c, [\theta]_c) = (\sigma^t \circ D_M)([\omega])([\phi]_c, [\theta]_c).$$

For the commutativity of the middle square let $\omega_1 \in A^k(U)$, $\omega_2 \in A^k(V)$ and $\eta \in A_c^{n-k}(U \cap V)$ be closed. Then

$$D_{U\cap V}([j^*\omega_2 - i^*\omega_1])([\eta]_c) = \int_{U\cap V} (\omega_2 - \omega_1) \wedge \eta = \int_V \omega_2 \wedge j_*\eta - \int_U \omega_1 \wedge i_*\eta$$

$$= D_U([\omega_1])(-[i_*\eta]_c) + D_V([\omega_2])([j_*\eta]_c) = \tau^t((D_U([\omega_1]), D_V(\omega_2]))([\eta]_c).$$

To prove the commutativity of the right square, we consider a smooth partition of unity $\{f_U, f_V\}$ subordinated to the open cover $\{U, V\}$ of M. If $\omega \in A^k(U \cap V)$ is closed, then $d^*[\omega]$ is represented by the closed differential $(k+1)$-form

$$d^*\omega = \begin{cases} -d(f_V\omega), & \text{on } U, \\ d(f_U\omega), & \text{on } V. \end{cases}$$

On the other hand, if $\phi \in A_c^{n-k-1}(M)$ is closed, then $d_*[\phi]_c$ is represented by $-d(f_U\phi)|_{U\cap V} = d(f_V\phi)|_{U\cap V}$. Now we compute

$$D_M(d^*([\omega]))([\phi]_c) = \int_M d^*\omega \wedge \phi = -\int_{U\cap V} d(f_V\omega) \wedge \phi = -\int_{U\cap V} df_V \wedge \omega \wedge \phi$$

$$= (-1)^{k+1} \int_{U\cap V} \omega \wedge d_*\phi = (-1)^{k+1} d_*^t(D_{U\cap V}([\omega]))(\phi]_c). \quad \square$$

An immediate consequence of the above Lemma 5.4.4 and the five lemma is the following.

Corollary 5.4.5. *Let U, $V \subset M$ be two open subsets of an oriented smooth n-manifold M. If D_U, D_V and $D_{U\cap V}$ are linear isomorphisms, so is $D_{U\cup V}$.* $\square$

Recall that the algebraic dual of the direct sum of a family $\mathcal{V}$ of vector spaces is isomorphic to the direct product of their algebraic duals. Indeed, the map

$$G : \prod_{V\in\mathcal{V}} V^* \to \Big(\bigoplus_{V\in\mathcal{V}} V\Big)^*$$

defined by

$$G((a_V)_{V\in\mathcal{V}})((x_V)_{V\in\mathcal{V}}) = \sum_{V\in\mathcal{V}} a_V(x_V)$$

for $(x_V)_{V\in\mathcal{V}} \in \bigoplus_{V\in\mathcal{V}} V$ is a linear isomorphism.

Lemma 5.4.6. *If $\mathcal{U}$ is an open cover of a smooth manifold M consisting of mutually disjoint open sets, then $H^*(M) \cong \prod_{U\in\mathcal{U}} H^*(U)$ and $\bigoplus_{U\in\mathcal{U}} H_c^*(U) \cong H_c^*(M)$.*

Proof. It suffices to observe that if $i_U : U \hookrightarrow M$ is the inclusion, then the maps $L : A^*(M) \to \prod_{U\in\mathcal{U}} A^*(U)$ defined by $L(\omega) = (i_U^*\omega)_{U\in\mathcal{U}}$ and $T : \bigoplus_{U\in\mathcal{U}} A_c^*U) \to A_c^*(M)$ defined by $T((\omega_U)_{U\in\mathcal{U}}) = \sum_{U\in\mathcal{U}} (i_U)_*\omega_U$ are cochain isomorphisms of obvious cochain complexes. □

Corollary 5.4.7. *If $\mathcal{U}$ is an open cover of a smooth n-manifold M consisting of mutually disjoint open sets and D_U is a linear isomorphism for every $U \in \mathcal{U}$, then so is D_M.*

Proof. The assertion follows immediately from the commutative diagram

$$\begin{tikzcd} H^k(M) \arrow[r, "\cong"] \arrow[d, "D_M"'] & \prod\limits_{U\in\mathcal{U}} H^k(U) \arrow[d, "\prod\limits_{U\in\mathcal{U}} D_U"] \\ H_c^{n-k}(M)^* \arrow[r, "\cong"] & \prod\limits_{U\in\mathcal{U}} H_c^{n-k}(U)^* \end{tikzcd}$$

in which the horizontal maps are the isomorphisms of Lemma 5.4.6. □

The proof of Theorem 5.4.2 will be a combination of the above lemmas and corollaries and the following general proposition.

Proposition 5.4.8. *Let M be a smooth m-manifold and let $\mathcal{U}$ be a set of open subsets of M with the following properties:*
(i) $\varnothing \in \mathcal{U}$.
(ii) If U is an open subset of M diffeomorphic to $\mathbb{R}^m$, then $U \in \mathcal{U}$.
(iii) If U_1, $U_2 \in \mathcal{U}$ are such that $U_1 \cap U_2 \in \mathcal{U}$, then $U_1 \cup U_2 \in \mathcal{U}$.
(iv) If $\{U_n : n \in \mathbb{N}\}$ is a countable family of mutually disjoint elements of $\mathcal{U}$, then $\bigcup\limits_{n=1}^{\infty} U_n \in \mathcal{U}$.

Then, $M \in \mathcal{U}$.

The proof of Proposition 5.4.8 relies on the following lemma.

Lemma 5.4.9. *With the assumptions of Proposition 5.4.8, let $\{U_n : n \in \mathbb{N}\}$ be a locally finite countable family of open and relatively compact subsets of M such that $\bigcap\limits_{j\in J} U_j \in \mathcal{U}$ for every finite set $J \subset \mathbb{N}$. Then, $\bigcup\limits_{n=1}^{\infty} U_n \in \mathcal{U}$.*

Proof. First we show that finite unions of elements of the countable family belong to $\mathcal{U}$. Let $n \in \mathbb{N}$ and i_1,..., $i_n \in \mathbb{N}$. We shall show inductively that $U_{i_1} \cup \cdots \cup U_{i_n} \in \mathcal{U}$. For $n = 1, 2$ this is true by property (iii) and our assumption (in case J is a singleton). Let $n \geq 3$ and suppose that the assertion holds for finite subfamilies with $n - 1$ elements. If $V = U_{1_2} \cup \cdots \cup U_{i_n}$, then

$$U_{i_1} \cap V = \bigcup_{k=2}^{n} U_{i_1} \cap U_{i_k} \in \mathcal{U}$$

from the inductive hypothesis. Moreover, from our assumption (iii) we have

$$U_{i_1} \cup \cdots \cup U_{i_n} = U_{i_1} \cup V \in \mathcal{U}.$$

Since finite unions of elements of the countable family belong to $\mathcal{U}$, for every $n \in \mathbb{N}$ and indices i_1, j_1,..., i_n, $j_n \in \mathbb{N}$ we have

$$\bigcup_{k=1}^{n} U_{i_k} \cap U_{j_k} \in \mathcal{U}.$$

Now we define inductively $I_1 = \{1\}$, $W_1 = U_1$ and

$$I_n = \{n\} \cup \{i \in \mathbb{N} : i > n \quad \text{and} \quad U_i \cap W_{n-1} \neq \varnothing\} \setminus \bigcup_{k=1}^{n-1} I_k, \quad W_n = \bigcup_{i \in I_n} U_i,$$

for $n \geq 2$. If I_{n-1} is finite, then W_{n-1} is relatively compact and intersects at most finitely many of the elements of the countable family, since the latter is assumed to be locally finite. Thus, inductively I_n is finite and W_n is relatively compact and belongs to $\mathcal{U}$ for every $n \in \mathbb{N}$. Moreover, $W_n \cap W_{n+1} \in \mathcal{U}$ and $W_n \cap W_k = \varnothing$, if $k > n+1$, because otherwise there exists some $i \in I_k$ such that $W_n \cap U_i \neq \varnothing$ and thus $i \in I_j$ for some $j \leq n+1$, contradiction. From property (iv) of $\mathcal{U}$ we have

$$\left(\bigcup_{k=1}^{\infty} W_{2k}\right) \cap \left(\bigcup_{k=1}^{\infty} W_{2k-1}\right) = \bigcup_{n=1}^{\infty} W_n \cap W_{n+1} \in \mathcal{U}$$

and from property (iii) the proof is concluded. □

Proof of Proposition 5.4.8. In the beginning we consider the case where M is an open subset of $\mathbb{R}^m$. Then there exists a locally finite countable open cover of M which consists of open cubes (with edges parallel to the axis) and refines $\mathcal{U}$. Any finite intersection of open cubes is an open cube and thus again diffeomorphic to $\mathbb{R}^m$. From property (ii) and Lemma 5.4.9 follows that $M \in \mathcal{U}$.

In the general case, for every chart (U, ϕ) of M the family

$$\mathcal{U}^{\phi} = \{B \subset \phi(U) : \quad B \text{ is open and } \quad \phi^{-1}(B) \in \mathcal{U}\}$$

has the properties (i), (ii), (iii) and (iv). Hence $\phi(U) \in \mathcal{U}^{\phi}$ and therefore $U \in \mathcal{U}$. Now we take any locally finite countable open cover of M consisting of relatively compact open sets which are domains of charts. Lemma 5.4.9 gives immediately $M \in \mathcal{U}$. □

Proof of Theorem 5.4.2. It suffices to consider the family $\mathcal{U}$ of all open subsets of M such that D_U is an isomorphism for all $U \in \mathcal{U}$. Then, Lemma 5.3.3 and Corollaries 5.3,5 and 5.3.7 say that $\mathcal{U}$ satisfies the assumptions of Proposition 5.3.8 and therefore $D_M \in \mathcal{U}$. □

Corollary 5.4.10. *If M is a non-compact orientable smooth n-manifold, then $H^n(M) = \{0\}$.* □

We shall give some applications of the Roincaré Duality Isomorphism in the particular case of compact smooth manifolds.

Example 5.4.11. We shall compute the de Rham cohomology algebra of the complex projective n-space $\mathbb{C}P^n$ for $n \geq 1$. The Poincaré Duality Isomorphism gives a non-degenerate bilinear pairing

$$H^{2k}(\mathbb{C}P^n) \times H^{2n-2k}(\mathbb{C}P^n) \xrightarrow{\smile} H^{2n}(\mathbb{C}P^n) \xrightarrow{\int_{\mathbb{C}P^n}} \mathbb{R}$$

for every $0 \leq k \leq n$. Let X denote the generator of $H^2(\mathbb{C}P^n)$. For $k = 1$ this gives $X^2 = X \smile X \neq 0$ and inductively $X^k = X \smile \cdots \smile X \neq 0$, for all $0 \leq k \leq n$, while $X^{n+1} = 0$. This implies that the map $F : \mathbb{R}[X] \to H^*(\mathbb{C}P^n)$ defined by

$$F\left(\sum_{k=0}^{\infty} a_k X^k\right) = (a_0, ..., a_n) \in \bigoplus_{k=0}^{n} H^{2k}(\mathbb{C}P^n) = H^*(\mathbb{C}P^n)$$

is an epimorphiam of algebras and its kernel is the ideal in $\mathbb{R}[X]$ that is generated by the monomial X^{n+1}. Hence the de Rham cohomology algebra $H^*(\mathbb{C}P^n)$ is isomorphic to the truncated polynomial algebra $\mathbb{R}[X]/ < X^{n+1} >$.

