University of Crete
Department of Mathematics and Applied Mathematics

An introduction to Riemannian Geometry

Course notes

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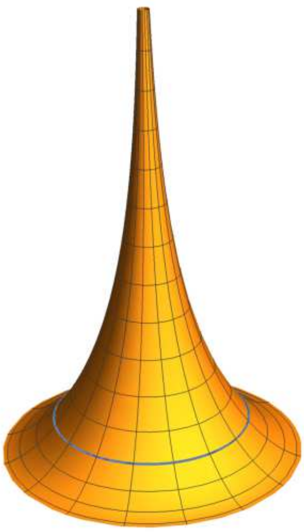

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Preface

These lecture notes correspond to the introductory graduate course on Riemannian Geometry that I have taught several times in the graduate program of the Department of Mathematics of the University of Crete. The reader is required to have a background in basic Algebra, basic Topology, Differential Calculus of functions of several variables and in the basic theory of Ordinary Differential Equations.

The first two chapters give an introduction to the basics of smooth manifolds. The next three chapters constitute the core of these notes. The third chapter is concerned with the metric space structure of Riemannian manifolds. The fourth chapter is devoted to the notion of curvature and its variants. The fifth chapter presents the elementary comparison theorems of Riemannian Geometry including the general description of spaces of constant sectional curvature. The last sixth chapter is devoted to the Riemannian volume comparison theorems and is optional. It can be taught according to the background and interests of the audience.

K. Athanassopoulos

Contents

Chapter 1

Manifolds

1.1 Topological and smooth manifolds

Problems of Classical Physics lead to the need for the development of differential and integral calculus on subsets of the phase space, like for instance level sets of constant energy, which are not open subsets of any euclidean space. Since differentiability of a function at a given point depends only on its local behaviour near the point, it is reasonable to try to develop differential calculus on topological spaces which are locally like euclidean space.

A topological space M is said to be a *topological n-manifold*, where $n \in \mathbb{Z}^+$, if it is a Hausdorff space with a countable basis for its topology and has the following property: there exists an open cover $\mathcal{U}$ of M every element of which is homeomorphic to some open subset of $\mathbb{R}^n$. Since the topology of M is assumed to have a countable basis, there exists a countable open cover $\mathcal{U}$ of M every element of which is homeomorphic to $\mathbb{R}^n$. If $U \in \mathcal{U}$, a homeomorphism $\phi : U \to \phi(U)$, where $\phi(U)$ is an open subset of $\mathbb{R}^n$, is called a *chart* of M and is usually denoted by (U, ϕ). The non-negative integer n is the *dimension* on M.

A topological manifold is a locally compact space, hence regular, and it follows from Uryshn's theorem that its topology is defined by some metric.

If now $f : M \to \mathbb{R}$ is a continuous function, it is reasonable to call f differentiable at a point $p \in M$, if there exists a chart $\phi : U \to \phi(U) \subset \mathbb{R}^n$ with $p \in U$ such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is differentiable at $\phi(p)$.

$$\begin{tikzcd}
U \arrow[r, "f"] & \mathbb{R} \\
\phi(U) \arrow[u, "\phi^{-1}"] \arrow[ur, "f\circ\phi^{-1}"'] &
\end{tikzcd}$$

However, in order such a definition to be good it must be independent of the choice of the chart ϕ. If $\psi : V \to \psi(V) \subset \mathbb{R}^n$ is another chart with $p \in V$, we have

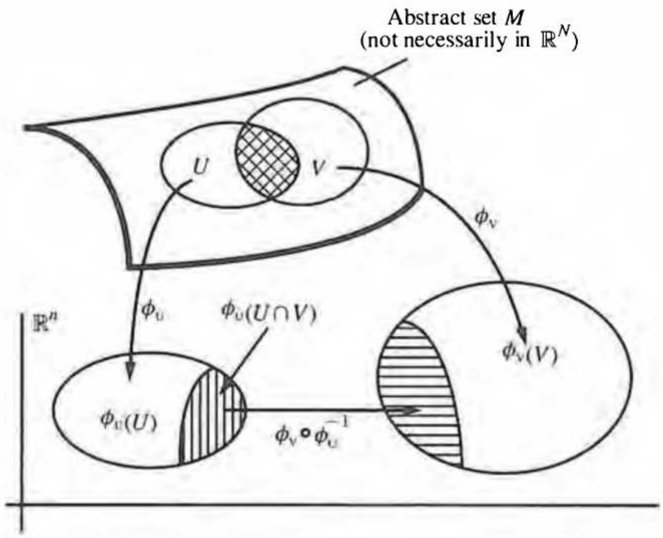
$$f \circ \phi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}).$$

Therefore, in order the differentiability of $f \circ \phi^{-1}$ at $\phi(p)$ to be equivalent to that of $f \circ \psi^{-1}$ at $\psi(p)$ it suffices $\psi \circ \phi^{-1}$ to be differentiable at $\phi(p)$ and $\phi \circ \psi^{-1}$ to be differentiable at $\psi(p)$. We are thus led to the following.

Definition 1.1.1. Two charts (U, ϕ_U) and (V, ϕ_V) of a topological n-manifold M are called *smoothly related* if $U \cap V \neq \varnothing$ and the transition map

$$\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$$

is a smooth diffeomorphism of open subsets of $\mathbb{R}^n$.



Definition 1.1.2. A *smooth atlas* of a topological n-manifold M is a family $\mathcal{A} = \{(U, \phi_U) : U \in \mathcal{U}\}$ consisting of smoothly related charts of M such that $\mathcal{U}$ is an open cover of M.

Two smooth atlases of M are called equivalent if their union is again a smooth atlas. Evidently, this is an equivalence relation on the set of smooth atlases of M. Every smooth atlas is contained in a unique maximal smooth atlas, which is the union of all smooth atlases in its equivalence class.

Definition 1.1.3. A *smooth structure* on a topological n-manifold is a maximal smooth atlas $\mathcal{A}$ of M. In this case the couple $(M, \mathcal{A})$ is called a *smooth n-manifold*. The smooth atlas $\mathcal{A}$ is usually omitted if it is clear which one is considered. The elements of $\mathcal{A}$ are called the smooth charts of M.

It is clear from the above that a smooth structure on a topological manifold can be described by a single, not necessarily maximal, smooth atlas. So, we can describe a smooth structure by defining a smooth atlas of minimum cardinality.

Examples 1.1.4. (a) The trivial example of a smooth n-manifold is an open subset M of $\mathbb{R}^n$, whose smooth structure is defined by the smooth atlas $\mathcal{A} = \{(M, id_M)\}$.

Also, if M is a smooth manifold, then any open set $X \subset M$ is a smooth manifold. If $\mathcal{A}$ is the smooth structure of M, the smooth structure of X is

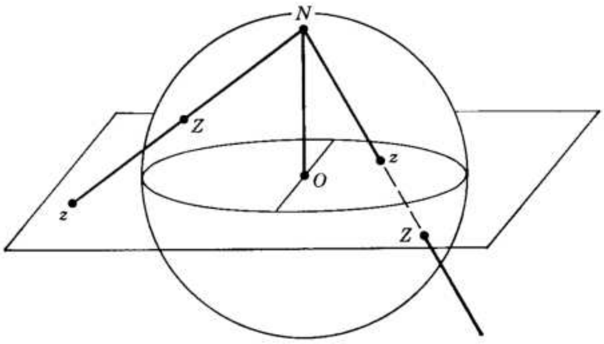
$$\mathcal{A}|_X = \{(X \cap U, \phi|_{X\cap U}) : (U, \phi) \in \mathcal{A}\}.$$

(b) The n-sphere $S^n_R = \{Z \in \mathbb{R}^{n+1} : \|Z\| = R\}$ of radius $R > 0$ is a smooth n-manifold. Its smooth structure is defined by the smooth atlas consisting of the stereographic projections with respect to the north and the south poles. More precisely, the stereographic projection with respect to the north pole is the homeomorphism $\pi_+ : S^n_R \setminus \{Re_{n+1}\} \to \mathbb{R}^n$ defined by

$$\pi_+(Z_1, ..., Z_n, Z_{n+1}) = \frac{R}{R - Z_{n+1}} \cdot (Z_1, ..., Z_n)$$

and the stereographic projection with respect to the south pole is the homeomorphism $\pi_- : S^n_R \setminus \{-Re_{n+1}\} \to \mathbb{R}^n$ defined by

$$\pi_-(Z_1, ..., Z_n, Z_{n+1}) = \frac{R}{R + Z_{n+1}} \cdot (Z_1, ..., Z_n).$$



Since the inverse π_+^{-1} is given by the formula

$$\pi_+^{-1}(z_1, ..., z_n) = \left(\frac{2R^2 z_1}{R^2 + \sum_{j=1}^n z_j^2}, ..., \frac{2R^2 z_n}{R^2 + \sum_{j=1}^n z_j^2}, \frac{R\big(-R^2 + \sum_{j=1}^n z_j^2\big)}{R^2 + \sum_{j=1}^n z_j^2} \right),$$

the transition map $\pi_- \circ \pi_+^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by

$$(\pi_- \circ \pi_+^{-1})(z) = \frac{R^2}{\|z\|^2} \cdot z.$$

In other words, $\pi_- \circ \pi_+^{-1}$ is the inversion with respect to S^{n-1}_R and is of course a smooth diffeomorphism. The standard smooth structure of S^n_R is defined by the smooth atlas $\mathcal{A} = \{(S^n_R \setminus \{Re_{n+1}\}, \pi_+), (S^n_R \setminus \{-Re_{n+1}\}, \pi_-)\}$. In case $R = 1$, we usually write S^n instead of S^n_1.

(c) If M_1 is a smooth n_1-manifold and M_2 is a smooth n_2-manifold, then their product $M_1 \times M_2$ is a smooth $(n_1 + n_2)$-manifold. Indeed, if $\mathcal{A}_j$ is a smooth atlas of M_j, $j = 1, 2$, then

$$\mathcal{A} = \{(U \times V, \phi \times \psi) : (U, \phi) \in \mathcal{A}_1, \quad (V, \psi) \in \mathcal{A}_2\}$$

is a smooth atlas of $M_1 \times M_2$.

In particular, the n-dimensional torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times) is a smooth n-manifold.

(d) The *complex projective space* $\mathbb{C}P^n$, $n \in \mathbb{Z}^+$, is the quotient space of the equivalence relation $\sim$ in $\mathbb{C}^{n+1} \setminus \{0\}$ such that $z \sim w$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ with $w = \lambda z$. In other words, the equivalence classes of $\sim$ are the complex 1-dimensional linear subspaces of $\mathbb{C}^{n+1}$ minus $0 \in \mathbb{C}^{n+1}$. Alternatively, $\mathbb{C}P^n$, can be defined as the quotient space of the equivalence relation $\sim$ on S^{2n+1} such that $z \sim w$ if and only if there exists $\lambda \in S^1$ with $w = \lambda z$. Thus, $\mathbb{C}P^n$ is the orbit space of the continuous action of the unit circle S^1 on the $(2n+1)$-sphere S^{2n+1} by scalar multiplication, whose orbits are great circles. The quotient map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a continuous, open, surjection and is called the *Hopf map*. We usually write $\pi(z_0, z_1, ..., z_n) = [z_0, z_1, ..., z_n]$ and call the complex numbers z_0, z_1,..., z_n the homogeneous coordinates od the point $[z_0, z_1, ..., z_n] \in \mathbb{C}P^n$. Obviously, $[z_0, z_1, ..., z_n] = [w_0, w_1, ..., w_n]$ if and only if

$$\begin{vmatrix} z_j & w_j \\ z_k & w_k \end{vmatrix} = 0$$

for every $j,\ k = 0, 1, ..., n$.

If $[z_0, z_1, ..., z_n] \neq [w_0, w_1, ..., w_n]$, there exist $0 \leq j, k \leq n$ such that $z_j w_k \neq z_k w_j$. The sets

$$U = \{[u_0, u_1, ..., u_n] \in \mathbb{C}P^n : |u_k z_j - u_j z_k| < |u_k w_j - u_j w_k|\},$$

$$W = \{[u_0, u_1, ..., u_n] \in \mathbb{C}P^n : |u_k z_j - u_j z_k| > |u_k w_j - u_j w_k|\}$$

are open, disjoint and $[z_0, z_1, ..., z_n] \in U$, $[w_0, w_1, ..., w_n] \in W$. This shows that $\mathbb{C}P^n$ is a Hausdorff space. Since the Hopf map is a continuous, open surjection, $\mathbb{C}P^n$ is a connected, compact space with a countable basis for its topology, hence metrizable.

For every integer $0 \leq k \leq n$ the set

$$U_k = \{[z_0, z_1, ..., z_n] \in \mathbb{C}P^n : z_k \neq 0\}$$

is open and the map $\phi_k : U_k \to \mathbb{C}^n$ with

$$\phi_k([z_0, z_1, ..., z_n]) = (\frac{z_0}{z_k}, ..., \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, ..., \frac{z_n}{z_k})$$

is a homeomorphism whose inverse is given by

$$\phi_k^{-1}(t_1, ..., t_n) = [t_1, ..., t_{k-1}, 1, t_k, ..., t_n].$$

Thus, $\mathbb{C}P^n$ is a topological $2n$-manifold, since

$$\mathbb{C}P^n = U_0 \cup U_1 \cup \cdots \cup U_n.$$

Moreover, if $U_j \cap U_k \neq \varnothing$ and $j \neq k$, then

$$\phi_k(U_j \cap U_k) = \begin{cases} \{(t_1, ..t_n) \in \mathbb{C}^n : t_j \neq 0\} & \text{if } j < k \\ \{(t_1, ..t_n) \in \mathbb{C}^n : t_{j-1} \neq 0\} & \text{if } j > k. \end{cases}$$

Thus, for $j < k$ we have

$$(\phi_j \circ \phi_k^{-1})(t_1, ..., t_n) = (\frac{t_1}{t_j}, ..., \frac{t_{j-1}}{t_j}, \frac{t_{j+1}}{t_j}, ..., \frac{t_{k-1}}{t_j}, \frac{1}{t_j}, \frac{t_k}{t_j}, ..., \frac{t_n}{t_j})$$

and for $j > k$ we have

$$(\phi_j \circ \phi_k^{-1})(t_1, ..., t_n) = (\frac{t_1}{t_{j-1}}, ..., \frac{t_{k-1}}{t_{j-1}}, \frac{1}{t_{j-1}}, \frac{t_k}{t_{j-1}}, ..., \frac{t_{j-2}}{t_{j-1}}, \frac{t_j}{t_{j-1}}, , ..., \frac{t_n}{t_{j-1}}).$$

So, $\mathcal{A} = \{(U_k, \phi_k) : k = 0, 1, ...n\}$ is a smooth atlas which defines a smooth structure and is called the canonical atlas of $\mathbb{C}P^n$.

(e) The *real projective space* $\mathbb{R}P^n$, $n \in \mathbb{Z}^+$, is defined in the same way simply by replacing the field $\mathbb{C}$ with the field $\mathbb{R}$. Now $\mathbb{R}P^n$ is the quotient space of the equivalence relation $\sim$ on S^n such that $x \sim -x$ for every $x \in S^n$. Again $\mathbb{R}P^n$ is a connected, compact metrizable space and a smooth n-manifold.

Definition 1.1.5. Let M be a smooth m-manifold and N be a smooth n-manifold. A continuous map $f : M \to N$ is clalled *smooth* if for every $p \in M$ there exist a smooth chart (U, ϕ) of M and smooth chart (V, ψ) of N such that $p \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is a smooth map of open subsets of euclidean spaces. We call $\psi \circ f \circ \phi^{-1}$ the local representation of f with respect to the smooth charts (U, ϕ) and (V, ψ).

The sbove definition is independent of the choice of the smooth charts (U, ϕ) and (V, ψ), because if (U_1, ϕ_1) and (V_1, ψ_1) is another pair of smooth charts with $p \in U_1$ and $f(U_1) \subset V_1$, then

$$\psi_1 \circ f \circ \phi_1^{-1} = (\psi_1 \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \phi_1^{-1})$$

and thus $\psi \circ f \circ \phi^{-1}$ is smooth if and only if $\psi_1 \circ f \circ \phi_1^{-1}$.

The class of smooth manifolds are the objects of a category whose morphisms are the smooth maps between smooth manifolds. The isomorphisms in the category are called diffeomorphisms. More precisely, a smooth map $f : M \to N$ as in Definition 1.1.5 is called a *smooth diffeomorphism* if there exists a smooth map $g : N \to M$ such that $g \circ f = id_M$ and $f \circ g = id_N$.

Definition 1.1.6. Two smooth manifolds M and N are called *(smoothly) diffeomorphic* if there exists a smooth diffeomorphism $f : M \to N$.

Obviously, two diffeomorphic manifolds must have the same dimension. If (U, ϕ) is a smooth chart of a smooth manifold M, then $\phi : U \to \phi(U)$ is a smooth diffeomorphism.

It is not true in general that any topological manifold admits a smooth structure. Also, a topological manifold may carry many non-diffeomorphic smooth structures (with the same underlying topology). J. Milnor proved in 1956 that on the 7-sphere S^7 there are non-diffeomorphic smooth structures. His work was the birth of Differential Topology. In 1982 S. Donaldson showed that already on $\mathbb{R}^4$ there exist uncountably many non-diffeomorphic smooth structures. On any topological n-manifold for $n = 1, 2, 3$ there exists a unique up to diffeomorphism smooth structure. In dimension 1 this is easy to prove. In dimension 2 this follows from the classification of topological surfaces and the uniformization theorem. In dimension 3 it was proved by J. Munkres in 1960. In both cases of dimensions 2 and 3 an important step in the proof is the non-trivial fact that topological 2- and 3-manifolds can be triangulated.

1.2 The tangent space

In order to define the derivative of a smooth map between manifolds, we shall describe the derivative of a map defined on a open subset of euclidean space in a suitable way that it can be carried over to smooth manifolds.

Let $A \subset \mathbb{R}^n$ be an open set and $p = (p^1, ..., p^n) \in A$. We denote by $C^\infty(A, p)$ the set of smooth real functions defined on some open neighbourhood of p contained in A. Let also

$$S(A, p) = \{\gamma | \gamma : (-\epsilon, \epsilon) \to A \quad \text{is smooth for some} \quad \epsilon > 0, \quad \text{with} \quad \gamma(0) = p\}.$$

Two curves γ_1, $\gamma_2 \in S(A, p)$ are tangent at p if and only if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every $f \in C^\infty(A, p)$. Tangency at p is an equivalence relation $\sim_p$ on $S(A, p)$. The quotient set $T_pA = S(A, p)/ \sim_p$ is called the *tangent space* of A at p and carries a vector space structure which is defined as follows. If $[\gamma_1]_p$, $[\gamma_2]_p \in T_pA$ and λ_1, $\lambda_2 \in \mathbb{R}$, then $\lambda_1[\gamma_1]_p + \lambda_2[\gamma_2]_p$ is the element of T_pA represented by

$$\gamma(t) = \lambda_1\gamma_1(t) + \lambda_2\gamma_2(t) - (\lambda_1 + \lambda_2 - 1)p.$$

The zero element of T_pA is represented by the constant curve at p. The elements of T_pA are called *tangent vectors* of A at p. If $\gamma_j(t) = p + te_j$, $j = 1, 2, ..., n$, then $\{[\gamma_1]_p, [\gamma_2]_p, ...[\gamma_n]_p\}$ is a basis of T_pA.

We shall give an alternative "algebraic" description of the tangent space. To every tangent vector $[\gamma]_p \in T_pA$ corresponds a linear operator $D_{[\gamma]_p} : C^\infty(A, p) \to \mathbb{R}$ which is defined by

$$D_{[\gamma]_p}(f) = (f \circ \gamma)'(0).$$

This is a fancy way to consider the directional derivative with respect to the velocity of γ at p. Recall that two functions f, $g \in C^\infty(A, p)$ are said to define the same germ at p if they agree on some small neighbourhood of p and this is an equivalence relation on $C^\infty(A, p)$ whose classes are called the germs of smooth functions at p. Note that if two functions f, $g \in C^\infty(A, p)$ define the same germ at p, then $D_{[\gamma]_p}(f) = D_{[\gamma]_p}(g)$.

The set $\mathcal{G}_p$ of germs of smooth functions at p can be endowed with the structure of a commutative, associative real algebra with a unity in the obvious way. The unity is the germ of the constant function with value 1. It is evident now that to every tangent vector $[\gamma]_p \in T_pA$ corresponds a linear operator $D_{[\gamma]_p} : \mathcal{G}_p \to \mathbb{R}$, as above, and this correspondence is injective by definition. Moreover, it satisfies the Leibniz rule for the derivative of a product of functions. Thus, we are led to the following.

Definition 1.2.1. A *derivation* on the algebra $\mathcal{G}_p$ of germs of smooth functions at p is a linear operator $D : \mathcal{G}_p \to \mathbb{R}$ which satisfies the Leibniz rule

$$D(\alpha \cdot \beta) = e_p(\beta)D(a) + e_p(\alpha)D(\beta)$$

for every α, $\beta \in \mathcal{G}_p$, where $e_p : \mathcal{G}_p \to \mathbb{R}$ denotes the evaluation at p.

A derivation of $\mathcal{G}_p$ vanishes on the germs of constant functions, because

$$D(1) = D(1 \cdot 1) = 1 \cdot D(1) + 1 \cdot D(1) = 2D(1).$$

The set T_p of all derivations of $\mathcal{G}_p$ is obviously a linear subspace of the algebraic dual of the vector space $\mathcal{G}_p$ and the map $F : T_pA \to T_p$ defined by

$$F([\gamma]_p) = D_{[\gamma]_p}$$

is a linear monomorphism, because if $D_{j,p} = F([\gamma_j]_p)$, then

$$D_{j,p}(f) = \frac{\partial f}{\partial x^j}(p)$$

for $j = 1, 2, ..., n$ and the set $\{D_{1,p}, D_{2,p}, ..., D_{n,p}\}$ is linearly independent, since

$$D_{j,p}(x^k) = \delta_{jk}$$

where $x^k : \mathbb{R}^n \to \mathbb{R}$ denotes the projection onto the k-th coordinate.

It is a non-trivial fact that F is actually a linear isomorphism. Its proof is based on the following lemma from advanced calculus.

Lemma 1.2.2. *For every $f \in C^\infty(A, p)$ there exist g_1,..., $g_n \in C^\infty(A, p)$ and a convex open neighbourhood W of p such that*

$$f(x) = f(p) + \sum_{k=1}^{n} (x^k - p^k)g_k(x)$$

for every $x = (x^1, ..., x^n) \in W$, and

$$g_k(p) = \frac{\partial f}{\partial x^k}(p)$$

for every $k = 1, 2, ..., n$.

Proof. Let W be any convex open neighbourhood of p on which f is defined and let

$$g_k(x) = \int_0^1 \frac{\partial f}{\partial x^k}(tx + (1-t)p)dt$$

for every $x = (x^1, ..., x^n) \in W$ and $k = 1, 2, ...n$. From the Fundamental Theorem of Calculus and the chain rule we have

$$f(x) - f(p) = \int_0^1 \frac{d}{dt}(f(tx + (1-t)p))dt$$

$$= \int_0^1 \left[\sum_{k=1}^n (x^k - p^k) \frac{\partial f}{\partial x^k}(tx + (1-t)p) \right] dt = \sum_{k=1}^n (x^k - p^k) g_k(x).$$

The rest is obvious. $\square$

Proposition 1.2.3. *The set $\{D_{1,p}, D_{2,p}, ..., D_{n,p}\}$ is a basis of T_p and therefore F is a linear isomorphism.*

Proof. It suffices to prove that $\{D_{1,p}, D_{2,p}, ..., D_{n,p}\}$ generates T_p. Let $D \in T_p$ and $a_k = D(x^k)$, $k = 1, 2, ..., n$. For every $f \in C^\infty(A, p)$ we apply Lemma 1.2.2 and then we have

$$D(f) = D(f(p)) + \sum_{k=1}^n D((x^k - x^k(p))g_k) = \sum_{k=1}^n D(x^k) g_k(p) + \sum_{k=1}^n (x^k(p) - x^k(p)) D(g)$$
$$= \sum_{k=1}^n a_k \frac{\partial f}{\partial x^k}(p) = \left(\sum_{k=1}^n a_k D_{k,p} \right)(f). \quad \square$$

Thus, henceforth we shall identify the linear space T_p with T_pA.

Let now $f = (f_1, f_2, ..., f_m) : A \to \mathbb{R}^m$ be a smooth map. The linear map $f_* : T_pA \to T_{f(p)}\mathbb{R}^m$ defined by

$$f_*([\gamma]_p) = [f \circ \gamma]_{f(p)}$$

is just the derivative of f at p, since $(f \circ \gamma)'(0) = Df(p) \cdot \gamma'(0)$ for every $\gamma \in S(A, p)$. This is a convenient way to consider the derivative of a smooth function that can be carried over to smooth manifolds.

