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Mathematical Note

Cohomology of Surface Minimal Sets

Konstantin Athanassopoulos

One of the main problems in the qualitative theory of dynamical systems is to study the topological and dynamical structure of the limit sets of a flow and describe the behavior of the orbits around them. Of particular interest is the study of minimal sets, as every compact limit set contains a minimal set. A subset of a topological space carrying a continuous flow is called *minimal* if it is nonempty, closed, flow invariant and has no proper subset with these properties. A *surface minimal set* is a compact minimal set such that the restricted flow on it is topologically equivalent to the restricted flow on a minimal set of a flow on a 2-manifold. A surface minimal set is called *trivial* if it is either a fixed point, a periodic orbit or is homeomorphic to the 2-torus. A nontrivial surface minimal set is a 1-dimensional continuum, which is locally homeomorphic to the cartesian product of an interval with a Cantor set, hence it is not locally connected. Its path connected components are precisely the orbits contained in it, which are uncountably many. Thus, its singular co-homology groups vanish in positive dimensions.

In this note we prove that the first integral Cech cohomology of a nontrivial surface minimal set is free abelian of rank at least 2. The result on the rank can be viewed as an alternative short proof of the more general fact that a 1-dimensional continuum X carrying a flow without singular points is a periodic orbit if and only $\check{H}^1(X) \cong \mathbb{Z}$, where $\check{H}^*$ denotes Cech cohomology with integer coefficients. This has been proved in [2]. Since an isolated 1-dimensional compact minimal set of a C^1 flow on a 3-manifold is a surface minimal set [6], the Cech cohomology of such a set is free abelian. In general the Cech cohomology of a nonsurface 1-dimensional compact minimal set may not be free abelian. Consider for example the dyadic solenoid

$$X = \{(z_n)_{n\geq 0} : z_n \in S^1 \text{ and } z_n = z_{n+1}^2, n \geq 0\}$$

with the minimal flow

$$\phi_t((z_n)_{n\geq 0}) = (z_n e^{it/2^n})_{n\geq 0}, \quad t \in \mathbb{R}.$$

It is known that $\check{H}^1(X)$ is isomorphic to the additive group of the dyadic rationals, which is a nonfree abelian group of rank 1.

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The proof is based on a topological and dynamical embedding of a nontrivial surface minimal set in a flow on a closed 2-manifold so that its complement consists of disjoint open discs. The result then is an immediate consequence of Poincaré-Lefschetz duality and Poincaré-Bendixson theory according to which every minimal set of a flow on the 2-sphere, the projective plane or the Klein bottle is trivial [7]. On every other closed 2-manifold there exist flows with nontrivial minimal sets.

In the proof of the Lemma below we make use of the end point compactification of a 2-manifold of finite type. The end point compactification Y^+ of a connected 2-manifold without boundary Y is obtained from the Stone-Cech compactification βY of Y by indentifying the connected components of $\beta Y \setminus Y$ to points and is the maximal compactification of Y with totally disconnected remainder [1], [4]. It follows from [1, Satz 2.3] that every flow on Y has an extension to a flow on Y^+ that fixes $Y^+ \setminus Y$ pointwise.

Suppose that Y is a connected 2-manifold without boundary of finite type n. This means that Y is constructed as follows. Let F be a closed and totally disconnected subset (maybe empty) of S^2. If Y is orientable, from $S^2 \setminus F$ we remove the interiors of $2n$ pairwise disjoint closed discs and identify their boundaries pairwise to form n handles $h_1, ..., h_n$. Then Y is homeomorphic to $S^2 \cup h_1 \cup ... \cup h_n \setminus F$ [8, Proposition 4.1]. If Y is nonorientable, from $S^2 \setminus F$ we remove the interiors of n pairwise disjoint closed discs and sew in n cross caps $c_1, ..., c_n$. Then Y is homeomorphic to $S^2 \cup c_1 \cup ... \cup c_n \setminus F$ [9, Theorem 2.1]. In both cases adding F to Y we obtain a closed 2-manifold of type n which is the end point compactification of Y [4, Proposition 3.12].

Lemma. *Let X be a nontrivial compact minimal set of a flow on a 2-manifold N. There exists a closed 2-manifold M carrying a flow with a mininal set which is a homeomorphic copy of X such that the connected components of $M \setminus X$ are (homeomorphic to) open discs.*

Proof. Let V be a connected open neighbourhood of X whose closure in N is a compact 2-manifold with boundary. We can reparametrize the local flow in V to obtain a global flow under which X is again a nontrivial minimal set [3]. If we identify the boundary components of $\overline{V}$ to points we get a closed 2-manifold V_1 of finite type say n, which is the end point compactification of V. The flow on V has an extension to a flow on V_1. If the connected components of $V_1 \setminus X$ are open discs, we take $M = V_1$. If not, there exists a simple closed curve C in $V_1 \setminus X$ which does not bound a disc in $V_1 \setminus X$. If C bounds a closed disc D in V_1, then X is not contained in D by the Poincaré-Bendixson theorem. Since $X \cap C = \emptyset$ and X is connected, it follows that $D \subset V_1 \setminus X$, which is a contradiction. Therefore, C cannot bound a disc in V_1. If G is the connected component of $V_1 \setminus C$ which contains X, then G is a 2-manifold of type at most $n-1$ [9, Theorem 2.2]. The end point compactification V_2 of G is a closed 2-manifold of type at most $n-1$.

Reparametrizing the flow in G and extending to V_2 we obtain a flow on V_2 with a nontrivial minimal set topologically equivalent to X. If the connected components of $V_2 \setminus X$ are open discs, we take $M = V_2$. If not, we repeat the above process as many times as it may require. Since at each step the type of the obtained closed 2-manifold strictly decreases, the process terminates after a finite number of steps.

Remark. The neighbourhood V in the beginning of the proof of the Lemma can be chosen so that the resulting flow on M has no periodic orbit and only finitely many singularities.

Theorem. *If X is a nontrivial surface minimal set, then $\check{H}^1(X)$ is a free abelian group of rank at least 2.*

Proof. According to the Lemma, X can be embedded in a flow on a closed 2-manifold M so that $M \setminus X$ is a disjoint union of open discs. Thus, $\check{H}^1(M, X)$ is trivial and $\check{H}^2(M, X) \cong H_0(M \setminus X)$ is free abelian, by Lefschetz duality [5, Ch. VIII, Proposition 7.14]. The exact sequence of the pair (M, X) in Cech cohomology reduces to

$$0 \to H^1(M) \to \check{H}^1(X) \xrightarrow{\delta} \check{H}^2(M, X) \to H^2(M) \to 0$$

where δ is the connecting homomorphism. Since $\check{H}^2(M, X)$ is free abelian, so is Imδ and $\check{H}^1(X) \cong H^1(M) \oplus \mathrm{Im}\delta$. If M is orientable, then $H^1(M) \cong \mathbb{Z}^{2k}$, where $k \geq 1$, because M is not the 2-sphere, since X is nontrivial. If M is nonorientable, $H^1(M) \cong \mathbb{Z}^{k-1}$, where $k \geq 3$ because M is not the projective plane or the Klein bottle, for the same reason. Hence $\check{H}^1(X)$ is always a free abelian group of rank at least 2.

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Konstantin Athanassopoulus
Department of Mathematics
University of Crete
GR-71409 Iraklion
Greece