Recall that if V is a real vector space and $A \subset V$ is a basis of V, then $V \cong \bigoplus_{a \in A} \mathbb{R}$. Since

$$\left(\bigoplus_{a \in A} \mathbb{R}\right)^* \cong \prod_{a \in A} \mathbb{R}$$

is follows that if V^* has finite dimension, then V necessarily has finite dimension. This simple algebraic observation combined with the Poincaré Duality Isomorphism and Proposition 5.3.4 gives immediately the following.

Corollary 5.4.12. *If M is an orientable smooth n-manifold of finite type, then $H_c^*(M)$ has finite dimension and $H^k(M)^* \cong H_c^{n-k}(M)$ for every $0 \leq k \leq n$.* $\square$

If M is a compact orientable smooth n-manifold, the integer

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M)$$

is the *Euler characteristic of M*. Suppose that M is also connected and $n = 2m$ is even. The Poincaré Duality Isomorphism gives a non-degenerate bilinear form

$$\langle ., . \rangle : H^m(M) \times H^m(M) \to \mathbb{R}$$

which is skew-symmetric if m is odd, and symmetric if m is even. In the latter case its signature is usually called the signature of M.

Proposition 5.4.13. *Let M be a connected, compact, oriented smooth n-manifold.*
(a) If n is odd, then $\chi(M) = 0$.
(b) If $n = 2m$ and m is odd, then $\chi(M) = 0 \mod 2$.
(c) If $n = 2m$ and m is even, then $\dim H^m(M) = \chi(M) \mod 2$.

Proof. Using the Poincaré Duality Isomorphism and Corollary 4.3.12 we compute

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M) = \sum_{k=0}^{n} (-1)^k \dim H^{n-k}(M) = (-1)^n \chi(M)$$

and so $\chi(M) = 0$, if n is odd.

If $n = 2m$, we have

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M) = 2\sum_{k=0}^{m-1} (-1)^k \dim H^k(M) + (-1)^m \dim H^m(M).$$

In case m is odd, $\dim H^m(M)$ is even, since the real vector space $H^m(M)$ carries the non-degenerate skew-symmetric bilinear form $\langle .,. \rangle$. The rest is obvious. □

5.5 The Künneth formula

In this section we shall compute the de Rham cohomology with compact supports of the cartesian product of two smooth manifolds. Let M, N be two smooth manifolds and let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ denote the natural projections. There is a well defined cochain map $\gamma : A^*(M) \otimes A^*(N) \to A^*(M \times N)$ by

$$\gamma(\omega \otimes \theta) = \pi_M^* \omega \wedge \pi_N^* \theta$$

which induces a linear map $\gamma : H^*(A^*(M) \otimes A^*(N)) \to H^*(M \times N)$. Composing with the algebraic isomorphism $\mu : H^*(M) \otimes H^*(N) \to H^*(A^*(M) \otimes A^*(N))$ with $\mu([\omega] \otimes [\theta]) = [\omega \otimes \theta]$, we get a linear map $\psi : H^*(M) \otimes H^*(N) \to H^*(M \times N)$ defined by

$$\psi(\alpha \otimes \beta) = \pi_M^* \alpha \smile \pi_N^* \beta$$

which is natural.

We observe that γ has a restriction $\gamma_c : A_c^*(M) \otimes A_n^*(N) \to A_c^*(M \times N)$, from which as above we take a well defined linear map $\psi_c : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N)$ with

$$\psi_c([\omega]_c \otimes [\theta]_c) = [\pi_M^* \omega \wedge \pi_N^* \theta]_c$$

since the support of $\pi_M^* \omega \wedge \pi_N^* \theta$ is contained in $\mathrm{supp}\omega \times \mathrm{supp}\theta$.

Theorem 5.5.1. *If M and N are two smooth manifolds, then*

$$\psi_c : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N)$$

with $\psi_c([\omega]_c \otimes [\theta]_c) = [\pi_M^* \omega \wedge \pi_N^* \theta]_c$ is an isomorphism.

Corollary 5.5.2. *If M and N are two compact smooth manifolds, then*

$$\psi : H^*(M) \otimes H^*(N) \to H^*(M \times N)$$

with $\psi(\alpha \otimes \beta) = \pi_M^* \alpha \smile \pi_N^* \beta$ is a natural isomorphism. $\square$

The procedure of the proof is similar to that of Theorem 5.4.2. We begin with the case $M = \mathbb{R}^n$, $n \geq 1$. Of course it suffices to prove that

$$\psi_c : H_c^*(\mathbb{R}) \otimes H_c^*(N) \to H_c^*(\mathbb{R} \times N)$$

is an isomorphism. From Proposition 5.4.1 follows however that

$$(H_c^*(\mathbb{R} \otimes H_c^*(N))^k \cong H_c^1(\mathbb{R}) \otimes H_c^{k-1}(N) \cong H_c^{k-1}(N)$$

for every $k \in \mathbb{Z}$, because $H_c^1(\mathbb{R}) \cong \mathbb{R}$, the isomorphism being integration over $\mathbb{R}$. Taking into account this isomorphism, we have to show that

$$\psi_c : H_c^{k-1}(N) \to H_c^k(\mathbb{R} \times N)$$

defined by

$$\psi_c([\theta]_c) = [e(t)dt \wedge \pi_N^* \theta]_c$$

is an isomorphism for every $k \in \mathbb{Z}$, where $e \in C_c^\infty(\mathbb{R})$ is such that $\displaystyle\int_{\mathbb{R}} e(t)dt = 1$. This is a version of the Poincaré Lemma for the de Rham cohomology with compact supports. Of course ψ_c can be defined at the level of the cochain complexes $A_c^*(N)$ and $A_c^*(\mathbb{R} \times N)$ where it is a cochain map of degree 1.

Theorem 5.5.3. *The map $\psi_c : H_c^{k-1}(N) \to H_c^k(\mathbb{R} \times N)$ is an isomorphism for every $k \in \mathbb{Z}$.*

Proof. As we did in the proof of Corollary 5.1.2, we shall construct a cochain map $\pi : A_c^*(\mathbb{R} \times N) \to A_c^*(N)$ of degree -1 and a cochain homotopy K such that $\pi \circ \psi_c = \pm id$ and $id - \psi_c \circ \pi = \pm(d \circ K - K \circ d)$. We define the linear map $\pi : A_c^k(\mathbb{R} \times N) \to A_c^{k-1}(N)$ by

$$\pi(\omega) = \left(\int_{\mathbb{R}} g(t,x)dt \right) \cdot \eta$$

if $\omega = f(t,x)\pi_N^* \theta + g(t,x)\pi_N^* \eta \wedge dt$, where f, $g \in C_c^\infty(\mathbb{R} \times N)$ $\theta \in A_c^k(N)$ and $\eta \in A_c^{k-1}(N)$. Now on the one hand we have

$$d(\pi(\omega)) = \left(\int_{\mathbb{R}} \frac{\partial g}{\partial x} dt \right) dx \wedge \eta + \left(\int_{\mathbb{R}} g(t,x)dt \right) d\eta$$

and on the other

$$\begin{aligned} \pi(d\omega) &= \pi(df \wedge \pi_N^* \theta + f \pi_N^*(d\theta) + dg \wedge \pi_N^* \eta \wedge dt + g \pi_N^*(d\eta) \wedge dt) \\ &= \pm \left(\int_{\mathbb{R}} \frac{\partial f}{\partial t} dt \right) \theta + \left(\int_{\mathbb{R}} \frac{\partial g}{\partial x} dt \right) dx \wedge \eta + \left(\int_{\mathbb{R}} g(t,x)dt \right) d\eta \end{aligned}$$

$$= \left(\int_{\mathbb{R}} \frac{\partial g}{\partial x} dt\right) dx \wedge \eta + \left(\int_{\mathbb{R}} g(t,x)dt\right) d\eta$$

from the Fundamental Theorem of Calculus, since f has compact support. Hence π is a cochain map. It is also obvious from the definitions that

$$\pi(\psi_c(\eta)) = \pi(e(t)dt \wedge \pi_N^* \eta) = (-1)^{k-1}\eta.$$

Now we define the linear map $K : A_c^k(\mathbb{R} \times N) \to A_c^{k-1}(\mathbb{R} \times N)$ by

$$K(\omega) = \left(\int_{-\infty}^{t} g(s,x)ds\right)\pi_N^*\eta - \left(h(t)\int_{\mathbb{R}} g(t,x)dt\right)\pi_N^*\eta$$

where $h(t) = \displaystyle\int_{-\infty}^{t} e(s)ds$. Again from the Fundamental Theorem of Calculus we have

$$(d \circ K - K \circ d)(f\pi_N^*\theta) = (-1)^{k-1}\left[\left(\int_{-\infty}^{t} \frac{\partial f}{\partial t} dt\right)\pi_N^*\theta - \left(h(t)\int_{\mathbb{R}} \frac{\partial f}{\partial t} dt\right)\pi_N^*\theta\right]$$

$$= (-1)^{k-1} f\pi_N^*\theta = (id - (-1)^{k-1}\psi_c \circ \pi)(f\pi_N^*\theta).$$

Also,

$$(id - (-1)^{k-1}\psi_c \circ \pi)(g\pi_N^*\eta \wedge dt) = g\pi_N^*\eta \wedge dt - \left(\int_{\mathbb{R}} g(t,x)dt\right)e(t)\pi_N^*\eta \wedge dt$$

and

$$(d \circ K)(g\pi_N^*\eta \wedge dt) = d\left[\left(\int_{-\infty}^{t} g(s,x)ds - h(t)\int_{\mathbb{R}} g(t,x)dt\right)\pi_N^*\eta\right]$$

$$= \left(\int_{-\infty}^{t} g(s,x)ds - h(t)\int_{\mathbb{R}} g(t,x)dt\right)\pi_N^*(d\eta) + (-1)^{k-1}\pi_N^*\eta \wedge \left(\int_{-\infty}^{t} \frac{\partial g}{\partial x} ds\right)dx$$

$$+(-1)^{k-1} g\pi_N^*\eta \wedge dt - (-1)^{k-1}\pi_N^*\eta \wedge \left[\left(\int_{\mathbb{R}} g(t,x)dt\right)e(t)dt + h(t)\left(\int_{\mathbb{R}} \frac{\partial g}{\partial x} dt\right)dx\right]$$

while

$$(K \circ d)(g\pi_N^*\eta \wedge dt) = K\big(g\pi_N^*(d\eta) \wedge dt + (-1)^{k-1}\frac{\partial g}{\partial x}\pi_N^*\eta \wedge dx \wedge dt\big)$$

$$= \left(\int_{-\infty}^{t} g(s,x)ds - h(t)\int_{\mathbb{R}} g(t,x)dt\right)\pi_N^*(d\eta)$$

$$+(-1)^{k-1}\left[\left(\int_{-\infty}^{t} \frac{\partial g}{\partial x} ds - h(t)\int_{\mathbb{R}} \frac{\partial g}{\partial x} dt\right)\pi_N^*\eta \wedge dx.\right.$$

Hence

$$(d \circ K - K \circ d)(g\pi_N^*\eta \wedge dt) = (-1)^{k-1} g\pi_N^*\eta \wedge dt - (-1)^{k-1}\left(\int_{\mathbb{R}} g(t,x)dt\right)e(t)\pi_N^*\eta \wedge dt$$

$$= (-1)^{k-1}(id - (-1)^{k-1}\psi_c \circ \pi)(g\pi_N^*\eta \wedge dt).$$

This shows that $(id - (-1)^{k-1}\psi_c \circ \pi) = (-1)^{k-1}(d \circ K - K \circ d)$. It follows immediately the $\psi_c : H_c^{k-1}(N) \to H_c^k(\mathbb{R} \times N)$ is an isomorphism. Moreover, its inverse is $(-1)^{k-1}\pi : H_c^k(\mathbb{R} \times N) \to H_c^{k-1}(N)$ for every $k \in \mathbb{Z}$. $\square$

Lemma 5.5.4. *Let U, $V \subset M$ be two open subsets of the smooth manifold M such that $M = U \cup V$ and N be a smooth manifold. If*

$$\psi_c : H_c^*(U) \otimes H_c^*(N) \to H_c^*(U \times N),$$

$$\psi_c : H_c^*(V) \otimes H_c^*(N) \to H_c^*(V \times N),$$

$$\psi_c : H_c^*(U \cap V) \otimes H_c^*(N) \to H_c^*((U \cap V) \times N)$$

are isomorphisms, then so is $\psi_c : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N)$.

