Math. Z. 223, 643–649 (1996)

Mathematische Zeitschrift

One-dimensional chain recurrent sets of flows in the 2-sphere

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Received 19 September 1994; in final form 27 February 1995

1. Introduction

The subject of the classical Poincare-Bendixson theory is the study of the structure of the limit sets of flows in the 2-sphere S^2 and the behavior of the orbits near them. A fairly complete account of the theory is given in [3]. A limit set of a flow in S^2 which contains at least one nonsingular point is 1-dimensional, compact, connected, invariant and the restricted flow on it is chain recurrent. The motivation of this note was to examine what properties of limit sets can be extended to the class of 1-dimensional invariant chain recurrent continua for flows in S^2. It seems that some basic properties do extend. For instance, an assertion similar to the Poincare-Bendixson theorem is true in this wider class. Precisely, if a 1-dimensional invariant chain recurrent continuum of a flow in S^2 contains no singular point, then it is a periodic orbit (see Corollary 3.5).

As far as the topological structure is concerned, it is well known that any 1-dimensional invariant chain recurrent continuum of a flow in S^2 separates S^2, if it contains at least one nonsingular point (see [4]). On the other hand, such a set may not be locally an arc at each of its nonsingular points, as simple examples show, while a limit set of a flow in S^2 always is (see [3, Ch. VIII, Lemma 1.8]). It turns out that the additional assumptions needed are the maximality and the existence of finitely many singular points. Precisely, a 1-dimensional chain component Y of a flow in S^2 with finitely many singularities is locally an arc at each of its nonsingular points (see Theorem 4.1). Moreover, in this case Y consists of finitely many orbits and is topologically a finite graph (see Corollary 4.4). The assumption that there are finitely many singular points is essential. In a final remark we describe a 1-dimensional continuum in S^2 which is not locally an arc at some of its points and is a chain component of a flow in S^2 whose set of singular points is countably infinite. The points at which this 1-dimensional continuum is not an arc are nonsingular.

2. Chain recurrence

Let X be a compact metrisable space with a compatible metric d and $v : \mathbf{R}\times X \to X$ a continuous flow. We shall usually write $v(t,x) = tx$ and $v(I \times A) = IA$, if $I \subset \mathbf{R}$ and $A \subset X$. The orbit of the point $x \in X$ will be denoted by $C(x)$, the positive semiorbit by $C^+(x)$ and the negative by $C^-(x)$. The positive limit set of x will be denoted by $L^+(x)$ and the negative by $L^-(x)$.

Given $\epsilon, T > 0$ an (ϵ, T)-chain from x to y is a pair of finite sets of points $\{x_0, ..., x_{p+1}\}$ and times $\{t_0, ..., t_p\}$ such that $x = x_0$, $y = x_{p+1}$, $t_j \geq T$ and $d(t_j x_j, x_{j+1}) < \epsilon$ for every $j = 0, 1, ..., p$. If for every $\epsilon, T > 0$ there is an (ϵ, T)-chain from x to y, we write xPy. The binary relation P is closed, transitive, flow invariant and depends only on the topology of X. The set $\Omega^+(x) = \{y \in X : xPy\}$ is called the positive chain limit set of x and the set $\Omega^-(x) = \{y \in X : yPx\}$ the negative chain limit set of x. Clearly $L^+(x) \subset \Omega^+(x)$. A point $x \in X$ is called chain recurrent if xPx and the set $R(v)$ of all chain recurrent points is closed and invariant. If $X = R(v)$, the flow v is called chain recurrent. It is well known (see [1, Theorem 3.6D]) that the connected components of $R(v)$ are the classes of the following equivalence relation in $R(v)$: $x \sim y$ if and only if xPy and yPx. Moreover the restricted flow on each connected component of $R(v)$ is chain recurrent. The connected components of $R(v)$ will be called chain components in the sequel. It is also well known that the restricted flow on a positive or negative limit set in X is chain recurrent (see [2, Theorem 3.1]). In the next section we shall use the following:

Lemma 2.1. *Let A be a nonempty, positively Lyapunov stable compact invariant set. Suppose that there is a neighbourhood base $\{V_n : n \in \mathbf{N}\}$ of A consisting of open, positively invariant sets and times $T_n > 0$ such that $T_n\overline{V}_n \subset V_n$ for all $n \in \mathbf{N}$. Then $\Omega^+(x) \subset A$ for every $x \in A$.*

Proof. Let $x \in A$ and $y \in \Omega^+(x)$. It suffices to prove that $y \in V_n$ for all $n \in \mathbf{N}$. Since V_n is supposed to be an open neighbourhood of the compact set $T_n\overline{V}_n$, there exists $\epsilon > 0$ such that $S(z, \epsilon) \subset V_n$ whenever $z \in X$ and $S(z, \epsilon) \cap T_n\overline{V}_n \neq \emptyset$, where $S(z, \epsilon)$ denotes the open ball of radius ϵ centered at z. Now let $\{x_0, ..., x_{p+1}\}$ be an (ϵ, T)-chain from x to y with times $\{t_0, ..., t_p\}$. Then $t_0x_0 = t_0x \in A \subset T_nV_n$ and $d(t_0x_0, x_1) < \epsilon$. Therefore $x_1 \in V_n$ and $t_1x_1 \in T_nV_n$, because V_n is positively invariant. Since $d(t_1x_1, x_2) < \epsilon$, we have $x_2 \in V_n$. Inductively, after a finite number of steps we have $y = x_{p+1} \in V_n$.

The assumptions of Lemma 2.1 are satisfied if A is positively asymptotically stable and in this case the conclusion is true for every point in the region of attraction of A.

3. The Poincare-Bendixson theorem for chain recurrent sets

In this section we shall generalise the Poincare-Bendixson theorem to 1-dimensional invariant chain recurrent continua. The proofs are not independent of the classical theory. In fact we shall make extensive use of Chapter VIII of [3]. In what follows we fix a flow v in S^2.

Proposition 3.1. *Let $x \in S^2$ be a nonperiodic point such that $L^+(x)$ contains at least one nonsingular point. If D is the connected component of $S^2 \setminus L^+(x)$ which contains x, then $\Omega^+(z) \subset S^2 \setminus D$ for every $z \in S^2 \setminus D$.*

Proof. Let $y \in L^+(x)$ be a nonsingular point. There is a local section Σ at y of some extent $\epsilon > 0$, which can be chosen to be an open arc (see [3,Ch. VIII, Theorem 1.6]). There is also a sequence $t_n \to +\infty$ such that $\{t_n x_n : n \in \mathbf{N}\}$ is a sequence of points of Σ which monotonically converges to y and $(t_n, t_{n+1})x \cap \Sigma = \emptyset$ for every $n \in \mathbf{N}$ (see [3, Ch. VII, Theorem 4.10]). If $[t_n x, t_{n+1} x]$ denotes the closed interval in Σ with endpoints $t_n x$ and $t_{n+1} x$, then the set $C_n = [t_n x, t_{n+1} x] \cup (t_n, t_{n+1})x$ is a simple closed curve and is the common boundary of two discs D_n and E_n such that $S^2 = D_n \cup E_n$, by the Jordan-Schoenflies theorem (see [5, p.71]). Moreover, D_n is positively invariant, E_n is negatively invariant, $L^+(x) \subset intD_n$, $C_n \subset D$ and $\partial D = L^+(x)$ (see [3, Ch. VIII, Proposition 1.18]). The set $S^2 \setminus D$ is compact, invariant and positively Lyapunov stable, because $\{intD_n : n \in \mathbf{N}\}$ is a neighbourhood base of $S^2 \setminus D$ consisting of open, positively invariant sets. Since $(t_{n+1} - t_n + \epsilon)D_n \subset intD_n$ for every $n \in \mathbf{N}$, Lemma 2.1 applies and gives the conclusion.

Corollary 3.2. *If $x \in S^2$ is a nonperiodic chain recurrent point, then $L^+(x)$ and $L^-(x)$ consist of singular points.*

Proof. Suppose that $L^+(x)$ contains a nonsingular point y. If D is the connected component of $S^2 \setminus L^+(x)$ which contains x, then $\Omega^+(y) \subset S^2 \setminus D$, by Proposition 3.1. On the other hand, y is chain recurrent and belongs to the same chain component which contains x. This means that $x \in \Omega^+(y)$, and we have a contradiction.

