(Non-)Existence of periodic orbits in dynamical systems

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Some History

Dynamical Systems is the field of science that studies the evolution with time of all kind of systems (physical, mathematical, biological, economical,...). It has empirical roots. A prototypical problem is the study of the motion of planets around the sun. People have studied this problem since ancient or even prehistoric times for practical reasons such as navigation and timekeeping. The many body problem is its mathematical formulation. The two-body problem has been solved since the 17th century by Kepler and Newton. In the end of the 19th century Poincaré took up the study of the three body problem. His study is considered to be the beginning of the modern theory of dynamical systems. Although his work had been analyzed by top scientists, such as Birkhoff for instance, his discovery of the phenomena which are nowdays called "recurrence" and "sensitive dependence on initial conditions" was ignored by the great majority of mathematicians for almost 70 years.

General methods

"Recurrence" and "sensitive dependence on initial conditions" are the main ingredients of "Chaos". Although chaotic dynamical systems are completely deterministic, the most useful description of their properties can be done using probabilities.

In general, the methods used in the study of dynamical systems are:

• Topological/geometrical (Topological and Differentiable Dynamics)

• Probabilistic (Ergodic Theory)

In both cases we often have to use methods of

• Hard Analysis

Topological Dynamics

In this lecture I will refer to a classical problem of Topological Dynamics. An abstract topological dynamical system is defined by the action of a group or semigroup on a topological space X, the phase space. The semigroup is taken to be $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$ or $\mathbb{R}^+$.
If V is a smooth vector field on a compact manifold, then the solutions of the differential equation

$$\frac{dx}{dt} = V(x)$$

define a smooth flow on M, that is a smooth action of the additive group $\mathbb{R}$ on M. In case the manifold is not compact and the solutions are not defined for all time, they can be reparametrized in a nice way to yield a flow.

Topological Equivalence

Two dynamical systems are isomorphic if they are topologically conjugate as (semi-)group actions. In the case of flows, if we are primarily interested in qualitative(=geometric) properies of the orbits in the long run of time, they are preseved under the weaker topological equivalences.
A topological equivalence is a homeomorphism sending time oriented orbits onto time oriented orbits.

A strategy to study dynamical systems topologically

The study of compact invariant sets plays a central role in the qualitative theory of differential equations and dynamical systems. There are basic difficulties in this study.

• The compact invariant sets are global objects and so one needs to develope global methods and tools for their study.

• Their structure may be extremely complicated.

• Even in the case of a simple compact invariant set, its structure may change dramatically under small perturbations of the system.

• The structurally stable dynamical systems are not dense.

In practice, when studying a parametrized family of differential equations one has to handle all these four problems simultaneously.

Minimal sets

The simplest compact invariant sets are:

- fixed points and
- periodic orbits.

Periodic orbits are finite sets in the discrete time case and simple closed curves in the case of flows. They are also the simplest kind of minimal compact invariant sets.

An invariant set A is called **minimal** if it is non-empty, closed, invariant and has no proper subset with these properties. Equivalently, every orbit passing from a point of A is contained and is dense in A.

Minimal sets are the cornerstones of the phase portrait of a dynamical system.

The Poincaré-Bendixson Theorem

Perhaps the first famous result describing the limit behavior of the orbits of a flow on the plane $\mathbb{R}^2$ (or the 2-sphere S^2) is the Poincaré-Bendixson Theorem.

Let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a separable, locally compact, metrizable space M. The positive limit set of (the orbit of) $x \in M$ is the closed, invariant set

$$L^+(x) = \{y \in M : \phi_{t_n}(x) \to y \quad \text{for some} \quad t_n \to +\infty\}.$$

The orbit of x is called positively recurrent if $x \in L^+(x)$.

Theorem. *Let $(\phi_t)_{t\in\mathbb{R}}$ be a flow on $\mathbb{R}^2$ (or the 2-sphere S^2). If $L^+(x)$ is compact and does not contain fixed points, then it is a periodic orbit.*

In particular, every compact minimal set of a plane flow is either a fixed point or a periodic orbit.

The Poincaré-Bendixson Theorem on other surfaces

Question: Does the Poincaré-Bendixson Theorem hold for flows on all surfaces, i.e 2-dimensional manifolds?

It is easy to see that it holds for flows on the real projective plane $\mathbb{R}P^2$, since it is doubly covered by S^2. The following theorem due to N. Markley (*Trans. Amer. Math. Soc.* 135 (1969), 159-165) is much harder to prove.

Theorem. Every positively recurrent point of a flow on the Klein bottle K^2 is either fixed or periodic.