Proof. From the Mayer-Vietoris exact sequences

$$0 \longrightarrow A_c^*(U \cap V) \xrightarrow{\tau} A_c^*(U) \oplus A_c^*(V) \xrightarrow{\sigma} A_c^*(M) \longrightarrow 0$$

and

$$0 \to A_c^*((U \cap V) \times N) \xrightarrow{\tau} A_c^*(U \times N) \oplus A_c^*(V \times N) \xrightarrow{\sigma} A_c^*(M \times N) \to 0$$

we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A_c^*(U \cap V) \otimes A_c^*(N) \arrow[r] \arrow[d, "\gamma"] & A_c^*(U) \otimes A_c^*(N) \oplus A_c^*(V) \otimes A_c^*(N) \arrow[r] \arrow[d, "\gamma \oplus \gamma"] & A_c^*(M) \otimes A_c^*(N) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A_c^*((U \cap V) \times N) \arrow[r] & A_c^*(U \times N) \oplus A_c^*(V \times N) \arrow[r] & A_c^*(M \times N) \arrow[r] & 0.
\end{tikzcd}$$

This gives an analogous commutative diagram for the corresponding long exact sequences in cohomology. The assertion follows then from the five lemma. $\square$

Lemma 5.5.5. *Let $\mathcal{U}$ be a countable open cover of the smooth manifold M by mutually disjoint sets and N be a smooth manifold. If $\psi_c : H_c^*(U) \otimes H_c^*(N) \to H_c^*(U \times N)$ is an isomorphism for every $U \in \mathcal{U}$, then so is $\psi_c : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N)$.*

Proof. The assertion follows from the obvious isomorphism

$$\bigoplus_{U \in \mathcal{U}} H_c^*(U) \otimes H_c^*(N) \cong H_c^*(M) \otimes H_c^*(N)$$

and the commutative diagram

$$\begin{tikzcd}
\bigoplus_{U \in \mathcal{U}} H_c^*(U) \otimes H_c^*(N) \arrow[r, "\bigoplus_{U \in \mathcal{U}} \psi_C"] \arrow[d, "\cong"'] & \bigoplus_{U \in \mathcal{U}} H_c^*(U \times N) \arrow[d, "\cong"] \\
H_c^*(M) \otimes H_c^*(N) \arrow[r, "\psi_c"] & H_c^*(M \times N). \quad \square
\end{tikzcd}$$

Proof of Theorem 5.5.1. Let $\mathcal{U}$ be the family of all open subsets U of M such that $\psi_c : H_c^*(U) \otimes H_c^*(N) \to H_c^*(U \times N)$ is an isomorphism. Then $\mathcal{U}$ fulfils the assumptions of Proposition 4.4.8, by Theorem 5.5.3, Lemma 5.5.4 and Lemma 5.5.5. Therefore, $M \in \mathcal{U}$. This completes the proof. □

Example 5.5.6. As an illustration we shall compute the de Rham vohomology algebra of the connected compact orientable 6-manifold $S^2 \times S^4$. Using Example 5.3.1 and Corollary 5.5.2, we have

$$H^1(S^2 \times S^4) \cong H^0(S^2) \otimes H^1(S^4) \oplus H^1(S^2) \otimes H^0(S^4) = \{0\},$$

$$H^2(S^2 \times S^4) \cong H^0(S^2) \otimes H^2(S^4) \oplus H^1(S^2) \otimes H^1(S^4) \oplus H^2(S^2) \otimes H^0(S^4) \cong \mathbb{R},$$

and similarly $H^3(S^2 \times S^4) = \{0\}$, $H^4(S^2 \times S^4) \cong \mathbb{R}$. Of course $H^0(S^2 \times S^4) \cong \mathbb{R}$ and $H^6(/S^2 \times S^4) \cong \mathbb{R}$. The generator of $H^2(S^2 \times S^4)$ is $\pi_{S^2}^* o_{S^2} = \psi(o_{S^2} \otimes 1)$. Thus,

$$(\pi_{S^2}^* o_{S^2})^2 = \pi_{S^2}^* o_{S^2} \smile \pi_{S^2}^* o_{S^2} = \pi_{S^2}^*(o_{S^2} \smile o_{S^2}) = 0$$

in $H^4(S^2 \times S^4)$. In other words the cup product

$$\smile : H^2(S^2 \times S^4) \times H^2(S^2 \times S^4) \to H^4(S^2 \times S^4)$$

is trivial

We observe now that although $H^k(S^2 \times S^4) \cong H^k(\mathbb{C}P^3)$ for all k, the de Rham cohomology algebras $H^*(S^2 \times S^4)$ and $H^*(\mathbb{C}P^3)$ are not isomorphic, since the cup product

$$\smile : H^2(\mathbb{C}P^3) \times H^2(\mathbb{C}P^3) \to H^4(\mathbb{C}P^3)$$

is non-trivial. This illustrates the fact that the de Rham cohomology algebra is a much finer invariant than the de Rham cohomology vector space.

5.6 Intersection theory

Let M be a compact connected oriented smooth n-manifold. A *k-cycle* in M is a pair (S, σ), where S is a compact oriented (possibly not connected) smooth k-manifold and $\sigma : S \to M$ is a smooth map. Such a k-cycle induces a well defined element of $H^k(M)^*$ which sends each $a \in H^k(M)$ to the integral of $\sigma^*\alpha$ over S. Indeed, if ω, $\theta \in A^k(M)$ and $\eta \in A^{k-1}(M)$ are such that $\omega = \theta + d\eta$, then

$$\int_S \sigma^*\omega = \int_S \sigma^*\theta + \int_S d(\sigma^*\eta) = \int_S \sigma^*\theta$$

by Theorem 4.5.1. By Poincaré Duality, there exists a unique $\delta_{(S,\sigma)} \in H^{n-k}(M)$ such that

$$\int_M \alpha \smile \delta_{(S,\sigma)} = \int_S \sigma^*\alpha$$

for every $\alpha \in H^k(M)$, which is called the *Poincaré dual de Rham cohomology class of the k-cycle* (S, σ). We will usually write simply δ_S instead of $\delta_{(S,\sigma)}$ if there no danger of confusion.

Examples 5.6.1. (a) The Poincaré dual cohomology class of a point in a compact connected oriented smooth n-manifold M is o_M.

(b) If M is a compact, connected, oriented smooth n-manifold, then the Poincaré dual cohomology class of the n-cycle (M, id_M) in M is 1.

(c) If N is a compact oriented k-dimensional smooth submanifold of a compact connected oriented smooth n-manifold M and $i : N \hookrightarrow M$ is the inclusion, then (N, i) is a k-cycle in M.

(d) Let M be a compact connected oriented smooth m-manifold and N be a compact connected oriented smooth n-manifold. If (S, σ) is a k-cycle in M and (T, τ) is a l-cycle in N, then $(S \times T, \sigma \times \tau)$ is a $(k + l)$-cycle in $M \times N$ and

$$\delta_{S\times T} = (-1)^{(m-k)l} \pi_M^* \delta_S \smile \pi_N^* \delta_T$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the projections. Indeed, for every $\alpha \in H^k(M)$ and $\beta \in H^l(M)$ we have

$$\int_{S\times T} (\sigma \times \tau)^* (\pi_M^a \smile \pi_N^* \beta) = \Big(\int_S \sigma^* \alpha\Big) \cdot \Big(\int_T \tau^* \beta\Big) = \Big(\int_M \alpha \smile \delta_S\Big) \cdot \Big(\int_N \beta \smile \delta_T\Big)$$

$$= \int_{M\times N} \pi_M^* (\alpha \smile \delta_S) \smile \pi_N^* (\eta \smile \delta_T)$$

$$= (-1)^{(m-k)l} \int_{M\times N} \pi_M^* (\alpha \smile \pi_N^* \eta) \smile (\pi_M^* \delta_S \smile \pi_N^* \delta_T).$$

This computation and Corollary 4.5.2 prove the assertion.

(e) Let M be a compact connected oriented smooth n-manifold. The diagonal map

$$\Delta : M \to M \times M$$

gives a n-cycle (M, Δ) in the smooth $2n$-manifold $M \times M$. If $\pi_j : M \times M \to M$ denotes the projection onto the j-th coordinate, $j = 1, 2$, then

$$\int_M \Delta^* (\pi_1^* \alpha \smile \pi_2^* \beta) = \int_M \alpha \smile \beta$$

for every $a \in H^k(M)$ and $\beta \in H^{n-k}(M)$, $0 \le k \le n$. Let $\{\alpha_i\}$ be a basis of $H^*(M)$ and let $\{\alpha^i\}$ be its Poincaré dual, that is

$$\int_M \alpha^i \smile \alpha_j = \delta_{ij}.$$

Every $a \in H^*(M)$ can be written as

$$a = \sum_i \Big(\int_M \alpha^i \smile \alpha\Big)\alpha_i \quad \text{and} \quad a = \sum_i \Big(\int_M \alpha \smile \alpha_i\Big)\alpha^i$$

and so

$$\int_M \alpha \smile \beta = \sum_{i,j} \left(\int_M \alpha^i \smile \alpha\right) \cdot \left(\int_M \beta \smile \alpha_j\right) \cdot \left(\int_M \alpha_i \smile a^j\right)$$

$$= \sum_i \left(\int_M \alpha \smile \alpha^i\right) \cdot \left(\int_M \beta \smile \alpha_i\right) = \sum_i \int_{M\times M} \pi_1^*(\alpha \smile \alpha^i) \smile \pi_2^*(\beta \smile a_i)$$

$$= \int_{M\times M} (\pi_1^*\alpha \smile \pi_2^*\beta) \smile \left(\sum_i (-1)^{\deg a^i} \pi_1^*\alpha^i \smile \pi_2^*\alpha_i\right).$$

It follows from Corollary 4.5.2 that

$$\delta_\Delta = \sum_i (-1)^{\deg a^i} \pi_1^*\alpha^i \smile \pi_2^*\alpha_i.$$

Note that

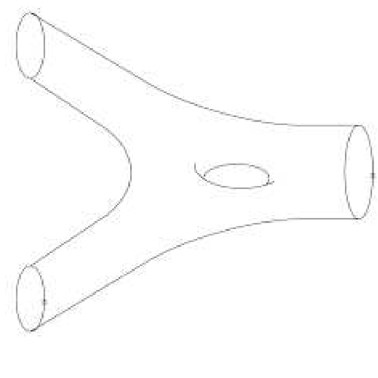
$$\int_M \Delta^*\delta_\Delta = \sum_i (-1)^{\deg \alpha^i} \int_M \Delta^*(\pi_1^*\alpha^i \smile \pi_2^*\alpha_i) = \sum_i (-1)^{\deg \alpha^i} \int_M \alpha^i \smile \alpha_i$$

$$= \sum_i (-1)^{\deg \alpha^i} = \sum_{k=0}^n (-1)^k \dim H^k(M) = \chi(M).$$

Two k-cycles (S_1, σ_1) and (S_2, σ_2) are called *cobordant* if there exists a relatively compact connected domain with smooth boundary D in an oriented smooth $(n+1)$-manifold P such that

$$\partial D = (-S_1) \coprod S_2$$

and a smooth map $\sigma : P \to M$ such that $\sigma|_{S_j} = \sigma_j$, $j = 1, 2$, where we have denoted by $-S_1$ the smooth k-manifold S_1 endowed with the reverse orientation.



