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Volume preserving flows with cyclic winding numbers groups and without periodic orbits on 3-manifolds

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Abstract. We construct examples of volume preserving non-singular C^1 vector fields on closed orientable 3-manifolds, which have cyclic winding numbers groups with respect to the preserved volume element, but have no periodic orbits.

1. Introduction

A great amount of work in dynamical systems has been oriented towards finding conditions which guarantee the existence of periodic orbits for homeomorphisms and flows. The best known condition concerning orientation preserving homeomorphisms of the unit circle is the rationality of the Poincaré rotation number. A generalization to continuous flows on closed orientable 2-manifolds is given in [2], based on the notions of asymptotic cycle and winding numbers group of a flow with respect to an invariant Borel probability measure. The rational rotation numbers are replaced in the general case by the cyclic winding numbers groups, i.e. winding numbers groups which are isomorphic to **Z**.

In the same paper it is shown that for any $n \geq 3$ there is a smooth flow on a closed orientable n-manifold which has no singular point or periodic orbit and has an invariant Borel probability measure with cyclic winding numbers group. In dimension 3 the invariant measure of the example presented in [2] is concentrated on two invariant 2-tori. The aim of the present note is to construct an example of a non-singular flow without periodic orbits on a closed orientable 3-manifold, which posesses an invariant Borel probability measure that is positive on non-empty open sets and has cyclic winding numbers group. Another source of motivation for constructing such an example is the Conley-Zehnder-Franks theorem, which in our terminology says that if a homeomorphism h of the 2-torus is homotopic to the identity, preserves a Borel probability measure that is positive on non-empty open sets and its suspension flow has cyclic winding numbers group with respect to the induced

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invariant measure, then h has a periodic point or equivalently the suspension flow has a periodic orbit [4]. One could ask whether this can be generalized to any continuous flow on a closed orientable 3-manifold. More precisely, the question is whether a continuous flow on a closed orientable 3-manifold which preserves a Borel probability measure that is positive on non-empty open sets with cyclic winding numbers group has a periodic orbit. Our main result gives negative answer to this question and can be stated as follows.

Theorem 1.1. *There exists a closed orientable 3-manifold with first Betti number 3 carrying a volume preserving, non-singular C^1 flow without periodic orbits and with cyclic winding numbers group.*

The construction is given in section 4 and is based on the technique of plugging flows used in G. Kuperberg's volume preserving counterexample to the Seifert conjecture [5]. First we construct an example on a closed orientable 3-manifold with first Betti number 1 and then we use this to construct another example on a closed orientable 3-manifold with first Betti number 3. The latter doubly covers the former.

2. Asymptotic cycles

Let X be a compact metrizable space carrying a continuous flow $(\phi_t)_{t\in\mathbf{R}}$. Let tx denote the translation of the point $x \in X$ along its orbit in time $t \in \mathbf{R}$. For every continuous function $f : X \to S^1$ there is a unique continuous function $g : \mathbf{R} \times X \to \mathbf{R}$, called the *1-cocycle* of f, such that $f(tx) = f(x)\exp(2\pi i g(t,x))$ and $g(t+s,x) = g(s,tx) + g(t,x)$ for every $x \in X$ and $t, s \in \mathbf{R}$. The Ergodic Theorem of Birkhoff implies that for every ϕ-invariant Borel probability measure μ on X the limit

$$g^*(x) = \lim_{t\to+\infty} \frac{g(t,x)}{t}$$

exists μ-almost for every $x \in X$. Moreover, g^* is an μ-almost everywhere defined measurable flow invariant function, that is $g^*(tx) = g^*(x)$ for every $t \in \mathbf{R}$, whenever $g^*(x)$ is defined and $\int_X g^* d\mu = \int_X g(1,.)d\mu$. This integral describes the μ-average rotation of points moving along their orbits with respect to the projection f. If the measure μ is ergodic, then g^* is constant μ-almost everywhere. In case the flow is uniquely ergodic, then g^* is an everywhere defined constant.