Let M be a smooth n-manifold and $p \in M$. We can define

$$S(M, p) = \{\gamma | \gamma : (-\epsilon, \epsilon) \to M \quad \text{is smooth for some} \quad \epsilon > 0, \quad \text{with} \quad \gamma(0) = p\}$$

and consider the set $C^\infty(M, p)$ of smooth real functions defined on some open neighbourhood of p in M. As before we call γ_1, $\gamma_2 \in S(M, p)$ tangent at p if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every $f \in C^\infty(M, p)$ and define the *tangent space* T_pM of M at p to be the quotient set of this equivalence relation. Let (U, ϕ_U) be a smooth chart of M such that $p \in U$. The map $\widetilde{\phi_U} : T_pM \to T_{\phi_U(p)}\phi(U)$ defined by $\widetilde{\phi_U}([\gamma]_p) = [\phi_U \circ \gamma]_{\phi_U(p)}$ is a bijection whose inverse is given by $\widetilde{\phi_U}^{-1}([\zeta]_{\phi_U(p)}) = [\phi_U^{-1} \circ \zeta]_p$. We transfer the vector space structure of $T_{\phi_U(p)}\phi_U(U)$ to T_pM so that $\widetilde{\phi_U}$ becomes a linear isomorphism. This vector space structure does not depend on the choice of the smooth chart (U, ϕ_U), because if (V, ϕ_V) is another smooth chart of M with $p \in V$, then $\widetilde{\phi_U} \circ \widetilde{\phi_V}^{-1} = (\phi_U \circ \phi_V^{-1})_{*\phi_V(p)}$ is a linear isomorphism, since it is the derivative at $\phi_V(p)$ of the transition map $\phi_U \circ \phi_V^{-1}$, which is a smooth diffeomorphism.

$$\begin{tikzcd}
T_pM \arrow[r, "id"] \arrow[d, "\widetilde{\phi_V}"'] & T_pM \arrow[d, "\widetilde{\phi_U}"] \\
T_{\phi_V(p)}\phi_V(V) \arrow[r, "(\phi_U \circ \phi_V^{-1})_{*\phi_V(p)}"] & T_{\phi_U(p)}\phi_U(U)
\end{tikzcd}$$

The elements of the tangent space T_pM are called *tangent vectors* of M at the point p. From the above discussion, the tangent vectors of M at p can be considered as derivations of the algebra of germs $\mathcal{G}_p(M)$ of real smooth functions defined on some open neighbourhood of p in M. If (U, ϕ_U) is a smooth chart of M, where $\phi_U = (x^1, x^2, ..., x^n)$, and

$$\left(\frac{\partial}{\partial x^j}\right)_p = \widetilde{\phi_U}^{-1}(D_{j,\phi_U(p)})$$

for $j = 1, 2, ..., n$, then the set of tangent vectors

$$\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, ..., \left(\frac{\partial}{\partial x^n}\right)_p\right\}$$

is a basis of T_pM which depends on ϕ_U and is called the *canonical basis* of T_pM with respect to the chart ϕ_U.

If now $f : M \to P$ is a smooth map into a smooth m-manifold P, the *derivative* of f at the point $p \in M$ is defined to be the linear map $f_{*p} : T_pM \to T_{f(p)}P$ with

$$f_{*p}([\gamma]_p) = [f \circ \gamma]_{f)p)}$$

for every $[\gamma]_p \in T_pM$. In particular, $\widetilde{\phi_U} = (\phi_U)_{*p}$ by definition.

Let (U, ϕ) be a smooth chart of M with $p \in U$ and (W, ψ) be a smooth chart of P with $f(U) \subset W$. If $\phi = (x^1, x^2, ..., x^n)$ and $\psi = (y^1, y^2, ..., y^m)$, then

$$\psi_{*f(p)}\left(f_{*p}\left(\left(\frac{\partial}{\partial x^j}\right)_p\right)\right) = (\psi \circ f \circ \phi^{-1})_{*\phi(p)}(D_{j,\phi(p)})$$

for $j = 1, 2, ..., n$ and therefore the matrix of f_{*p} with respect to the ordered basis

$$\left[\left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, ..., \left(\frac{\partial}{\partial x^n}\right)_p\right]$$

of T_pM and

$$\left[\left(\frac{\partial}{\partial y^1}\right)_p, \left(\frac{\partial}{\partial y^2}\right)_p, ..., \left(\frac{\partial}{\partial y^m}\right)_p\right]$$

of $T_{f(p)}P$ is the Jacobian matrix at $\phi(p)$ of the local representation $\psi \circ f \circ \phi^{-1}$ of f.

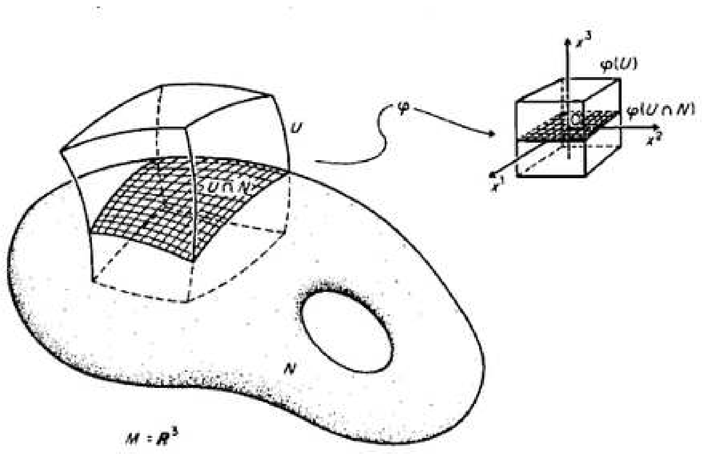
$$\begin{tikzcd}
T_pM \arrow[r, "f_{*p}"] \arrow[d, "\phi_{*p}"'] & T_{f(p)}P \arrow[d, "\psi_{*f(p)}"] \\
T_{\phi(p)}\phi(U) \arrow[r, "(\psi\circ f\circ\phi^{-1})_{*\phi(p)}"] & T_{\psi(f(p))}\psi(W)
\end{tikzcd}$$

1.3 Submanifolds

Let M be a smooth m-manifold and $0 \leq n \leq m$ be an integer. A set $N \subset M$ is said to be a *(regular or embedded) n-dimensional smooth submanifold* of M if for every $p \in N$ there exists smooth chart (U, ϕ) of M such that $p \in N$ and

$$\phi(N \cap U) = Q \cap (\mathbb{R}^n \times \{0\})$$

for some open set $Q \subset \mathbb{R}^m$. The smooth chart (U, ϕ) of M is said to be *N-straightening.*



If we denote by $\pi : \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ the projection onto the first n coordinates and by $i : \mathbb{R}^n \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^m$ the inclusion, then the map

$$(\pi \circ |_{N \cap U})^{-1} = \phi^{-1} \circ i : i^{-1}(Q) \to M$$

is smooth and is usually called local parametrization of N.

Obviously, a n-dimensional smooth submanifold N of M is a topological n-manifold, with respect to the subspace topology which it inherits from M. Moreover,

$$\mathcal{A}|_N = \{(N \cap U, \pi \circ \phi|_{N \cap U}) : (U, \phi) \quad \text{is a } N\text{-straightening smooth chart of } M\}$$

is a smooth atlas of N. If (U, ϕ) and (V, ψ) are two N-straightening smooth charts of M with $N \cap U \cap V \neq \varnothing$, the transition map of the corresponding elements of $\mathcal{A}|_N$ is $\pi \circ (\psi \circ \phi^{-1}) \circ i$ defined on an open subset of $\mathbb{R}^n$. Thus N becomes a smooth n-manifold.

The local representation of the inclusion $i_N : N \hookrightarrow M$ with respect to a N-straightening smooth chart (U, ϕ) of M and the corresponding smooth chart of N in $\mathcal{A}|_N$, as above, is

$$\phi \circ i_N \circ (\pi \circ \phi|_{N \cap U})^{-1} = i|_{i^{-1}(Q)} : i^{-1}(Q) \to \mathbb{R}^m.$$

Therefore, i_N is smooth and its derivative at every point of N is a linear monomorphism. Generalizing, we give the following.

Definition 1.3.1. Let N be a smooth n-manifold and M be a smooth m-manifold, with $n \leq m$. A smooth map $f : N \to M$ is called *immersion* if its derivative $f_{*q} : T_qN \to T_{f(q)}M$ is a linear monomorphism for every $q \in N$. If moreover f is a topological embedding, then f is called a *smooth embedding*.

Perhaps the most important examples of submanifolds are the level sets of smooth maps. Conditions which ensure that this kind of subsets of a given smooth manifold are smooth submanifolds are provided from the Implicit Function Theorem or the more general Constant Rank Theorem of advanced calculus, which we shall prove as a consequence of the Inverse Map Theorem.

Theorem 1.3.2. *Let $A \subset \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}^m$ be a smooth map. If $p \in A$ and the Jacobian matrix $Df(x)$ has constant rank k for every x in some open neighbourhood of p in A, then there exist an open neighbourhood $U \subset A$ of p and a smooth diffeomorphism $\phi : U \to \phi(U)$ onto an open set $\phi(U) \subset \mathbb{R}^n$, and an open neighbourhood V of $f(p)$ and a smooth diffeomorphism $\psi : V \to \psi(V)$ onto an open set $\psi(V) \subset \mathbb{R}^m$ such that the smooth map*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \subset \mathbb{R}^m$$

is given by the formula

$$(\psi \circ f \circ \phi^{-1})(x^1, ..., x^k, x^{k+1}, ..., x^n) = (x^1, ..., x^k, 0, ..., 0)$$

for every $(x^1, .., x^n) \in \phi(U)$.

Proof. Up to translations and linear isomorphisms of $\mathbb{R}^n$ and $\mathbb{R}^m$, which are of course diffeomorphisms, we may assume that $p = 0$, $f(p) = 0$ and

$$\begin{vmatrix} \frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^k} \\ \frac{\partial f_2}{\partial x^1} & \cdots & \frac{\partial f_2}{\partial x^k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x^1} & \cdots & \frac{\partial f_k}{\partial x^k} \end{vmatrix} \neq 0$$

on an open neighbourhood $A_0 \subset A$ of 0, where $f = (f_1, ..., f_k, f_{k+1}, ..., f_m)$.

We consider the smooth map $F : A_0 \to \mathbb{R}^n$ defined by

$$F(x^1, ..., x^n) = (f_1(x^1, ..., x^n), ..., f_k(x^1, ..., x^n), x^{k+1}, ..., x^n).$$

Then, $F(0) = 0$ and

$$\det DF(0) = \begin{vmatrix} \frac{\partial f_1}{\partial x^1}(0) & \cdots & \frac{\partial f_1}{\partial x^k}(0) \\ \frac{\partial f_2}{\partial x^1}(0) & \cdots & \frac{\partial f_2}{\partial x^k}(0) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x^1}(0) & \cdots & \frac{\partial f_k}{\partial x^k}(0) \end{vmatrix} \neq 0.$$

Applying the Inverse Map Theorem, there exist an open neighbourhood $U_0 \subset A_0$ of 0 such that $F(U_0)$ is an open subset of $\mathbb{R}^n$ and $\phi = F|_{U_0}$ is a smooth diffeomorphism.

Shrinking, we can take U_0 such that $\phi(U_0)$ is an open cube in $\mathbb{R}^n$ with center 0. Now there exist smooth functions g_{k+1},..., $g_m : \phi(U_0) \to \mathbb{R}$ such that

$$(f \circ \phi^{-1})(z^1, ..., z^n) = (z^1, ..., z^k, g_{k+1}(z^1, ..., z^n), ..., g_m(z^1, ..., z^n))$$

for every $(z^1, ..., z^n) \in \phi(U_0)$ and $g_{k+1}(0) = \cdots = g_m(0) = 0$. Moreover,

$$Df(\phi^{-1}(z)) \cdot D(\phi^{-1})(z) = D(f \circ \phi^{-1})(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ * & * & \cdots & * & \frac{\partial g_{k+1}}{\partial x^{k+1}}(z) & \cdots & \frac{\partial g_{k+1}}{\partial x^n}(z) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & \frac{\partial g_m}{\partial x^{k+1}}(z) & \cdots & \frac{\partial g_m}{\partial x^n}(z) \end{pmatrix}$$

for every $z = (z^1, ..., z^n) \in \phi(U_0)$. Since $Df(\phi^{-1}(z))$ has constant rank k and $D(\phi^{-1})(z)$ is invertible for every $z = (z^1, ..., z^n) \in \phi(U_0)$, we must have

$$\frac{\partial g_j}{\partial x^l} = 0$$

on $\phi(U_0)$ for every $j = k+1, ..., m$ and $l = k+1, ..., n$. This implies that the smooth functions g_{k+1},..., g_m do not depend on the variables x^{k+1},.., x^n and descent to smooth functions (again denoted by) g_{k+1},..., $g_m : P \to \mathbb{R}$, where the open cube $P \subset \mathbb{R}^k$ is the image of $\phi(U_0)$ under the projection onto the first k coordinates.

If now $\psi : P \times \mathbb{R}^{m-k} \to \mathbb{R}^m$ is the smooth map defined by

$$\psi(y^1, ..., y^m) = (y^1, ..., y^k, y^{k+1} - g_{k+1}(y^1, ..., y^k), ..., y^m - g_m(y^1, ..., y^k)),$$

then

$$D\psi(0) = \begin{pmatrix} I_k & 0 \\ * & I_{m-k} \end{pmatrix}$$

and by the Inverse Map Theorem there exists an open neighbourhood V of 0 in $\mathbb{R}^m$ such that $\psi(V)$ is an open neighbourhood of $\psi(0) = 0$ and $\psi|_V$ is a smooth diffeomorphism. Let $U \subset U_0$ be an open neighbourhood of 0 such that $f(U) \subset V$. Then,

$$(\psi \circ f \circ \phi^{-1})(z^1, ..., z^k, z^{k+1}, ..., z^n) = (z^1, ..., z^k, 0, ..., 0)$$

for every $(z^1, .., z^n) \in \phi(U)$. □

Corollary 1.3.3. *Let N be a smooth n-manifold, M be a smooth m-manifold, with $n \leq m$, and let $f : N \to M$ be an immersion. Then, for every $p \in N$ there exist a smooth chart (U, ϕ) of N with $p \in U$ and a smooth chart (V, ψ) of M with $f(U) \subset V$ such that the corresponding local representation of f is*

$$(\psi \circ f \circ \phi^{-1})(x^1, ..., x^n) = (x^1, ..., x^n, 0, ..., 0). \quad \square$$

Corollary 1.3.4. *Let N be a smooth n-manifold and M be a smooth m-manifold, with $n \leq m$. If $f : N \to M$ is a smooth embedding, then $f(N)$ is a n-dimensional* *smooth submanifold of M.* □

Let M be a smooth m-manifold, P be a smooth n-manifold, with $n \leq m$, and let $f : M \to P$ be a smooth map. We call $p \in M$ a *critical point* of f if the derivative $f_{*p} : T_pM \to T_{f(p)}P$ is not a linear epimorphism. Note that if $p \in M$ is a non-critical point of f, then f_{*q} has constant maximal rank n for every point q in some open neighbourhood of p in M. A point $c \in P$ is called a *regular value* of f if the level set $f^{-1}(c)$ does not contain any critical point of f.

Corollary 1.3.5. *Let M be a smooth m-manifold, P be a smooth n-manifold, with $n \leq m$, and let $f : M \to P$ be a smooth map. If $c \in P$ is a regular value of f, then the level set $f^{-1}(c)$ is a $(m-n)$-dimensional smooth submanifold of M, if non-empty.*

Proof. By Theorem 1.3.2, for every point $p \in f^{-1}(c)$ there exists a smooth chart (U, ϕ) of M with $p \in U$ and a smooth chart (V, ψ) of P with $f(U) \subset V$ such that the corresponding local representation of f is

$$(\psi \circ f \circ \phi^{-1})(x^1, ..., x^m) = (x^1, ..., x^n)$$

for every $(x^1, .., x^m) \in \phi(U)$. Now we have

$$\phi(f^{-1}(c) \cap U) = \phi(U) \cap (\{\psi(c)\} \times \mathbb{R}^{m-n})$$

and therefore (U, ϕ) is a $f^{-1}(c)$-straightening chart of M. □

Definition 1.3.6. Let M be a smooth m-manifold and P be a smooth n-manifold, with $n \leq m$. A smooth map $f : M \to P$ onto P is called *submersion* if its derivative $f_{*p} : T_pM \to T_{f(p)}P$ is a linear epimorphism for every $p \in M$.

Thus, if $f : M \to P$ is a submersion, then $f^{-1}(c)$ is a $(m-n)$-dimensional smooth submanifold of M for every $c \in P$.

Example 1.3.7. The determinant is a smooth function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ and the general linear group $GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$ is an open subset of $\mathbb{R}^{n\times n}$. Let $A \in GL(n, \mathbb{R})$ and $\gamma(t) = (1 + t)A$. Then, $\gamma(0) = A$ and

$$(\det)_{*A}([\gamma]_A) = [\det \circ \gamma]_{\det A}.$$

Also, $(\det \circ \gamma)(t) = (1 + t)^n \det A$, and so $(\det \circ \gamma)'(0) = n \det A \neq 0$. This implies that $(\det)_{*A}$ is non-zero, and hence an epimorphism. This shows that $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is a submersion. In particular, the special linear group $SL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$ is a $(n^2 - 1)$-dimensional smooth submanifold of $\mathbb{R}^{n\times n}$.

1.4 Smooth partitions of unity

Our requirement a smooth manifold to have a countable basis for its topology implies the existence of technically very useful families of smooth functions, the construction of which will be the subject of this section.

Definition 1.4.1. Let M be a smooth manifold and let $\mathcal{U}$ be an open cover of M. A *smooth partition of unity* subordinated to $\mathcal{U}$ is a family of smooth functions $f_U : M \to [0,1]$, $U \in \mathcal{U}$, with the following properties:
(i) $\mathrm{supp} f_U = \overline{\{p \in M : f_U(p) \neq 0\}} \subset U$ for every $U \in \mathcal{U}$.
(ii) The family $\{\mathrm{supp} f_U : U \in \mathcal{U}\}$ of closed subsets of M is a locally finite cover of M.
(iii) $\sum_{U \in \mathcal{U}} f_U(p) = 1$ for every $p \in M$.

Recall that a family $\mathcal{F}$ of subsets of a topological space X is called *locally finite* if every point $x \in X$ has an open neighbourhood V in X such that the set

$$\{F \in \mathcal{F} : F \cap V \neq \varnothing\}$$

is finite. A family $\mathcal{S}$ of subsets of X is called a *refinement* of $\mathcal{F}$ if for every $F \in \mathcal{F}$ there exists some $S \in \mathcal{S}$ such that $S \subset F$.

In order to prove the existence of smooth partitions of unity we shall need some preliminary lemmas. In the sequel we shall denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$.

Lemma 1.4.2. *For every $0 < \rho < r$ there exists a smooth function $f : \mathbb{R}^n \to [0,1]$ such that $\overline{B(0,\rho)} \subset f^{-1}(1)$ and $\mathbb{R}^n \setminus B(0,r) \subset f^{-1}(0)$.*

Proof. It suffices to consider the smooth function $g : \mathbb{R} \to \mathbb{R}$ with

$$g(t) = \begin{cases} e^{-\frac{1}{t}}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0 \end{cases}$$

and take $f : \mathbb{R}^n \to [0,1]$ defined by

$$f(x) = \frac{g(r^2 - \|x\|^2)}{g(r^2 - \|x\|^2) + g(\|x\|^2 - \rho^2)}. \quad \square$$

Functions like f in Lemma 1.4.2 are usually called bump functions.

Lemma 1.4.3. *Let M be a smooth n-manifold and let $\mathcal{U}$ be an open cover of M. There exists a countable smooth atlas $\mathcal{A}$ of M with the following properties:*
(a) The open cover $\mathcal{V} = \{V : (V, \phi_V) \in \mathcal{A}\}$ is a locally finite refinement of $\mathcal{U}$.
(b) $\phi_V(V) = B(0,3) \subset \mathbb{R}^n$, for every $(V, \phi_V) \in \mathcal{A}$.
(c) $\{\phi_V^{-1}(B(0,1)) : (V, \phi_V) \in \mathcal{A}\}$ is an open cover of M.

Proof. There exists a countable open cover $\{A_k : k \in \mathbb{N}\}$ of M such that $\overline{A_k} \subset A_{k+1}$ and $\overline{A_k}$ is compact for every $k \in N$, because M is locally compact and its topology has a countable basis. This sort of cover can be constructed inductively, starting with any countable open cover $\{C_k : k \in \mathbb{N}\}$ such that $\overline{C_k}$ is compact for every $k \in N$. First we choose any open set $A_1 \subset M$ with compact closure such that

$\overline{C_1} \subset A_1$ and once A_{k-1} has been defined we choose $A_k \subset M$ to be any open set with compact closure such that $\overline{A_{k-1}} \cup C_k \subset A_k$.

The set $\overline{A_{k+1}} \setminus A_k$ is compact and contained in the open set $A_{k+2} \setminus \overline{A_{k-1}}$. For every $p \in \overline{A_{k+1}} \setminus A_k$ there exist $U_p \in \mathcal{U}$ and a smooth chart $(V_{k,p}, \phi_{V_{k,p}})$ of M such that $p \in V_{k,p} \subset U_p \cap A_{k+2} \setminus \overline{A_{k-1}}$ and $\phi_{V_{k,p}}(V_{k,p}) = B(0,3)$ with $\phi_{V_{k,p}}(p) = 0$. By compactness of $\overline{A_{k+1}} \setminus A_k$, there exist p_1,...,$p_{m_k} \in \overline{A_{k+1}} \setminus A_k$, for some $m_k \in \mathbb{N}$, such that

$$\overline{A_{k+1}} \setminus A_k \subset \phi^{-1}_{V_{k,p_1}}(B(0,1)) \cup \cdots \cup \phi^{-1}_{V_{k,p_{m_k}}}(B(0,1)).$$

It suffices now to take

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \{(V_{k,p_1}, \phi_{V_{k,p_1}}), ..., (V_{k,p_{m_k}}, \phi_{V_{k,p_{m_k}}})\}. \quad \square$$

Theorem 1.4.4. *If M is a smooth n-manifold and $\mathcal{U}$ is an open cover of M, then there exists a smooth partition of unity subordinated to $\mathcal{U}$.*

Proof. Let $\mathcal{A}$ be the smooth atlas of M provided by Lemma 1.4.3. By Lemma 1.4.2, there exists a smooth function $f : \mathbb{R}^n \to [0,1]$ such that $\overline{B(0,1)} \subset f^{-1}(1)$ and $\mathbb{R}^n \setminus B(0,2) \subset f^{-1}(0)$. For every $(V, \phi_V) \in \mathcal{A}$ we consider the smooth function $g_V : M \to [0,1]$ defined by

$$g_V(p) = \begin{cases} f(\phi_V(p)), & \text{if } p \in V, \\ 0, & \text{if } p \in M \setminus V. \end{cases}$$

According to Lemma 1.4.3, $\mathcal{V} = \{V : (V, \phi_V) \in \mathcal{A}\}$ is a locally finite open cover of M. So the function $\sum\limits_{V \in \mathcal{V}} g_V : M \to [0, +\infty)$ is well defined and smooth. Since $\mathcal{V}$ is also a refinement of $\mathcal{U}$, there exists a function $\sigma : \mathcal{V} \to \mathcal{U}$ such that $V \subset \sigma(V)$ for every $V \in \mathcal{V}$. For every $U \in \mathcal{U}$ we define now

$$f_U = \frac{1}{\sum_{V \in \mathcal{V}} g_V} \cdot \sum_{\sigma(V)=U} g_V : M \to [0,1].$$

In case $\sigma^{-1}(U) = \varnothing$ we put $f_U = 0$. It follows from Lemma 1.4.3(c) that f_U is a well defined smooth function Obviously,

$$\mathrm{supp} f_U \subset \bigcup_{\sigma(V)=U} \mathrm{supp} g_V \subset \bigcup_{\sigma(V)=U} V \subset U.$$

and $\{\mathrm{supp} f_U : U \in \mathcal{U}\}$ is locally finite, because $\mathcal{V}$ is locally finite. Finally,

$$\sum_{U \in \mathcal{U}} f_U = \frac{1}{\sum_{V \in \mathcal{V}} g_V} \cdot \sum_{U \in \mathcal{U}} \sum_{\sigma(V)=U} g_V = \frac{1}{\sum_{V \in \mathcal{V}} g_V} \cdot \sum_{V \in \mathcal{V}} g_V = 1. \quad \square$$

Corollary 1.4.5. *Let M be a smooth manifold and $F \subset A \subset M$, where F is closed in M and A is open in M. Then, then exists a smooth function $f : M \to [0,1]$ such that $F \subset f^{-1}(1)$ and $M \setminus A \subset f^{-1}(0)$.*

Proof. From Theorem 1.4.4, there exists a smooth partition of unity $\{f_{M\setminus F}, f_A\}$ subordinated to the open cover $\{M \setminus F, A\}$ of M. It suffices to take $f = f_A$. □

As an application of the existence of smooth partitions of unity we shall give a partial answer to the following question. Is a smooth manifold diffeomorphic to a smooth submanifold of some $\mathbb{R}^N$ for sufficiently large $N \in \mathbb{N}$ and what is the minimum value of N for which this is possible?