Proposition 5.6.2. *If two k-cycles (S_1, σ_1) and (S_2, σ_2) in M are cobordant, then $\delta_{S_1} = \delta_{S_2}$.*

Proof. Using the above notations, by Stokes' formula we have

$$\int_{S_2} \sigma_2^*\omega - \int_{S_1} \sigma_1^*\omega = \int_{\partial D} \sigma^*\omega = \int_{\overline{D}} d(\sigma^*\omega) = \int_{\overline{D}} \sigma^*(d\omega) = 0$$

for every closed $\omega \in A^k(M)$. $\square$

An observation that is often useful in computations involving Poincaré dual cohomology classes of cycles is the following. Let (S, σ) be a k-cycle in a compact connected oriented smooth n-manifold M. If U is any open neighbourhood of $\sigma(S)$ in M, then U contains a smaller open neighbourhood W of $\sigma(S)$ which as a smooth manifold is of finite type. Let $i : W \hookrightarrow M$ denote the inclusion. There exists then a Poincaré dual $\delta_S^W \in H_c^{n-k}(W)$ of (S, σ) in W, by Corollary 5.4.12, and

$$\int_M \alpha \smile i_*\delta_S^W = \int_W i^*\alpha \smile \delta_S^W = \int_S \sigma^*\alpha$$

for every $\alpha \in H^k(M)$. This shows that the Poincaré dual cohomology class of (S, σ) in M is $\delta_S = i_*\delta_S^W$. In other words δ_S can be represented by closed differential $(n-k)$-forms in M with compact supports in arbitrarily small neighbourhoods of $\sigma(S)$. This is the localization principle for Poincaré dual classes.

Let now N be an compact oriented k-dimensional smooth submanifold of M. If S is a smooth manifold, a smooth map $\sigma : S \to M$ is said to be *transverse* to N if

$$T_{\sigma(x)}M = T_{\sigma(x)}N + \sigma_{*x}(T_xS)$$

for every $x \in N$. We shall restrict ourselves to the case where the dimension of S is $n-k$ and then the above sum of vector spaces is direct. It follows that if in addition S is compact, then $\sigma^{-1}(N)$ is a finite set. This is a consequence of the elementary observation that if $f : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map and there exists a sequence $(x_l)_{l\in\mathbb{N}}$ converging to some point $x \in \mathbb{R}^m$ such that $f(x_l) \in \mathbb{R}^k \times \{0\}$ for every $l \in \mathbb{N}$, there exists some $v \in S^{m-1}$ such that $Df(x)v \in \mathbb{R}^k \times \{0\}$.

Suppose that S is oriented. The orientations of $T_{\sigma(x)}N$ and T_xS induce an orientation on $T_{\sigma(x)}M = T_{\sigma(x)}N \oplus \sigma_{*x}(T_xS)$. If it coincides with the orientation of M, we put $i_x(N, S) = +1$. If not, we put $i_x(N, S) = -1$. The integer

$$N \bullet S = \sum_{x \in \sigma^{-1}(N)} i_x(N, S)$$

is called the *intersection number* of N with S.

Lemma 5.6.3. *Let M be a compact connected oriented smooth n-manifold and N be a compact oriented k-dimensional smooth submanifold of M. Let $B = (-1, 1)^{n-k}$ and let $\sigma : B \to M$ be a smooth map which is transverse to N and $\sigma^{-1}(N) = \{0\}$. Then,*

$$N \bullet B = \int_B \sigma^*\delta_N.$$

Proof. Since σ is assumed to be transverse to N, we have

$$T_{\sigma(0)}M = T_{\sigma(0)}N \oplus \sigma_{*0}(\mathbb{R}^{n-k})$$

and so $\sigma_{*0} : \mathbb{R}^{n-k} \to T_{\sigma(0)}M$ is a monomorphism. Then $\sigma_{*x} : \mathbb{R}^{n-k} \to T_{\sigma(x)}M$ is a monomorphism for x in an open neighbourhood of 0. There is no loss of generality

if we assume the σ is an immersion. By the Constant Rank Theorem 1.3.2 or rather its proof and its Corollary 1.3.3, we may further assume that there exists a smooth chart (U, ϕ) of M with $\phi = (x^1, ..., x^n)$ with the following properties:
(i) $\sigma(B) \subset U$ and $\phi(U) = (-1, 1)^n$.
(ii) $\sigma(0) \in U$ and $\phi(\sigma(0)) = 0$.
(iii) (U, ϕ) is N-straightening, that is $\phi(N \cap U) = (-1, 1)^k \times \{0\}$.
(iv) The orientation on $N \cap U$ is defined by $dx^1 \wedge \cdots \wedge dx^k$.
(v) σ has a local representation

$$(\phi \circ \sigma)(t^1, ..., t^{n-k}) = (0, ..., 0, t^1, ..., t^{n-k}).$$

By a previous observation, the dual cohomology class δ_N is represented by a differential $(n-k)$-form on M with compact support contained in an open neighbourhood W of N such that $W \cap U = \phi^{-1}((-1/2, 1/2)^n)$.

By definition, $\epsilon = N \bullet B = i_x(N, B) = \pm 1$ and so $\epsilon dx^1 \wedge \cdots \wedge dx^n$ defines an orientation on U. For every $p \in N \cap U$, let $\sigma_p : B \to U$ be the smooth map defined by

$$\sigma_p(t^1, ..., t^{n-k}) = \phi^{-1}(x^1(p), ..., x^k(p), t^1, ..., t^{n-k}).$$

It suffices to prove now that $\displaystyle\int_B \sigma_p^* \delta_N = \epsilon$.

If $g : \mathbb{R}^k \to \mathbb{R}$ is a smooth function with compact support contained in $(-1/3, 1/3)^k$ and $\omega = (g \circ \phi) dx^1 \wedge \cdots \wedge dx^k$ then ω is closed and

$$\int_U [\omega] \smile \delta_N = \int_{N \cap U} \omega|_{N \cap U} = \int_{(-1,1)^k} g(t^1, ..., t^k) dt^1 \cdots dt^k.$$

The left hand side can be computed by assuming that the restriction of δ_N in U is represented by a differential $(n-k)$-form $f dx^{n+1} \wedge \cdots \wedge dx^n$ for some $f \in C_c^\infty(W)$, because in the wedge product with ω all other terms involving dx^j for $1 \le j \le k$ will disappear. Then,

$$\int_U [\omega] \smile \delta_N = \epsilon \int_{(-1,1)^n} g \cdot (f \circ \phi^{-1})$$

$$= \epsilon \int_{(-1,1)^k} g \cdot \left(\int_{\{(t^1, ..., t^k)\} \times (-1,1)^{n-k}} (f \circ \phi^{-1}) dt^{k+1} \cdots dt^n \right) dt^1 \cdots dt^k$$

$$= \epsilon \int_{(-1,1)^k} g \cdot \left(\int_B \sigma_p^* \delta_N \right) dt^1 \cdots dt^k.$$

Thus,

$$\int_{(-1,1)^k} g(t^1, ..., t^k) dt^1 \cdots dt^k = \int_{(-1,1)^k} g(t^1, ..., t^k) \cdot \epsilon \left(\int_B \sigma_p^* \delta_N \right) dt^1 \cdots dt^k$$

for any such g. This implies that

$$\epsilon \int_B \sigma_p^* \delta_N = 1. \quad \square$$

Theorem 5.6.4. *Let M be a compact connected oriented smooth n-manifold and let $N \subset$ be a compact oriented k-dimensional smooth submanifold of M. If (S, σ) is a $(n-k)$-cycle in M which is transverse to N, then*

$$N \bullet S = \int_M \delta_N \smile \delta_S.$$

Proof. Since S is compact and the $(n-k)$-cycle (S, σ) is transverse to N, the set $\sigma^{-1}(N)$ is finite. Suppose that $\sigma^{-1}(N) = \{p_1, ..., p_m\}$ for some $m \in \mathbb{N}$. By transversality, each p_j has an open neighbourhood B_j in S which is diffeomorphic to $(-1,1)^k$ and such that $\sigma_j = \sigma|_{B_j}$ is a smooth embedding with $\sigma_j^{-1}(N) = \{p_j\}$. From Lemma 5.6.3 we have

$$N \bullet S = \sum_{j=1}^{m} N \bullet B_j = \sum_{j=1}^{m} \int_{B_j} \sigma_j^* \delta_N.$$

The Poincaré dual cohomology class δ_N can be represented by a differential $(n-k)$-form with compact support contained in an open neighbourhood W of N such that $W \cap \sigma\big(S \setminus \bigcup_{j=1}^{m} B_j\big) = \varnothing$. So, $\sigma^* \delta_N$ can be represented by a differential $(n-k)$-form with compact support contained in $B_1 \cup \cdots \cup B_m$ and

$$N \bullet S = \int_{B_1 \cup \cdots \cup B_m} \sigma^* \delta_N = \int_S \sigma^* \delta_N = \int_M \delta_N \smile \delta_S. \quad \square$$

This can be seen as a geometric interpretation of the wedge product of closed differential forms in terms of submanifolds which intersect transversally. From Proposition 5.6.2 and Theorem 5.6.4 we get the invariance of the intersection number under cobordism.

Corollary 5.6.5. *Let M be a compact connected oriented smooth n-manifold and let $N_j \subset M$, $j = 1, 2$, be compact oriented k-dimensional smooth submanifolds of M. Let (S_j, σ_j) is a $(n-k)$-cycle in M which is transverse to N_j, $j = 1, 2$. If N_1 is cobordant to N_2 and (S_1, σ_1) is cobordant to (S_2, σ_2), then $N_1 \bullet S_1 = N_2 \bullet S_2$.*

A compact k-dimensional smooth submanifold N of M intersects transversally a compact $(n-k)$-dimensional smooth submanifold S of M if $T_pM = T_pN \oplus T_pS$ for every $p \in N \cap S$.