If $f_1, f_2 : X \to S^1$ are homotopic continuous functions with cocycles g_1, g_2, respectively, then $\int_X g_1^* d\mu = \int_X g_2^* d\mu$. Since the first Cech cohomology group with integer coefficients $\check{H}^1(X;\mathbf{Z})$ of X is isomorphic to the group of homotopy classes of continuous functions of X to S^1, there is a group homomorphism $A_\mu : \check{H}^1(X; \mathbf{Z}) \to \mathbf{R}$ defined by

$$A_\mu[f] = \int_X g(1, .)d\mu$$

where g is the 1-cocycle of $f : X \to S^1$ and $[f]$ the homotopy class of f. The homomorphism A_μ was defined by S.Schwartzman in [8] and is called the μ*-asymptotic cycle* of the flow. It describes how a μ-average orbit homologically winds around X. The image of A_μ is called the μ*-winding numbers group* of the flow ϕ and will be denoted by W_μ. An exposition of the basic theory of asymptotic cycles with details is given in [1].

Examples 2.1. (a) Let C be a periodic orbit in X of prime period $T > 0$ and let μ be the uniformly distributed Borel probability measure along C. Then,

$$A_\mu[f] = \frac{1}{T}\deg(f|C)$$

for every continuous function $f : X \to S^1$. Thus, $A_\mu = 0$, if C is null homologous in X or represents a torsion element in homology. Otherwise, $W_\mu \cong \mathbf{Z}$, in case X is a compact manifold.

(b) Each continuous one-parameter group $(\phi_t)_{t\in\mathbf{R}}$ of translations of the n-torus T^n has the form

$$\phi_t((x_1, ..., x_n) \text{ (mod } \mathbf{Z}^n)) = (x_1 + a_1 t, ..., x_n + a_n t) \text{ (mod } \mathbf{Z}^n),$$

for some $a_1, ..., a_n \in \mathbf{R}$ and preserves the Haar measure μ of T^n. It is uniquely ergodic if and only if $a_1, ..., a_n$ are linearly independent over $\mathbf{Q}$, by Kronecker's theorem. The winding numbers group with respect to μ is $W_\mu = a_1\mathbf{Z} + ... + a_n\mathbf{Z}$.

3. Winding numbers of volume preserving flows

Let ξ be a C^r vector field, $1 \le r \le \infty$, on a connected compact smooth n-manifold M with flow $(\phi_t)_{t\in\mathbf{R}}$ and let μ be a ϕ-invariant Borel probability measure on M. By de Rham's theorem and the Universal Coefficient theorem, $H^1_{DR}(M) \cong H^1(M; \mathbf{Z}) \otimes \mathbf{R}$. So, there is a basis of $H^1_{DR}(M)$ every element of which can be represented by $f^*(d\theta/2\pi)$, where $f : M \to S^1$ is a smooth function and $d\theta/2\pi$ is the representative of the natural generator of $H^1_{DR}(S^1) \cong \mathbf{R}$. We can extend A_μ to $H^1_{DR}(M)$ linearly in the obvious way.

It is easy to see that the 1-cocycle $g : \mathbf{R}\times M \to \mathbf{R}$ of a smooth function $f : M \to S^1$ is given by

$$g(t, x) = \int_0^t f^*(\frac{d\theta}{2\pi})(\xi(sx))ds.$$

So, by Fubini's theorem we have

$$A_\mu[f] = \int_0^1 \Big(\int_M f^*(\frac{d\theta}{2\pi})(\xi(tx))d\mu\Big)dt = \int_M f^*(\frac{d\theta}{2\pi})(\xi)d\mu.$$

Consequently, $A_\mu[\alpha] = \int_M (i_\xi\alpha)d\mu$ for every smooth closed 1-form α on M, where $i_\xi\alpha$ denotes the interior product of α with ξ.

Since $H^1(M;\mathbf{Z})$ is a finitely generated free group, so is the μ-winding numbers group W_μ. If $W_\mu = 0$, then ξ is said to be *μ-homologically trivial*. If $W_\mu \cong \mathbf{Z}$, then $W_\mu = \lambda\mathbf{Z}$, for some $\lambda > 0$, and ξ is said to be *μ-homologically rational*. In any other case $W_\mu = \lambda(\mathbf{Z}+\alpha_1\mathbf{Z}+...+\alpha_k\mathbf{Z})$, for some $\lambda > 0$ and $0 < \alpha_i < 1$, $i = 1, 2, ..., k$, such that $1, \alpha_1, ..., \alpha_k$ are linearly independent over $\mathbf{Q}$ and $1 \le k \le \operatorname{rank} H^1(M;\mathbf{Z}) - 1$.