Theorem 1.4.6. *If M is a compact smooth n-manifold, there exist $N \in \mathbb{N}$ and a smooth embedding $g : M \to \mathbb{R}^N$.*

Proof. From Lemma 1.4.3 and the compactness of M, there exist some $m \in \mathbb{N}$, a finite family $\{(U_i, \phi_i) : 1 \leq i \leq m\}$ of smooth charts of M and a finite family $\{V_i : 1 \leq i \leq m\}$ of open subsets of M such that $\overline{V}_i \subset U_i$ for all $1 \leq i \leq m$ and

$$M = U_1 \cup \cdots \cup U_m = V_1 \cup \cdots \cup V_m.$$

For each $1 \leq i \leq m$ there exists a smooth function $f_i : M \to [0,1]$ such that $\overline{V}_i \subset f_i^{-1}(1)$ and $\mathrm{supp} f_i \subset U_i$, from Corollary 1.4.5. The map $\psi_i : M \to \mathbb{R}^n$ defined by

$$\psi_i(p) = \begin{cases} f_i(p)\phi_i(p), & \text{if } p \in U_i, \\ 0, & \text{otherwise,} \end{cases}$$

is smooth. The map $g : M \to (\mathbb{R}^n)^m \times \mathbb{R}^m$ defined by

$$g(p) = (\psi_1(p), ..., \psi_m(p), f_1(p), ..., f_m(p))$$

is smooth and actually an immersion, because for every $p \in M$ there exists some $1 \leq i \leq m$ with $p \in V_i$ and $\psi_i|_{V_i} = \phi_i|_{V_i}$ maps V_i diffeomorphically onto an open subset of $\mathbb{R}^n$. To see that g is injective, let p, $q \in M$ be such that $g(p) = g(q)$. Then, $\psi_i(p) = \psi_i(q)$ and $f_i(p) = f_i(q)$ for every $1 \leq i \leq m$. There exists however some $1 \leq j \leq m$ with $p \in V_j$ and so $f_j(q) = f_j(p) = 1$. Therefore, $q \in U_j$ and $\phi_j(p) = \psi_j(p) = \psi_j(q) = \phi_j(q)$, hence $p = q$. Finally, g is a topological embedding, since M is compact. □

It has been proved by H. Whitney that a compact smooth n-manifold can be smoothly embedded in $\mathbb{R}^{2n}$. Also any smooth n-manifold can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset. The presentation of these topics are beyond the scope of these notes.

1.5 Exercises

1. On $\mathbb{R}$ we consider the smooth structure $\mathcal{B}$ defined by the smooth atlas $\{(\mathbb{R}, \psi)\}$, where $\psi : \mathbb{R} \to \mathbb{R}$ is the map $\psi(t) = t^3$. Let $\mathcal{A}$ denote the standard smooth structure of $\mathbb{R}$.
(a) Prove that $\mathcal{A} \neq \mathcal{B}$.
(b) Prove that $id : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is not a smooth diffeomorphism.

(c) Are the smooth 1-manifolds $(\mathbb{R}, \mathcal{A})$, $(\mathbb{R}, \mathcal{B})$ diffeomorphic?

2. For every $t > 0$ we consider the map $h_t : \mathbb{R} \to \mathbb{R}$ with $h_t(x) = x$, if $x \leq 0$ and $h_t(x) = tx$, if $x \geq 0$. Let $\mathcal{A}_t$ be the smooth structure on $\mathbb{R}$ defined by the smooth atlas $\{(\mathbb{R}, h_t)\}$, $t > 0$.
(a) Prove that $\mathcal{A}_t \neq \mathcal{A}_s$ for $t \neq s$.
(b) Are the smooth 1-manifolds $(\mathbb{R}, \mathcal{A}_t)$ and $(\mathbb{R}, \mathcal{A}_s)$ diffeomorphic for all t, $s > 0$?

3. Let $U_i^+ = \{(x_1, ..., x_{n+1}) \in S^n : x_i > 0\}$, $U_i^- = \{(x_1, ..., x_{n+1}) \in S^n : x_i < 0\}$, and let $h_i^\pm : U_i^\pm \to \mathbb{R}^n$ be the map with

$$h_i^\pm(x_1, ..., x_{n+1}) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}), \qquad 1 \leq i \leq n+1.$$

(a) Prove that $\mathcal{B} = \{(U_i^\pm, h_i^\pm) : 1 \leq i \leq n+1\}$ is a smooth atlas on S^n.
(b) Prove that $\mathcal{B}$ is equivalent to the smooth atlas

$$\mathcal{A} = \{(S^n \setminus \{e_{n+1}\}, \pi_+), (S^n \setminus \{-e_{n+1}\}, \pi_-)\},$$

where $\pi_\pm : S^n \setminus \{\pm e_{n+1}\} \to \mathbb{R}^n$ is the stereographic projection.

4. Let $(V, \langle , \rangle)$ be a finite dimensional inner product real vector space and let

$$S(V) = \{x \in V : \|x\| = 1\},$$

where $\|x\| = \langle x, x\rangle^{1/2}$.
(a) If $p \in S(V)$, prove that for every $x \in S(V) \setminus \{p\}$ the intersection point of the line through p and x with the orthogonal complement $\langle p\rangle^\perp$ is

$$\phi(x) = \frac{x - \langle x, p\rangle p}{1 - \langle x, p\rangle}.$$

The map $\phi : S(V) \setminus \{p\} \to \langle p\rangle^\perp$ is the stereographic projection with respect to p.
(b) Compute $\phi^{-1} : \langle p\rangle^\perp \to S(V) \setminus \{p\}$.
(c) If $\psi : S(V) \setminus \{-p\} \to \langle p\rangle^\perp$ is the stereographic projection with respect to $-p$, compute $\psi \circ \phi^{-1} : \langle p\rangle^\perp \to \langle p\rangle^\perp$.

5. Consider the canonical smooth atlas $\{(U_0, \phi_0), (U_1, \phi_1)\}$ of $\mathbb{C}P^1$ and observe that $\mathbb{C}P^1 \setminus U_0 = \{[0, 1]\}$ and $\mathbb{C}P^1 \setminus U_1 = \{[1, 0]\}$. Prove that $g : \mathbb{C}P^1 \to S^2$ defined by

$$g[z_0, z_1] = \begin{cases} (\pi_+^{-1} \circ \phi_0)[z_0, z_1], & \text{if } z_0 \neq 0 \\ (0, 0, 1), & \text{if } z_0 = 0. \end{cases}$$

is a smooth diffeomorphism, where $\pi_+ : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$ denotes the stereographic projection with respect to the north pole.

6. Let X be a Hausdorff topological space and $H(X)$ be the group of the homeomorphisms of X onto itself. A subgroup G of $H(X)$ defines on X the following equivalence relation: $x \sim y$ if and only if there exists some $g \in G$ with $y = g(x)$.

The equivalence classes are called the orbits of G. Let $\pi : X \to X/G$ denote the quotient map. We say that G acts properly discontinuously on X if every point $x \in X$ has some open neighbourhood U in X such that $U \cap g(U) = \varnothing$, for every $g \in G$, $g \neq id$.
(a) If G acts properly discontinuously, prove that every point $[x] \in X/G$ has an open neighbourhood V^* such that

$$\pi^{-1}(V^*) = \bigcup_{g \in G} g(V),$$

where V is a suitable open neighbourhood of $x \in X$, so that $g_1(V) \cap g_2(V) = \varnothing$, for $g_1 \neq g_2$ and $\pi|_V : V \to V^*$ is a homeomorphism.
(b) Let M be a smooth n-manifold and G be a group of smooth diffeomorphisms which acts properly discontinuously on M. If the quotient space M/G is Hausdorff, prove that it is a smooth n-manifold.
(c) Let M be a smooth n-manifold and G be a finite group of smooth diffeomorphisms of M. If $g(x) \neq x$ for every $x \in M$, $g \in G$, $g \neq id$, prove that G acts properly discontinuously on M, the quotient space M/G is Hausdorff and therefore a smooth n-manifold.
(d) On S^n the antipodal map $a : S^n \to S^n$ with $a(x) = -x$ is a smooth diffeomorphism. If $G = \{id, a\}$, determine the smooth n-manifold S^n/G.
(e) On the 2-torus $T^2 = S^1 \times S^1$ let $f : T^2 \to T^2$ be the map

$$f(e^{2\pi ix}, e^{2\pi iy}) = (e^{-2\pi ix}, -e^{2\pi iy}).$$

If $G = \{id, f\}$, Prove that $K^2 = T^2/G$ is a smooth 2-manifold. This manifold is called Klein bottle.
(f) Prove that the group of translations by vectors with integer coordinates, which is isomorphic to $\mathbb{Z}^n$, acts properly discontinuously on $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomorphic to the n-torus T^n.

7. Prove that the 1-dimensional real projective space $\mathbb{R}P^1$ is deffeomorphic to the circle S^1.

8. Let $f : M \to N$ be a bijective smooth map of smooth manifolds. If its derivative $f_{*p} : T_pM \to T_{f(p)}N$ is a linear isomorphism for every $p \in M$, prove that f is a smooth diffeomorphism.

9. Let $f : M \to Q$ be a smooth map of smooth manifolds and $q \in Q$ be a regular value of f with $N = f^{-1}(q) \neq \varnothing$. If $i_N : N \hookrightarrow M$ is the inclusion, show that $(i_N)_{*p}(T_pN) = \mathrm{Ker} f_{*p}$ for every $p \in N$.

10. Prove that $T_pS^n = \{[\gamma]_p \in T_p\mathbb{R}^{n+1} : \langle \gamma'(0), p\rangle = 0\}$ for every $p \in S^n$, where $\langle , \rangle$ is the euclidean inner product.

11. Let $n > 1$ and $p : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree $m \in \mathbb{N}$. Prove that $p^{-1}(c)$ is a $(n-1)$-dimensional smooth submanifold of $\mathbb{R}^n$ for every $c \neq 0$.

12. Let M be a smooth m-manifold, N be a smooth n-manifold and let $f : M \to N$ be a smooth map. If $q \in N$ is such that $f^{-1}(q) \neq \varnothing$ and f has constant rank k on some open neighbourhood of $f^{-1}(q)$, prove that the level set $f^{-1}(q)$ is a $(m-k)$-dimensional smooth submanifld of M.

13. Prove that the set $N = \{A \in \mathbb{R}^{2\times 2} : A \text{ has rank } 1\}$ is a 3-dimensional smooth submanifold of $\mathbb{R}^{2\times 2}$.

14. The set S of all real $n \times n$ symmetric matrices is a vector subspace of $\mathbb{R}^{n\times n}$ of dimension $n(n+1)/2$. Let $f : GL(n, \mathbb{R}) \to S$ be the map $f(A) = A \cdot A^t$.
(a) Prove that $f_{*A}(H) = AH^t + HA^t$ for every $H \in T_A GL(n, \mathbb{R})$, $A \in GL(n, \mathbb{R})$.
(b) Prove that the identity $I_n \in S$ is a regular value of f.
(c) Prove that the orthogonal group $O(n, \mathbb{R})$ is a $\frac{n(n-1)}{2}$-dimensional smooth submanifold of $GL(n, \mathbb{R})$.
(d) Prove that $T_{I_n} O(n, \mathbb{R}) = \{H \in \mathbb{R}^{n\times n} : H + H^t = 0\}$.

15. Prove that the map $g : T^2 \to \mathbb{R}^3$ with

$$g(e^{2\pi i\phi}, e^{2\pi i\theta}) = ((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta)$$

is an embedding of the 2-torus T^2 into $\mathbb{R}^3$ and its image is

$$g(T^2) = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}.$$

16. Prove that the map $f : S^2 \to \mathbb{R}^6$ with

$$f(x, y, z) = (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy)$$

an immersion which induces an embedding of the real projective plane $\mathbb{R}P^2$ into $\mathbb{R}^6$.

17. Prove that the map $f : \mathbb{R}P^2 \to \mathbb{R}^3$ with $f([x, y, z]) = (yz, zx, xy)$ is an immersion and the map $g : \mathbb{R}P^2 \to \mathbb{R}^4$ with $g([x, y, z]) = (yz, zx, xy, x^2 + 2y^2 + 3z^2)$ is an embedding.

18. Let M, N be two smooth n-manifolds and let $f : M \to N$ be an immersion.
(a) Prove that f is an open map.
(b) If M is compact and N is connected, prove that $f(M) = N$.

19. Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the orthogonal transformation (complex structure of $\mathbb{R}^{2n}$) with $J(x, y) = (-y, x)$ for every $(x, y) \in \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.
(a) Prove that the set $S = \{A \in \mathbb{R}^{2n\times 2n} : A^t JA = J\}$ is a smooth submanifold of $\mathbb{R}^{2n\times 2n}$.
(b) Describe $T_{I_{2n}} S$ as a vector subspace of $\mathbb{R}^{2n\times 2n}$.
(c) Find the dimension of S.
(Hint : Prove that $J \in \mathbb{R}^{2n\times 2n}$ is a regular value of the smooth map $f : GL(2n, \mathbb{R}) \to \{H \in \mathbb{R}^{2n\times 2n} : H + H^t = 0\}$ with $f(A) = A^t JA$.)

20. Let $d \in \mathbb{N}$, $n \geq 2$ and denote by V_d^{2n} the set of points $(z_0, z_1, ..., z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ which are solutions of the equation

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0.$$

(a) Prove that V_d^{2n} is a smooth $2n$-manifold.
(b) Prove that the set $W_d^{2n-1} = V_d^{2n} \cap S^{2n+1}$ is a smooth $(2n-1)$-manifold. W_d^{2n-1} is called Brieskorn manifold.

21. The unit tangent bundle of the 2-sphere S^2 is the subset

$$T^1S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|p\| = 1, \|v\| = 1, \langle p, v \rangle = 0\}$$

of $\mathbb{R}^6$, where $\langle , \rangle$ is the euclidean inner product on $\mathbb{R}^3$.
(a) Prove that T^1S^2 is a 3-dimensional smooth submanifold of $\mathbb{R}^6$.
(b) Prove that $F : SO(3, \mathbb{R}) \to T^1S^2$ with $F(A) = (Ae_3, Ae_1)$ is a smooth diffeomorphism.
(c) Let $D^3 = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ and let $g : D^3 \to SO(3, \mathbb{R})$ be the map with $g(0) = I_3$ and such that if $x \in D^3 \setminus \{0\}$ then $g(x)$ is the rotation with respect to the axis generated by x by the angle $\|x\| \cdot \pi$. Prove that g induces a smooth diffeomorphism from $\mathbb{R}P^3$ onto $SO(3, \mathbb{R})$.
(Hint : Observe that $T^1S^2 = f^{-1}(0)$, where $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the smooth map $f(p, v) = (\|p\|^2 - 1, \|v\|^2 - 1, \langle p, v \rangle)$.)

Chapter 2

Vector fields

2.1 The tangent bundle and vector fields

In this section we shall define the notion of vector field on a smooth manifold, which is a generalization and globalization of the notion o ordinary differential equation on an open subset of euclidean space. A continuous vector field is a map which to a point p assigns a tangent vector with point of application p and varies continuously with p. So, first we need to consider the set of all tangent vectors.

Let M be a smooth n-manifold and consider the disjoint union of all tangent spaces at points of M, that is the set

$$TM = \bigcup_{p \in M} \{p\} \times T_pM.$$

Let $\pi : TM \to M$ denote the natural projection $\pi(p, v) = p$, for $v \in T_pM$, $p \in M$. We shall endow TM with the structure of a smooth manifold, so that π becomes smooth and a submersion.

If $\mathcal{A}$ is a smooth atlas of M, we define the class

$$\tilde{\mathcal{A}} = \{(\pi^{-1}(U), \tilde{\phi}_U) : (U, \phi_U) \in \mathcal{A}\}$$

where $\tilde{\phi}_U : \pi^{-1}(U) \to \phi_U(U) \times \mathbb{R}^n$ is the bijection defined by

$$\tilde{\phi}_U(p, v) = (\phi_U(p), (\phi_U)_{*p}(v))$$

for every $p \in U$, $v \in T_pM$. In other words, if $\phi_U = (x^1, ..., x^n)$, then for $p \in M$ and

$$v = \sum_{k=1}^{n} v^k \left(\frac{\partial}{\partial x^k}\right)_p \in T_pM$$

we have $\tilde{\phi}_U(v, v) = (x^1(p), ..., x^n(p), v^1, ..., v^n)$.

If now (U, ϕ_U), $(V, \phi_V) \in \mathcal{A}$ are such that $U \cap V \neq \varnothing$, then the transition map $\tilde{\phi}_U \circ \tilde{\phi}_V^{-1} : \phi_V(U \cap V) \times \mathbb{R}^n \to \phi_U(U \cap V) \times \mathbb{R}^n$ is given by the formula

$$(\tilde{\phi}_U \circ \tilde{\phi}_V^{-1})(x, y) = ((\phi_U \circ \phi_V^{-1})(x), D(\phi_U \circ \phi_V^{-1})(x)(y))$$

and is thus a smooth diffeomorphism. This means that $\tilde{\mathcal{A}}$ would be a smooth atlas of TM, if we had a topology on TM making it a topological $2n$-manifold in such a way the the sets $\pi^{-1}(U)$ were open and the maps $\tilde{\phi}_U$ homeomorphisms. This topology is provided by the following.

Lemma 2.1.1. *Let X be a non-empty set and $\mathcal{U}$ be a family of subsets of X which covers X. We assume that for every $U \in \mathcal{U}$ there exist a topological space X_U and a bijection $\psi_U : U \to X_U$ such that for $U, V \in \mathcal{U}$ with $U \cap V \neq \varnothing$ the set $\psi_V(U \cap V)$ is open in X_V and the map $\psi_U \circ \psi_V^{-1} : \psi_V(U \cap V) \to X_U$ is continuous.*

Then there exists a unique topology on X with respect to which every element of $\mathcal{U}$ becomes an open set and every map ψ_U becomes a homeomorphism.

Proof. Our assumptions imply that $\psi_U \circ \psi_V^{-1} : \psi_V(U \cap V) \to \psi_U(U \cap V)$ is a homeomorphism for every $U, V \in \mathcal{U}$ with $U \cap V \neq \varnothing$. The family

$$\mathcal{T} = \{A \subset X : \psi_U(U \cap A) \quad \text{is open in } X_U \text{ for every } U \in \mathcal{U}\}$$

is a topology on X which contains the family $\mathcal{U}$. By the definition of $\mathcal{T}$, each ψ_U is an open map. For the continuity of ψ_U let $W \subset X_U$ be an open set. Then,

$$(\psi_U \circ \psi_V^{-1})(\psi_V(\psi_U^{-1}(W) \cap V)) = W \cap \psi_U(U \cap V)$$

is open in X_U for every $U, V \in \mathcal{U}$ with $U \cap V \neq \varnothing$. Since $\psi_U \circ \psi_V^{-1}$ is a homeomorphism, $\psi_V(\psi_U^{-1}(W) \cap V))$ must be open in X_V. This shows that $\psi_U^{-1}(W) \in \mathcal{T}$ and that ψ_U is continuous. The uniqueness of the topology $\mathcal{T}$ is obvious. □

Applying now Lemma 2.1.1, we obtain a unique topology on TM with respect to which each set $\pi^{-1}(U)$ is open and each map $\tilde{\phi}_U$ is a homeomorphism for $(U, \phi_U) \in \mathcal{A}$. Since M and $\mathbb{R}^n$ are Hausdorff spaces and have countable basis for their topologies, the same is true for TM. Thus, TM becomes a smooth $2n$-manifold. For every $(U, \phi_U) \in \mathcal{A}$ the corresponding local representation $\phi_U \circ \pi \circ \tilde{\phi}_U^{-1} : \phi_U(U) \times \mathbb{R}^n \to \phi_U(U)$ of π is the projection $(\phi_U \circ \pi \circ \tilde{\phi}_U^{-1})(x, y) = x$. Hence π is a submersion.

The triple (TM, π, M) is the *tangent bundle* of M. The natural projection π is the bundle map and M is the base space of the bundle. The total space of the bundle is TM. Abusing terminology, we shall also use the term tangent bundle for TM itself.

Definition 2.1.2. A smooth *vector field* on a smooth n-manifold M is a smooth map $X : M \to TM$ which to every $p \in M$ assigns a tangent vector $X(p) \in T_pM$. Briefly, $X \circ \pi = id_M$ or in other words X is a smooth section of π.

The set $\mathcal{X}(M)$ of all smooth vector fields of a smooth manifold M is an infinite dimensional real vector space. It is also a module over the commutative ring $C^\infty(M)$ of all real valued smooth functions defined on M. Every smooth diffeomorphism $f : M \to M$ induces a linear isomorphism $f_* : \mathcal{X}(M) \to \mathcal{X}(M)$ defined by $(f_*X)(f(p)) = f_{*p}(X(p))$ for every $p \in M$. The smooth vector field X of M is called f-invariant if $f_*X = X$.

Let X be a smooth vector field on a smooth n-manifold M. If $\mathcal{A}$ is a smooth atlas of M anf $\tilde{\mathcal{A}}$ is the corresponding smooth atlas of TM, then $X(U) \subset \pi^{-1}(U)$ for every $(U, \phi_U) \in \mathcal{A}$. There exists a smooth map $F_U : \phi_U(U) \to \mathbb{R}^n$, which is called the principal part of X with respect to (U, ϕ_U), such that the corresponding local representation $\tilde{\phi}_U \circ X \circ \phi_U^{-1} : \phi_U(U) \to \phi_U(U) \times \mathbb{R}^n$ of X is

$$(\tilde{\phi}_U \circ X \circ \phi_U^{-1})(x) = (x, F_U(x)).$$

Thus, if $\phi_U = (x^1, ..., x^n)$ and $F_U = (F^1, ...F^n)$, then

$$X(p) = \sum_{k=1}^{n} F^k(\phi(p)) \left(\frac{\partial}{\partial x^k} \right)_p$$

for every $p \in U$ and the smoothness of X is equivalent to the smoothness of F_U. In particular, on U we have the basic smooth vector fields

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$$

defined by the smooth chart ϕ_U.

Apart for the notion of tangent vector field on a smooth manifold we need to have a notion of tangent vector field along a smooth curve.

Definition 2.1.3. A smooth *vector field along a smooth curve* $\gamma : I \to M$ on a smooth n-manifold M, for $I \subset \mathbb{R}$ an open interval, is a smooth map $X : I \to TM$ which to every $s \in I$ assigns a tangent vector $X(s) \in T_{\gamma(s)}M$.

If $\gamma : I \to M$ is a smooth curve on a smooth n-manifold M, then for every $s \in I$ the tangent vector

$$\dot{\gamma}(s) = \gamma_{*s}\left(\left(\frac{d}{dt} \right)_s \right)$$

is the *velocity of* γ *at* $\gamma(s)$, where $\dfrac{d}{dt}$ is the basic vector field on $\mathbb{R}$. Thus, $\dot{\gamma} : I \to TM$ is a smooth vector field along γ, which is called the *velocity field* of γ.

Recall that $\left(\dfrac{d}{dt} \right)_s$ is the usual derivation at s. Using the notation of section 1.4, note that $[\gamma]_p$ and $\dot{\gamma}(0)$ denote one and the same vector in T_pM for $p \in M$ and $\gamma \in S(M, p)$, namely the velocity of γ at $p = \gamma(0)$.

If $\gamma(I) \subset U$ for the smooth chart (U, ϕ_U) of M and $\phi_U \circ \gamma = (\gamma^1, ..., \gamma^n)$ is the corresponding local representation of γ, then

$$\dot{\gamma}(s) = \sum_{k=1}^{n} (\gamma^k)'(s) \left(\frac{\partial}{\partial x^k} \right)_{\gamma(s)}$$

for every $s \in I$.

2.2 Flows of smooth vector fields

Let M be a smooth n-manifold and let X be a smooth vector field on M. An *integral curve* of X is a smooth curve $\gamma : I \to M$, defined on an open interval $I \subset \mathbb{R}$, such that

$$\dot{\gamma}(s) = X(\gamma(s))$$

for every $s \in I$.