This strengthens an old result of H. Kneser from 1924 stating that a flow on K^2 without fixed points always has a periodic orbit (*Math. Ann.* 91 (1924), 135-154).

The role of the degree of smoothness

The Poincaré-Bendixson Theorem does not hold on compact surfaces other than S^2, $\mathbb{R}P^2$ and K^2. However, A.J. Schwartz proved (*Amer. J. Math.* 85 (1963), 453-458) the following:

Theorem. *A compact minimal set of a C^2 flow on a smooth 2-manifold M is either a fixed point, a periodic orbit or M is diffeomorphic to the 2-torus T^2 and the flow is topologically equivalent to a linear flow with irrational slope.*

The degree of smoothnesss in Schwartz's result cannot be lowered. This follows from work of A. Denjoy (*J. Math. Pures Appl.* 11 (1932), 333-375) on the dynamics of orientation preserving C^1 diffeomorphisms of the circle, which Schwartz actually generalized.

The mapping torus and suspensions

Counterexamples to Schwartz's Theorem can be constructed on T^2 from orientation preserving C^1 diffeomorphisms of the the circle S^1 using the mapping torus construction. Let $f : Y \to Y$ be a homeomorphism of the space Y and $X = [0,1] \times Y/\sim$, where $(1,y) \sim (0, f(y))$. The flow $(\phi_t)_{t\in\mathbb{R}}$ on X defined by

$$\phi_t[s,y] = [t+s-n, f^n(y)], \quad \text{for } n \leq t+s < n+1$$

is called the suspension of f. Obiously, $\{0\} \times Y$ is a global cross section to this flow and the return map is precisely f. So the dynamics of the suspension are completely determined by those of f.
In case $Y = S^1$ and f is an orientation preserving homeomorphism of S^1, then X is homeomorphic to T^2, because f is isotopic to the identity. Any flow on T^2 without fixed points is topologically equivalent to a suspension.

Homeomorphisms of the circle

His study of the three-body problem led H. Poincaré to search for periodic orbits of orientation preseving homeomorphisms of S^1. He found the following existence criterion.

Theorem. *An orientation preserving homeomorphism $f : S^1 \longrightarrow S^1$ has a periodic orbit if and only if its rotation number $\rho(f)$ is rational. If $\rho(f)$ is irrational, then either every orbit is dense in S^1, in which case f is topologically conjugate to the corresponding irrational rotation, or f has a unique Cantor minimal set which is the positive and negative limit set of every orbit.*

The Poincaré rotation number

If $f : S^1 \to S^1$ be an orientation preserving homeomorphism, there is a increasing homeomorphism $F : \mathbb{R} \to \mathbb{R}$, called a lift of f, such that $f(e^{2\pi it}) = e^{2\pi iF(t)}$ for every $t \in \mathbb{R}$. Poincaré proved the following:

Proposition. *There exists a constant* $\rho(F) \in \mathbb{R}$ *such that*

$$\lim_{n\to+\infty} \frac{1}{n}(F^n - id) = \rho(F)$$

uniformly on $\mathbb{R}$.

The number $\rho(f) = e^{2\pi i\rho(F)} \in S^1$ does not depend on the choice of the particular lift F of f. It is called the *Poincaré rotation number* of f.

C^1 diffeomorphisms of the circle

Schwartz's Theorem was inspired by the following theorem of A. Denjoy.

Theorem. *Let $f : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number. If f' has bounded variation, then f is topologically conjugate to the corresponding irrational rotation.*

Denjoy also constructed examples of orientation preserving C^1 diffeomorphism with irrational rotation number having a unique Cantor minimal set, according to Poincaré's dichotomy, which are usually called Denjoy C^1 diffeomorphisms. The suspension of such an example gives a C^1 vector field on T^2 whose flow has a unique minimal 1-dimensional continuum, that is not a simple closed curve, i.e. a periodic orbit.

Asymptotic Cycles

The notion of Poincaré rotation number has been generalized to flows on compact metric spaces by S. Schwartzman (*Ann. of Math.* 66 (1957), 270-284). For a flow on a compact manifold M and a flow invariant probability μ the μ-asymptoric cycle is an element of $H_1(M;\mathbb{R})$ and describes how a μ-average orbit winds around the holes of M. It is useful if the phase space has sufficiently large first homology group.
Using asymptotic cycles, K. Athanassopoulos generalized the criterion of Poincaré for the existence of periodic orbits from orientation preserving homeomorphisms, or equivalently fixed point free flows on T^2, to flows on arbitrary compact orientable 2-manifolds (*J. reine angew. Math.* 481 (1996), 207-215).