Suppose that M is oriented by a volume element ω and the flow of ξ preserves volume. Then, $\phi_t^*\omega = \omega$ for every $t \in \mathbf{R}$ and $i_\xi\omega$ is a C^r closed $(n-1)$-form, called the *flux form*. The map sending ξ to its flux form is a linear isomorphism between the space of volume preserving C^r vector fields and the space of C^r closed $(n-1)$-forms. Since $\alpha \wedge \omega = 0$, for every closed 1-form α, we have

$$0 = i_\xi(\alpha \wedge \omega) = (i_\xi\alpha)\omega - \alpha \wedge (i_\xi\omega)$$

and therefore

$$A_\omega[\alpha] = \int_M (i_\xi\alpha)\omega = \int_M \alpha \wedge (i_\xi\omega).$$

So, if everything is C^∞, the asymptotic cycle A_ω is the Poincaré dual of the de Rham cohomology class represented by the flux form $i_\xi\omega$.

We shall give now another description of the winding numbers of a volume preserving C^1 flow using submanifolds of codimension 1. The Pontryagin construction shows that there is a one-to-one correspondence between the classes in $H^1(M;\mathbf{Z})$ and the framed cobordism classes of framed compact smooth submanifolds of M of codimension 1. We shall briefly review some aspects of interest to us. Let S be a compact, smooth submanifold of M of codimension 1 with trivial normal bundle and with a fixed normal orientation, the framing. So S is also oriented. Let N be a tubular neighbourhood of S in M, $T : N \to \mathbf{R} \times S$ be a trivialization compatible with the framing and $B = T^{-1}([0,1] \times S)$. Let $f : M \to S^1$ be the smooth function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \notin B \\ \exp\Big(2\pi i\chi(p(T(x)))\Big), & \text{if } x \in B, \end{cases}$$

where $p : \mathbf{R} \times S \to \mathbf{R}$ is the projection and $\chi : \mathbf{R} \to \mathbf{R}$ is a smooth function such that $\chi^{-1}(0) = (-\infty, 0]$, $\chi^{-1}(1) = [1, +\infty)$ and $\chi'(t) > 0$ for $0 < t < 1$. The homotopy class of f does not depend on the particular choice of the tubular neighbourhood and its trivialization (see [7]) and is called the Pontryagin class of S with the chosen framing. It can be proved that each element of $H^1(M;\mathbf{Z})$ is the Pontryagin class of some compact, smooth, framed submanifold of M of codimension 1 and two such submanifolds define the same Pontryagin class if and only if they are framed cobordant in M (see [7]).

If S is transverse to ξ, then we have a framing defined by ξ and S is a local section to the flow. In this case the Pontryagin class coincides with the flow class of a compact local section defined in [2] and [8].

Let $\sigma \in H_{n-1}(S;\mathbf{Z})$ be the fundamental class of S, that is σ is the unique class such that

$$\int_\sigma \psi = \int_S \psi \text{ for every } \psi \in H^{n-1}_{DR}(S).$$

If $f : M \to S^1$ is a smooth function, which represents the Pontryagin class of the compact, smooth, framed submanifold S and $j : S \hookrightarrow M$ is the inclusion, then $f^*(d\theta/2\pi)$ is Poincaré dual to $j_*(\sigma)$, that is

$$\int_{j_*(\sigma)} \alpha = \int_M f^*(\frac{d\theta}{2\pi}) \wedge \alpha$$

for every C^1 closed $(n-1)$-form α on M. Thus, we arrive at the following.

Proposition 3.1. *Let ξ be a C^1 vector field on a closed orientable smooth manifold M whose flow preserves a volume element ω. Let $f : M \to S^1$ be a continuous function and let S be a compact, smooth, framed submanifold of M of codimension 1, whose Pontryagin class is represented by f. If $j : S \hookrightarrow M$ is the inclusion, then*

$$A_\omega[f] = \int_S j^*(i_\xi \omega).$$

The value of the asymptotic cycle A_ω of ξ on the Pontryagin class of a compact, smooth, framed submanifold S of M of codimension 1 is called the *flux* of the flow of ξ through S. If we reverse the framing of S, the flux just changes sign.