If (U, ϕ_U) is a smooth chart of M with $\phi_U = (x^1, ..., x^n)$ and $F_U = (F^1, ..., F^n)$ is the principal part of X on U with respect to ϕ_U, the discussion in the preceding section 2.1 shows that a smooth curve $\gamma : I \to U$ is an integral curve of X on U if and only if its local representation $\phi_U \circ \gamma = (\gamma^1, ..., \gamma^n)$ is a solution of the autonomous n-dimensional ordinary differential equation $x'(s) = F_U(x(s))$, which means that it satisfies the system of ordinary differential equations

$$(\gamma^k)'(s) = F_U^k((\gamma^1(s), ..., \gamma^n(s)), \quad s \in I, \quad k = 1, 2, ..., n.$$

Thus, locally on M the integral curves of smooth vector fields on M are the solutions of autonomous ordinary differential equations. The standard existence and uniqueness theorems combined with continuous and differentiable dependence on initial conditions imply that if X is a smooth vector field on M, then for every point $p \in M$ there exist an open neighbourhood V of p in M, some $\epsilon > 0$ and a smooth map $\Phi^V : (-\epsilon, \epsilon) \times V \to M$ such that $\Phi^V(0, q) = q$ for every $q \in V$ and

$$\frac{\partial \Phi^V}{\partial t}(s, q) = X(\Phi^V(s, q))$$

for every $(s, q) \in (-\epsilon, \epsilon) \times V$. Moreover, the map Φ^V is unique, in the sense that if W, $\delta > 0$ and $\Phi^W : (-\delta, \delta) \times W \to M$ is another triple like V, ϵ and Φ^V, then $\Phi^V = \Phi^W$ on $(-\epsilon, \epsilon) \times V \cap (-\delta, \delta) \times W$. Thus, for every $q \in V$ the smooth curve $\Phi^V(\cdot, q) : (-\epsilon, \epsilon) \to M$ is the unique integral curve of X defined on the interval $(-\epsilon, \epsilon)$ and satisfying the initial condition $\Phi^V(0, q) = q$. The map Φ^V is called the *local flow of X on the open set V*.

The existence of maximal integral curves globally on M can be established in the usual way.

Proposition 2.2.1. *If X is a smooth vector field on M, then for every $p \in M$ there exist $a_p < 0 < b_p$ and a maximal integral curve $\Phi^p : (a_p, b_p) \to M$ of X with $\Phi^p(00 = p$ in the sense that if $\gamma : I \to M$ is any other integral curve of X defined on an open interval $I \subset \mathbb{R}$ which contains 0 such that $\gamma(0) = p$ then $I \subset (a_p, b_p)$ and $\gamma = \Phi^p|_I$.*

Proof. Let $\gamma_j : I_j \to M$, $j = 1, 2$, be integral curves of X defined on open intervals such that $0 \in I_1 \cap I_2$, with $\gamma_1(0) = \gamma_(0) = p$. Then, $I_1 \cap I_2$ is a non-empty open interval and the set $I^* = \{s \in I_1 \cap I_2 : \gamma_1(s) = \gamma_2(s)\}$ is non-empty and closed in $I_1 \cap I_2$, by continuity. If $s \in I^*$, there exists $\delta > 0$ such hat $(s - \delta, s + \delta) \subset I_1 \cap I_2$. The smooth curves $\beta_j : (-\delta, \delta) \to M$ defined by $\beta_j(t) = \gamma(t + s)$, $j = 1, 2$, are integral curves of X with $\beta_1(0) = \gamma_1(s) = \gamma_2(s) = \beta_2(0)$. By uniqueness of solutions, there exists some $0 < \eta \leq \delta$ such hat $\beta_1 = \beta_2$ on $(-\eta, \eta)$. Therefore, $(s-\eta, s+\eta) \subset I^*$, which shows that I^* is open in $I_1 \cap I_2$. By connectedness now we must have $I^* = I_1 \cap I_2$. This shows that the union of all open intervals I containing 0 on which there is an integral curve $\gamma : I \to M$ of X with $\gamma(0) = p$, is an open interval (a_p, b_p) on which a maximal integral curve $\Phi^p : (a_p, b_p) \to M$ of X with $\Phi^p(00 = p$ is well defined. □

Recall that the open interval on which a maximal integral curve is defined is not necessarily the whole real line $\mathbb{R}$. For instance, the maximal solution of the autonomous ordinary differential equation $x'(s) = (x(s))^2$ on $\mathbb{R}$ with initial condition $x(0) = 1$ is $\Phi : (-\infty, 1) \to \mathbb{R}$ given by the formula

$$\Phi(s) = \frac{1}{1-s}.$$

Lemma 2.2.2. *Let $p \in M$ and $\Phi^p : (a_p, b_p) \to M$ be a maximal integral curve of a smooth vector field X om M with $\Phi^p(0) = p$. If $t \in (a_p, b_p)$ and $q \in \Phi^p(t)$, then the maximal integral curve Φ^q with $\Phi^q(0) = q$ is defined on the open interval $(a_p - t, b_p - t)$ and $\Phi^q(s) = \Phi^p(s+t)$.*

Proof. Since the smooth curve $\gamma : (a_p - t, b_p - t) \to M$ with $\gamma(s) = \Phi^p(s+t)$ is an integral curve of X with $\gamma(0) = q$, the maximal integral curve Φ^q with $\Phi^q(0) = q$ is defined at least on $(a_p - t, b_p - t)$. Conversely, if the interval of definition of Φ^q is the open interval (a_q, b_q), then $a_q \leq a_p - t$, $b_p - t \leq b_q$ and $\delta : (a_q + t, b_q + t) \to M$ defined by $\delta(s) = \Phi^q(s-t)$ is an integral curve with $\delta(0) = p$. Hence $a_p \leq a_q + t$, $b_q + t \leq a_p$. □

Using the notation of Lemma 2.2.2 for a smooth vector field X on M, we define

$$D = \bigcup_{p \in M} (a_p, b_p) \times \{p\}$$

and $\Phi : D \to M$ by $\Phi(s, p) = \Phi^p(s)$, which has the following properties:
(i) $\Phi(0, p) = p$ for every $p \in M$ and
(ii) $\Phi(t, \Phi(s, p)) = \Phi(t+s, p)$ for every $p \in M$ and s, $t \in \mathbb{R}$ such that at least one side of this equality is defined.

Theorem 2.2.3. *The set D is open in $\mathbb{R} \times M$ and $\Phi : D \to M$ is smooth.*

Proof. For $p \in M$ we consider the set I^* consisting of all $a_p < t < b_p$ for which there exist $\delta > 0$ and an open neighbourhood U of p in M such that $(t-\delta, t+\delta) \times U \subset D$ and Φ is smooth on $(t-\delta, t+\delta) \times U$. Then, $0 \in I^*$ and I^* is an open set. Thus, it suffices to prove that I^* is closed in the interval (a_p, b_p), by connectedness. Suppose that $a_p < s < b_p$ lies in the closure of I^*. There exist an open neighbourhood V of $\Phi(s.p)$ in M, some $\epsilon > 0$ and a local flow $\Phi^V : (-\epsilon, \epsilon) \times V \to M$, so that $\Phi^V = \Phi|_{(-\epsilon,\epsilon)\times V}$. By continuity, there exists some $t \in I^*$ with $|t - s| < \dfrac{\epsilon}{3}$ and $\Phi(t, p) \in V$. Since $t \in I^*$, there exist $0 < \delta < \dfrac{\epsilon}{3}$ and an open neighbourhood U of

p in M such that $(t-\delta, t+\delta)\times U \subset D$ and Φ is smooth on $(t-\delta, t+\delta)\times U$. By continuity of $\Phi(t,.) : U \to M$ and the fact that $\Phi(t,p) \in V$, shrinking U if necessary, we may take U so that $\Phi(\{t\}\times U) \subset V$. So, from Lemma 2.2.2 we have

$$(-\epsilon, \epsilon) \subset (a_{\Phi(t,q)}, b_{\Phi(t,q)}) = (a_q - t, b_q - t)$$

for every $q \in U$, which implies that $(t-\epsilon, t+\epsilon)\times U \subset D$, and Φ is smooth on $(t-\epsilon, t+\epsilon)\times U$, because

$$\Phi(r,q) = \Phi^V(r-t, \Phi(t,q))$$

for every $(r,q) \in (t-\epsilon, t+\epsilon)\times U$. Now

$$(s,p) \in (s-\delta, s+\delta)\times U \subset (t-\epsilon, t+\epsilon)\times U \subset D,$$

which means that $s \in I^*$. $\square$

The fact that D is an open subset of $\mathbb{R}\times M$ is equivalent to saying that the function $a : M \to [-\infty, 0)$ is upper semicontinuous and $b : M \to (0, +\infty]$ is lower semicontinuous.

The smooth map $\Phi : D \to M$ is called the *flow* of the smooth vector field X. The vector field X can be reconstructed from its flow by setting

$$X(p) = \frac{\partial \Phi}{\partial t}(0,p)$$

for every $p \in M$. The image $\Phi((a_p, b_p)\times\{p\})$ of the maximal integral curve of X through the point $p \in M$ is called the *orbit* of p with respect to X.

A smooth vector field X on M is called *complete* if every maximal integral curve of X is defined on the whole real line $\mathbb{R}$ or $D = \mathbb{R}\times M$, using the above notation. In this case, the flow $\Phi : \mathbb{R}\times M \to M$ is a smooth action of the additive group of real numbers $\mathbb{R}$ on M. For every $t \in \mathbb{R}$ the map $\Phi_t = \Phi(t,.) : M \to M$ is a smooth diffeomorphism. Moreover, $\Phi_0 = id_M$ and $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for every t, $s \in \mathbb{R}$ and the family $(\Phi_t)_{t\in\mathbb{R}}$ is called the *one-parameter group of diffeomorphisms* defined by X. For every $t \in \mathbb{R}$ and $p \in M$ we have

$$(\Phi_t)_{*p}(X(p)) = (\Phi_t)_{*p}\left(\frac{\partial \Phi}{\partial t}(0,p)\right) = \frac{\partial(\Phi_t \circ \Phi^p)}{\partial t}(0).$$

However,

$$(\Phi_t \circ \Phi^p)(s) = \Phi(t, \Phi(s,p)) = \Phi(t+s, p) = \Phi(s, \Phi(t,p))$$

for every $s \in \mathbb{R}$ and therefore

$$(\Phi_t)_{*p}(X(p)) = X(\Phi_t(p)).$$

This means that X is Φ_t-invariant for every $t \in \mathbb{R}$.

In case the smooth vector field X is not complete, the smooth diffeomorphisms Φ_t are defined on suitable open subsets of M.

The integral curves of a smooth vector field X which are not defined on the whole real line must necessarily explode at infinity. This is made more precise in the following.

Lemma 2.2.4. *Let X be a smooth vector field with flow $\Phi : D \to M$ and $p \in M$. If $b_p < +\infty$, then for every compact set $K \subset M$ there exists $0 < T < b_p$ such that $\Phi(t,p) \in M \setminus K$ for every $T < t < b_p$.*

Proof. For every $q \in K$ there exist $\delta_q > 0$ and an open neighbourhood V_q of q such that $(-\delta_q, \delta_q) \times V_q \subset D$. By compactness of K, there exist q_1,..., $q_m \in K$, for some $m \in \mathbb{N}$, such that $K \subset V_{q_1} \cup \cdots \cup V_{q_m}$. If now $\delta = \min\{\delta_{q_1}, ..., \delta_{q_m}\}$, then $(-\delta, \delta) \times K \subset D$. Thus, if there exists a sequence $t_k \nearrow b_p$ such that $\Phi(t_k, p) \in K$ for every $k \in \mathbb{N}$, we arrive at the contradiction $0 < \delta < b_p - t_k$ for all $k \in \mathbb{N}$. □

This implies the following important fact.

Corollary 2.2.5. *Every smooth vector field on a compact smooth manifold is complete.* □

It is possible to find all integral curves of a given smooth vector field only in very rare cases. The aim of the qualitative (or geometric) theory of dynamical systems is to find the distribution of the time oriented orbits of vector fields studying their asymptotic behaviour. In this point of view, we may replace X with $f \cdot X$ where $f : M \to (0, +\infty)$ is a smooth function, because both vector fields have the same orbits. Indeed, if $\Phi : D \to M$ is the flow of X, for every $p \in M$ the smooth map $h : (a_p, b_p) \to \mathbb{R}$ defined by

$$h(s,p) = \int_0^s \frac{1}{f(\Phi(t,p))} dt$$

is strictly increasing and $h((a_p, b_p))$ is an open interval. Also, $(h^{-1})'(s) = f(\Phi(h^{-1}(s)), p)$. It follows now that the maximal integral curve of $f \cdot X$ through p is just $\Phi^p \circ h^{-1} : h((a_p, b_p)) \to M$. In other words, the maximal integral curves of $f \cdot X$ are reparametrizations of the maximal integral curves of X.

The following can be obtained as a consequence of the existence of smooth partitions of unity.

Theorem 2.2.6. *If X is a smooth vector field of a smooth manifold M, then there exists a smooth function $f : M \to (0, 1]$ such that the smooth vector field $f \cdot X$ is complete.*

Proof. Let $\Phi : D \to M$ be the flow of X as above. Since D is an open subset of $\mathbb{R} \times M$, the function $g : M \to (0, 1]$ defined by

$$g(p) = \min\{1, -a_p, b_p\}$$

is lower semicontinuous. Thus, every $p \in M$ has an open neighbourhood W_p such that $g(q) > \frac{1}{2} g(p)$ for every $q \in W_p$. By Theorem 1.4.4, there exists a smooth partition of unity $\{f_p : p \in M\}$ subordinated to the open cover $\{W_p : p \in M\}$. The function $f : M \to (0,1]$ defined by

$$f(q) = \frac{1}{2}\sum_{p\in M} g(p) f_p(q)$$

is smooth and for every $q \in M$ there exist p_1,..., $p_k \in M$, for some $k \in \mathbb{N}$, such that $q \in \mathrm{supp} f_{p_1} \cap \cdots \cap \mathrm{supp} f_{p_k}$ and $f_p(q) = 0$ for $p \neq p_1, ..., p_k$. It follows that

$$f(q) = \frac{1}{2}\sum_{j=1}^{k} g(p_j) f_{p_j}(q) < \sum_{j=1}^{k} g(q) f_{p_j}(q) = g(q) = \min\{1, -a_q, b_q\}$$

for every $q \in M$.

Let now $\psi : D \to \mathbb{R}$ be the smooth function defined by

$$\psi(s,p) = \int_0^s \frac{1}{f(\Phi(t,p))} dt.$$

The smooth map $h : D \to \mathbb{R} \times M$ with $h(s,p) = (\psi(s,p), p)$ is obviously injective, since

$$\frac{\partial \psi}{\partial t}(s,p) = \frac{1}{f(\Phi(s,p))} \geq 1.$$

Moreover, $\psi(s,p) \geq s$ for $0 \leq s < b_p$ and $\psi(s,p) \leq s$ for $a_p < s \leq 0$. Thus, $\lim\limits_{s\to b_p} \psi(s,p) = +\infty$, if $b_p = +\infty$. In case $b_p < +\infty$, for every $0 < s < b_p$ we have

$$\psi(s,p) > \int_0^s \frac{1}{b_{\Phi(t,p)}} dt = \int_0^s \frac{1}{b_p - t} dt = -\log\left(1 - \frac{s}{b_p}\right)$$

and therefore again $\lim\limits_{s\to b_p} \psi(s,p) = +\infty$. Similarly, $\lim\limits_{s\to a_p} \psi(s,p) = -\infty$ for all $p \in M$. This shows that h is surjective.

Since h is a bijection and its derivative $h_{*(s,p)}$ is a linear isomorphism at every point $(s,p) \in D$, it follows from the Inverse Map Theorem that h is a smooth diffeomorphism.

$$\begin{tikzcd}
D \arrow[rr, "h"] \arrow[dr, "\Phi"'] & & \mathbb{R} \times M \arrow[dl, "\Psi"] \\
& M &
\end{tikzcd}$$

The proof is now concluded by the observation that $\Psi = \Phi \circ h^{-1} : \mathbb{R} \times M \to M$ is the flow of $f \cdot X$, because

$$\frac{\partial \Psi}{\partial t}(0,p) = f(\Phi(h^{-1}(0,p))) \cdot \frac{\partial \Phi}{\partial t}(h^{-1}(0,p)) = f(p) \cdot \frac{\partial \Phi}{\partial t}(0,p) = f(p) \cdot X(p)$$

for every $p \in M$. $\square$

2.3 The Lie bracket

Let M be a smooth n-manifold and let X be a smooth vector field on M. At every point $p \in M$ the value $X(p) \in T_pM$ of X is a derivation on the algebra of germs $\mathcal{G}_p(M)$ of smooth functions defined on neighbourhoods of p and

$$X(p)(f) = \lim_{t \to 0} \frac{f(\Phi(t,p)) - f(p)}{t}$$

for every smooth function f which is defined on some open neighbourhood of p in M, where Φ is the flow of X.

Apart from functions, it is possible to define a special kind of derivation of another smooth vector field Y with respect to X, by transporting Y along the integral curves of X by the flow of X. The result can be defined in a purely algebraic way as follows.

Let $p \in M$. If $f \in C^\infty(M,p)$, then $Yf(q) = Y(q)(f)$ is a smooth function $Yf \in C^\infty(M,p)$ for every $Y \in \mathcal{X}(M)$. We define

$$[X,Y](p)(f) = X(p)(Yf) - Y(p)(Xf)$$

for every $f \in C^\infty(M,p)$ and X, $Y \in \mathcal{X}(M)$. We observe that

$$[X,Y](p)(f \cdot g) = X(p)(f \cdot Yg + g \cdot Yf) - Y(p)(f \cdot Xf + g \cdot Xf)$$

$$= f(p)X(p)(Yg) + Y(p)(g)X(p)(f) + Y(p)(f)X(p)(g) + g(p)X(p)(Yf)$$

$$-f(p)Y(p)(Xg) - Y(p)(f)X(p)(g) - Y(p)(g)X(p)(f) - g(p)Y(p)(Xf)$$

$$= f(p) \cdot [X,Y](p)(g) + g(p) \cdot [X,Y](p)(f).$$

Therefore, $[X,Y](p)$ is a derivation of the algebra of germs $\mathcal{G}_p(M)$ and so is a tangent vector in T_pM.

Let (U,ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$. Then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^i}\right) = 0$$

on U for all i, $j = 1, 2, ..., n$. If now X, $Y \in \mathcal{X}(U)$ and

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j},$$

then for every $p \in U$ and $f \in C^\infty(M,p)$ we have

$$[X,Y](p)(f) = \sum_{i,j=1}^n X^i(p)\left(\frac{\partial}{\partial x^i}\right)_p \left(Y^j \frac{\partial f}{\partial x^j}\right) - \sum_{i,j=1}^n Y^j(p)(p)\left(\frac{\partial}{\partial x^j}\right)_p \left(X^i \frac{\partial f}{\partial x^i}\right)$$

$$= \sum_{i,j=1}^n X^i(p)\frac{\partial Y^j}{\partial x^i}(p)\frac{\partial f}{\partial x^j}(p) + \sum_{i,j=1}^n X^i(p)Y^j(p)\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)(p)$$

$$-\sum_{i,j=1}^{n} Y^j(p)\frac{\partial X^i}{\partial x^j}(p)\frac{\partial f}{\partial x^i}(p) - \sum_{i,j=1}^{n} Y^j(p)X^i(p)\frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)(p)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} X^i(p)\frac{\partial Y^j}{\partial x^i}(p) - Y^i(p)\frac{\partial X^j}{\partial x^i}(p)\right)\frac{\partial f}{\partial x^j}(p).$$

This means that

$$[X,Y] = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$$

on U.

The above show that $[X,Y] \in \mathcal{X}(M)$ for every X, $Y \in \mathcal{X}(M)$, and is called the *Lie derivative* of Y with respect to X. The so defined function

$$[.,.] : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

is called the *Lie bracket* and has the following rather obvious properties:
(i) It is bilinear and alternating.
(ii) It satisfies the Jacobi identity, that is

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for every X, Y, $Z \in \mathcal{X}(M)$.
(iii) $[X, fY] = f[X,Y] + Xf \cdot Y$ for every $f \in C^\infty(M)$ and X, $Y \in \mathcal{X}(M)$.
(iv) If $F : M \to M$ is a smooth diffeomorphism, then $[F_*X, F_*Y] = F_*[X,Y]$ for every X, $Y \in \mathcal{X}(M)$. More generally, let M be a smooth n-manifold, L be a smooth k-manifold, $k \leq n$, and let $g : L \to M$ be an injective immersion. Let X, $Y \in \mathcal{X}(M)$ be such that $X(g(x))$, $Y(g(x)) \in g_{*x}(T_xL)$ for every $x \in L$. Then, there exist unique $\tilde{X}(x)$, $\tilde{Y}(x) \in T_xL$ such that $g_{*x}(\tilde{X}(x)) = X(g(x))$ and $g_{*x}(\tilde{Y}(x)) = Y(g(x))$ and it follows from the local presentation of immersions provided by the Constant Rank Theorem 1.3.2 that $\tilde{X}$, $\tilde{Y} \in \mathcal{X}(L)$. Now we have

$$g_{*x}([\tilde{X},\tilde{Y}](x)) = [X,Y](g(x))$$

for every $x \in L$. Indeed, let $x \in L$ and let f be a smooth function defined on some open neighbourhood of $g(x)$. Note first that the chain rule implies that

$$\tilde{Y}(f \circ g) = Yf \circ g.$$

From the definitions now we have

$$g_{*x}([\tilde{X},\tilde{Y}](x))f = [\tilde{X},\tilde{Y}](x)(f \circ g) = \tilde{X}(x)(\tilde{Y}(f \circ g)) - \tilde{Y}(x)(\tilde{X}(f \circ g))$$

$$= \tilde{X}(x)(Yf \circ g) - \tilde{Y}(x)(Xf \circ g) = X(g(x))(Yf) - Y(g(x))(Xf) = [X,Y](g(x))f.$$

The structure on a vector space E imposed by an alternating, bilinear map $[.,.] : E \times E \to E$, which satisfies the Jacobi identity is called a *Lie algebra*. The following formula reveals the true nature of the Lie bracket.

Theorem 2.3.1. *Let M be a smooth n-manifold and X, $Y \in \mathcal{X}(M)$. If $\Phi : D \to M$ is the flow of X, then*

$$[X,Y](p) = \lim_{t\to 0} \frac{1}{t}\big((\Phi_{-t})_{*\Phi(t,p)}(Y(\Phi(t,p)) - Y(p)\big)$$

for every $p \in M$.

For the proof we shall need the following technical lemma.

Lemma 2.3.2. *Let U, $V \subset M$ be two open neighbourhoods of the point $p \in M$ for which there exists $\epsilon > 0$ such that $\Phi((-\epsilon,\epsilon)\times V) \subset U$. Then, for every smooth function $f : U \to \mathbb{R}$ there exists a smooth function $g : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ with the following properties:*
(i) $f(\Phi(-t,q)) = f(q) - tg(t,q)$ for every $(t,q) \in (-\epsilon,\epsilon)\times V$.
(ii) $X(q)(f) = g(0,q)$ for every $q \in V$.

Proof. If $h : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ is the smooth function defined by $h(s,q) = f(\Phi(-s,q)) - f(q)$, and if we define $g : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ by

$$g(t,q) = -\int_0^1 \frac{\partial h}{\partial s}(ts,q)ds,$$

then

$$-tg(t,q) = \int_0^t \frac{\partial h}{\partial s}(s,q)ds = h(t,q).$$

By continuity, we also have

$$g(0,q) = \lim_{t\to 0} g(t,q) = \lim_{t\to 0} \frac{f(\Phi(-t,q)) - f(q)}{-t} = X(q)(f). \quad \square$$

Proof of Theorem 2.3.1. Let $f : U \to \mathbb{R}$ be a smooth function defined on an open neighbourhood U of the point $p \in M$. There exist an open neighbourhood V of p and $\epsilon > 0$ such that $\Phi((-\epsilon,\epsilon)\times V) \subset U$. Let g be the smooth function supplied by Lemma 2.3.2 and let $g_t = g(t,.)$. Then, $Xf = g_0$ and

$$\lim_{t\to 0} \frac{1}{t}\big((\Phi_{-t})_{*\Phi(t,p)}(Y(\Phi(t,p)) - Y(p)\big)(f)$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[f_{*p}\big((\Phi_{-t})_{*\Phi(t,p)}(Y(\Phi(t,p)))\big) - Y(p)(f)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Y(\Phi(t,p))(f\circ\Phi_{-t}) - Y(p)(f)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Y(\Phi(t,p))(f - tg_t) - Y(p)(f)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Y(\Phi(t,p))(f) - Y(p)(f)\Big] - \lim_{t\to 0} Y(\Phi(t,p))(g_t)$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[Yf(\Phi(t,p)) - Yf(p)\Big] - Y(p)(Xf)$$
$$= X(p)(Yf) - Y(p)(Xf) = [X,Y](p)(f). \quad \square$$

Definition 2.3.3. Two complete smooth vector fields X, Y on a smooth manifold M *commute* if $[X,Y]=0$.