Lemma 3.3. *Let C_1 and C_2 be two periodic orbits which bound an annulus K with no singular point. If C_1 and C_2 belong to an invariant chain recurrent continuum X, then the flow in K is periodic and $K \subset X$.*

Proof. By the Jordan-Schoenflies theorem, C_1 and C_2 bound invariant discs D_1 and D_2 respectively in S^2 such that $D_2 = K \cup D_1$ and $K \cap D_1 = C_1$. Suppose that $x \in intK$ were a nonperiodic point. Then, $C_1 = L^+(x)$ and $C_2 = L^-(x)$ are periodic orbits by the Poincare-Bendixson theorem, since K contains no singular point. For the same reason C_1 and C_2 are not nullhomotopic in K and therefore divide K into three subannuli (some may be trivial) K_1, K_2 and K_3 which have no interior point in common and are such that either $\partial K_1 = C_1 \cup C_1$, $\partial K_2 = C_1 \cup C_2$ and $\partial K_3 = C_2 \cup C_2$ or $\partial K_1 = C_1 \cup C_2$, $\partial K_2 = C_1 \cup C_2$ and $\partial K_3 = C_1 \cup C_2$. In the former case, $K_1 \cup D_1$ is a positively asymptotically stable invariant disc, which contains C_1 but not C_2. Hence no point of C_1 is chained to a point of C_2. In the later case $K_1 \cup D_1$ is a negatively asymptotically stable invariant disc and no point of C_2 is chained to a point of C_1. Thus in both cases C_1 and C_2 cannot belong to the same invariant chain recurrent continuum. This proves that the flow in K is periodic. The connectedness of X implies now that $K \subset X$.

Theorem 3.4. *Let X be a 1-dimensional invariant chain recurrent continuum in S^2. If X contains a periodic orbit C, then $X = C$.*

Proof. By the Jordan-Schoenflies theorem, C is the boundary of an invariant disc D in S^2 and $E = \overline{S^2 \setminus intD}$ is also a disc. Suppose that $X \cap intD \neq \emptyset$. If there is a point $x \in intD$ such that $L^+(x) = C$, respectively $L^-(x) = C$, then C is one-sided positively, respectively negatively, asymptotically stable and therefore $X \subset \Omega^+(C) \subset E$, respectively $X \subset \Omega^-(C) \subset E$, contradiction. So, according to [3, Ch. VIII, Theorem 3.3], E is bilateraly Lyapunov stable and there is a sequence of periodic orbits $\{C_n : n \in \mathbf{N}\}$ in $intD$ such that C_n together with C bound an annulus $A_n \subset D$ with no singular point and $\{E \cup A_n : n \in \mathbf{N}\}$ is a decreasing neighbourhood base of E. Since X is connected, $X \cap A_n \neq \emptyset$ for every $n \in \mathbf{N}$. If $z_n \in X \cap A_n$, then $L^-(z_n)$ is a periodic orbit in $A_n \setminus C$ with C bounding an annulus $B_n \subset A_n$. But since $L^-(z_n) \subset X$, it follows from Lemma 3.3 that $B_n \subset X$ and hence X is not 1-dimensional.This contradiction shows that $X \cap intD = \emptyset$ and it is similarly proved that $X \cap intE = \emptyset$. Hence $X = C$.

Corollary 3.5. *Let X be a 1-dimensional invariant chain recurrent continuum of a flow in S^2. If X contains no singular point, then X is a periodic orbit.*

4. The structure of 1-dimensional chain components

Throughout this section we assume that v is a flow in S^2 with finitely many singular points. Our purpose is to examine the topological structure of the 1-dimensional chain components of v.