Corollary 5.6.6. *Let M be a compact connected oriented smooth n-manifold. If a compact k-dimensional smooth submanifold N intersects transversally a compact $(n-k)$-dimensional smooth submanifold S of M, then*

$$N \bullet S = (-1)^{n-k} \int_{M \times M} \delta_{N \times S} \smile \delta_\Delta.$$

Proof. From Example 5.6.1(d) we have

$$\delta_{N\times S} = (-1)^{n-k}\pi_1^*\delta_N \smile \pi_2^*\delta_S$$

where $\pi_1 : M\times M \to M$ and $\pi_2 : M\times M \to N$ are the projections onto the first and second coordinate, respectively. Since $\pi_1\circ\Delta = \pi_2\circ\Delta = id_M$, we compute

$$\int_M \delta_N \smile \delta_S = \int_M \Delta^*(\pi_1^*\delta_N \smile \pi_2^*\delta_S)$$

$$= \int_{M\times M} \pi_1^*\delta_N \smile \pi_2^*\delta_S \smile \delta_\Delta = (-1)^{n-k}\int_{M\times M} \delta_{N\times S} \smile \delta_\Delta. \quad \square$$

5.7 The Lefschetz formula

The aim of this section is to give a proof of the Leschetz Fixed Point Theorem for smooth maps of compact oriented smooth manifolds and some of its numerous applications. We shall need some algebraic preliminaries.

Let V, W be two real vector spaces and let $g : V^*\otimes W \to \mathrm{Hom}(V,W)$ be the linear map defined by

$$g(a\otimes w)(v) = a(v)w$$

for every $v\in V$, $a\in V^*$ and $w\in W$. Then g is a linear monimorphism. Indeed, let $\{a_i\}$ be a basis of V^* and let $\{w_j\}$ be a basis of W. Then $\{a_i\otimes w_j\}$ is a basis of $V^*\otimes W$ and each element $z\in V^*\otimes W$ has a unique expansion

$$z = \sum_{i,j}\lambda_{ij}a_i\otimes w_j$$

for some $\lambda_{ij}\in\mathbb{R}$. If $g(z)=0$, then

$$\sum_j\left(\sum_i\lambda_{ij}a_i(v)\right)w_j = 0$$

for every $v\in V$. Therefore,

$$\sum_i\lambda_{ij}a_i(v) = 0$$

for every $v\in V$ and every j, which means that $\lambda_{ij}=0$ for all i,j.

In case W is finite dimensional, g is an isomorphism. To see this, let $\{w_1,...,w_k\}$ be a basis of W. For each $h\in\mathrm{Hom}(V,W)$ there are ϕ_1,..., $\phi_k\in V^*$ such that

$$h(v) = \phi_1(v)w_1 + \cdots + \phi_k(v)w_k$$

for every $v\in V$. For each $1\le j\le$ there are a_{1j},..., a_{nj}, for some $n\in\mathbb{N}$, and some λ_{1j},..., $\lambda_{nj}\in\mathbb{R}$ such that

$$\phi_j = \sum_{l=1}^n\lambda_{ij}a_{ij}.$$

Substituting,

$$h(v) = \sum_{j=1}^{k}\sum_{l=1}^{n} \lambda_{ij} a_{ij}(v) w_j = g\Big(\sum_{j=1}^{k}\sum_{l=1}^{n} \lambda_{ij} a_{ij}(v) \otimes w_j\Big)(v).$$

This shows that g is an epimorphism.

Lemma 5.7.1. *Let V be a finite dimensional real vector space. If $a \in V^*$ and $v \in V$, then $\mathrm{Tr}g(a \otimes v) = a(v)$.*

Proof. Let $\dim V = n$ and $\{v_1, ..., v_n\}$ be a basis of V. Let $(a_{ij})_{1 \le i,j \le n}$ be the matrix of $g(a \otimes v)$ with respect to this basis. For every $1 \le j \le n$ we have

$$a(v_j)v = g(a \otimes v)(v_j) = \sum_{i=1}^{n} a_{ij} v_i$$

and hence

$$v = \sum_{i=1}^{n} \frac{a_{ij}}{a(v_j)} v_i$$

for every $j \in I = \{1 \le k \le n : a(v_k) \neq 0\}$. The expansion of a with respect to the dual basis $\{v_1^*, ..., v_n^*\}$ is $a = \sum_{j \in I} a(v_j) v_j^*$. It follows that

$$a(v) = \sum_{j \in I} a(v_j) v_j^*(v) = \sum_{j \in I} a(v_j) \frac{a_{jj}}{a(v_j)} = \sum_{j \in I} a_{jj} = \mathrm{Tr}g(a \otimes v)$$

because if $a(v_j) = 0$, then $g(a \otimes v)(v_j) = a(v_j)v = 0$ and so $a_{ij} = 0$ for all $1 \le i \le n$.
□

Let M be a compact connected oriented smooth n-manifold. For brevity we shall use the notation

$$E^k(M) = \mathrm{Hom}(H^k(M), H^k(M)), \quad 0 \le k \le n.$$

and $E(M) = \bigoplus_{k=0}^{n} E^k(M)$. By Corollary 5.4.12 and the above considerations, we have isomorphisms $g_k : H^k(M)^* \otimes H^k(M) \to E^k(M)$, $0 \le k \le n$ and the isomorphism

$$g = \sum_{k=0}^{n} (-1)^k g_k : \bigoplus_{k=0}^{n} H^k(M)^* \otimes H^k(M) \to E(M).$$

From the Poincaré Duality Isomorphism D_M we get the isomorphism

$$D_M \otimes id : \bigoplus_{k=0}^{n} H^{n-k}(M) \otimes H^k(M) \to \bigoplus_{k=0}^{n} H^k(M)^* \otimes H^k(M).$$

We shall also need the Künneth isomorphism

$$\psi : \bigoplus_{k=0}^{n} H^{n-k}(M) \otimes H^k(M) \to H^n(M \times M)$$

of Corollary 5.5.2 defined by $\psi(\alpha \otimes \beta) = \pi_1^*\alpha \smile \pi_2^*\beta$, where $\pi_j : M \times M \to M$ denotes the projection onto the j-th coordinate, $j = 1, 2$. Composing, we get the isomorphism

$$\lambda = \psi \circ (D_M^{-1} \otimes id) \circ g^{-1} : E(M) \to H^n(M \times M).$$

Lemma 5.7.2. *If $\sigma = (\sigma_0, \sigma_1, ..., \sigma_n) \in E(M)$, then*

$$\sum_{k=0}^{n} (-1)^k \mathrm{Tr}\sigma_k = \int_M \Delta^*(\lambda(\sigma))$$

where $\Delta : M \to M \times M$ is the diagonal map.

Proof. Let $0 \le k \le n$. There are unique $\alpha \in H^{n-k}(M)$ and $\beta \in H^k(M)$ such that $\sigma_k = g_k(D_M(\alpha) \otimes \beta$ and therefore $\lambda(\sigma_k) = (-1)^k \pi_1^*\alpha \smile \pi_2^*\beta$, because $g^{-1}(\sigma_k) = (-1)^k D_M(\alpha) \otimes \beta$. On the other hand, from Lemma 4.7.1 we get

$$\mathrm{Tr}\sigma_k = D_N(\alpha)(\beta) = \int_M \alpha \smile \beta = \int_M \Delta^*(\pi_1^*\alpha \smile \pi_2^*\beta) = \int_M \Delta^*(\lambda(\sigma_k)). \quad \square$$

A smooth map $f : M \to M$ induces for each $0 \le k \le n$ a transpose linear map $f_k^* : H^k(M) \to H^k(M)$ and so an element $f^* = (f_0^*, f_1^*, ..., f_n^*) \in E(M)$. We call

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{Tr} f_k^*$$

the *Lefschetz number* of f. According to Lemma 5.7.2,

$$L(f) = \int_M \Delta^*(\lambda(f^*)).$$

Obviously, two smoothly homotopic maps of M have the same Lefschetz number.

Note that $L(id_M) = \chi(M)$, the Euler characteristic of M. Actually,

$$\lambda(id) = (-1)^n \delta_\Delta$$

where δ_Δ is the Poincaré dual cohomology class of the diagonal in $M \times M$. To see this, recall from Example 5.6.1(e) that

$$\delta_\Delta = \sum_i (-1)^{\deg a^i} \pi_1 \alpha^i \smile \pi_2^* \alpha_i$$

where $\{\alpha_i\}$ is a basis of $H^*(M)$ and is $\{\alpha^i\}$ is its Poincaré dual basis that is

$$\int_M \alpha^i \smile \alpha_j = \delta_{ij}.$$

So,

$$\lambda^{-1}(\delta_\Delta)(\alpha_j) = \sum_i (-1)^{\deg a^i} g(D_M(\alpha^i) \otimes \alpha_i)(\alpha_j)$$

$$\sum_i (-1)^{\deg a^i + \deg a_i} D_M(\alpha^i)(\alpha_j)\alpha_i = (-1)^n a_j.$$

Lemma 5.7.3. *If $f : M \to M$ is a smooth map, then*

$$\lambda(f^*) = (-1)^n (id_M \times f)^*(\delta_\Delta).$$

Proof. Suppose that $id = g_k(D_M(\alpha) \otimes \beta)$ for some $\alpha \in H^{n-k}(M)$ and $\beta \in H^k(M)$. Then, $\lambda(id) = (-1)^k \pi_1^*\alpha \smile \pi_2^*\beta$ and for every $\theta \in H^k(M)$ we have

$$f_k^*(\theta) = f_k^*(D_M(\alpha)(\theta)\beta) = D_M(\alpha)(\theta) f_k^*\beta = g_k(D_M(\alpha) \otimes f_k^*\beta)(\theta).$$

This means that $f_k^* = g_k(D_M(\alpha) \otimes f_k^*\beta)$ and consequently

$$\lambda(f_k^*) = (-1)^k \pi_1^*\alpha \smile \pi_2^*(f^*\beta) = (id \times f)^*((-1)^k \pi_1^*\alpha \smile \pi_2^*\beta)$$

$$= (id \times f)^*(\lambda(id)) = (-1)^n (id \times f)^*(\delta_\Delta). \quad \square$$

We are now ready to state and prove the following.

Theorem 5.7.4. *Let M be a compact connected oriented smooth n-manifold and $f : M \to M$ be a smooth map.*
(a) If $\Gamma : M \to M \times M$ is the smooth map $\Gamma(p) = (p, f(p))$, then

$$L(f) = (-1)^n \int_M \Gamma^*\delta_\Delta.$$

(b) If $L(f) \neq 0$, then f has at least one fixed point.