This terminology is justified by the following.

Proposition 2.3.4. *Let X and Y be two smooth vector fields on a smooth manifold M. Let $(\Phi_t)_{t\in\mathbb{R}}$ be the one-parameter group of smooth diffeomorphisms of M defined by the flow of X and $(\Psi_t)_{t\in\mathbb{R}}$ be the one-parameter group of smooth diffeomorphisms defined by the flow of Y. Then $[X,Y]=0$ if and only if $\Phi_t\circ\Psi_s = \Psi_s\circ\Phi_t$ for every t, $s\in\mathbb{R}$.*

Proof. If $\Phi_t\circ\Psi_s = \Psi_s\circ\Phi_t$ for every t, $s\in\mathbb{R}$, differentiating with respect to s at 0, we get $(\Phi_t)_*Y = Y$ for every $t\in\mathbb{R}$. It follows now from Theorem 2.3.1 that $[X,Y]=0$.

Conversely, let $[X,Y]=0$ and let $p\in M$ and $s\in\mathbb{R}$. The velocity of the smooth curve $\gamma:\mathbb{R}\to T_{\Psi_s(p)}M$ defined by $\gamma(t) = (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}(Y((\Phi_t(\Psi_s(p))))$ is

$$\dot\gamma(t) = \lim_{h\to 0}\frac{1}{h}\Big[(\Phi_{-t+h})_{*\Phi_{t+h}(\Psi_s(p))}(Y(\Phi_{t+h}(\Psi_s(p))) - (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}(Y((\Phi_t(\Psi_s(p))))\Big]$$
$$= (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}\left(\lim_{h\to 0}\frac{1}{h}\Big[(\Phi_{-h})_{*\Phi_{t+h}(\Psi_s(p))}(Y(\Phi_h(\Phi_t(\Psi_s(p))))) - Y(\Phi_t(\Psi_s(p)))\Big]\right)$$
$$= (\Phi_{-t})_{*\Phi_t(\Psi_s(p))}([X,Y](\Phi_t(\Psi_s(p)))) = 0.$$

Thus, γ is constant, which means that $(\Phi_{-t})_{*\Phi_t(\Psi_s(p))}(Y((\Phi_t(\Psi_s(p)))) = Y(\Psi_s(p))$ or equivalently
$$Y(\Phi_t(\Psi_s(p))) = (\Phi_t)_{*\Psi_s(p)}(Y(\Psi_s(p)))$$
for every $p\in M$ and t, $s\in\mathbb{R}$. In other words, Y is Φ_t-invariant for every $t\in\mathbb{R}$. This implies that $\Phi_t\circ\Psi^p$ is an integral curve of Y and since $(\Phi_t\circ\Psi^p)(0) = \Phi_t(p)$, we must necessarily have $\Phi_t\circ\Psi^p = \Psi^{\Phi_t(p)}$, hence $\Phi_t(\Psi_s(p)) = \Psi_s(\Phi_t(p))$. $\square$

If X and Y are two commuting complete smooth vector fields on a smooth manifold M with corresponding one-parameter groups of smooth diffeomorphisms $(\Phi_t)_{t\in\mathbb{R}}$ and $(\Psi_t)_{t\in\mathbb{R}}$, respectively, then $F:\mathbb{R}^2\times M\to M$ defined by
$$F(t,s,p) = (\Phi_t\circ\Psi_s)(p)$$
is a smooth action of the abelian group $(\mathbb{R}^2,+)$ on M. More generally, a finite family of mutually commuting complete smooth vector fields X_1,..., X_k with corresponding one-parameter groups of smooth diffeomorphisms $(\Phi^1_t)_{t\in\mathbb{R}}$,..., $(\Phi^k_t)_{t\in\mathbb{R}}$, respectively, defines a smooth action $F:\mathbb{R}^k\times M\to M$ of the abelian group $(\mathbb{R}^k,+)$ by the formula
$$F(t_1,...,t_k,p) = (\Phi^1_{t_1}\circ\cdots\Phi^k_{t_k})(p).$$

2.4 Exercises

1. Let M be a smooth n-manifold, $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ be a smooth atlas of M and $\bar{\mathcal{A}} = \{(\pi^{-1}(U_i), \bar{\phi}_i) : i \in I\}$ be the corresponding smooth atlas of TM, where $\pi : TM \to M$ is the tangent bundle projection. Prove that

$$\det D(\bar{\phi}_i \circ \bar{\phi}_j^{-1})(x, v) > 0$$

for every i, $j \in I$ with $U_i \cap U_j \neq \varnothing$ and $(x, v) \in \phi_j(U_i \cap U_j) \times \mathbb{R}^n$.

2. Let M be a smooth manifold and G be a group of diffeomorphisms of M which acts properly discontinuously on M. If $X \in \mathcal{X}(M)$ and $g_*X = X$ for every $g \in G$, prove that there exists a unique $\tilde{X} \in \mathcal{X}(M/G)$ such that $p_{*p}(X(p)) = \tilde{X}(\pi(p)))$ for every $p \in M$, where $\pi : M \to M/G$ is the quotient map. Construct a smooth vector field on the real projective plane $\mathbb{R}P^2$, which vanishes at exactly one point and every other maximal integral curve is periodic.

3. A smooth n-manifold M is called parallelizable if there are X_1,X_2,...,$X_n \in \mathcal{X}(M)$ such that $\{X_1(p), X_2(p), ..., X_n(p)\}$ is a basis of T_pM for every $p \in M$. Prove that M is parellelizable if and only if its tangent bundle is trivial, which means that there exists a smooth diffeomorphism $f : TM \to M \times \mathbb{R}^n$ such that the following diagram commutes

$$\begin{tikzcd} TM \arrow[rr, "f"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^n \arrow[dl, "\text{projection}"] \\ & M & \end{tikzcd}$$

and f maps linearly T_pM onto $\{p\} \times \mathbb{R}^n$ for every $p \in M$. Prove that the circle S^1 and the n-torus T^n are parallelizable.

4. On $\mathbb{R}^{2n}$ the nowhere vanishing smooth vector field

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + ... + x^{2n} \frac{\partial}{\partial x^{2n-1}} - x^{2n-1} \frac{\partial}{\partial x^{2n}}$$

is tangent to S^{2n-1}. In case $n = 2$, complete this vector field with two other vector fields to prove that the 3-sphere S^3 are parallelizable.

5. Let M be a smooth manifold and $f : M \to M$ be a diffeomorphism. If $X \in \mathcal{X}(M)$ has flow $\Phi : D \to M$, prove that the flow Ψ of f_*X is given by the formula $\Psi(t, f(p)) = f(\Phi(t, p))$.

6. Let $h : [0, 1] \to [0, \pi]$ be a smooth function with $h^{-1}(0) = [0, 1/5] \cup [4/5, 1]$ and $h^{-1}(\pi/2) = [2/5, 3/5]$. We extend h on $\mathbb{R}$ periodically by $h(x + 1) = h(x)$. Prove that the smooth vector fields

$$X(t) = t^2 \cos^2 h(t) \frac{d}{dt} \text{ and } Y(t) = t^2 \sin^2 h(t) \frac{d}{dt}$$

on $\mathbb{R}$ are complete, but $X+Y$ is not complete.

7. Let M be a smooth manifold, $X \in \mathcal{X}(M)$ with flow $\phi : D \to M$, where

$$D = \bigcup_{p \in M} (a_p, b_p) \times \{p\}.$$

If $f : M \to (0,1]$ is a smooth function such that $f(p) < \min\{-a_p, b_p\}$ for every $p \in M$, prove that the smooth vector field $f \cdot X$ is complete.

8. On $\mathbb{R}^3$ we consider the smooth vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Prove that the map $g : \mathbb{R}^3 \to \mathcal{X}(\mathbb{R}^3)$ with

$$g(a,b,c) = aX + bY + cZ$$

is a linear monomorphism which has the additional property $g(A \times B) = [g(A), g(B)]$ for every A, $B \in \mathbb{R}^3$, where $\times$ is the usual exterior product on $\mathbb{R}^3$.

9. Let M be a smooth manifold and X, $Y \in \mathcal{X}(M)$ be complete with flows Φ and Ψ, respectively. If there exists a smooth function $h : M \to \mathbb{R}$ such that $[X,Y] = hX$, prove

$$(\Psi_t \circ \Phi_s)(p) = (\Phi_{T_p(t,s)} \circ \Psi_t)(p)$$

for every $p \in M$, t, $s \in \mathbb{R}$, where $T_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the smooth function

$$T_p(t,s) = \int_0^s \left(\exp\Big(\int_0^t h(\psi_\tau(\phi_\sigma(p))) d\tau \Big) \right) d\sigma.$$

Chapter 3

Riemannian manifolds

3.1 Connections

A straight line segment in euclidean n-space $\mathbb{R}^n$ is the unique piecewise smooth curve of minimum length between its endpoints. Equivalently, straight lines in $\mathbb{R}^n$ are the smooth curves whose acceleration vanishes identically. One way to define a notion of "straight line" on a smooth manifold is by defining first the notion of acceleration. The difficulty now lies in the fact that if M is a smooth manifold, $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth curve, the velocity vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(s)$ belong to different vector spaces for $t \neq s$ and their difference has no meaning. This difference can become meaningful if we have a way to connect the tangent spaces of M at the points $\gamma(t)$, $t \in I$. This requires the endowment of M with an extra structure. This structure can be described elegantly in an algebraic way.

Definition 3.1.1. A (linear) *connection* on a smooth n-manifold M is a map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

with the following properties, writing $\nabla_X Y$ instead of $\nabla(X, Y)$:
(i)$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$, for every f_1, $f_2 \in C^\infty(M)$ and X_1, X_2, $Y \in \mathcal{X}(M)$.
(ii) $\nabla_X(a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2$ for every a_1, $a_2 \in \mathbb{R}$ and X, Y_1, $Y_2 \in \mathcal{X}(M)$.
(iii) $\nabla_X(fY) = f \nabla_X Y + Xf \cdot Y$ for every $f \in C^\infty(M)$ and X, $Y \in \mathcal{X}(M)$.

The smooth vector field $\nabla_X Y$ is called the *covariant derivative of* Y *in the direction of* X. Some immediate consequences of the above definition are given in the following lemmas.

Lemma 3.1.2. *If ∇ is a connection on a smooth n-manifold M and $p \in M$, then for every X, $Y \in \mathcal{X}(M)$ the vector $(\nabla_X Y)(p) \in T_p M$ depends only on the values of X and Y in arbitrarily small open neighbourhoods of p.*

Proof. By bilinearity, it suffices to prove that $(\nabla_X Y)(p) = 0$ in case there exists an open neighbourhood V of p such that $X|_V = 0$ or $Y|_V = 0$. By Corollary 1.4.5, there exists a smooth function $f : M \to [0,1]$ such that $f(p) = 1$ and $\mathrm{supp} f \subset V$.

If $Y|_V = 0$, then $fY = 0$ on M and so

$$0 = \nabla_X(fY)(p) = f(p)(\nabla_X Y)(p) + (Xf)(p) \cdot Y(p) = (\nabla_X Y)(p).$$

If $X|_V = 0$, we have $fX = 0$ on M, and

$$0 = (\nabla_{fX} Y)(p) = f(p)(\nabla_X Y)(p) = (\nabla_X Y)(p). \quad \square$$

Lemma 3.1.3. *If ∇ is a connection on a smooth n-manifold M and $p \in M$, then for every X, $Y \in \mathcal{X}(M)$ the vector $(\nabla_X Y)(p) \in T_pM$ depends only on the tangent vector $X(p)$ and the values of Y in arbitrarily small open neighbourhoods of p.*

Proof. It suffices to prove that $(\nabla_X Y)(p) = 0$ if $X(p) = 0$. In view of the preceding Lemma 3.1.2, we can work locally in the domain of a smooth chart (U, ϕ) of M with $p \in U$. If $\phi = (x^1, ..., x^n)$, there exist X^1,..., $X^n \in C^\infty(U)$ such that

$$X|_U = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}.$$

If $X(p) = 0$, we have $X^k(p) = 0$ for $1 \le k \le n$ and

$$(\nabla_X Y)(p) = \sum_{k=1}^{n} X^k(p)\big(\nabla_{\frac{\partial}{\partial x^k}} Y\big)(p) = 0. \quad \square$$

According to the above Lemma 3.1.3, it is legitimate to write henceforth $\nabla_{X(p)} Y$ instead of $(\nabla_X Y)(p)$. The same argument of the proof shows that if

$$S : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{X}(M)$$

is a $C^\infty(M)$-m-multilinear map, then for every X_1,..., $X_m \in \mathcal{X}(M)$ and $p \in M$ the value $S(X_1, ..., X_m)(p)$ depends only on the values $X_1(p)$,..., $X_m(p)$ and so we can write $S(X_1(p), ..., X_m(p))$ instead.

Lemma 3.1.4. *If ∇ is a connection on a smooth n-manifold M and $p \in M$, then for every X, $Y \in \mathcal{X}(M)$ the vector $(\nabla_X Y)(p) \in T_pM$ depends only on the tangent vector $X(p)$ and the values $Y(\gamma(t))$ for any smooth curve $\gamma : (-\epsilon, \epsilon) \to M$, $\epsilon > 0$, such that $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$.*

Proof. According to the preceding Lemmas 3.1.2 and 3.1.3, we may assume that $\gamma((-\epsilon, \epsilon)) \subset U$ for some smooth chart (U, ϕ) of M with $p \in U$. Let $\phi = (x^1, ..., x^n)$. There exist Y^1,..., $Y^n \in C^\infty(U)$ such that

$$Y|_U = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x^k}$$

and

$$\nabla_{X(p)}Y = \sum_{k=1}^{n} Y^k(p)\nabla_{X(p)}\frac{\partial}{\partial x^k} + \sum_{k=1}^{n}(Y^k \circ \gamma)'(0)\left(\frac{\partial}{\partial x^k}\right)_p.$$

If $Y(\gamma(t)) = 0$ for all $|t| < \epsilon$, then obviously $\nabla_{X(p)}Y = 0$. □

We can now find a local formula for a given connection ∇ in the domain of a smooth chart (U, ϕ) of M with $\phi = (x^1, ..., x^n)$. There exist unique $\Gamma_{ij}^k \in C^\infty(U)$, $1 \leq i, j, k \leq n$, such that

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \sum_{k=1}^{n}\Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

for every $1 \leq i, j \leq n$. The smooth functions Γ_{ij}^k are called *the Christoffel symbols* of ∇ with respect to the smooth chart (U, ϕ). If now

$$X = \sum_{k=1}^{n} X^k\frac{\partial}{\partial x^k} \quad \text{and} \quad Y = \sum_{k=1}^{n} Y^k\frac{\partial}{\partial x^k},$$

a routine computation shows that on U we have

$$\nabla_X Y = \sum_{k=1}^{n}\left(X(Y^k) + \sum_{i,j=1}^{n}\Gamma_{ij}^k X^i Y^j\right)\frac{\partial}{\partial x^k}.$$

Conversely, given smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$, $1 \leq i, j, k \leq n$, the above formula defines a connection on U, because for every $f \in C^\infty(U)$ we have

$$\nabla_X(fY) = \sum_{k=1}^{n}\left(X(fY^k) + \sum_{i,j=1}^{n}\Gamma_{ij}^k X^i fY^j\right)\frac{\partial}{\partial x^k}$$

$$= \sum_{k=1}^{n}\left(Xf \cdot Y^k + fX(Y^k) + f\sum_{i,j=1}^{n}\Gamma_{ij}^k X^i Y^j\right)\frac{\partial}{\partial x^k} = Xf \cdot Y + f\nabla_X Y.$$

The connection on $\mathbb{R}^n$ with Christoffel symbols identically equal to zero is called the *euclidean connection* and is given by the formula

$$\nabla_X Y = \sum_{k=1}^{n} X(Y^k)\frac{\partial}{\partial x^k}.$$

In other words, the covariant derivative of Y in the direction of X with respect to the euclidean connection is the directional derivative of Y in the direction of X.

Example 3.1.5. A $(n-1)$-dimensional smooth submanifold M of $\mathbb{R}^n$ is usually called *hypersurface*. We identify the tangent space T_pM at a point $p \in M$ with its image under the derivative of the inclusion and consider it a vector subspace of $T_p\mathbb{R}^n$. The euclidean connection ∇ on $\mathbb{R}^n$ induces a connection on any hypersurface M in $\mathbb{R}^n$. We observe first that if $p \in M$ and (U, ϕ) is a M-straightening chart of $\mathbb{R}^n$ with $\phi(U \cap M) \subset \mathbb{R}^{n-1} \times \{0\}$ and $p \in U \cap M$, then for every $X \in \mathcal{X}(M)$ there exists an extension $\tilde{X} \in \mathcal{X}(U)$, that is $\tilde{X}|_{U\cap M} = X|_{U\cap M}$. For every X, $Y \in \mathcal{X}(M)$ we put now

$$\overline{\nabla}_{X(p)}Y = \pi_p(\nabla_{X(p)}\tilde{Y})$$

where $\pi_p : T_p\mathbb{R}^n \to T_pM$ is the projection with respect to the orthogonal splitting $T_p\mathbb{R}^n = T_pM \oplus (T_pM)^\perp$. By Lemma 3.1.4, this definition does not depend on the choice of the extension $\tilde{Y}$. Obviously, $\overline{\nabla}$ is a connection on M and is called the *euclidean connection* of the hypersurface M.

Proposition 3.1.6. *On every smooth manifold M there are connections.*

Proof. From the above, there are connections locally on M. Let $\mathcal{A}$ be a smooth atlas of M. For every $(U, \phi_U) \in \mathcal{A}$ there is a connection ∇^U on U. Let $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ be a smooth partition of unity subordinated to the open cover $\{U : (U, \phi_U) \in \mathcal{A}\}$ of M. Then, the formula

$$\nabla_X Y = \sum_{(U,\phi_U)\in\mathcal{A}} f_U \nabla^U_X Y$$

for X, $Y \in \mathcal{X}(M)$, defines a connection on M because if $f \in C^\infty(M)$, we have

$$\nabla_X(fY) = \sum_{(U,\phi_U)\in\mathcal{A}} f_U \nabla^U_X(fY) = \sum_{(U,\phi_U)\in\mathcal{A}} f_U(Xf \cdot Y + f\nabla^U_X Y)$$

$$= \Bigg(\sum_{(U,\phi_U)\in\mathcal{A}} f_U\Bigg) Xf \cdot Y + f \sum_{(U,\phi_U)\in\mathcal{A}} f_U \nabla^U_X Y = Xf \cdot Y + f\nabla_X Y. \quad \square$$

In view of Lemma 3.1.4, given a connection it is possible to define a covariant differentiation of smooth vector fields along a smooth curve. Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. The set $\mathcal{X}(\gamma)$ of smooth vector fields along γ is a vector space.

Proposition 3.1.7. *Let ∇ be a connection on a smooth n-manifold M. For every smooth curve $\gamma : I \to M$ there exists a unique linear operator*

$$\frac{D}{dt} : \mathcal{X}(\gamma) \to \mathcal{X}(\gamma)$$

with the following properties:
(i) $\dfrac{D}{dt}(fX) = f'X + f\dfrac{DX}{dt}$ *for every $X \in \mathcal{X}(\gamma)$ and smooth function $f : I \to \mathbb{R}$.*
(ii) If $X \in \mathcal{X}(\gamma)$ has a smooth extension $\tilde{X} \in \mathcal{X}(V)$ on an open set V which contains $\gamma(I)$, then

$$\frac{DX}{dt}(t) = \nabla_{\dot\gamma(t)}\tilde{X}, \quad t \in I.$$

The vector field $\dfrac{DX}{dt}$ *along γ is called the covariant derivative of X along γ.*

Proof. We shall prove uniqueness first. As in the proof of Lemma 3.1.2 we see that for every $t_0 \in I$ the value $\dfrac{DX}{dt}(t_0)$ depends only on the values of X on an arbitrarily small open interval with center t_0. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and $\gamma(t_0) \in U$. There exist $\epsilon > 0$ such that $\gamma((t_0-\epsilon, t_0+\epsilon)) \subset U$ and smooth functions X^1,..., $X^n : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}$ such that

$$X(t) = \sum_{k=1}^{n} X^k(t) \left(\frac{\partial}{\partial x^k} \right)_{\gamma(t)}$$

for $|t - t_0| < \epsilon$. By linearity and properties (i), (ii) we compute

$$\frac{DX}{dt}(t) = \sum_{k=1}^{n} (X^k)'(t) \left(\frac{\partial}{\partial x^k} \right)_{\gamma(t)} + \sum_{k=1}^{n} X^k(t) \nabla_{\dot\gamma(t)} \frac{\partial}{\partial x^k}$$

$$= \sum_{k=1}^{n} \left((X^k)'(t) + \sum_{i,j=1}^{n} \Gamma^k_{ij}(\gamma(t))(\gamma^i)'(t) X^j(t) \right) \left(\frac{\partial}{\partial x^k} \right)_{\gamma(t)},$$

where $(\phi \circ \gamma)(t) = (\gamma^1(t), ..., \gamma^n(t))$ for every $|t - t_0| < \epsilon$. This proves the uniqueness.

The existence follows covering $\gamma(I)$ by the domains of smooth charts of M and defining $\frac{D}{dt}$ locally by the above formula. By uniqueness, the local definitions coincide on overlapping intervals. $\square$

In the rest of the section we shall see that the algebraic definition of a connection indeed gives a mechanism of "connecting" tangent spaces at various points of a smooth manifold. Let ∇ be a connection on a smooth n-manifold M.

Definition 3.1.8. If $\gamma : I \to M$ is a smooth curve defined on an open interval $I \subset \mathbb{R}$, a smooth vector field $X \in \mathcal{X}(\gamma)$ is said to be *parallel along* γ, if $\frac{DX}{dt} = 0$ on I. A smooth vector field $X \in \mathcal{X}(M)$ is called *parallel* if $\nabla_Y X = 0$ on M for every $Y \in \mathcal{X}(M)$.

Example 3.1.9. The parallel vector fields on $\mathbb{R}^n$ with respect to the euclidean connection are the constant ones, that is the vector fields

$$\sum_{k=1}^{n} a^k \frac{\partial}{\partial x^k}$$

for a^1,..., $a^n \in \mathbb{R}$.

Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and let $\gamma : I \to U$ be a smooth curve with local representation $\phi \circ \gamma = (\gamma^1, ..., \gamma^n)$. From the formula of the covariant differentiation along γ derived in the proof of Proposition 3.1.7 follows that a smooth vector field

$$X(t) = \sum_{k=1}^{n} X^k(t) \left(\frac{\partial}{\partial x^k} \right)_{\gamma(t)}, \quad t \in I$$

along γ is parallel if and only if the smooth functions X^1,..., X^n satisfy the system of linear ordinary differential equations

$$(X^k)'(t) = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)X^j(t), \quad t \in I, \quad 1 \le k \le n.$$

From the existence and uniqueness of solutions for linear ordinary differential equations we have that for every $t_0 \in I$ and every $v \in T_{\gamma(t_0)}M$ there exists a unique parallel vector field X along γ satisfying the initial condition $X(t_0) = v$.

Proposition 3.1.10. *Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. For every $t_0 \in I$ and every $v \in T_{\gamma(t_0)}M$ there exists a unique parallel vector field X along γ such that $X(t_0) = v$.*

Proof. From the above there exists $b > t_0$ such that there exists a unique parallel vector field along $\gamma|_{[t_0,b]}$ with $X(t_0) = v$. It suffices to prove that the supremum T of all such b does not belong to I. Suppose that it does. Choosing a smooth chart (V, ψ) of M with $\gamma(T) \in V$, there exists $\delta > 0$ such that $\gamma((T-\delta, T+\delta)) \subset V$. From the above, there exists a unique parallel vector field $\tilde{X}$ along $\gamma|_{(T-\delta,T+\delta)}$ satisfying the initial condition $\tilde{X}(T - \frac{\delta}{2}) = X(T - \frac{\delta}{2})$. From the uniqueness of solutions we get $\tilde{X} = X$ on $(T - \delta, T)$ and so X has a smooth extension on $[t_0, T + \delta)$. This contradicts the definition of T. $\square$

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. The preceding Proposition 5.1.9 implies that for every a, $b \in I$ with $a < b$ there is a well defined map $\tau_{b,a} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ where $\tau_{b,a}(u)$ is the value $X(b)$ of the unique parallel vector field X along γ with $X(a) = u$. Since the parallel vector fields along γ are the solutions of a system of linear ordinary differential equations, $\tau_{b,a}$ is a linear isomorphism and it is called the *parallel translation along γ form $\gamma(a)$ to $\gamma(b)$*.