Remark 3.2. If ξ is ω-homologically rational, there is some $\lambda > 0$ such that $\frac{1}{\lambda} A_\omega$ is represented by an integral 1-cycle γ. If $f : M \to S^1$ represents the Pontryagin class of some compact smooth, framed submanifold S of M of codimension 1, we have

$$A_\omega[f] = \lambda \int_\gamma f^*(\frac{d\theta}{2\pi}) = \lambda(\text{intersection number of } S \text{ with } \gamma).$$

4. The proof of Theorem 1.1

The technique used to prove theorem 1.1 consists of inserting measured plugs constructed in [5] into other flows. The reader is refered to [6] and [5] for detailed background on the technique of plugging flows. According to the first of the two main constructions described in [5], there exists a measured, integrally Dehn twisted C^∞ plug $\mathcal{D}$ with two periodic orbits. More precisely, $\mathcal{D}$ is supported on the solid torus $[1, 3]\times S^1\times[-1, 1]$, with base $[1, 3]\times S^1$, is divergenceless with respect to $dr\wedge d\theta\wedge dz$ and $a(m) = m-l$ homologically, where a is the attaching map, m is the meridian and l is the longitude.

On the 3-torus T^3 we consider the uniquely ergodic flow generated by the one-parameter subgroup with slopes a_1, a_2 and a_3, which are linearly independent over $\mathbf{Q}$. The Poincaré dual of the 2-torus $S = S^1 \times S^1 \times \{1\}$ is the element of $H^1(T^3; \mathbf{Z})$ represented by the projection onto the third factor. So, the flux of the flow through S is a_3. The simple closed curves $\gamma_1 = S^1 \times \{1\} \times \{i\}$ and $\gamma_2 = \{1\} \times S^1 \times \{-i\}$ are disjoint, tranverse to the flow and represent the first two generators of $H_1(T^3; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

The transversality of γ_1 and γ_2 to the flow guarantees that there exist insertion maps of copies of $\mathcal{D}$ into the flow on T^3. Plugging in two (disjoint) copies of $\mathcal{D}$ using these insertion maps, we get a volume preserving C^∞ flow with exactly four periodic orbits on a closed orientable 3-manifold M. Topologically, M has been obtained by performing surgery along γ_1 and γ_2. Since γ_1 and γ_2 generate direct summands of $H_1(T^3; \mathbf{Z})$, we have $H_1(M; \mathbf{Z}) \cong \mathbf{Z}$ and $H^1(M; \mathbf{Z}) \cong \mathbf{Z}$ (see [3], Theorem IV.2.13). Of course all these can be done far away from the 2-torus S, which will still be embedded in M. Note also that the generator of $H_1(M; \mathbf{Z})$ is represented by an integral 1-cycle whose intersection number with S is 1. It follows by Poincaré duality that $H^1(M; \mathbf{Z})$ is generated by the Pontryagin class of S in M. The flow and the volume element in a sufficiently small tubular neighbourhood of S in M have not changed (see section 2.1 in [5]) and the new flux of the flow on M through S equals the old one, i.e. a_3, by Proposition 3.1. Hence the flow is homologically rational.

We use now the measured C^1 semi-plug constructed in paragraph 4.3 of [5] in the same way as in Schweitzer's counterexample to the Seifert conjecture, in order to break the four periodic orbits. In doing this, the topology of M is unchanged. More precisely, there exists a measured C^1 semi-plug $\mathcal{E}$ supported on $T^2 \times [-1, 1]$, with base T^2 and without periodic orbits, which on the level $T^2 \times \{0\}$ has a minimal set $X \times \{0\}$, homeomorphic to a Denjoy continuum. An orbit passing through a point of $(T^2 \setminus X) \times \{0\}$ is an unknotted segment with two endpoints on $T^2 \times \{-1\} \cup T^2 \times \{1\}$. The mirror image construction applied to $\mathcal{E}$ yields a measured C^1 plug $\mathcal{F}$ without periodic orbits supported on $T^2 \times [-1, 1]$ with base T^2. Since T^2 is closed, it cannot be C^1 embedded into $\mathbf{R}^3$ transversely to the vertical lines, and so $\mathcal{F}$ is not insertible.

