UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXVI

1998

PERIODICITY CRITERIA OF POINCARÉ–BENDIXSON TYPE

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The subject of the classical Poincaré–Bendixson theory is the study of the structure of the limit sets of continuous flows in the 2-sphere S^2 and the behavior of the orbits near them. A fairly complete account of the theory is given in [2]. A limit set of a flow in S^2 which contains at least one non-singular point is 1-dimensional, compact, connected, invariant and the restricted flow on it is chain recurrent. In this note we summarize the main results of the forthcoming paper [1], the motivation of which was to examine what properties of limit sets can be extended to the class of 1-dimensional invariant and internally chain recurrent continua of continuous flows in S^2.

It seems that some basic properties do extend. For instance, an assertion similar to the Poincaré–Bendixson theorem is true in this wider class.

THEOREM 1. *Let X be a 1-dimensional invariant chain recurrent continuum of a continuous flow on S^2. If X contains a periodic orbit C, then $X = C$.*

COROLLARY 2. *If a 1-dimensional invariant chain recurrent continuum of a continuous flow on S^2 contains no singular point, then it is a periodic orbit.*

In the course to prove the above, the following useful property of non-periodic chain reccurent points is proved.

PROPOSITION 3. *If $x \in S^2$ is a non-periodic chain recurrent point of a continuous flow on S^2, then the limit sets $L^+(x)$ and $L^-(x)$ consist of singular points.*

This generalizes a well known property of non-periodic limit sets of continuous flows on S^2 [2, Ch. VIII, Proposition 1.11].

As far as the topological structure is concerned, it is well known that any 1-dimensional invariant chain recurrent continuum of a continuous flow on S^2 separates S^2, if it contains at least one non-singular point [3]. On the other hand, such a set may not be locally an arc at each of its non-singular points, as simple examples show, while a limit set of a continuous flow in S^2 always is [2, Ch. VIII, Lemma 1.8]. It turns out that the additional assumptions needed are the maximality and the existence of finitely many singular points.

THEOREM 4. *Every 1-dimensional chain component Y of a continuous flow on S^2 with finitely many singularities is locally an arc at its non-singular points.*

Moreover, in this case each singularity in Y is an isolated invariant set in S^2. It follows from this and Proposition 3 that a singularity in Y can be the positive (or negative) limit set of finitely many orbits in Y. Thus, we obtain the following.

COROLLARY 5. *Every 1-dimensional chain component of a continuous flow in S^2 with finitely many singularities consists of finitely many orbits and is homeomorphic to a finite graph.*

The assumption in Theorem 4 (and Corollary 5) that there are finitely many singular points is essential. It is not hard to describe a continuous flow on S^2 with countably many singularities and having a 1-dimensional chain component which is not an arc at some of its non-singular points.

References

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Received July 19, 1996

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