Proof. (a) Prom the preceding Lemma 5.7.2 and Lemma 5.7.3 we have

$$L(f) = \int_M \Delta^*(\lambda(f^*)) = \int_M \Delta^*((id \times f)^*(\lambda(id))) = \int_M \Gamma^*(\lambda(id)) = (-1)^n \int_M \Gamma^*\delta_\Delta.$$

(b) If f has no fixed point, then $M \times M \setminus \Gamma(M)$ is an open neighbourhood of the diagonal $\Delta(M)$ and so δ_Δ can be represented by a differential n-form with compact support contained in $M \times M \setminus \Gamma(M)$. Therefore $\Gamma^*\delta_\Delta = 0$ and $L(f) = 0$, by (a). $\square$

Corollary 5.7.5. *Let M be a compact connected oriented smooth n-manifold. If $\chi(M) \neq 0$, then every smooth vector field $X \in \mathcal{X}(M)$ vanishes at some point of M and so has some constant integral curve.*

Proof. Since M is compact, a smooth vector field X on M is complete, by Corollary 2.2.5. Let $(\Phi_t)_{t\in\mathbb{R}}$ be the one-parameter group of diffeomorphisms of M defined by the flow $\Phi : \mathbb{R} \times M \to M$ of X. Note that Φ is a smooth homotopy and thus each Φ_t is smoothly homotopic to $\Phi_0 = id_M$. Therefore, $L(\Phi_t) = L(id_M) = \chi(M)$. From our assumption and Theorem 5.7.4, every Φ_t has at least one fixed point. Let F_k

denote the fixed point set of $\Phi_{1/2^k}$, $k \in \mathbb{N}$. Since $\Phi_{1/2^{k+1}} \circ \Phi_{1/2^{k+1}} = \Phi_{1/2^k}$, we have $F_{k+1} \subset F_k$ for every $k \in \mathbb{N}$. By compactness of M, we have

$$F = \bigcap_{k=1}^{\infty} F_k \neq \varnothing.$$

Thus, there exists $p \in M$ such that $\Phi_{1/2^k}(p) = p$ and hence

$$\Phi\big(\frac{m}{2^k}, p\big) = \Phi_{m/2^k}(p) = (\Phi_{1/2^k})^m(p) = p$$

for every $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. This implies that $\Phi(t,p) = p$ for every $t \in \mathbb{R}$, because the set of dyadic rational numbers is dense in $\mathbb{R}$. This is equivalent to saying that $X(p) = 0$. □

Example 5.7.6. Let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a smooth map, $n \geq 1$. Let $X \in H^2(\mathbb{C}P^n)$ be a generator so that $\{1, X, ..., X^n\}$ is a basis of $H^*(\mathbb{C}P^n)$, where powers are taken with respect to the cup product, according to Example 5.4.11. There exists a unique $t \in \mathbb{R}$ such that $f^*(X) = tX$. Then, $f^*(X^k) = (f^*(X))^k = t^k X^k$, $0 \leq k \leq n$, and so the Lefschetz number of f is

$$L(f) = 1 + t + \cdots + t^n.$$

If $t = 1$, then $L(f) = n + 1$ and f has at least one fixed point. If $t \neq 1$ and n is even, then

$$L(f) = \frac{t^{n+1} - 1}{t - 1} \neq 0$$

and f has a fixed point. Thus in any case, if n is even, then every smooth map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point.

5.8 Exercises

1. If $\pi : S^{2n+1} \to \mathbb{C}P^n$, $n \geq 1$, is the Hopf map prove that there is no smooth map $s : \mathbb{C}P^n \to S^{2n+1}$ such that $\pi \circ s = id$.

2. Prove that there is no smooth map $r : \mathbb{R}^{n+1} \to S^n$ such that $r|_{S^n} = id_{S^n}$, $n \in \mathbb{N}$.

3. Prove the Fundamental Theorem of Algebra.

4. If $n \in \mathbb{N}$ is odd, prove that the quotient map $\pi : \mathbb{R}P^n \to S^n$ has degree 2.

5. Compute the de Rham cohomology of the real projective spaces $\mathbb{R}P^n$, $n \geq 0$.

6. Let M be a compact, connected, oriented smooth n-manifold with cohomological fundamental class $o_M \in H^n(M)$.
(a) Prove that for every non-zero $\alpha \in H^k(M)$, $0 \leq k \leq n$, there exists a unique non-zero $\beta \in H^{n-k}(M)$ such that $\alpha \smile \beta = o_M$.

(b) Prove that every non-trivial ideal of the de Rham cohomology algebra $H^*(M)$ of M contains o_M.
(c) Let N be a smooth manifold and let $f : N \to M$ be a smooth map. If $f^* o_M \neq 0$, prove that the transpose $f^* : H^*(M) \to H^*(N)$ is a monomorphism.

7. Let M be a smooth n-manifold, $n \geq 1$, and let θ be a closed differential 1-form on M. We consider the linear map $d_\theta : A^*(M) \to A^*(M)$ with $d_\theta(\omega) = d\theta - \theta \wedge \omega$ for every $\omega \in A(M)$.
(a) Prove that $d_\theta \circ d_\theta = 0$.
We denote by $H^*_\theta(M)$ the cohomology of the cochain complex $(A^*(M), d_\theta)$.
(b) If $f \in C^\infty(M)$, prove that the map $F : (A^*(M), d_{\theta+df}) \to (A^*(M), d_\theta)$ with $F(\omega) = e^{-f}\omega$ is a cochain isomorphism, which therefore induces an isomorphism $H^k_{\theta+df}(M) \cong H^k_\theta(M)$ for every $k \geq 0$.
(c) If θ is exact, prove that $H^*_\theta(M) \cong H^*(M)$.
(d) If the closed differential 1-form $\theta \in A^1(S^1)$ is not exact, prove that $H^0_\theta(S^1) = 0$.

8. Let k, $l \in \mathbb{N}$ and let $\sigma : S^1 \to S^1 \times S^1$ be the smooth map $\sigma(z) = (z^k, z^l)$. Compute the Poincaré dual de Rham cohomology class of the 1-cycle (S^1, σ) in the 2-torus $S^1 \times S^1$.

9. Let M and N be two compact connected oriented smooth n-manifolds and $f : M \to N$ be a smooth map. Prove that

$$D_M(f^*(\alpha))(f^*(\beta)) = (\deg f) \cdot D_N(\alpha)(\beta)$$

for every $\alpha \in H^k(N)$, $\beta \in H^{n-k}(N)$ and $0 \leq k \leq n$. Deduce from this that if $\deg f \neq 0$, then $f^* : H^*(N) \to H^*(M)$ is a monomorphism.

10. Let M be compact connected oriented smooth n-manifold. If there exists a smooth map $f : S^n \to M$ such that $\deg f \neq 0$, prove that $H^k(M) = \{0\}$ for all $0 < k < n$.

11. Let M be compact connected oriented smooth n-manifold and $f : M \to M$ be a smooth map. If the smooth map $\Gamma : M \to M \times M$ with $\Gamma(p) = (p, f(p))$, which parametrizes the graph $\Gamma(M)$ of f, is transverse to the diagonal $\Delta(M)$ in $M \times M$, prove that $L(f) = \Gamma(M) \bullet \Delta(M)$.

12. Prove that the Lefschetz number of a smooth map $f : S^n \to S^n$ is

$$L(f) = 1 + (-1)^n \deg f.$$

Deduce from this that every orientation preserving diffeomorphism $f : S^2 \to S^2$ has at least one fixed point and give an example of an orientation reversing diffeomorphism of S^2 with no fixed point.

13. Let $f : S^3 \to S^2$ be a smooth map and let $\omega \in A^2(S^2)$ with $\displaystyle\int_{S^2} \omega = 1$. If

$\theta \in A^1(S^3)$ is such that $f^*\omega = d\theta$, prove that the integral

$$h(f) = \int_{S^3} \theta \wedge d\theta$$

does not depend on the choice of the primitive θ and it depends only on the homotopy class of f. This integral is called *the Hopf invariant* of f.

14. (a) Prove that the differential 2-form

$$\Omega = \frac{i}{2\pi} \cdot \frac{1}{(|z_0|^2 + |z_1|^2)^2} \cdot (z_1 dz_0 - z_0 dz_1) \wedge (\bar{z}_1 d\bar{z}_0 - \bar{z}_0 d\bar{z}_1)$$

on $\mathbb{C}^{n+1} \setminus \{0\}$ induces a well-defined differential 2-form ω on $\mathbb{C}P^1$, so that the pull-back of ω under the natural quotient map is Ω.
(b) Prove that

$$\int_{\mathbb{C}P^1} \omega = 1.$$

(c) Let $f : S^3 \to S^2 \approx \mathbb{C}P^1$ denote the Hopf fibration. Prove that

$$f^*\omega = \frac{1}{\pi} \cdot d(x^1 dx^2 + x^3 dx^4)$$

where $z_0 = x^1 + ix^2$ and $z_1 = x^3 + ix^4$ for $(z_0, z_1) \in S^3$.
(d) Compute that the Hopf invariant of the Hopf fibration is equal to 1.

Chapter 6

Čech-de Rham theory

6.1 Generalized Mayer-Vietoris exact sequences

In this section we shall generalize the Mayer-Vietoris argument for the computation of the de Rham cohomology of a smooth n-manifold M to countable open covers.

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of M, where we assume that the index set I is countable and ordered. For simplicity, if $k \in \mathbb{N}$ and i_0,..., $i_k \in I$ we shall use the notation $U_{i_0 \cdots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$. The generalized Mayer-Vietoris sequence corresponding to the open cover $\mathcal{U}$ is the following sequence of vector spaces and linear maps

$$A^*(M) \xrightarrow{\ r\ } \prod_{i \in I} A^*(U_i) \xrightarrow{\ \delta\ } \prod_{i_0 < i_1} A^*(U_{i_0 i_1}) \xrightarrow{\ \delta\ } \prod_{i_0 < i_1 < i_2} A^*(U_{i_0 i_1 i_2}) \xrightarrow{\ \delta\ } \cdots$$

where $r(\omega) = (\omega|_{U_i})_{i \in I}$ for every $\omega \in A^*(M)$ and for every $m \in \mathbb{Z}^+$ and every $\omega = (\omega_{i_0 \cdots i_m})_{i_0 < \cdots < i_m} \in \displaystyle\prod_{i_0 < \cdots < i_m} A^*(U_{i_0 \cdots i_m})$ the coordinates of $\delta\omega$ are

$$(\delta\omega)_{i_0 \cdots i_m i_{m+1}} = \sum_{k=0}^{m+1} (-1)^k \omega_{i_0 \cdots i_{k-1} i_{k+1} \cdots i_{m+1}}.$$

We observe that

$$(\delta(\delta\omega))_{i_0 \cdots i_m i_{m+2}} = \sum_{k=0}^{m+2} (-1)^k (\delta\omega)_{i_0 \cdots i_{k-1} i_{k+1} \cdots i_{m+2}}$$

$$\sum_{l<k} (-1)^{l+k} \omega_{i_0 \cdots i_{-1} i_{l+1} \cdots l i_{k-1} i_{k+1} \cdots i_{m+2}} + \sum_{k<l} (-1)^{k+(l-1)} \omega_{i_0 \cdots i_{k-1} i_{k+1} \cdots i_{l-1} i_{l+1} \cdots i_{m+2}} = 0.$$

Thus, the above generalized Mayer-Vietoris sequence of vector spaces and linear maps is a cochain complex. If now $\{f_i : i \in I\}$ is a smooth partition of unity subordinated to the open cover $\mathcal{U}$, for each $\omega = (\omega_{i_0 \cdots i_m})_{i_0 < \cdots < i_m} \in \displaystyle\prod_{i_0 < \cdots < i_m} A^*(U_{i_0 \cdots i_m})$

we define the element $L\omega \in \prod_{i_0<\cdots<i_{m-1}} A^*(U_{i_0\cdots i_{m-1}})$ with coordinates

$$(L\omega)_{i_0\cdots i_{m-1}} = \sum_{i\in I} f_i \omega_{ii_0\cdots i_{m-1}}.$$

It follows from the definitions that

$$(\delta(L\omega))_{i_0\cdots i_m} = \sum_{i\in I}\sum_{k=1}^{m} (-1)^k f_i \omega_{ii_0\cdots i_{k-1}i_{k+1}\cdots i_m}$$

and

$$(L(\delta\omega))_{i_0\cdots i_m} = \sum_{i\in I} f_i\omega_{i_0\cdots i_m} + \sum_{i\in I}\sum_{k=1}^{m}(-1)^{k-1} f_i\omega_{ii_0\cdots i_{k-1}i_{k+1}\cdots i_m}.$$

Consequently, $\delta(L\omega) + L(\delta\omega)) = \omega$, which means that L is a cochain homotopy between id and 0. This shows that the generalized Mayer-Vietoris sequence is exact.