However, if N is a sufficiently small invariant tubular neighbourhood of an unknotted orbit of $\mathcal{F}$ with two endpoints as above, the restriction $\mathcal{R}$ of $\mathcal{F}$ to $T^2 \times [-1, 1] \setminus \mathrm{int}N$ is an untwisted measured C^1 plug with base a 2-torus minus the interior of a disc and is insertible. Taking now four disjoint flow boxes in M around points of the four periodic orbits and not intersecting S, we plug in copies of $\mathcal{R}$ to break the periodic orbits. Thus, we get a non-singular, volume preserving C^1 vector field ζ without periodic orbits on M. The flux of ζ through S is again a_3, for the same reasons as before, and therefore the flow is again homologically rational.

From M we can also construct a closed oriented 3-manifold P having first Betti number 3, with a volume element Ω and carrying a Ω-homologically rational volume preserving non-singular C^1 vector field ξ without periodic orbits. Since the 2-torus S is transverse to ζ in M, there exists $\epsilon > 0$ such that $B = [-\epsilon, \epsilon]S$ is a tubular neighbourhood of S in M. If $N = M \setminus \mathrm{int}B$, on the disjoint union $N \times \{0\} \cup N \times \{1\}$ consider the equivalence relation $(\epsilon x, 0) \sim ((-\epsilon)x, 1)$ and $(\epsilon x, 1) \sim ((-\epsilon)x, 0)$. The quotient space defined by $\sim$ is a closed orientable 3-manifold P. In other words, P is obtained by doubling N interchanging its two boundary components. Note that there is a fixed point free involution $h : P \to P$ whose orbit space is diffeomorphic to M. Thus, there is a two-sheeted covering map $q : P \to M$. The volume element ω and the vector field ζ on M constructed above can be uniquely lifted to the volume element $\Omega = q^*\omega$ and to an h-invariant, non-singular C^1 vector field ξ on P which preserves Ω. By construction, ξ has no periodic orbit, since ζ has no periodic orbit.

Using a Mayer–Vietoris exact sequence and Poincaré–Lefschetz duality we find that $H_1(P; \mathbf{Z}) \cong \mathbf{Z}^3$ and $H^1(P; \mathbf{Z}) \cong \mathbf{Z}^3$. The two boundary components of N correspond to two framed cobordant embedded copies of S in P, which are mapped to each other by h and are transverse to ξ. Let them be denoted by Σ and $h(\Sigma)$. Their Pontryagin classes in $H^1(P; \mathbf{Z})$ coincide. From the homology exact sequence of the pair (P, Σ) and Poincaré–Lefschetz duality we obtain the split short exact sequence

$$0 \to H_2(\Sigma; \mathbf{Z}) \xrightarrow{\delta} H^1(P; \mathbf{Z}) \xrightarrow{j^*} H^1(\Sigma; \mathbf{Z}) \to 0,$$

where $j : \Sigma \hookrightarrow P$ is the inclusion and δ sends the fundamental class of Σ to its Pontryagin class in P, with respect to the framing defined by ξ. This means that if Λ and Γ are any two framed closed surfaces in P that intersect Σ transversely, such that $\Lambda \cap \Sigma$ is (homologous in Σ to) the longitude of Σ and $\Gamma \cap \Sigma$ is (homologous in Σ to) the meridian of Σ, then the Pontryagin classes of Σ, Λ and Γ form a basis of $H^1(P; \mathbf{Z})$.

Recalling the construction of M by surgery in T^3, we observe that there exist two framed double tori L and G in M that intersect S transveresly, and such that $L \cap S$ consists of two disjoint longitudes of S and $G \cap S$ consists of two disjoint meridians. Following the construction of P we see that L and G are lifted in P to framed closed surfaces Λ and Γ, respectively, that intersect Σ transversely, and such that $\Lambda \cap \Sigma$ consists of two disjoint longitudes of Σ and $\Gamma \cap \Sigma$ consists of two disjoint meridians. By construction of the volume element Ω and the vector field ξ on P, if the element $[f] \in H^1(P;\mathbf{Z})$ corresponds via the above splitting to the homotopy class of the projection onto the longitude on Σ, then

$$A_\Omega[f] = \frac{1}{2}(\text{flux of } \xi \text{ through } \Gamma) = \text{flux of } \zeta \text{ through } G,$$

which is an integer multiple of a_3. Similarly if $[f]$ corresponds to the homotopy class of the projection onto the meridian on Σ. Since obviously the flux of ξ through Σ is again a_3, this shows that ξ is Ω-homologically rational.

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