Theorem 4.1. *Every 1-dimensional chain component Y is locally an arc at its nonsingular points.*

Proof. In view of Theorem 3.4 we consider only the case where Y contains no periodic point. Let $x \in Y$ be a nonsingular point. There is a local section S at x of some extent $\epsilon > 0$, homeomorphic to an open interval, such that $S \cap C(x) = \{x\}$. Suppose that Y is not locally an arc at x. Then there is a sequence $\{x_n : n \in \mathbf{N}\}$ of points of $S \cap Y$ which monotonically converges to x on S. Since there are finitely many singular points, by Corollary 3.2 we may assume that there are singular points z_1, z_2 (possibly identical) such that $L^+(x_n) = \{z_1\}$ and $L^-(x_n) = \{z_2\}$ for every $n \in \mathbf{N}$. We may moreover assume that $C(x_n) \cap C(x_m) = \emptyset$, if $n \neq m$, again by Corollary 3.2. Each orbit $C(x_n)$ meets S in a finite number of points. Let s_n and t_n be the first and last time respectively, the orbit $C(x_n)$ meets S. Passing to a subsequence if necessary, we may assume that the sequences $\{s_n x_n : n \in \mathbf{N}\}$ and $\{t_n x_n : n \in \mathbf{N}\}$ are monotone in S. For any $a,b \in S$ let $[a, b]$ and (a, b) denote the closed and open interval in S, respectively, with endpoints a,b. From [6, Lemma 2.8] we may assume that for every $n \in \mathbf{N}$ the simple closed curve $[t_n x_n, t_{n+1} x_{n+1}] \cup C^+(t_n x_n) \cup C^+(t_{n+1} x_{n+1})$ bounds a positively invariant disc D_n such that $D_n \cap [-\epsilon, 0]S = \emptyset$. Similarly, the simple closed curve $[s_n x_n, s_{n+1} x_{n+1}] \cup C^-(s_n x_n) \cup C^-(s_{n+1} x_{n+1})$ bounds a negatively invariant disc E_n such that $E_n \cap [0, \epsilon]S = \emptyset$.

It follows now that $C(x_n) \cap intD_m = C(x_n) \cap E_m = \emptyset$ for every $n,m \in \mathbf{N}$. For if $C(x_n) \cap intD_m \neq \emptyset$, there is some $s \in \mathbf{R}$ such that $sx_n \in \partial D_m$, because $x_n \notin intD_m$. If $sx_n \in (t_m x_m, t_{m+1} x_{m+1})$, then $(0, +\infty)(sx_n) \subset intD_m$ and hence $s = t_n$, which contradicts the monotonicity. If $sx_n \in C^+(t_m x_m)$, then $sx_n = (t + t_m)x_m$ for some $t \geq 0$ and hence $(s - t)x_n = t_m x_m$. Since x_n and x_m do not belong to the same orbit unless $n = m$, we conclude that $x_n = x_m$ and $t_n = s - t = t_m$. Similarly, if $sx_n \in C^+(t_{m+1} x_{m+1})$, then $x_n = x_{m+1}$ and $t_n = s - t = t_{m+1}$. In both cases this is a contradiction, because obviously $C(x_m) \cap intD_m = \emptyset$ for every $m \in \mathbf{N}$.

We claim that $intD_n \cap intD_m = intE_n \cap intE_m = \emptyset$ for $n \neq m$. This follows from the fact that $intD_n \cap intD_m$ is an open and closed set in $intD_n$ and $intD_m$. Indeed, let $\{y_k : k \in \mathbf{N}\}$ be a sequence in $intD_n \cap intD_m$ converging to some point $y \in intD_n$. Then, $y \in D_m \setminus S$ and since $C(x_m) \cap intD_n = C(x_{m+1}) \cap intD_n = \emptyset$, it follows that $y \in intD_m$. This shows that $intD_n \cap intD_m$ is open and closed in $intD_n$ and similarly in $intD_m$. Thus, if it were nonempty, we would have $D_n = D_m$, contradiction.

Since there are finitely many singular points and $intD_n$, $n \in \mathbf{N}$, are pairwise disjoint, we may assume that z_1 is the only singular point in D_n and similarly that z_2 is the only singular point in E_n, for every $n \in \mathbf{N}$. It follows that D_n and E_n contain no periodic orbit either, because they are discs. Consequently, $z_1 \in L^+(p)$ and $z_2 \in L^-(q)$ for every $p \in D_n$ and $q \in E_n$. It suffices to consider now the following two cases:

(a) $C(x_n) \cap S = \{x_n\}$ for all $n \in \mathbf{N}$. Then, $t_n = s_n = 0$ and $z_1 \in L^+(p)$, $z_2 \in L^-(p)$, for every $p \in [x_n, x_{n+1}]$. Hence $[x_n, x_{n+1}] \subset \Omega^-(z_1) \cap \Omega^+(z_2) = Y$, which implies that $\dim Y = 2$.