Theorem 3.1.11. *If $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ is a smooth curve, then for every $X \in \mathcal{X}(\gamma)$ and $s \in I$ we have*

$$\frac{DX}{dt}(s) = \lim_{h \to 0} \frac{1}{h}[\tau_{s,s+h}(X(s+h)) - X(s)].$$

Proof. It suffices to prove the assertion in case there exists a smooth chart (U, ϕ) and $\gamma(I) \subset U$. Since the parallel vector fields along γ are the solutions of a system of linear ordinary differential equations, there are parallel vector fields E_1,..., E_n along γ such that $\{E_1(t), ..., E_n(t)\}$ is a basis of $T_{\gamma(t)}M$ for every $t \in I$. Now there are unique smooth functions f_1,..., $f_n : I \to \mathbb{R}$ such that

$$X(t) = \sum_{k=1}^{n} f_k(t)E_k(t), \quad t \in I.$$

Therefore,

$$\frac{DX}{dt} = \sum_{k=1}^{n} f_k' \cdot E_k.$$

On the other hand, $\tau_{s,s+h}(E_k(s+h)) = E_k(s)$, because E_k is parallel along γ, $1 \leq k \leq n$, and hence

$$\tau_{s,s+h}(X(s+h)) - X(s) = \sum_{k=1}^{n} f_k(s+h)\tau_{s,s+h}(E_k(s+h)) - \sum_{k=1}^{n} f_k(s)E_k(s)$$
$$= \sum_{k=1}^{n} (f_k(s+h) - f_k(s))E_k(s).$$

It follows that

$$\lim_{h\to 0} \frac{1}{h}[\tau_{s,s+h}(X(s+h)) - X(s)] = \lim_{h\to 0} \sum_{k=1}^{n} \frac{f_k(s+h) - f_k(s)}{h} \cdot E_k(s) = \sum_{k=1}^{n} f'_k(s) \cdot E_k(s). \quad \square$$

3.2 Geodesics and exponential map

Let M be a smooth n-manifold and ∇ a connection on M. The *acceleration* of a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, is the smooth vector field $\dfrac{D\dot{\gamma}}{dt}$ along γ.

Definition 3.2.1. A smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, is called *geodesic* of the connection ∇ if $\dfrac{D\dot{\gamma}}{dt} = 0$.

Note that the differential equation of geodesics is independent of local coordinates of M. Its expression in the local coordinates of a smooth chart (U, ϕ) of M with $\phi = (x^1, ..., x^n)$, where $\phi \circ \gamma = (\gamma^1, ..., \gamma^n)$, is

$$(\gamma^k)''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)(\gamma^j)'(t) = 0, \quad 1 \leq k \leq n.$$

In the particular case of the euclidean connection on $\mathbb{R}^n$, where the Christoffel symbols vanish, it follows that the geodesics are the euclidean straight lines.

The geodesics in U are the projections under the tangent bundle projection $\pi : TM \to M$ of the integral curves of the smooth vector field

$$\sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} + \sum_{k=1}^{n} \Big(- \sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j \Big) \frac{\partial}{\partial v^k}$$

on $\pi^{-1}(U)$, where $\tilde{\phi} = (x^1, ..., x^n, v^1, ..., v^n)$ is the smooth chart of TM corresponding to (U, ϕ). Since the differential equation of geodesics does not depend on smooth charts, we conclude that this is the local representation in the smooth chart $(\pi^{-1}(U), \tilde{\phi})$ of a smooth vector field G which is globally defined on TM and is called the *geodesic vector field of the connection* ∇. Its flow is called the *geodesic flow of* ∇.

The homogeneity of the differential equation of geodesics implies the following property.

Lemma 3.2.2. *If $\gamma : I \to M$ is the geodesic of the connection ∇ defined on the open interval I and satisfying the initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, then for every $\lambda \in \mathbb{R} \setminus \{0\}$ the maximal geodesic γ_λ satisfying the initial conditions $\gamma_\lambda(0) = p$ and $\dot{\gamma}_\lambda(0) = \lambda v$ is defined on the open interval $\frac{1}{\lambda} I$ and is given by $\gamma_\lambda(t) = \gamma(\lambda t)$.*

Proof. Indeed $\dot{\gamma}_\lambda = \lambda \dot{\gamma}$ and therefore $\frac{D\dot{\gamma}_\lambda}{dt} = \lambda^2 \frac{D\dot{\gamma}}{dt}$. Hence γ_λ is a geodesic if and only if γ is. □

In the rest of the section we fix a connection ∇ on a smooth n-manifold M. Let $E \subset TM$ denote the set of all points $(p, v) \in TM$ such that the geodesic $\gamma_{(p,v)}$ from p with initial velocity v is defined on the unit interval $[0, 1]$. Let $\exp : E \to M$ be the smooth map $\exp(p, v) = \gamma_{(p,v)}(1)$. From Lemma 3.2.2, for every $p \in M$ the set $E_p = E \cap T_pM$ is an open neighbourhood of $0 \in T_pM$ and the map $\exp_p(v) = \exp(p, v)$ is smooth.

Lemma 3.2.3. *For every $p \in M$ the set E_p is star-shaped with respect to $0 \in T_pM$ and the geodesic $\gamma_{(p,v)}$ from p with initial velocity v is given by the formula*

$$\gamma_{(p,v)}(t) = \exp_p(tv)$$

for all $t \in \mathbb{R}$ for which at least one of the two sides is defined.

Proof. From Lemma 3.2.2. we have $\gamma_{(p,v)}(t) = \gamma_{(p,v)}(t \cdot 1) = \exp_p(tv)$ for every $t \in \mathbb{R}$ such that at least one of the two sides is defined. Moreover, if $v \in E_p$, then $\gamma_{(p,v)}$ is defined at least on $[0, 1]$ and hence $tv \in E_p$ for all $0 \leq t \leq 1$. This means that E_p is star-shaped with respect to $0 \in T_pM$. □

Proposition 3.2.4. *For every point $p \in M$ there exist an open neighbourhood V of $0 \in T_pM$ and an open neighbourhood U of p in M such that $\exp_p(V) = U$ and $\exp_p : V \to U$ is a smooth diffeomorphism.*

Proof. According to the Inverse Map Theorem it suffices to prove that the derivative $(\exp_p)_{*0} : T_0(T_pM) \cong T_pM \to T_pM$ is a linear isomorphism. If $v \in T_pM$ and $\sigma : \mathbb{R} \to T_pM$ is the straight line $\sigma(t) = tv$, and $\gamma_{(p,v)}$ is the geodesic from p with initial velocity v, we have

$$(\exp_p)_{*0}(v) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(\sigma(t)) = \dot{\gamma}_{(p,v)}(0) = v.$$

Hence $(\exp_p)_{*0} = id_{T_pM}$. □

Choosing a basis of T_pM, that is a linear isomorphism $h : T_pM \to \mathbb{R}^n$, the pair $(U, h \circ (\exp_p|_V)^{-1})$ is a smooth chart of M and is called a *normal chart* of M at p (with respect to the connection ∇). The neighbourhood U of p in Proposition 5.2.4 is called normal. Observe that the local representations of geodesics emanating from p with respect to a normal chart at p are straight lines through 0. Thus, if $(\gamma^1, ..., \gamma^n)$

is the local representation of any geodesic γ emanating from p with respect to a normal chart at p, then

$$\sum_{i,j=1}^{n} \Gamma_{ij}^{k}(p)(\gamma^i)'(0)\gamma^j)'(0) = 0, \quad 1 \leq k \leq n.$$

This means that the polynomial

$$\sum_{i,j=1}^{n} \Gamma_{ij}^{k}(p) v^i v^j$$

vanishes identically on some open neighbourhood of $0 \in \mathbb{R}^n$. Therefore,

$$\Gamma_{ij}^{k}(p) + \Gamma_{ji}^{k}(p) = 0$$

for every $1 \leq i, j, k \leq n$.

Given a connection ∇ on a smooth n-manifold M, we define its *torsion* to be the $C^\infty(M)$-bilinear map $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ with

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Thus the value of $T(X, Y)$ at a point $p \in M$ depends only on the values $X(p)$ and $Y(p)$.

The connection ∇ is said to be *symmetric* if its torsion vanishes. This terminology is justified as follows. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$. If X, $Y \in \mathcal{X}(M)$ and

$$X|_U = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k} \quad \text{and} \quad Y|_U = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x^k},$$

we have

$$T(X, Y)|_U = \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} (\Gamma_{ij}^{k} - \Gamma_{ji}^{k}) X^i Y^j \right) \frac{\partial}{\partial x^k}.$$

Hence ∇ is symmetric if and only if the Christoffel symbols with respect to any smooth chart are symmetric with respect to the lower indices, that is $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$ for every $1 \leq i, j, k \leq n$.

It follows from the above that if ∇ is a symmetric connection and $p \in M$, then the Christoffel symbols with respect to a normal chart at p vanish at the point p.

Proposition 3.3.5. *For every connection ∇ on a smooth n-manifold M there exists a unique symmetric connection $\overline{\nabla}$ on M which has the same geodesics as ∇.*

Proof. If T is the torsion of ∇, we define the connection $\overline{\nabla}$ by

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T(X, Y).$$

Since $T(X,X) = 0$ for every $X \in \mathcal{X}(M)$, it follows that $\overline{\nabla}$ and ∇ have the same geodesics. The uniqueness is the fact that two symmetric connections with the same geodesics coincide. Indeed, if ∇^1 and ∇^2 are two symmetric connections, then

$$S = \nabla^1 - \nabla^2 : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

is a symmetric $C^\infty(M)$-bilinear map. If ∇^1 and ∇^2 have the same geodesics, $S(X,X) = 0$ for every $X \in \mathcal{X}(M)$ and therefore

$$2S(X,Y) = S(X+Y, X+Y) = 0$$

for every X, $Y \in \mathcal{X}(M)$. □

3.3 Riemannian metrics

A *Riemannian metric* on a smooth n-manifold M is a family $g = (g_p)_{p \in M}$ of inner products

$$g_p : T_pM \times T_pM \to T_pM$$

which depend smoothly on p in the sense that if $U \subset M$ is an open set and X, $Y \in \mathcal{X}(U)$, then the function $f : U \to \mathbb{R}$ with $f(p) = g_p(X(p), Y(p))$ is smooth. A *Riemannian manifold* is a smooth manifold endowed with a Riemannian metric.

Let (M,g) and (N,h) be two Riemannian manifolds. A smooth map $f : M \to N$ is called *(Riemannian) isometry* if it is a smooth diffeomorphism and its derivative at each point preserves the Riemannian metrics, that is

$$h_{f(p)}(f_{*p}(v), f_{*p}(w)) = g_p(v,w)$$

for every v, $w \in T_pM$ and $p \in M$. The isometries are the isomorphisms of the category with objects the Riemannian manifolds and the aim of Riemannian Geometry is the classification of Riemannian manifolds up to isometry.

In the sequel we shall use in any case the symbol $\langle .,. \rangle$ to denote the Riemannian metric and the symbol $\|.\|$ for its corresponding norm on tangent spaces, if there is no danger of confusion.

If M is a Riemannian manifold, the set $I(M)$ of all isometries of M onto itself is a subgroup of its group of diffeomorphisms and is called the *isometry group* of M. If the action of $I(M)$ on M by evaluation is transitive, M is called *homogeneous*. Recall that the *isotropy group (or stabilizer)* at a point p is the subgroup

$$I_p(M) = \{f | f \in I(M) \quad \text{and } f(p) = p\}$$

of $I(M)$. The derivative of an element $f \in I_p(M)$ is an orthogonal transformation, that is linear isometry, $f_{*p} : T_pM \to T_pM$. It follows from the chain rule, that the assignment of f_{*p} to $f \in I_p(M)$ is a homomorphism of $I_p(M)$ into the group of the orthogonal transformations of T_pM which is usually called the *isotropic representation at p*. The point p is called *isotropic* if the action of $I_p(M)$ on the unit sphere in T_pM via the isotropic representation at p is transitive. Thus $p \in M$ is isotropic if for every v, $w \in T_pM$ with $\|v\| = \|w\| = 1$ there exists $f \in I_p(M)$ such that $f_{*p}(v) = w$. A Riamannian manifold M is called *isotropic* if every point of M is isotropic.

Example 3.3.1. On every open set $M \subset \mathbb{R}^n$, $n \geq 1$ the euclidean inner product of $\mathbb{R}^n$ defines a Riemannian metric in the obvious way which is called the *euclidean Riamannian metric.* Evidently, the euclidean n-space $\mathbb{R}^n$ is a homogeneous and isotropic Riemannian manifold.

Proposition 3.3.2. *On every smooth n-manifold there are Riemannian metrics.*

Proof. Let M be a smooth n-manifold and let $\mathcal{A}$ be a smooth atlas of M. For every $(U, \phi_U) \in \mathcal{A}$ there is a Riemannian metric g^U on U defined by

$$g_p^U(v, w) = \langle (\phi_U)_{*p}(v), (\phi_U)_{*p}(w) \rangle$$

for v, $w \in T_pM$, $p \in U$, where $\langle .,. \rangle$ is the euclidean inner product in $\mathbb{R}^n$. Let $\{f_U : (U, \phi_U) \in \mathcal{A}\}$ be a smooth partition of unity subordinated to the open cover $\mathcal{U} = \{U : (U, \phi_U) \in \mathcal{A}\}$ of M. For every $p \in M$ and v, $w \in T_pM$ we define

$$g_p(v, w) = \sum_{(U,\phi_U)\in\mathcal{A}} f_U(p) g_p^U(v, w).$$

Since g is locally a convex combination of Riemannan metrics, it is a Riemannian metric itself. □

In the rest of the section we shall give in some detail several examples of Riemannian manifolds.

Example 3.3.3. Let (M, g) be a Riemannian manifold and let $i : N \to M$ be an immersion of the smooth manifold N into M. There is an induced by i Riemannian metric g^N on N defined by

$$g_p^N(v, w) = g_{i(p)}(i_{*p}(v), i_{*p}(w))$$

for every v, $w \in T_pN$ and $p \in N$. In particular, every smooth submanifold of M inherits a Riemannian metric.

The n-sphere $S_R^n = \{p \in \mathbb{R}^{n+1} : \|p\| = R\}$ of radius $R > 0$ inherits a Riamannian metric from the euclidean Riemannian metric $\langle .,. \rangle$ of $\mathbb{R}^{n+1}$. Obviously, the orthogonal group $O(n+1, \mathbb{R})$ is contained in the isometry group of $I(S_R^n)$. Actually, it can be proved that $O(n+1, \mathbb{R})$ coincides with $I(S_R^n)$, but we will not need this for the time being. We shall show that S_R^n is homogeneous and isotropic with one strike. Let $p \in S_R^n$ and let $\{E_1, ..., E_n\}$ be an orthonormal basis of $T_pS_R^n$. Then,

$$\{E_1, ..., E_n, \frac{1}{R}p\}$$

is an orthonormal basis of $T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ and there exists $f \in O(n+1, \mathbb{R})$ such that

$$f(e_k) = E_k, \quad 1 \leq k \leq n, \quad f(Re_{n+1}) = p.$$

This implies that S_R^n is homogeneous and isotropic, since every point p is the image of the north pole Re_{n+1} and $I_{Re_{n+1}}(S_R^n)$ acts transitively on the set of orthonormal basis of $T_{Re_{n+1}}S_R^n$.

Example 3.3.4. The hyperbolic metric on the upper half plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$$

is defined by

$$g_z(v, w) = \frac{1}{(\mathrm{Im} z)^2}\langle v, w\rangle = \frac{1}{(\mathrm{Im} z)^2}\mathrm{Re}(v\overline{w})$$

for $v,\ w \in T_z\mathbb{H}^2$, $z \in \mathbb{H}^2$, where $\langle v, w\rangle = \mathrm{Re}(v\overline{w})$ is the euclidean inner product in complex notation.

The reflection with respect to the imaginary semi-axis $\ell = \{it : t > 0\}$ is the map $\tau : \mathbb{H}^2 \to \mathbb{H}^2$ with $\tau(z) = -\overline{z}$ and is an orientation reversing isometry of $\mathbb{H}^2$.

If $a,\ b,\ c,\ d \in \mathbb{R}$ and $ad - bc = 1$, for the Möbius transformation $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with

$$T(z) = \frac{az + b}{cz + d}$$

we have

$$\mathrm{Im}(T(z)) = \frac{\mathrm{Im} z}{|cz + d|^2}$$

and

$$T'(z) = \frac{1}{(cz + d)^2}.$$

Therefore, $T(\mathbb{H}^2) = \mathbb{H}^2$ and

$$g_{T(z)}(T_{*z}(v), T_{*z}(w)) = g_{T(z)}(T'(z)v, T'(z)w) = \frac{1}{(\mathrm{Im} T(z))^2}\mathrm{Re}(|T'(z)|^2 v\overline{w})$$

$$= \frac{1}{(\mathrm{Im} z)^2}\mathrm{Re}(v\overline{w}) = g_z(v, w)$$

for every $v,\ w \in T_z\mathbb{H}^2$ and $z \in \mathbb{H}^2$. Therefore the group of Möbius transformations with real coefficients, which is isomorphic to $PSL(2, \mathbb{R})$, is a subgroup of the isometry group $I(\mathbb{H}^2)$. It can be proved that this is the group of orientation preserving isometries of $\mathbb{H}^2$ and it has index 2 in $I(\mathbb{H}^2)$, but we will not need this now.

The action of $PSL(2, \mathbb{R})$ on $\mathbb{H}^2$ by Möbius transformations is transitive because if $z_0 = a + ib$, $a \in \mathbb{R}$, $b > 0$, then $z_0 = T(i)$, where T is the Möbius transformation

$$T(z) = \frac{\sqrt{b}z + \frac{a}{\sqrt{b}}}{0z + \frac{1}{\sqrt{b}}} = bz + a.$$

Thus, $\mathbb{H}^2$ is homogeneous. It is isotropic as well. Indeed, if $v \in T_i\mathbb{H}^2$ and $g_i(v, v) = 1$, there exists $0 \le \theta < 2\pi$ such that $v = e^{-2i\theta}$. If

$$T(z) = \frac{\cos\theta \cdot z - \sin\theta}{\sin\theta \cdot z + \cos\theta},$$

then $T(i) = i$ and $T'(i) = e^{-2i\theta}$. Hence $v = T_{*i}(1)$.

The Riemannian manifold $\mathbb{H}^2$ is the Poincaré upper half-plane model of the *hyperbolic plane.*

Example 3.3.5. We shall describe three models of the higher dimensional version of the hyperbolic plane. The first one resembles the case of the sphere. Let $n \geq 2$, $R > 0$ and

$$\mathbb{H}^n_R = \{(x_1,,,,.x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -R^2, \quad x_{n+1} > 0\}$$

be the upper connected component of the two-sheeted hyperboloid in $\mathbb{R}^{n+1}$. On $\mathbb{H}^n_R$ we consider the Riemannian metric which on each tangent space is the restriction of the Minkowski non-degenerate symmetric bilinear form

$$\langle x, y\rangle = -x_{n+1}y_{n+1} + \sum_{k=1}^{n} x_k y_k$$

where $x = (x_1, ..., x_{n+1})$, $y = (y_1, ..., y_{n+1})$. Although the Minkowski form is not positive definite, its restriction on each tangent space $T_p\mathbb{H}^n_R$, $p \in \mathbb{H}^n_R$, is. To see this, suppose that $p = (p_1, ..., p_{n+1})$. If $v = (v_1, ..., v_{n+1}) \in T_p\mathbb{H}^n_R$, then

$$p_1v_1 + \cdots + x_nv_n - p_{n+1}v_{n+1} = 0$$

and

$$\langle v, v\rangle = \sum_{k=1}^{n} v_k^2 - \frac{1}{p_{n+1}^2}\left(\sum_{k=1}^{n} p_k v_k\right)^2 \geq \left(1 - \frac{p_{n+1}^2 - R^2}{p_{n+1}^2}\right)\sum_{k=1}^{n} v_k^2 \geq 0$$

from the Cauchy-Schwartz inequality, and $\langle v, v\rangle = 0$ if and only if $v_1 = \cdots = v_n = 0$ and therefore $v_{n+1} = 0$ as well, since $p_{n+1} > 0$.

The Riemannian manifold $\mathbb{H}^n_R$ is called the *hyperbolic n-space of radius $R > 0$.* An alternative model is the upper half n-space, which we denote temporarily by $\mathbb{U}^n_R = \{(p_1, ..., p_n) \in \mathbb{R}^n : p_n > 0\}$, endowed with the Riemannian metric

$$g_p(v, w) = \frac{R^2}{p_n^2}\sum_{k=1}^{n} v_k w_k$$

where $p = (p_1, ..., p_n) \in \mathbb{U}^n_R$ and $v = (v_1, ..., v_n)$, $w = (w_1, ..., w_n) \in T_p\mathbb{U}^n_R$. A tedious calculation shows that the map $F : \mathbb{H}^n_R \to \mathbb{U}^n_R$ defined by

$$F(x_1, ..., x_n, x_{n+1}) = \left(\frac{x_1(R + x_{n+1})}{x_{n+1} - x_n}, ..., \frac{x_{n-1}(R + x_{n+1})}{x_{n+1} - x_n}, \frac{R^2}{x_{n+1} - x_n}\right)$$

is an isometry. So we use henceforth the notation $\mathbb{H}^n_R$ for both models.

The group $O_+(n, 1)$ of linear automorphisms of $\mathbb{R}^{n+1}$ which preserve the Minkowski form and send $\mathbb{H}^n_R$ onto itself is contained in the isometry group $I(\mathbb{H}^n_R)$. In this case too, it can be proved that this is the entire isometry group, but we will not need this fact now. In a similar way as in the case of the n-sphere S^n_R we can prove that $\mathbb{H}^n_R$ is homogeneous and isotropic. Let $p = (p_1, ..., p_n) \in \mathbb{H}^n_R$, so $\langle p,p\rangle = -R^2$, $p_{n+1} > 0$. and let $\{E_1,...,E_n\}$ be an orthonormal basis of $T_p\mathbb{H}^n_R$. Then, $\langle E_k, p\rangle = 0$, $1 \le k \le n$ and so

$$\{E_1,...,E_n,\frac{1}{R}p\}$$

is a basis of $\mathbb{R}^{n+1}$. If now $A \in O_+(n,1)$ is the matrix with columns E_1,..., E_n, $\frac{1}{R}p$, then $A(Re_{n+1}) = p$, which shows that $O_+(n,1)$ acts transitively on $\mathbb{H}^n_R$, and $Ae_k = E_k$, $1 \le k \le n$, which shows that $\mathbb{H}^n_R$ is isotropic, since $\{e_1,...,e_n\}$ is an orthonormal basis of $T_{Re_{n+1}}\mathbb{H}^n_R$.

There is a third convenient model of the hyperbolic n-space of radius $R > 0$. The affine diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$f(x) = x + (\frac{1}{2} - 2x^n)e_n$$

for $x = (x^1,...,x^n)$ maps the upper-half space $\mathbb{H}^n_R$ onto the open half-space $E = \{(y^1,...,y^n) \in \mathbb{R}^n : y^n < \frac{1}{2}\}$. The hyperbolic Riemannian metric is mapped by f to the Riemannian metric

$$\langle u,v\rangle_y = \frac{R^2}{(\frac{1}{2} - y^n)^2}\langle u,v\rangle$$

for u,$v \in T_yE$, $y = (y^1,...,y^n) \in E$, where $\langle .,.\rangle$ on the right hand side denotes the euclidean inner product. The diffeomorphism $g : \mathbb{R}^n \setminus \{e_n\} \to \mathbb{R}^n \setminus \{e_n\}$ defined by

$$g(y) = e_n + \frac{1}{\|y - e_n\|^2}(y - e_n)$$

is the inversion with respect to the sphere of radius 1 with center e_n and maps E onto the open unit n-ball $\mathbb{D}^n = \{z \in \mathbb{R}^n : \|z\| < 1\}$. Note that $g = g^{-1}$. Differentiating,

$$g_{*y}(u) = \frac{1}{\|y - e_n\|^2}u - \frac{2\langle y - e_n, u\rangle}{\|y - e_n\|^4}(y - e_n)$$

for every $u \in T_y\mathbb{R}^n$, $y \in \mathbb{R}^n \setminus \{e_n\}$. The hyperbolic Riemannian metric on $\mathbb{H}^n_R$ is now mapped by $g \circ f$ to the Riemannian metric on $\mathbb{D}^n$ given by the formula

$$\langle u,w\rangle_z = \langle g^{-1}_{*z}(u), g^{-1}_{*z}(w)\rangle_{g^{-1}(z)} = \frac{4R^2}{(1 - \|z\|^2)^2}\langle u,w\rangle$$

for $z \in \mathbb{R}^n$ with $\|z\| < 1$ and u, $w \in T_z\mathbb{D}^n$. The open unit n-ball endowed with this Riemannian metric is thus an alternative model of the hyperbolic n-space of radius $R > 0$ and will be denoted by $\mathbb{D}^n_R$.