The flow around complicated minimal sets

Since complicated non-periodic minimal sets exist, the following question arises:

Question: How does the complexity of a compact minimal set affects the behavior of the flow around it.

The simplest behavior occurs near a stable attractor.

Stable attractors

Let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a separable, locally compact, metrizable space M and let $A \subset M$ be a compact invariant set. The invariant set

$$W^+(A) = \{x \in M : \varnothing \neq L^+(x) \subset A\}$$

is called the region of attraction of A. If $W^+(A)$ is an open neighbourhood of A, then A is called an attractor. A compact invariant set A is called (positively) Lyapunov stable if every neighbourhood of A contains a positively invariant open neighbourhood of A.

Lyapunov functions

If $A \subset M$ is a stable attractor, there exists a Lyapunov function for A. More precisely, there exists a continuous function $f : M \to [0,1]$ such that
(i) $f^{-1}(0) = A$ and $f^{-1}(1) = M \setminus W^+(A)$, and
(ii) $f(\phi_t(x)) < f(x)$ for every $t > 0$ and $x \in W^+(A) \setminus A$.
If $0 < c < 1$, for every $x \in W^+(A) \setminus A$ there exists a unique $\tau(x) \in \mathbb{R}$ such that $f(\phi_{\tau(x)}(x)) = c$. Actually,
$\tau(x) = \sup\{t \in \mathbb{R} : \phi_t(x) \in M \setminus f^{-1}([0,c])\}$.

The structure of the region of attraction

The flow in $W^+(A)$ can be described as follows:
(a) For every $0 < c < 1$, the set $f^{-1}([0,c])$ is compact and for every open neighbourhood V of A there exists $0 < c < 1$ such that $f^{-1}([0,c]) \subset V$.
(b) If we put $\tau(A) = -\infty$, then the function $\tau : W^+(A) \to [-\infty, +\infty)$ is continuous. Moreover, $\tau(\phi_t(x)) = \tau(x) - t$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$.
(c) The map $h : W^+(A) \setminus A \to \mathbb{R} \times f^{-1}(c)$ defined by

$$h(x) = (-\tau(x), \phi_{\tau(x)}(x))$$

is a homeomorphism such that $h(\phi_t(x)) = (-\tau(x) + t, \phi_{\tau(x)}(x))$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. In other words, h conjugates the restricted flow on $W^+(A) \setminus A$ with the parallel flow on $\mathbb{R} \times f^{-1}(c)$.

Uniformly unbounded Lyapunov function

Note that $F : W^+(A) \to [0, +\infty)$ defined by

$$F(x) = \begin{cases} e^{\tau(x)}, & \text{if } x \in W^+(A) \setminus A \\ 0, & \text{if } x \in A \end{cases}$$

is a uniformly unbounded continuous Lyapunov function for A and $F(\phi_t(x)) = e^{-t}F(x)$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. Thus, $F^{-1}([0, a])$ is homeomorphic to $F^{-1}([0, b])$ for every a, $b > 0$, because $\phi_{\log(b/a)}(F^{-1}([0, b]) = F^{-1}([0, a])$.

The cohomology of the region of atraction

Also $F^{-1}([0,c])$ is a deformation retract of $W^+(A)$ and the inclusion $i : F^{-1}([0,c]) \hookrightarrow W^+(A)$ induces an isomorphism $i^* : \bar{H}^*(W^+(A);\mathbb{Z}) \cong \bar{H}^*(F^{-1}([0,c]);\mathbb{Z})$ in Alexander-Spanier cohomology. Similarly, the inclusion $F^{-1}([0,a]) \hookrightarrow F^{-1}([0,b])$ induces an isomorphism $\bar{H}^*(F^{-1}([0,b]);\mathbb{Z}) \cong \bar{H}^*(F^{-1}([0,a]);\mathbb{Z})$ for every $0 < a < b$. From the continuity property of the Alexander-Spanier cohomology follows that

$$\bar{H}^*(A;\mathbb{Z}) \cong \varinjlim \bar{H}^*(F^{-1}([0,c]);\mathbb{Z}) \cong \bar{H}^*(W^+(A);\mathbb{Z}),$$

since $A = \bigcap_{c>0} F^{-1}([0,c])$ and $F^{-1}([0,c])$ is compact for every $c > 0$.

Stable attractors of smooth flows

In case M is a smooth manifold and the flow is smooth, there is a smooth Lyapunov function F for A. From the implicit function theorem follows that $F^{-1}([0,c])$ is a compact, smooth submanifold with boundary $\partial F^{-1}([0,c]) = F^{-1}(c)$. Moreover, τ is smooth on $W^+(A) \setminus A$.