(b) $C(x_n) \cap S \neq \{x_n\}$ for all $n \in \mathbf{N}$. Then, the Poincare map r is defined for S and $s_n x_n$ belongs to the domain of some power r^k, $k \in \mathbf{N}$, such that $r^k(s_n x_n) = t_n x_n$. Since S^2 is orientable, r is increasing and by continuity there is a (nontrivial) interval $I \subset [s_n x_n, s_{n+1} x_{n+1}]$ in S with one endpoint $s_n x_n$ which is mapped by r^k to an interval in $[t_n x_n, t_{n+1} x_{n+1}]$ with one endpoint $t_n x_n$. It follows that $z_1 \in L^+(p)$ for every $p \in I$ and as in case (a) we have $I \subset Y$. Hence again $\dim Y = 2$. This contradiction proves the Theorem.

Finally, we shall investigate the structure of a 1-dimensional chain component Y of v near its nonsingular points. Note that a singular point of Y cannot be positively or negatively asymptotically stable.

Theorem 4.2. *If Y is a 1-dimensional chain component and $z \in Y$ is a singular point, then $\{z\}$ is an isolated invariant set in S^2.*

Proof. Suppose that $\{z\}$ is not isolated in S^2. Then, there are a neighbourhood base $\{V_n : n \in \mathbf{N}\}$ of z consisting of interiors of discs, so that $\overline{V}_{n+1} \subset V_n$ and orbits $C(x_n) \subset V_n$, where $z \neq x_n$, for every $n \in \mathbf{N}$. Since there are finitely many singular points, we may assume that z is the only singular point in $\overline{V}_1$. If $L^+(x_n)$ is a periodic orbit for infinitely many values of n, then passing to a subsequence we may assume it is for all. In this case, $L^+(x_n)$ bounds a disc $D_n \subset \overline{V}_n$ containing z in its interior and $\{D_n : n \in \mathbf{N}\}$ is a neighbourhood base of z. Since $Y \neq \{z\}$, the connectedness of Y implies that $L^+(x_n) \subset Y$ for some $n \in \mathbf{N}$ and therefore Y is a periodic orbit, by Theorem 3.4. This contradiction shows that we may assume that for every $n \in \mathbf{N}$ the limit sets $L^+(x_n)$, and similarly $L^-(x_n)$, are not periodic. If $L^+(x_n)$ and $L^-(x_n)$ consist of singular points, then $L^+(x_n) = L^-(x_n) = \{z\}$. If $L^+(x_n)$ (or $L^-(x_n)$) contains a nonsingular point y_n, then $L^+(y_n) = L^-(y_n) = \{z\}$ (see [3, Ch. VIII, Proposition 1.11]). Thus, considering the point y_n instead of x_n if necessary, we may assume that $L^+(x_n) = L^-(x_n) = \{z\}$ for every $n \in \mathbf{N}$. The simple closed curve $\overline{C(x_n)}$ bounds an invariant disc $E_n \subset \overline{V}_n$. Then *int*$E_n$ contains no singular point and hence no periodic orbit either. It follows that $z \in L^+(x) \cap L^-(x)$ for every $x \in E_n$ and therefore $E_n \subset Y$. This contradicts $\dim Y = 1$.

Corollary 4.3. *If Y is a 1-dimensional chain component and $z \in Y$ is a singular point, then the set of orbits in $Y \setminus \{z\}$ whose positive or negative limit set is $\{z\}$ is nonempty and finite.*

Proof. Suppose that there is a sequence $\{x_n : n \in \mathbf{N}\}$ in $Y \setminus \{z\}$ such that $L^+(x_n) = \{z\}$ and $C(x_n) \cap C(x_m) = \emptyset$ for every $n \neq m$. Since there are finitely many singular points, we may assume that there is a singular point $z_1 \in Y$ such that $L^-(x_n) = \{z_1\}$ for every $n \in \mathbf{N}$, by Corollary 3.2. By Theorem 4.2 there exists an isolating neighbourhood V of z in S^2. Then $C(x_n) \not\subset V$ and hence for each $n \in \mathbf{N}$ there exists a point $y_n \in C(x_n) \cap \partial V$. Since ∂V is compact, the sequence $\{y_n : n \in \mathbf{N}\}$ has a limit point $y \in \partial V$. Then y is a nonsingular point of Y and Y is not locally an arc at y. This contradicts Theorem 4.1.