We consider now the double cochain complex $(K^{m,l})_{m,l\in\mathbb{Z}^+}$, with

$$K^{m,l} = \prod_{i_0<\cdots<i_m} A^l(U_{i_0\cdots i_m})$$

and differentials δ, d. As it is usual, from this we obtain a cochain complex (K, D), if we put $K^s = \bigoplus_{m+l=s} K^{m,l}$ and $D = \delta + (-1)^s d$ on K^s. Thus, if $\theta = (\theta_0, ..., \theta_s) \in K^s$, where $\theta_m \in K^{m,s-m}$, $0 \le m \le s$, then

$$D\theta = (d\theta_0, \delta\theta_0 - d\theta_1, ..., \delta\theta_{s-1} + (-1)^s d\theta_s, \delta\theta_s).$$

There is a product $\smile\colon K^{s_1} \times K^{s_2} \to K^{s_1+s_2}$, s_3, $s_2 \in \mathbb{Z}^+$, on K defined as follows. If $\omega \in K^{m_1+l_1}$ and $\theta \in K^{m_1+l_2}$, where $m_1 + l_1 = s_1$, $m_2 + l_2 = s_2$, then

$$(\omega \smile \theta)_{i_0\cdots i_{m_1+m_2}} = (-1)^{l_1 l_2}(\omega|_{U_{i_0\cdots i_{m_1}}}) \wedge (\theta|_{U_{i_{m_1}\cdots i_{m_1+m_2}}})$$

on their common domain of definition $U_{i_0\cdots i_{m_1+m_2}} = U_{i_0\cdots i_{m_1}} \cap U_{i_{m_1}\cdots i_{m_1+m_2}}$. From this definition and the definition of δ we have

$$\begin{aligned}
(\delta(\omega \smile \theta))_{i_0\cdots i_{m_1+m_2+1}} &= \delta((-1)^{l_1l_2}\omega_{i_0\cdots i_{m_1}} \wedge \theta_{i_{m_1}\cdots i_{m_1+m_2}}) \\
&= (-1)^{l_1l_2}\Big[\sum_{k\le m_1} (-1)^k \omega_{i_0\cdots i_{k-1}i_{k+1}\cdots i_{m_1+1}} \wedge \theta_{i_{m_1+1}\cdots i_{m_1+m_2+1}} \\
&\quad +(-1)^{m_1}\sum_{k\ge m_1}(-1)^{k+m_1}\omega_{i_0\cdots i_{m_1}} \wedge \theta_{i_{m_1}\cdots i_{k-1}i_{k+1}\cdots i_{m_1+m_2+1}}\Big] \\
&= (\delta\omega \smile \theta)_{i_0\cdots i_{m_1+m_2+1}} + (-1)^{m_1+l_1l_2}(-1)^{l_1(l_2+1)}(\omega \smile \delta\theta)_{i_0\cdots i_{m_1+m_2+1}} \\
&= (\delta\omega \smile \theta)_{i_0\cdots i_{m_1+m_2+1}} + (-1)^{s_1}(\omega \smile \delta\theta)_{i_0\cdots i_{m_1+m_2+1}}.
\end{aligned}$$

Hence

$$D(\omega \smile \theta) = D\omega \smile \theta + (-1)^{s_1}\omega \smile D\theta.$$

This implies that there is an induced product on the cohomology $H^*_D(K)$ of K, which we denote again by $\smile$. In this way $H^*_D(K)$ becomes a graded algebra.

Note that
$$D(r(\omega)) = (\delta + d)(r(\omega)) = d(r(\omega)) = r(d\omega)$$
for every $\omega \in A^*(M)$, which means that $r : A^*(M) \to K$ is a cochain map and hence induces an algebra homomorphism $r^* : H^*(M) \to H^*_D(K)$ in cohomology.

Proposition 6.1.1. *The map $r^* : H^*(M) \to H^*_D(K)$ is an algebra isomorphism.*

Proof. We shall show first that r^* is surjective. Let $\theta = (\theta_0, ..., \theta_s) \in K^s$, where $\theta_m \in K^{m,s-m}$, $0 \leq m \leq s$, and $D\theta = 0$. Then, $\delta\theta_s = 0$ and by the exactness of the generalized Mayer-Vietoris sequence there exists $\psi_{n-1} \in K^{s-1,0}$ such that $\delta\psi_{s-1} = \theta_s$. If $u = (0, ..., 0, \psi_{s-1}) \in K^{s-1}$, we have
$$Du = (0, ..., 0, (-1)^{s-1} d\psi_{s-1}, \delta\psi_{s-1}) \in K^s.$$
Thus, $\theta - Du$ and θ represent the same element of $H^s_D(K)$ and
$$\theta - Du = (\omega_0, ..., \omega_{s-1}, 0)$$
for some $\omega_m \in K^{m,s-m}$, $0 \leq m \leq s-1$. Since $D\omega = 0$, we have $\delta\omega_{s-1} = 0$. Repeating the above argument $s-1$ times we arrive at an element $\tau = (\tau_0, 0, ..., 0) \in K^s$, for some $\tau_0 \in K^{0,s}$ with $D\tau = 0$, or equivalently $\delta\tau_0 = 0$ and $d\tau_0 = 0$, which is cohomologous to θ in K. Since $\delta\tau_0 = 0$, the coordinates of τ_0 are restrictions to the elements of the open cover $\mathcal{U}$ of a differential s-form on M, which we denote again by τ_0 and which is closed. Obviously, $r^*[\tau_0] = [\tau]_D = [\theta]_D$.

To see that r^* is injective, let $\omega \in A^{s+1}(M)$ be closed and such that $r(\omega) = D\theta$ for some $\theta \in K^s$. Then $\theta = (\theta_0, ..., \theta_s) \in K^s$, for some $\theta_m \in K^{m,s-m}$, $0 \leq m \leq s$. Since $D\theta \in K^{0,s+1}$, we must have $\delta\theta_s = 0$. As above, there exists an element $\sigma = (\sigma_0, 0, ..., 0) \in K^s$ for some $\sigma_0 \in K^{0,s}$ such that $\delta\sigma_0 = 0$ and $D\sigma = D\theta$. In other words, σ_0 defines a differential s-form on M and
$$r(\omega) = D\theta = D\sigma = (d\sigma_0, 0, ..., 0)$$
which means that $\omega = d\sigma_0$. □

We denote now by $\check{C}^m(\mathcal{U}; \mathbb{R})$ the kernel of $d|_{K^{m,0}} : K^{m,0} \to K^{m,1}$, for $m \in \mathbb{Z}^+$. Note that the coordinates of the elements of $\check{C}^m(\mathcal{U}; \mathbb{R})$ are locally constant functions on the open sets $U_{i_0 \cdots i_m}$, $i_0 < \cdots < i_m$. The cohomology $\check{H}^*(\mathcal{U}; \mathbb{R})$ of the cochain complex $(\check{C}^*(\mathcal{U}; \mathbb{R}), \delta)$ is called the *Čech cohomology of the open cover* $\mathcal{U}$ of M (with real coefficients). The restriction of the product $\smile$ on K restricts to a product of $(\check{C}^*(\mathcal{U}; \mathbb{R}), \delta)$ defined by
$$(\omega \smile \theta)_{i_0 \cdots i_{m_1 + im_2}} = \omega_{i_) \cdots i_{m_1}} \cdot \theta_{i_{m_1} \cdots i_{m_1 + m_2}}$$
for $\omega = (\omega_{i_0 \cdots i_{m_1}})_{i_0 < \cdots < i_{m_1}}$ and $\theta = (\theta_{i_0 \cdots i_{m_2}})_{i_0 < \cdots < i_{m_2}}$. This turns $\check{H}^*(\mathcal{U}; \mathbb{R})$ into a graded commutative algebra with unity.

Recall that from Theorem 5.6.5 the set of admissible open covers is non-empty and cofinal in the directed family of all open covers of M.

Theorem 6.1.2. *If $\mathcal{U}$ is an admissible open cover of M, then we have algebra isomorphisms*

$$\check{H}^*(\mathcal{U};\mathbb{R}) \cong H_D^*(K) \cong H^*(M).$$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^2(M) \arrow[r, "r"] \arrow[u, "d"] & K^{0,2} \arrow[r, "\delta"] \arrow[u, "d"] & K^{1,2} \arrow[r, "\delta"] \arrow[u, "d"] & K^{2,2} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & A^1(M) \arrow[r, "r"] \arrow[u, "d"] & K^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & K^{1,1} \arrow[r, "\delta"] \arrow[u, "d"] & K^{2,1} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & A^0(M) \arrow[r, "r"] \arrow[u, "d"] & K^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & K^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & K^{2,0} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & & \check{C}^0(\mathcal{U};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "\text{inclusion}"] & \check{C}^1(\mathcal{U};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "\text{inclusion}"] & \check{C}^2(\mathcal{U};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "\text{inclusion}"] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Proof. The rows in the above diagram are the Mayer-Vietoris exact sequences in the corresponding degrees. If the columns of the augmented double complex are exact, then the assertion is proved using exactly the same argument of the proof of Proposition 6.1.1. The obstructions for this are the de Rham cohomologies

$$\prod_{i_0<\cdots<i_m} H^*(U_{i_0\cdots i_m}), \quad m \in \mathbb{Z}^+.$$

In case the open cover $\mathcal{U}$ is admissible, the open sets $U_{i_0\cdots i_m}$ are contractible and these de Rham cohomologies are trivial, by Corollary 5.1.7. □

Corollary 6.1.3. *The Čech cohomologies of any two admissible open covers of a smooth manifold are isomorphic.*

6.2 Čech cohomology

Let X be a topological space and R be a commutative ring with unity. A *presheaf* of R-modules on X is a contravariant functor Γ from the category with objects the open subsets of X and morphisms the inclusions to the category $\mathcal{M}_R$ of R-modules, which sends the empty subset of X to the trivial R-module. In other words, to each open set $U \subset X$ corresponds a R-module $\Gamma(U)$ and to an inclusion $U \subset V$ of open subsets of X corresponds a morphism $\rho_{UV} : \Gamma(V) \to \Gamma(U)$ of R-modules, which is usually called *restriction*, such that $\rho_{UU} = id_U$ and if $U \subset V \subset W$, then

$$\rho_{UW} = \rho_{UV} \circ \rho_{VW}.$$

Examples 6.2.1. (a) If G is a R-module, the constant preshef, denoted again by G, sends to every non-empty open set $U \subset X$ the R-module G and to every inclusion $U \subset V$ of open subsets of X the identity map of G, that is $\rho_{UV} = id_G$.

(b) Let M be a smooth manifold and let $\mathcal{GA}_{\mathbb{R}}$ denote the category of graded commutative, associative algebras with unity over $\mathbb{R}$. The contravariant functor A which to a non-empty open set $U \subset M$ assigns the exterior algebra $A^*(U)$ of differential forms of U and to an inclusions $U \subset V$ of open subsets of M assigns the usual restriction, which is the transpose of the inclusion map, is a presheaf on M, which is called the de Rham presheaf on M.