Example 3.3.6. Let $n \ge 1$ and $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ be the quotient map. Recall that in the canonical atlas $\{(V_j, \phi_j) : 0 \le j \le n\}$ of $\mathbb{C}P^n$ we have

$$V_j = \{[z_0,...,z_n] \in \mathbb{C}P^n : z_j \neq 0\}$$

and
$$\phi_j[z_0,...,z_n] = (\frac{z_0}{z_j},...,\frac{z_{j-1}}{z_j},\frac{z_{j+1}}{z_j},...,\frac{z_n}{z_j}).$$
The quotient map π is a submersion. To see this note first that its local representation $\phi_0 \circ \pi : \pi^{-1}(V_0) \to \mathbb{C}^n$ with respect to the smooth chart (V_0, ϕ_0) is given by the formula
$$(\phi_0 \circ \pi)(z_0,...,z_n) = (\frac{z_1}{z_0},...,\frac{z_n}{z_0}).$$
Let $z = (z_0,...,z_n) \in \pi^{-1}(V_0)$ and $v = (v_0,...,v_n) \in T_z\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ be non-zero. Then $v = \dot\gamma(0)$, where $\gamma(t) = z + tv$, and
$$(\phi_0 \circ \pi \circ \gamma)(t) = \left(\frac{z_1+tv_1}{z_0+tv_0},...,\frac{z_n+tv_n}{z_0+tv_0}\right)$$
so that
$$(\phi_0 \circ \pi \circ \gamma)'(0) = \left(\frac{v_1}{z_0} - \frac{z_1v_0}{z_0^2},...,\frac{v_n}{z_0} - \frac{z_nv_0}{z_0^2}\right).$$
This implies that $v \in \text{Ker } \pi_{*z}$ if and only if $[v_0,...,v_n] = [z_0,...,z_n]$. In other words $\text{Ker } \pi_{*z} = \{\lambda z : \lambda \in \mathbb{C}\}$. Obviously, for every $(\zeta_0,...,\zeta_n) \in \mathbb{C}^n$ there exists $v = (v_0,...,v_n) \in \mathbb{C}^{n+1}$ such that
$$\zeta_j = \frac{v_j}{z_0} - \frac{z_jv_0}{z_0^2}.$$
Since the same holds for any other chart (V_j, ϕ_j) instead of (V_0, ϕ_0), this shows that π is a submersion.

The inclusion $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}\setminus\{0\}$ is an embedding and so its derivative at every point of S^{2n+1} is a linear monomorphism. For every $z \in S^{2n+1}$ we have
$$\text{Ker}(\pi|_{S^{2n+1}})_{*z} = \text{Ker}\pi_{*z} \cap T_zS^{2n+1} = \{\lambda z : \lambda \in \mathbb{C} \text{ and } \text{Re}\lambda = 0\}$$
which is a real line. On the other hand, $\pi^{-1}(\pi(z)) \cap S^{2n+1}$ is the trace of the smooth curve $\sigma : \mathbb{R} \to S^{2n+1}$ with $\sigma(t) = e^{it}z$ for which $\sigma(0) = z$ and $\dot\sigma(0) = iz$. Therefore $\text{Ker}(\pi|_{S^{2n+1}})_{*z}$ is generated by $\dot\sigma(0)$.

Let h be the usual hermitian product on $\mathbb{C}^{n+1}$. If
$$W_z = \{\eta \in T_z\mathbb{C}^{n+1} : h(\eta, z) = 0\},$$
then $\pi_{*z}|_{W_z} : W_z \to T_{[z]}\mathbb{C}P^n$ is a linear isomorphism for every $z \in \mathbb{C}^{n+1}\setminus\{0\}$. Indeed, for every $v \in T_z\mathbb{C}^{n+1}$ there are unique $\lambda \in \mathbb{C}$ and $\eta \in W_z$ such that $v = \lambda z + \eta$. Obviously,
$$\lambda = \frac{h(v,z)}{h(z,z)}, \qquad \eta = v - \frac{h(v,z)}{h(z,z)} \cdot z.$$
The restricted hermitian product on W_z can be transfered isomorphically by π_{*z} on $T_{[z]}\mathbb{C}P^n$. If now
$$g_{[z]}(v,w) = \text{Re } h((\pi_{*z}|_{W_z})^{-1}(v), (\pi_{*z}|_{W_z})^{-1}(w))$$
for v, $w \in T_{[z]}\mathbb{C}P^n$, then g is Riemannian metric on $\mathbb{C}P^n$ called the *Fubini-Study metric*. If $z \in S^{2n+1}$, then $W_z = \{v \in T_zS^{2n+1} : \langle v, \dot\sigma(0)\rangle = 0\}$.

Each element $A \in U(n+1)$ induces a diffeomorphism $\tilde{A} : \mathbb{C}P^n \to \mathbb{C}P^n$. Moreover, $A(W_z) = W_{A(z)}$ for every $z \in \mathbb{C}^{n+1} \setminus \{0\}$ and therefore $\tilde{A}$ is an isometry of the Fubini-Study metric. In this way, $U(n+1)$ acts on $\mathbb{C}P^n$ by isometries. The action is transitive and so $\mathbb{C}P^n$ is a homogeneous Riemannian manifold with respect to the Fubibi-Study metric. Indeed, $U(n+1)$ acts transitively on S^{2n+1}, because if $z \in S^{2n+1}$, there exist $E_1, \dots E_n \in \mathbb{C}^{n+1}$ such that $\{E_1, \dots E_n, z\}$ is an h-orthonormal basis of $\mathbb{C}^{n+1}$. The matrix U with columns $E_1, \dots, E_n$, z is an element of $U(n+1)$ such that $U(e_j) = E_j$ for $1 \le j \le n$ and $U(e_{n+1}) = z$. This last equality shows that $U(n+1)$ acts transitively on $\mathbb{C}P^n$.

The isotropy group of $[e_{n+1}] = [0, \dots, 0, 1]$ consists of all $A \in U(n+1)$ such that $\lambda A(e_{n+1}) = e_{n+1}$ for some $\lambda \in S^1$. This means that

$$\lambda A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $B \in U(n)$. Since $\tilde{A} = \widetilde{\lambda A}$, this implies that the isotropy group of $[e_{n+1}]$ is $U(n)$, considered as a subgroup of $U(n+1)$ as above, and therefore $\mathbb{C}P^n$ is diffeomorphic to the homogeneous space $U(n+1)/U(n)$.

If $A \in U(n+1)$, then $\det A \in S^1$ and taking $a \in S^1$ such that $a^n = \det A$ we have $a^{-1}A \in SU(n+1)$ and $\tilde{A} = \widetilde{a^{-1}A}$. Hence $SU(n+1)$ acts also transitively on $\mathbb{C}P^n$ and $\mathbb{C}P^n$ is diffeomorphic to $SU(n+1)/U(n)$, if we identify $U(n)$ with the subgroup of $SU(n+1)$ consisting of matrices of the form

$$\begin{pmatrix} B & 0 \\ 0 & \frac{1}{\det B} \end{pmatrix}$$

for $B \in U(n)$. If $A \in SU(n+1)$ belongs to the isotropy group of $[e_{n+1}]$ and λA has the above form, then $\det B = \lambda^{n+1}$ and putting $B' = \dfrac{1}{\lambda} B$, we have now

$$A = \begin{pmatrix} B' & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

where $\det B' = \lambda$. Therefore $A \in U(n)$, as a subgroup of $SU(n+1)$.

Example 3.3.7. If (M, g) and (N, h) are two Riemannian manifolds, on the product manifold $M \times N$ there is a Riemannian metric $\langle ., . \rangle$ defined by

$$\langle v, w \rangle_p = g_{p_1}(v_1, w_1) + h_{p_2}(v_2, w_2)$$

for $v = (v_1, v_2)$, $w = (w_1, w_2) \in T_p(M \times N) = T_{p_1}M \oplus T_{p_2}N$, $p = (p_1, p_2) \in M \times N$, which is called the product Riemannian metric.

Example 3.3.8. Let M be a Riemannian manifold and let G be a subgroup of its isometry group $I(M)$ which acts properly discontinuously on M, that is every point $p \in M$ has an open neighbourhood U in M such that $g(U) \cap U = \varnothing$ for all $g \in G$, $g \neq id_M$. If the orbit space M/G is Hausdorff, it is a smooth manifold and the quotient map $\pi : M \to M/G$ is a smooth covering map, in particular a local diffeomorphism as it maps each open neighbourhood like U above diffeomorphically onto $\pi(U)$.

Let $p \in M$, $g \in G$ and $q = g(p)$. Since $\pi \circ g = \pi$, from the chain rule we have $\pi_{*q} \circ g_{*p} = \pi_{*p}$, and since g is an isometry, it follows that

$$\langle \pi_{*q}^{-1}(v), \pi_{*q}^{-1}(w)\rangle_q = \langle g_{*p}^{-1}(\pi_{*q}^{-1}(v)), g_{*p}^{-1}(\pi_{*q}^{-1}(w))\rangle_p = \langle \pi_{*p}^{-1}(v), \pi_{*p}^{-1}(w)\rangle_p$$

for every v, $w \in T_{\pi(p)}(M/G)$. This means that there is a unique well defined Riemannian metric $\tilde{g}$ on M/G with respect to which π becomes a local isometry, as it maps each open neighbourhood U as above isometrically onto $\pi(U)$.

In the special case $M = S^n$ and $G = \{id_{S^n}, a\} \cong \mathbb{Z}_2$, where $a(x) = -x$ is the antipodal map, we obtain a Riemannian metric on the real projective n-space $\mathbb{R}P^n$ which is locally isometric to the euclidean Riemannian metric on S^n. Similarly, the group of translations of $\mathbb{R}^n$ by a vector in $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^n$ and acts properly discontinuously on $\mathbb{R}^n$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomotphic to the n-torus $T^n = S^1 \times \cdots \times S^1$, n-times. Since translations are euclidean isometries, we obtain a Riemannian metric on T^n such that the quotient map $\pi : \mathbb{R}^n \to T^n$ which is given by

$$\pi(t_1, ..., t_n) = (e^{it_1}, ..., e^{it_n})$$

becomes a local isometry. The n-torus T^n equipped with this Riemannian metric is usually called *flat n-torus.*

3.4 The Levi-Civita connection

In this section we shall prove that on a Riemannian manifold there exists a unique symmetric connection which is compatible with the Riemannian metric in the sense that parallel translation along smooth curves with respect to this connection is a linear isometry of inner product vector spaces. This result is sometimes called the Fundamental Theorem of Riemannian Geometry. Connections on a Riamannian manifold which are compatible with the Riemannian metric are characterized as follows.

Proposition 3.4.1. *Let M be a Riemannian smooth n-manifold. For a connection ∇ on M the following statements are equivalent.*
(i) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ for every X, Y, $Z \in \mathcal{X}(M)$.
(ii) If $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth curve, then

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$$

for every V, $W \in \mathcal{X}(\gamma)$.
(iii) If a, $b \in \mathbb{R}$, $a < b$, and $\gamma : [a, b] \to M$ is a smooth curve, then the parallel translation $\tau_{b,a} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ from $\gamma(a)$ to $\gamma(b)$ along γ with respect to ∇ is a linear isometry of inner product vector spaces.

Proof. The equivalence of (i) and (ii) is an immediate consequence of Lemma 5.1.4 and Proposition 3.1.7. If (ii) holds and V, W are parallel along γ then

$$\frac{d}{dt}\langle V, W\rangle = 0$$

and so $\langle V, W\rangle$ is constant on $[a, b]$. This implies (iii). Conversely, there are parallel E_1,..., $E_n \in \mathcal{X}(\gamma)$ such that $\{E_1(t_0), ..., E_n(t_0)\}$ is n orthonormal basis of $T_{\gamma(t_0)}M$ for some $t_0 \in I$. If (iii) holds, $\{E_1(t), ..., E_n(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M$ for every $t \in I$. If V, $W \in \mathcal{X}(\gamma)$, there are unique smooth functions f_k, $g_k : I \to \mathbb{R}$, $1 \le k \le n$, such that

$$V = \sum_{k=1}^{n} f_k E_k \quad \text{and} \quad \sum_{k=1}^{n} g_k E_k.$$

Then, $\langle V, W\rangle = f_1 g_1 + \cdots + f_n g_n$ and

$$\frac{d}{dt}\langle V, W\rangle = \sum_{k=1}^{n} f_k' g_k + \sum_{k=1}^{n} f_k g_k' = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle. \quad \square$$

Corollary 3.4.2. *Let M be a Riemannian smooth n-manifold and ∇ be a connection on M. If ∇ is compatible with the Riemannian metric, then the velocity field of each geodesic of ∇ has constant length.*

Proof. Indeed, if γ is a geodesic of ∇ and the latter is compatible with the Riemannian metric, we have

$$\frac{d}{dt}\|\dot{\gamma}\|^2 = \langle \frac{D\dot{\gamma}}{dt}, \dot{\gamma}\rangle + \langle \dot{\gamma}, \frac{D\dot{\gamma}}{dt}\rangle = 0. \quad \square$$

For every $c > 0$ the set

$$T^c M = \{(p, v) \in TM : p \in M, v \in T_p M, \|v\| = c\}$$

is a $(2n-1)$-dimensional smooth submanifold of TM, by Corollary 1.3.5, because $T^c M = f^{-1}(\frac{1}{2}c^2)$ and $\frac{1}{2}c^2$ is a regular value of the kinetic energy $f : TM \to \mathbb{R}$ defined by

$$f(p, v) = \frac{1}{2}\|v\|^2.$$

Indeed, if (U, ϕ) is a smooth chart of M and $(\pi^{-1}(U), \tilde{\phi})$ is the corresponding chart of TM, then the local representation of f is

$$(f \circ \tilde{\phi}^{-1})(x^1, ..., x^n, v^1, ..., v^n) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(\phi^{-1}(x^1, ..., x^n))v^i v^j$$

and differentiating

$$\frac{\partial(f \circ \tilde{\phi}^{-1})}{\partial v^i}(x^1, ..., x^n, v^1, ..., v^n) = \sum_{j=1}^{n} g_{ij}(\phi^{-1}(x^1, ..., x^n))v^j$$

because the matrix $(g_{ij})_{1\le i,j\le n}$ of the Riemannian metric is symmetric. Since it is invertible at every point as well,

$$\frac{\partial(f \circ \tilde{\phi}^{-1})}{\partial v^i}(x^1, ..., x^n, v^1, ..., v^n) = 0$$

for all $1 \leq i \leq n$ if and only if $v^1 = \cdots = v^n = 0$.

The tangent space $T_{(p,v)}T^cM$ is the $\mathrm{Ker} f_{*(p,v)}$ for every $(p, v) \in T^cM$. Now γ is a geodesic of a connection ∇ on M if and only if $(\gamma, \dot{\gamma})$ is an integral curve of the geodesic vector field G of ∇. If ∇ is compatible with the Riemannian metric, Corollary 3.4.2 says that $\|\dot{\gamma}\|$ takes on a constant value c. If γ is not constant, $c > 0$ and $(\gamma, \dot{\gamma})$ lies entirely on the constant kinetic energy level set T^cM. Thus, the geodesic vector field is tangent to constant kinetic energy level sets. In particular, T^1M is called the *unit tangent bundle* of M and from Lemma 3.2.2 every geodesic is a reparametrization of a geodesic whose velocities lie in T^1M.

Theorem 3.4.3. *On every Riemannian smooth n-manifold M there exists a unique symmetric connection which is compatible with the Riemannian metric.*

Proof. We shall prove first the uniqueness by finding an explicit formula for such a connection ∇. For every X, Y, $Z \in \mathcal{X}(M)$ we have

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle$$

since ∇ is symmetric and compatible with the Riemannian metric. From these we get

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle + \langle Z, [Y, X]\rangle - \langle X, [Z, Y]\rangle.$$

This equality uniquely determines ∇ because the Riemannian metric on each tangent space is a non-degenerate symmetric bilinear form.

The existence of ∇ will be proved locally by providing the Christoffel symbols from which it is determined. Due to uniqueness the local definitions will coincide on the overlapping domains. Let (U, ϕ) be a smooth chart of M with $\phi = (x^1, ..., x^n)$ and let

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad 1 \leq i, j \leq n.$$

By the above formula, a symmetric connection ∇ which is compatible with the Riemannian metric must satisfy

$$\sum_{k=1}^{n} \Gamma_{ij}^k g_{km} = \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m} \right\rangle = \frac{1}{2}\left[\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right]$$

on U, for every $1 \leq i, j, m \leq n$. The Christoffel symbols are uniquely determined from the above linear systems, because the Riemannian metric on each tangent space is a non-degenerate symmetric bilinear form and therefore the symmetric matrix $(g_{ij})_{1 \leq i,j \leq n}$ is invertible at each point of U. If we denote by g^{ij} the entries of the inverse matrix of the Riemannian metric $(g_{ij})^{-1}_{1 \leq i,j \leq n}$, the the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \quad 1 \leq i, j, k \leq n.$$

It remains to show that the connection on ∇ on U whose Christoffel symbols are the solutions of the above linear systems is symmetric and compatible with Riemannian metric. The first is obvious, because the matrix $(g_{ij})_{1\le i,j\le n}$ is symmetric and so the (i,j) linear system is the same as the (j,i) one. To prove compatibility, we let

$$X=\sum_{k=1}^n X^k\frac{\partial}{\partial x^k},\quad Y=\sum_{k=1}^n Y^k\frac{\partial}{\partial x^k},\quad Z=\sum_{k=1}^n Z^k\frac{\partial}{\partial x^k},$$

and then we have

$$\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle$$

$$=\sum_{k,l=1}^n\Big[g_{kl}\big(Z^lX(Y^k)+Y^kX(Z^l)\big)+\sum_{i,j=1}^n X^iY^j\Gamma^k_{ij}g_{kl}Z^l+\sum_{i,j=1}^n X^iZ^j\Gamma^l_{ij}g_{kl}Y^k\Big].$$

Since the matrix $(g_{ij})_{1\le i,j\le n}$ is symmetric, substituting we compute

$$\sum_{j,k,l=1}^n(Y^jZ^l\Gamma^k_{ij}g_{kl}+Z^jY^k\Gamma^l_{ij}g_{kl})=\sum_{j,k,l=1}^n Y^jZ^l\Gamma^k_{ij}g_{kl}+\sum_{j,k,l=1}^n Y^kZ^j\Gamma^k_{ij}g_{kl}$$

$$=\sum_{j,l=1}^n(Z^lY^j+Y^lZ^j)\Big(\sum_{k=1}^n\Gamma^k_{ij}g_{kl}\Big)$$

$$=\frac12\sum_{j.l=1}^n Z^lY^j\Big(\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{li}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\Big)+\frac12\sum_{j,l=1}^n Z^jY^l\Big(\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{li}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\Big)$$

$$=\sum_{j,l=1}^n Z^lY^j\frac{\partial g_{jl}}{\partial x^i}.$$

Therefore,

$$\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle=\sum_{k,l=1}^n g_{kl}\big(Z^lX(Y^k)+Y^kX(Z^l)\big)+\sum_{i,j,l=1}^n X^iZ^lY^j\frac{\partial g_{jl}}{\partial x^i}$$

$$=X\Big(\sum_{k,l=1}^n g_{kl}Y^kZ^l\Big)=X\langle Y,Z\rangle.\quad\square$$

The unique connection of a Riemannian manifold M which is symmetric and compatible with the Riemannian metric is called the *Levi-Civita connection* of M. The geodesics of the Levi-Civita sonnection of M will be simply called geodesics of M. It easy to see that if ∇ is a connection on M and $f: M\to M$ is a smooth diffeomorphism, then the formula

$$\overline{\nabla}_X Y=f_*^{-1}\big(\nabla_{f_*X}f_*Y\big)$$

for X, $Y\in\mathcal{X}(M)$ defines a new connection on M. If ∇ is symmetric, so is $\overline{\nabla}$. If ∇ is compatible with the Riemannian metric of M and f is an isometry, then $\overline{\nabla}$ is also compatible with the Riemannian metric. By uniqueness, if ∇ is the Levi-Civita connection of M, it is preserved by isometries, that is

$$f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$$

for every X, $Y \in \mathcal{X}(M)$ and $f \in I(M)$. In particular, every isometry sends geodesics to geodesics. This observation is crucial for the determination of the geodesics of a Riemennian manifold with sufficiently large isometry group.

Example 3.4.4. The Levi-Civita connection of the euclidean n-space $\mathbb{R}^n$ is the euclidean connection with vanishing Christoffel symbols. If $M \subset \mathbb{R}^n$ is a hypersurface, the induced euclidean connection on M defined in Example 3.1.5 is the Levi-Civita connection of M for the restricted euclidean Riemannian metric, as it is easily seen.

Example 3.4.5. We shall describe the geodesics on a n-sphere S^n_R of radius $R > 0$. Let $\gamma : I \to S^n_R$ be the geodesic satisfying the initial conditions $\gamma(0) = Re_{n+1}$ and $\dot\gamma(0) = e_1$, defined on some open interval $I \subset \mathbb{R}$ containing zero. Suppose that $\gamma(t) = (\gamma^1(t), ..., \gamma^{n+1}(t))$ for $t \in I$. For $2 \le j \le n$, the reflection $a_j : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with

$$a_j(x^1, ..., x^{n+1}) = (x^1, ..., x^{j-1}, -x^j, x^{j+1}, ..., x^{n+1})$$

is an isometry of S^n_R such that $a_j(Re_{n+1}) = Re_{n+1}$ and

$$(a_j)_{*Re_{n+1}}(\dot\gamma(0)) = a_j(e_1) = e_1 = \dot\gamma(0).$$

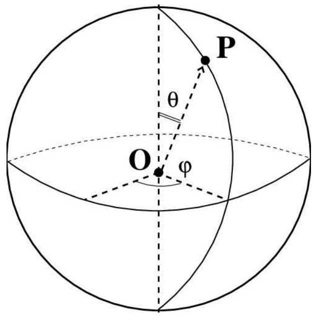
From the invariance of geodesics under isometries and uniqueness follows now that $a_j \circ \gamma = \gamma$ and hence $\gamma^j(y) = -\gamma^j(t)$, that is $\gamma^j(t) = 0$ for every $t \in I$ and $2 \le j \le n$. This means that $\gamma(I)$ is an arc on the great circle which is the intersection of S^n_R with the plane generated by $\{e_1, e_{n+1}\}$. Since S^n_R is homogeneous and isotropic, again the existence and uniqueness of geodesics implies that all geodesics are great circles. In particular, the geodesic vector field on TS^n_R is complete.

As an illustration we shall write down the system of differential equations of geodesics on S^2 with respect to the spherical coordinates (θ, ϕ), where the point $(x, y, z) \in S^2$ is written

$$x = \cos\phi \cdot \sin\theta, \quad y = \sin\phi \cdot \sin\theta, \quad z = \cos\theta.$$

The basic vector fields are

$$\frac{\partial}{\partial\theta} = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}, \quad \frac{\partial}{\partial\phi} = \begin{pmatrix} -\sin\phi\sin\theta \\ \cos\phi\sin\theta \\ 0 \end{pmatrix}$$



and so the matrix of the Riemannian metric is

$$(g_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

It follows that almost all Christoffel symbols vanish except

$$\Gamma^1_{22} = -\frac{1}{2}\sin 2\theta, \quad \Gamma^2_{12} = \cot\theta.$$

Therefore, the system of differential equations of geodesics in spherical coordinates is

$$\theta'' - \frac{1}{2}\sin 2\theta \cdot (\phi')^2 = 0,$$
$$\phi'' + 2\cot\theta \cdot \phi'\theta' = 0.$$

The meridians are obvious solutions of this system.

Example 3.4.6. The matrix of the hyperbolic Riemannian metric on the upper half plane $\mathbb{H}^2$ is

$$(g_{ij})_{1\le i,j\le 2} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

and so the Christoffel symbols are

$$\Gamma^1_{12} = -\frac{1}{y}, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^2_{22} = -\frac{1}{y},$$

and the rest are zero, at the point $z = x + iy \in \mathbb{H}^2$. So the system of differential equations of geodesics is

$$x'' - \frac{2}{y}x'y' = 0,$$
$$y'' + \frac{1}{y}[(x')^2 - (y')^2] = 0.$$

An obvious solution is $\ell(t) = ie^t$, $t \in \mathbb{R}$, whose image is the imaginary semi-axis. Since $\mathbb{H}^2$ is homogeneous and isotropic with respect to the subgroup $PSL(2,\mathbb{R})$ of its isometry group which acts by Möbius transformations, the geodesics are euclidean semi-circles with center on $\partial\mathbb{H}^2$ (the boundary taken in the Riemann sphere $\hat{\mathbb{C}}$), because the Möbius transformations send circles onto circles on $\hat{\mathbb{C}}$ and preserve angles.