Corollary 4.4. *Every 1-dimensional chain component of a flow in S^2 with finitely many singularities consists of finitely many orbits and is homeomorphic to a finite graph.*

Remark. The assumption that the flow has finitely many singularities is essential for the validity of the results of this section. For example let $z_0 = (-1,0)$, $z_\infty = (1,0)$, $z_n = (\cos(\pi/(n+1)), \sin(\pi/(n+1)))$, $n \in \mathbf{N}$ and let

$$Y = S^1 \cup [z_\infty, z_0] \cup \bigcup_{n=1}^{\infty} [z_n, z_0]$$

where S^1 is the unit circle in $\mathbf{R}^2$ and $[a,b]$ denotes the closed line segment with endpoints a, $b \in \mathbf{R}^2$ directed from a to b. Then Y is not an arc at any point of $[z_\infty, z_0]$. There is a continuous flow on $S^2 = \mathbf{R}^2 \cup \{\infty\}$ whose singular points are ∞, $(-1/2, 0)$, $(-2/3, 2/3)$, z_0, z_∞, z_n and $u_n = (1-\sin(\pi/(n+2)), \sin(\pi/(n+2)))$, $n \in \mathbf{N}$, which has the following properties:

1. The unit disc D^2 is invariant and positively asymptotically stable and $\{\infty\}$ is negatively asymptotically stable with region of attraction $\mathbf{R}^2 \setminus D^2$.
2. Every orbit in $\mathbf{R}^2 \setminus D^2$ has positive limit set $\{z_1\}$ except one whose positive limit set is $\{z_0\}$.
3. The clockwise directed open segment on S^1 from z_0 to z_1 and the counter-clockwise directed open segments on S^1 from z_0 to z_∞ and from z_{n+1} to z_n, $n \in \mathbf{N}$, are complete orbits.

4. The directed open line segments from z_∞ to z_0, from z_n to z_0 and from u_n to z_{n+1}, $n \in \mathbf{N}$, are complete orbits.
5. The positive limit set of every orbit in $D_n \setminus [u_n, z_{n+1}]$ is $\{z_0\}$, where D_n is the open "triangle" formed by $[z_n, z_0]$, $[z_{n+1}, z_0]$ and the segment on S^1 with endpoints z_n and z_{n+1}. The singular point u_n is negatively asymptotically stable with region of attraction D_n.
6. The singular point $(-1/2, 0)$ is negatively asymptotically stable with region of attraction the open lower half unit disc.
7. The singular point $(-2/3, 2/3)$ is negatively asymptotically stable with region of attraction the open area bounded by $[z_1, z_0]$ and the segment on S^1 with endpoints z_1 and z_0.

It follows from the above properties that Y is a 1-dimensional chain component of this flow.

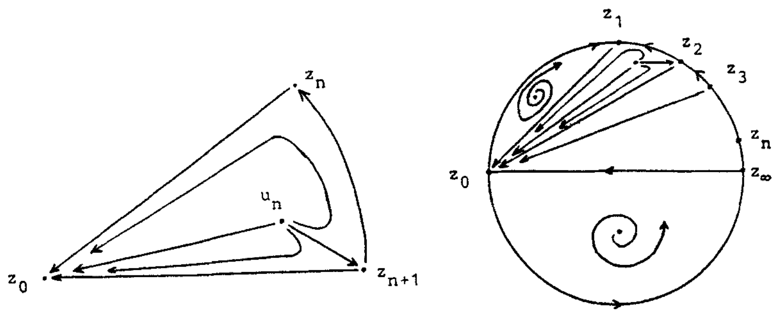

Fig. 1.

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