A *homomorphism of presheaves* Γ and Γ' on a topological space X is a natural transformation from Γ to Γ'. This is a family of homomorphisms $h_U : \Gamma(U) \to \Gamma'(U)$ of R-modules, where U runs over all open subsets of X, such that for each inclusion $U \subset V$ of open subsets of X the following diagram commutes.

$$\begin{array}{ccc} \Gamma(V) & \xrightarrow{h_V} & \Gamma'(V) \\ {\scriptstyle \rho_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle \rho'_{UV}} \\ \Gamma(U) & \xrightarrow{h_U} & \Gamma'(U) \end{array}$$

Let now Γ be a presheaf on a topological space X and let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X. For every $m \in \mathbb{Z}^+$ we put

$$\check{C}^m(\mathcal{U};\Gamma) = \prod_{i_0,\dots,i_m \in I} \Gamma(U_{i_0\cdots i_m})$$

where $U_{i_0\cdots i_m} = U_{i_0} \cap \cdots \cap U_{i_m}$, and define $\delta : \check{C}^m(\mathcal{U};\Gamma) \to \check{C}^{m+1}\mathcal{U};\Gamma)$ by the formula

$$(\delta\omega)_{i_0\cdots i_{m+1}} = \sum_{k=0}^{m+1} (-1)^k \rho_{U_{i_0\cdots i_{m+1}} U_{i_0\cdots i_{k-1} i_{k+1}\cdots i_{m+1}}} (\omega_{i_0\cdots i_{k-1} i_{k+1}\cdots i_{m+1}})$$

for $\omega = (\omega_{i_0,\dots,i_m})_{i_0,\cdots,i_m \in I} \in \check{C}^m(\mathcal{U};\Gamma)$. Then, $(\check{C}^*(\mathcal{U};\Gamma),\delta)$ is a cochain complex of R-modules, whose cohomology $\check{H}^*(\mathcal{U};\Gamma)$ is called the *Čech cohomology of the open cover* $\mathcal{U}$ *of* X *with coefficients in the presheaf* Γ.

Let now $\mathcal{V} = \{V_j : j \in J\}$ be an open cover of M which is a refinement of $\mathcal{U}$. There exists a function $\phi : J \to I$ such that $V_j \subset U_{\phi(j)}$ for every $j \in J$. This gives a cochain map $\phi^\sharp : \check{C}^*(\mathcal{U};\Gamma) \to \check{C}^*(\mathcal{V};\Gamma)$ defined by

$$(\phi^\sharp\omega)_{i_0\cdots i_m} = \omega_{\phi(i_0)\cdots\phi(i_m)}$$

if $\omega = (\omega_{i_0,\dots,i_m})_{i_0,\dots,i_m} \in \check{C}^m(\mathcal{U};\Gamma)$. In the above formula the restriction has been suppressed for notational simplicity. If $\psi : J \to I$ is another function such that

$V_j \subset U_{\psi(j)}$ for every $j \in J$, we obtain a cochain homotopy H between $\psi^\sharp$ and $\phi^\sharp$, if we define $H : \check{C}^m(\mathcal{U};\Gamma) \to \check{C}^{m-1}(\mathcal{V};\Gamma)$ by

$$(H\omega)_{j_0\cdots j_{m-1}} = \sum_{k=0}^{m-1} (-1)^k \omega_{\phi(j_0)\cdots\phi(j_k)\psi(j_k)\cdots\psi(j_{m-1})}$$

where restrictions have been suppressed again. Indeed, we compute

$$(\delta(H\omega))_{j_0\ldots j_m} = \sum_{k=0}^{m} (-1)^k (H\omega)_{j_0\ldots j_{k-1}j_{k+1}\cdots j_m}$$

$$= \sum_{k=0}^{m} (-1)^k \Bigg[\sum_{l=0}^{k-1} (-1)^l \omega_{\phi(j_0)\ldots\phi(j_l)\psi(j_l)\ldots\psi(j_{k-1})\psi(j_{k+1})\ldots\psi(j_m)}$$
$$+ \sum_{l=k+1}^{m} (-1)^{l-1} \omega_{\phi(j_0)\ldots\phi(j_{k-1})\phi(j_{k+1})\ldots\phi(j_l)\psi(j_l)\ldots\psi(j_m)}\Bigg]$$

and

$$(H(\delta\omega))_{j_0\ldots j_m} = \sum_{l=0}^{m} (-1)^l (\delta\omega)_{\phi(j_0)\ldots\phi(j_j)\psi(j_l)\ldots\psi(j_m)}$$

$$= \sum_{l=0}^{m} (-1)^l \Bigg[\sum_{k=0}^{l} (-1)^k \omega_{\phi(j_0)\ldots\phi(j_{k-1})\phi(j_{k+1})\ldots\phi(j_l)\psi(j_l)\ldots\psi(j_m)}$$
$$+ \sum_{k=l}^{m} (-1)^{k+1} \omega_{\phi(j_0)\ldots\phi(j_l)\psi(j_l)\ldots\psi(j_{k-1})\psi(j_{k+1})\ldots\psi(j_m)}\Bigg].$$

Therefore,

$$(\delta(H\omega) + H(\delta\omega))_{j_0\ldots j_m}$$
$$= \Bigg(\sum_{l=0}^{m}\sum_{k=0}^{m} (-1)^{k+l} - \sum_{k=0}^{m}\sum_{l=k+1}^{m} (-1)^{k+l}\Bigg) \omega_{\phi(j_0)\ldots\phi(j_{k-1})\phi(j_{k+1})\ldots\phi(j_l)\psi(j_l)\ldots\psi(j_m)}$$
$$+ \Bigg(\sum_{k=0}^{m}\sum_{l=0}^{k-1} (-1)^{k+l} - \sum_{k=0}^{m}\sum_{k=l}^{m} (-1)^{k+l}\Bigg) \omega_{\phi(j_0)\ldots\phi(j_l)\psi(j_l)\ldots\psi(j_{k-1})\psi(j_{k+1})\ldots\psi(j_m)}$$
$$= \sum_{k=0}^{m} \omega_{\phi(j_0)\ldots\phi(j_{k-1})\psi(j_k)\ldots\psi(j_m)} - \sum_{k=0}^{m} \omega_{\phi(j_0)\ldots\phi(j_k)\psi(j_{k+1})\ldots\psi(j_m)}$$
$$= \omega_{\psi(j_0)\ldots\psi(j_m)} - \omega_{\phi(j_0)\ldots\phi(j_m)} = (\psi^\sharp\omega - \phi^\sharp\omega)_{j_0\cdots j_m}.$$

This implies that there is a well defined homomorphism $\phi^\sharp : \check{H}^*(\mathcal{U};\Gamma) \to \check{H}^*(\mathcal{V};\Gamma)$ of graded R-modules, which does not depend on the choice of the function ϕ. It is obvious now that the family

$$\Big\{\check{H}^*(\mathcal{U};\Gamma) : \mathcal{U} \text{ open cover of } X\Big\}$$

is a direct system of graded R-modules. The *Čech cohomology of the topological space X with coefficients in the presheaf* Γ is the graded R-module

$$\check{H}^*(X;\Gamma) = \lim_{\rightarrow} \check{H}^*(\mathcal{U};\Gamma).$$

Especially, if Γ is the constant presheaf G for a R-module G, then $\check{H}^*(X;G)$ is the *Čech cohomology of the topological space X with coefficients in the R-module G.*

The content of the previous section 6.1 can be encoded in the following which is known as the Čech-de Rham theorem.

Theorem 6.2.2. *For every smooth manifold M there is an isomorphism*

$$\check{H}^*(M;\mathbb{R}) \cong H^*(M).$$

Proof. Since the countable admissible open covers of M constitute a cofinal subset of the directed set of open covers of M, by Theorem 5.3.3, we can consider only this sort of open covers. From Theorem 6.1.2, if $\mathcal{U}$ is a countable admissible open cover of M, then $\check{H}^*(\mathcal{U};\mathbb{R}) \cong H^*(M)$ and if $\mathcal{V}$ is another countable admissible open cover of M which refines $\mathcal{U}$, the inclusions $i_{\mathcal{U}} : \check{C}^*(\mathcal{U};\mathbb{R}) \to \check{C}^*(\mathcal{U};A)$ and $i_{\mathcal{V}} : \check{C}^*(\mathcal{U};\mathbb{R}) \to \check{C}^*(\mathcal{V};A)$ induce isomorphisms in cohomology so that the following diagram commutes.

$$\begin{tikzcd}
\check{H}^*(\mathcal{U};\mathbb{R}) \arrow[rr, "\phi^\sharp"] \arrow[d, "i^*_{\mathcal{U}}"'] & & \check{H}^*(\mathcal{V};\mathbb{R}) \arrow[d, "i^*_{\mathcal{V}}"] \\
H^*_D(C(\mathcal{U};A)) \arrow[rr] \arrow[dr, "r^*_{\mathcal{U}}"'] & & H^*_D(C(\mathcal{V};A)) \arrow[dl, "r^*_{\mathcal{V}}"] \\
& H^*(M) &
\end{tikzcd}$$

Since $i^*_{\mathcal{U}}$, $i^*_{\mathcal{V}}$, $r^*_{\mathcal{U}}$ and $r^*_{\mathcal{V}}$ are isomorphisms by Theorem 6.1.2, it follows that $\phi^\sharp$ is an isomorphism as well. Going to the direct limit the isomorphisms $r^*_{\mathcal{U}} \circ i^*_{\mathcal{U}}$ induce the desired isomorphism $\check{H}^*(M;\mathbb{R}) \cong H^*(M)$. $\square$

It is obvious from the definition that the Čech cohomology with coefficients in a preasheaf of a topological space is a topological invariant. In particular the Čech cohomology algebra $\check{H}^*(M;\mathbb{R})$ with real coefficients of a smooth manifold M is a purely topological invariant. Thus, the preceding Theorem 6.2.2 has the following very interesting consequence.

Corollary 6.2.3. *The de Rham cohomology algebra $H^*(M)$ of a smooth manifold M depends only on the underlying topology of M and not on the choice of the smooth structure.* $\square$

6.3 Exercises

1. If $d' = (-1)^m d : K^{m,l} \to K^{m,l+1}$, prove that

$$\delta \circ (d' \circ L)^i = (d' \circ L)^i \circ \delta - (d' \circ L)^{i-1} \circ d'$$

for every integer $i \geq 0$.

2. If $\theta = (\theta_0, ..., \theta_s) \in K^s$ and $D\theta = (\psi_0, ..., \psi_s, \psi_{s+1})$, so that $\psi_0 = d\theta_0$, $\psi_{s+1} = \delta\theta$ and $\psi_j = \delta + d'\theta_j$, $1 \leq j \leq s$, we define

$$f(\theta) = \sum_{j=0}^{s} (-d' \circ L)^j \theta_j - \sum_{j=1}^{s+1} L \circ (-d' \circ L)^{j+1} \psi_j.$$

(a) Prove that $f(\theta)$ defines a differential s-form by showing that $\delta f(\theta) = 0$.
(b) Prove that the so defined map $f : K \to A^*(M)$ is cochain and $f \circ r = id_{A^*(M)}$.
(c) If $L' : M^s \to K^{s-1}$ is defined by $L'\theta = ((L'\theta)_0, ..., (L'\theta)_{s-1})$, where

$$(L'\theta)_j = \sum_{i=j+1}^{s} L \circ (-d' \circ L)^{i-j-1} \theta_i, \quad 0 \leq j \leq s-1,$$

prove that $id_K - r \circ f = D \circ L' + L' \circ D$ and therefore the induced algebra homomorphism $f^* : H^*_D(K) \to H^*(M)$ is the inverse of the algebra isomorphism $r^* : H^*(M) \to H^*_D(K)$.
(d) If $\mathcal{U}$ is an admissible open cover of M and $\eta \in \check{C}^m(\mathcal{U}; \mathbb{R})$ with $\delta\eta = 0$, prove that the closed differential m-form which corresponds to η under the isomorphism $\check{H}^*(\mathcal{U}; \mathbb{R}) \cong H^*(M)$ is $f(\eta) = (-1)^m (d' \circ L)^m \eta$.

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