Minimal sets which are stable attractors

Complicated minimal sets affect to some extent the behavior of the flow around them. An indication is the following result of G. Allaud and E.S. Thomas (*J. Differential Equations* 15 (1974), 158-171).

Theorem. *Let M be an orientable, smooth n-manifold carrying a smooth flow and $A \subset M$ be an almost periodic compact minimal set. If A is a stable attractor, then A is a torus.*

This is not true for minimal sets which are not almost periodic, except in the following case, proved by K. Athanassopoulos (*Topol. Methods Nonlinear Anal.* 30 (2007), 397-406).

Theorem. *Let M be a locally connected, locally compact, separable, metric space carrying a continuous flow. If a 1-dimensional, compact minimal set A is a stable attractor, then A is a periodic orbit.*

Seifert's Conjecture

For flows on 3-manifolds or of higher dimension there is no general Poincaré-Bendixson Theorem. We are usually happy to have a result confirming the existence of a periodic orbit. One of the very first results was proved by H. Seifert (*Proc. Amer. Math. Soc.* 1 (1950), 287-302).

Theorem. *If a C^1 vector field on the 3-sphere S^3 is sufficiently close to the vector field whose orbits are the fibers of the Hopf fibration, then it has a periodic orbit.*

Seifert's Conjecture. A smooth vector field on S^3 has either a fixed point or a periodic orbit.

We know now that Seifert's Conjecture is false.

Counterexamples to Seifert's conjecture

It took more than 20 years for a counterexample to Seifert's conjecture to be given. This was done by P. Schweitzer (*Ann. of Math.* 100 (1974), 396-440), who described a C^1 counterexample. Unfortunately, his example could not be smoothed, because is was based on the Denjoy C^1 diffeomorphisms. In the mid 80s Jenny Harrison gave a C^2 counterexample (*Topology* 27 (1988), 249-278). In the early 90s K. Kuperberg finally gave a smooth counterexample (*Ann. of Math.* 140 (1994), 723-732) and soon after jointly with G. Kuperberg a real analytic counterexample (*Ann. of Math.* 144 (1996), 239-268).

Hamiltonian Systems

A very inportant class of dynamical systems coming from classical mechanics are Hamiltonian systems. If (M, ω) is a symplectic $2n$-manifold and $H : M \to \mathbb{R}$ is a smooth function, the Hamiltonian vector field X_H of H is determined by the equation $\omega(X_H, .) = dH$. Obviously, X_H preserves H. This is conservation of energy. So, if $c \in \mathbb{R}$ is a regular value of H, then $H^{-1}(c)$ is a $(2n-1)$-dimensional submanifold of M and X_H is tangent to $H^{-1}(c)$.

Question: Does a Hamiltonian vector field X_H have a periodic orbit on every regular level set $H^{-1}(c)$?

Periodic orbits for Hamiltonian Systems

H. Hofer and E. Zehnder answered this question in case $M = \mathbb{R}^{2n}$ with the standard symplectic structure (Symplectic invariants and Hamiltonian dynamics, Birkäuser, 1994).

Theorem. *If* $H : \mathbb{R}^{2n} \to \mathbb{R}$ *is a proper smooth function, then the Hamiltonian vector field* X_H *has a periodic orbit on* $H^{-1}(c)$ *for almost all regular values* c *such that* $H^{-1}(c) \neq \varnothing$*.*

The word "almost" cannot be removed. This is closely related to Weinstein's Conjecture.

Weinstein's Conjecture

Let M be an smooth oriented $(2n-1)$-manifold. A contact form on M is any smooth 1-form λ such that $\lambda \wedge (d\lambda)^{n-1}$ is a positive volume element. A contact form λ determines a unique Reeb vector field X defined by the equations

$$\lambda(X) = 1 \quad \text{and} \quad d\lambda(X, .) = 0.$$

Every orientable 3-manifold carries at least one contact form.

Weinstein's Conjecture. If M is a closed oriented odd dimensional manifold, then the Reeb vector field of any contact form on M must have a periodic orbit.

The 3-dimensional case of Weinstein's Conjecture

The answer to Weinstein's Conjecture is not known for odd dimensional manifolds of dimension at least 5. Recently, it has been proved for 3-manifolds by H.C. Taubes (*Geometry and Topology* 11 (2007), 2117-2202).

Theorem. *If M is a closed oriented 3-manifold, then the Reeb vector field of a contact form on M has a periodic orbit.*