The geodesics of the hyperbolic n-space $\mathbb{H}^n_R$ of radius $R > 0$, $n \geq 3$, have a similar description. First we observe that for $1 \leq j < n$ each euclidean reflection $a_j : \mathbb{R}^n \to \mathbb{R}^n$ with

$$a_j(x^1, ..., x^n) = (x^1, ..., -x^j, ..., x^n)$$

is a hyperbolic isometry which fixes e_n. As in Example 3.4.5, this implies that the trace of a hyperbolic geodesic emanating from e_n with initial velocity in the plane generated by $\{e_1, e_n\}$ is contained in the part of this plane in $\mathbb{H}^n_R$. Since the latter is clearly isometric to $\mathbb{H}^2_R$, it follows from the above that the trace of such a geodesic is either the positive semi-axis generated by e_n or a euclidean semi-circle passing through e_n with center on $\partial\mathbb{H}^n_R$. Moreover, every orthogonal transformation of $\mathbb{R}^n$ that fixes e_n is a hyperbolic isometry. This implies that the trace of any geodesic emanating from e_n is either the positive semi-axis generated by e_n or a euclidean semi-circle passing through e_n with center on $\partial\mathbb{H}^n_R$. Since $\mathbb{H}^n_R$ is homogeneous and isotropic in a strong sense, we conclude that the trace of any geodesic of $\mathbb{H}^n_R$ is either a euclidean half-line orthogonal to $\partial\mathbb{H}^n_R$ or a euclidean semi-circle with center on $\partial\mathbb{H}^n_R$. If f and g are the diffeomorphisms of Example 3.3.5, then $(g \circ f)(\frac{1}{2}e_n) = 0$ and it is easily seen that the geodesics through 0 in the open unit disc model $\mathbb{D}^n_R$ are the euclidean diameters. The geodesics through the other points of $\mathbb{D}^n_R$ are arcs of euclidean circles which intersect orthogonally the boundary sphere $\partial\mathbb{D}^n_R$.

Let M be a Riemannian smooth n-manifold. On M we shall always consider the Levi-Civita connection and all the related notions associated with it such as parallel translation, geodesics and exponential map. Let $p \in M$ and U be a normal neighbourhood of p, that is there exists an open neighbourhood V of $0 \in T_pM$ in T_pM such that $\exp : V \to U$ is a smooth diffeomorphism. We denote by $B_p(0, \epsilon)$ the open ball in T_pM of radius $\epsilon > 0$ and center $0 \in T_pM$. There exists $\epsilon_0 > 0$ such that $\overline{B_p(0, \epsilon_0)} \subset V$. The set $\exp_p(\overline{B_p(0, \epsilon)})$ will be called the *closed geodesic ball* of radius $0 < \epsilon \leq \epsilon_0$ and center p and its interior $\exp(B_p(0, \epsilon))$ *open geodesic ball.* Its boundary $\exp_p(\partial B_p(0, \epsilon))$ will be called *geodesic sphere.* Fixing an orthonormal basis $\{E_1, ..., E_n\}$ of T_pM we have a linear isometry of inner product spaces $\sigma : \mathbb{R}^n \to T_pM$ with $\sigma(e_k) = E_k$, $1 \leq k \leq n$, and a normal chart (U, ϕ) where $\phi = \sigma^{-1} \circ (\exp_p|_V)^{-1}$. Let $\phi = (x^1, ..., x^n)$ and

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad 1 \leq i, j \leq n.$$

Then $g_{ij}(p) = \delta_{ij}$, $1 \leq i, j \leq n$, Since the Levi-Civita connection is symmetric, the Christoffel symbols with respect to this normal chart vanish at p. From the formula in the proof of Theorem 3.4.3 giving the Christoffel symbols we compute

$$\sum_{k=1}^{n} \Gamma^k_{ij} g_{kl} + \sum_{k=1}^{n} \Gamma^k_{il} g_{kj} = \frac{\partial g_{jl}}{\partial x^i}$$

and in particular $\dfrac{\partial g_{jl}}{\partial x^i}(p) = 0$ for every $1 \leq i, j, l \leq n$.

In order a normal neighbourhood of p, in particular a geodesic ball, to be useful for local calculations near p, it is desirable to be a normal neighbourhood of nearby points also. An open set $W \subset M$ will be called *uniformly normal* if it is a normal neighbourhood of all its points. More precisely, W is uniformly normal if there exists some $\delta > 0$ such that $W \subset \exp_p(B_p(0,\delta))$ and $\exp_p : B_p(0,\delta) \to \exp_p(B_p(0,\delta))$ is a smooth diffeomorphism onto the open set $\exp_p(B_p(0,\delta)) \subset M$ for every $p \in W$. In order to prove the existence of uniformly normal neighbourhoods we shall need the following technical remark which is a parametrized version of the equivalence of norms in finite dimensional real vector spaces.

Lemma 3.4.7. *If M is a Riemannian smooth n-manifold and $p \in M$, for every open neighbourhood $A \subset TM$ of $(p,0)$ there exists an open neighbourhood U of p in M and some $\delta > 0$ such that*

$$U_\delta = \{(q,v) \in TM : q \in U, v \in B_q(0,\delta)\} \subset A.$$

Proof. Let (W,ψ) be a smooth chart of M with $p \in W$ and $\psi(p) = 0$. Let $\psi = (x^1, ..., x^n)$. We denote by r the euclidean norm on $\mathbb{R}^n$. If $(\pi^{-1}(W), \tilde{\psi})$ is the corresponding smooth chart of TM, where $\pi : TM \to M$ is the tangent bundle projection, we have $\tilde{\psi}(p,0) = 0$ and we may assume that $A \subset \pi^{-1}(W)$. Since $\tilde{\psi}(A) \subset \mathbb{R}^n \times \mathbb{R}^n$ is open, there exists $\epsilon > 0$ such that $B(0,2\epsilon) \times B(0,2\epsilon) \subset \tilde{\psi}(A)$. The set

$$K = \{ (q, \sum_{k=1}^n v_k \left(\frac{\partial}{\partial x^k}\right)_q) \in \pi^{-1}(W) : r(\psi(q)) \le \epsilon, \quad \sum_{k=1}^n v_k^2 = \epsilon^2 \}$$

is compact and so there exist $0 < \delta \le c$ such that

$$0 < \delta^2 \le \sum_{i,j=1}^n g_{ij}(q) v_i v_j \le c^2$$

for $(q, \sum_{k=1}^n v_k \left(\frac{\partial}{\partial x^k}\right)_q) \in K$. If now $r(\psi(q)) \le \epsilon$, then

$$(q, \frac{\epsilon}{(\sum_{k=1}^n v^2)^{1/2}} \cdot \sum_{k=1}^n v_k \left(\frac{\partial}{\partial x^k}\right)_q) \in K$$

and thus

$$\frac{\delta}{\epsilon}\left(\sum_{k=1}^n v_k^2\right)^{1/2} \le \left\| \sum_{k=1}^n v_k \left(\frac{\partial}{\partial x^k}\right)_q \right\| \le \frac{c}{\epsilon}\left(\sum_{k=1}^n v_k^2\right)^{1/2}$$

for every v_1,..., $v_n \in \mathbb{R}$. If we take $U = \psi^{-1}(B(0,\epsilon))$, we have

$$U_\delta \subset \tilde{\psi}^{-1}(B(0,\epsilon) \times B(0,\epsilon)) \subset A. \quad \square$$

Proposition 3.4.8. *If M is a Riemannian smooth n-manifold and $p \in M$, then every open neighbourhood of p contains a uniformly normal open neighbourhood of p.*

Proof. Let $E \subset TM$ be the domain of definition of the exponential map and let $F : E \to M \times M$ be the smooth map

$$F(p,v) = (p, \exp_p(v)).$$

For every $p \in M$, the derivative $F_{*(p,0)}$ is a linear isomorphism and from the Inverse Map Theorem there exists an open neighbourhood $A \subset E \subset TM$ of $(p, 0)$ such that $F(A) \subset M \times M$ is open and $F|_A : A \to F(A)$ is a smooth diffeomorphism. From the preceding Lemma 3.4.7 there exists an open neighbourhood U of p and some $\delta > 0$ such that $U_\delta \subset A$. Since $F(p, 0) = (p.p)$, there exists an open neighbourhood $W \subset U$ of p such that $W \times W \subset F(U_\delta)$. We shall show that W uniformly normal. We observe first that $\exp_q$ is defined on $B_q(0, \delta) \subset T_qM$ for all $q \in W$. Moreover, $(\exp_q |_{B_q(0,\delta)})^{-1} = (F|_{\{0\}\times B_q(0,\delta)})^{-1}$ is smooth for $q \in W$. Finally, if $(q, y) \in W \times W$, there exists $v \in B_q(0, \delta)$ such that $(q, y) = F(q, v)$, that is $y = \exp_q(v)$. This shows that $W \subset \exp_q(B_q(0, \delta))$ for every $q \in W$. $\square$

Note that if U is a (closed or open) geodesic ball with center $p \in M$, for every $q \in U$ there exists a unique geodesic path in U from p to q, but if p, q are two points in a uniformly normal open set W, there exists a geodesic path from p to q, which however may not lie entirely in W.

3.5 The Riemannian distance

On a Riemannian manifold M it is possible to define the length of curves as follows. Let a, $b \in \mathbb{R}$, $a < b$, and $\gamma : [a, b] \to M$ be a piecewise smooth parametrized curve. The non-negative real number

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$$

is defined to be the *length* of γ with respect to the Riemennian metric. By the change of variables formula, it is invariant by piecewise smooth reparametrizations.

If $\gamma : I \to M$ is a smooth parametrized curve defined on an open interval $I \subset \mathbb{R}$ such that $\dot{\gamma}(t) \neq 0$ for every $t \in I$, then taking any $t_0 \in I$ and putting

$$h(t) = \int_{t_0}^t \|\dot{\gamma}(s)\| ds$$

the smooth function $h : I \to \mathbb{R}$ is strictly increasing and maps I diffeomorphically onto an open interval $h(I) \subset \mathbb{R}$. The smooth parametrized curve

$$\sigma = \gamma \circ h^{-1} : h(I) \to M$$

is a reparametriztion of γ such that $\|\dot{\sigma}\| = 1$.

A smooth parametrized curve γ with $\|\dot{\gamma}\| = 1$ is said to be parametrized by arclength or unit speed. By Corollary 3.4.2, every non-constant geodesic is parametrized proportionally to arclength and from Lemma 3.2.2 every such geodesic can be reparametrized to a unit speed geodesic.

If M is connected, for every p, $q \in M$ the non-negative real number

$$d(p, q) = \inf\{L(\gamma) | \gamma : [a, b] \to M \quad \text{is a piecewise smooth parametrized curve}$$
$$\text{with } \gamma(a) = p \text{ and } \gamma(b) = q \text{ for some } a,\, b \in \mathbb{R},\, a < b\}$$

is called the *(Riemannian) distance* of p and q. The function $d : M \times M \to \mathbb{R}$ has the following obvious properties:
(i) $d(p, q) \geq 0$ and $d(p, p) = 0$,
(ii) $d(p, q) = d(q, p)$ and
(ii) $d(p, q) \leq d(p, z) + d(z, q)$
for every p, q, $z \in M$. In other words, d is a pseudo-distance on M. It can be proved directly that the topology defined by d coincides with the topology of M and hence d is actually a distance. However, we shall derive this from considerations showing the strong connection of d with geodesics, at least locally. We shall need a couple of lemmas, which are of independent interest.

Lemma 3.5.1. *let M be a smooth n-manifold endowed with a symmetric connection ∇ and let $A \subset \mathbb{R}^2$ be an open set. If $\sigma : A \to M$ is a smooth map then*

$$\frac{D}{dt}\left(\frac{\partial \sigma}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial \sigma}{\partial t}\right).$$

Proof. It suffices to prove the formula in case there is a smooth chart $U, \phi)$ of M such that $\sigma(A) \subset U$. If $\phi = (x^1, ..., x^n)$ and $\phi \circ \sigma = (\sigma_1, ..., \sigma_n)$, we have

$$\frac{\partial \sigma}{\partial s} = \sum_{k=1}^{n} \frac{\partial \sigma_k}{\partial s} \cdot \frac{\partial}{\partial x^k}$$

and

$$\frac{D}{dt}\left(\frac{\partial \sigma}{\partial s}\right) = \sum_{k=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial \sigma_k}{\partial s}\right) + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{\partial \sigma_i}{\partial t} \cdot \frac{\partial \sigma_j}{\partial s}\right] \frac{\partial}{\partial x^k}$$

and similarly

$$\frac{D}{ds}\left(\frac{\partial \sigma}{\partial t}\right) = \sum_{k=1}^{n} \left[\frac{d}{ds}\left(\frac{\partial \sigma_k}{\partial t}\right) + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{\partial \sigma_i}{\partial s} \cdot \frac{\partial \sigma_j}{\partial t}\right] \frac{\partial}{\partial x^k}.$$

Since ∇ is symmetric, $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$, $1 \leq i, j, k \leq n$, and the result follows from Schwartz's theorem. $\square$.

The next lemma is due to C.F. Gauss.

Lemma 3.5.2. *Let M be a Riemannian smooth n-manifold, $p \in M$ and let $V = \exp_p(B_p(0, \epsilon))$ be an open geodesic ball of radius $\epsilon > 0$ with center p. Then every geodesic emanating from p intersects orthogonally the geodesic spheres $\exp_p(\partial B_p(0, \delta))$, $0 < \delta < \epsilon$.*

Proof. Let $I \subset \mathbb{R}$ be an open interval and let $u : I \to T_pM$ be a smooth curve with $\|u(t)\| = 1$ for every $t \in I$. If $\sigma : I \times (-\epsilon, \epsilon) \to M$ is the smooth map

$$\sigma(t, s) = \exp_p(su(t)),$$

it suffices to prove that $\left\langle \frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s} \right\rangle = 0$.

We compute

$$\frac{\partial}{\partial s}\left\langle \frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial s}\right\rangle = \left\langle \frac{D}{ds}\left(\frac{\partial\sigma}{\partial t}\right), \frac{\partial\sigma}{\partial s}\right\rangle + \left\langle \frac{\partial\sigma}{\partial t}, \frac{D}{ds}\left(\frac{\partial\sigma}{\partial s}\right)\right\rangle = \left\langle \frac{D}{dt}\left(\frac{\partial\sigma}{\partial s}\right), \frac{\partial\sigma}{\partial s}\right\rangle + 0$$

by Lemma 3.5.1 and since $\sigma(t,.) : (-\epsilon,\epsilon) \to M$ is a geodesic for every $t \in I$. For the same reason,

$$\left\|\frac{\partial\sigma}{\partial s}\right\|^2 = 1$$

by Corollary 3.4.2, and differentiating

$$2\left\langle \frac{D}{dt}\left(\frac{\partial\sigma}{\partial s}\right), \frac{\partial\sigma}{\partial s}\right\rangle = 0.$$

Thus,

$$\frac{\partial}{\partial s}\left\langle \frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial s}\right\rangle = 0$$

and $\left\langle \frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial s}\right\rangle$ is independent of s. However $\sigma(t,0) = p$ for all $t \in I$ and so $\frac{\partial\sigma}{\partial t}(.,0) = 0$. Therefore,

$$\left\langle \frac{\partial\sigma}{\partial t}(t,s), \frac{\partial\sigma}{\partial s}(t,s)\right\rangle = \left\langle \frac{\partial\sigma}{\partial t}(t,0), \frac{\partial\sigma}{\partial s}(t,0)\right\rangle = 0. \quad \square$$

As in the situation of the preceding Lemma 3.5.2, let M be a Riemannian smooth n-manifold, $p \in M$ and $V = \exp_p(B_p(0,\epsilon))$ be an open geodesic ball of radius $\epsilon > 0$ with center p. A piecewise smooth parametrized curve $\gamma : [a,b] \to V \setminus \{p\}$, where a, $b \in \mathbb{R}$, $a < b$, is a the form

$$\gamma(t) = \exp_p(r(t)u(t))$$

where $r : [a,b] \to (0,\epsilon)$ is a unique piecewise smooth function and $u : [a,b] \to T_pM$ is a unique piecewise smooth parametrised curve with $\|u(t)\| = 1$ for $t \in [a,b]$. Using the notation of the proof of Lemma 3.5.2 we have $\gamma(t) = \sigma(t, r(t))$ and

$$\dot\gamma(t) = \frac{\partial\sigma}{\partial t} + r'(t)\frac{\partial\sigma}{\partial s}.$$

From Lemma 3.5.2 we have

$$\|\dot\gamma(t)\|^2 = \left\|\frac{\partial\sigma}{\partial t}\right\|^2 + (r'(t))^2\left\|\frac{\partial\sigma}{\partial s}\right\|^2 \geq (r'(t))^2$$

and the equality holds if and only if u is constant. This implies that

$$L(\gamma) \geq \int_a^b |r'(t)|dt \geq \left|\int_a^b r'(t)dt\right| = |r(b) - r(a)|$$

and the equality holds if and only if u is constant and r is monotone.

Proposition 3.5.3. *Let M be a Riemannian smooth n-manifold, $p \in M$ and let $V = \exp_p(B_p(0,\epsilon))$ be an open geodesic ball of radius $\epsilon > 0$ with center p. Let $\gamma : [0,\ell] \to V$ be a geodesic from $\gamma(0) = p$ to a point $q = \gamma(\ell) \in V$. If a, $b \in \mathbb{R}$, $a < b$, and $\sigma : [a,b] \to M$ is any piecewise smooth curve from $\sigma(a) = p$ to $\sigma(b) = q$, then $L(\gamma) \leq L(\sigma)$. Moreover, if $L(\gamma) = L(\sigma)$, then $\sigma([a,b]) = \gamma([0,\ell])$.*

Proof. We may assume that γ is parametrized by arclength, so that $\ell = L(\gamma)$ and γ is given by $\gamma(t) = \exp_p(tv)$, where $v = \dot{\gamma}(0)$ and $\|v\| = 1$. Obviously, $\ell < \epsilon$. We shall prove first that $L(\sigma) \geq \ell$. Let $0 < \delta < \ell$. By continuity and connectedness, there exist $a < c < d \leq b$ such that $\sigma(c) \in \exp_p(\partial B_p(0,\delta))$, $\sigma(d) \in \exp_p(\partial B_p(0,\ell))$ and $\sigma((c,d)) \subset \exp_p(B_p(0,\ell)) \setminus \exp_p(\overline{B_p(0,\delta)})$. Then,

$$L(\sigma) \geq L(\sigma|_{[c,d]}) \geq \ell - \delta$$

from the above considerations and letting δ go to zero this implies that $L(\sigma) \geq \ell$. This proves the first part.

Suppose now that $L(\sigma) = \ell$. Applying what we have already proved to $\sigma|_{[a,c]}$ we have $L(\sigma|_{[a,c]}) \geq \delta$ and therefore

$$L(\sigma|_{[c,d]}) \leq L(\sigma|_{[c,d]}) + L(\sigma|_{[d,b]}) = \ell - L(\sigma|_{[a,c]}) \leq \ell - \delta.$$

Hence $L(\sigma|_{[c,d]}) = \ell - \delta$ and from the above the trace $\sigma([c,d])$ is the same as the trace of a geodesic path $\exp_p(tv)$, $\delta \leq t \leq \ell$, for some $v \in T_pM$ with $\|v\| = 1$. Letting again δ go to zero we get a geodesic $\exp_p(tv)$, $0 \leq t \leq \ell$ whose trace is the same as $\sigma(|_{[a,d]}$. Thus, necessarily $L(\sigma|_{[d,b]}) = 0$ and $\gamma(l) = q = \exp_p(lv)$. It follows that $\gamma(t) = \exp_p(tv)$ for all $0 \leq t \leq \ell$. □

Corollary 3.5.4. *Let M be a Riemannian smooth n-manifold with Riemannian distance d. For every $p \in M$ there exists $\epsilon > 0$ such hat*

$$\exp_p(B_p(0,\delta)) = \{q \in M : d(p,q) < \delta\}$$

for every $0 < \delta < \epsilon$.

Proof. By Proposition 3.2.4, there exists $\epsilon > 0$ such that $\exp_p$ maps $B_p(0,\epsilon) \subset T_pM$ diffeomorphocally onto the open neighbourhood $\exp_p(B_p(0,\epsilon))$ of p. Obviously then

$$\exp_p(B_p(0,\delta)) \subset \{q \in M : d(p,q) < \delta\}$$

for every $0 < \delta < \epsilon$, since each geodesic path in the open geodesic ball $\exp_p(B_p(0,\delta))$ emanating from p has length $< \delta$.

Conversely, if $q \notin \exp_p(B_p(0,\delta))$, then every piecewise smooth parametrized curve σ from p to q intersects the geodesic sphere $\exp_p(\partial B_p(0,\rho))$ for all $0 < \rho < \delta$, and so $L(\sigma) \geq \rho$, by Proposition 3.5.3. Consequently, $L(\sigma) \geq \delta$. This shows that $d(p,q) \geq \delta$. □

Corollary 3.5.5. *On a Riemannian smooth manifold M the Riamannian distance d induces the original manifold topology and the pair (M,d) is a metric space.* □

In the sequel we shall denote by $B(p, \delta)$ the open d-ball in M with radius δ and center p.. According to Proposition 3.5.3, for every $p \in M$ there exists some $\epsilon > 0$ such that $B(p, \delta)$ is the geodesic open ball of radius δ and center p and for each $q \in B(p, \delta)$ the distance $d(p, q)$ is the length of the unique geodesic path in $B(p, \epsilon)$ from p to q for all $0 < \delta < \epsilon$. It follows from this that geodesics locally minimize length.

Proposition 3.5.6. *Let M be a Riamannian smooth manifold and $\gamma : [a, b] \to M$, where a, $b \in \mathbb{R}$, $a < b$, be a piecewise smooth parametrized curve from $\gamma(a) = p$ to $\gamma(b) = q$. If $L(\gamma) = d(p, q)$, then $\gamma([a, b])$ is the trace of a geodesic path. In particular, if γ is parametrized by arclength, it is a geodesic path and in particular smooth.*

Proof. Since being a geodesic is a local property, it suffices to show that the trace of γ is locally the same as that of a geodesic. Let $a < t_0 < b$. By Proposition 3.4.8, there exists a uniformly normal neighbourhood W of $\gamma(t_0)$. So there exists $\epsilon > 0$ such that $W \subset \exp_y(B_y(0, \epsilon))$ and $\exp_y|_{B_y(0,\epsilon)}$ is a diffeomorphism for every $y \in W$. There exists $\eta > 0$ such that $\gamma([[t_0 - \eta, t_0 + \eta]) \subset \exp_{\gamma(t_0)}(B_{\gamma(t_0)}(0, \epsilon))$. Our assumption implies that $L(\gamma|_{[t_0-\eta,t_0+\eta]}) = d(\gamma(t_0 - \eta), \gamma(t_0 + \eta))$ and thus, by Proposition 3.5.3, $\gamma([t_0 - \eta, t_0 + \eta])$ is the trace of a geodesic path. $\square$

Definition 3.5.7. A geodesic path $\gamma : [a, b] \to M$, a, $b \in \mathbb{R}$, $a < b$, on a Riemennian smooth manifold M with Riemannian distance d is called *minimizing* if $L(\gamma) = d(\gamma(a), \gamma(b))$.

Note that if γ is a minimizing geodesic path as in the above definition, then $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s))$, that is $\gamma|_{[t,s]}$ is minimizing, for every $a \leq t < s \leq b$. According to Proposition 3.5.3, every geodesic of a Riemannian manifold is locally minimizing. However, the example of the sphere shows that on a Riemennian manifold there may exist non-minimizing geodesic paths. The question now arises whether any two points on a connected Riemennian manifold can be joined by a minimizing geodesic path. This is answered by the following theorem which is due to H. Hopf and his student W. Rinow. The proof we present here is due G. de Rham.

Theorem 3.5.8. *Let M be a connected Riemannian smooth n-manifold. If the geodesic vector field of M is complete, then any two given points of M can be joined by a minimizing geodesic path.*

Proof. Let p, $q \in M$ and $r = d(p, q) > 0$. According to Corollary 3.5.4, there exists $0 < \epsilon < r$ such that $\exp_p(B_p(0, \delta)) = B(p, \delta)$ is a normal neighbourhood of p for every $0 < \delta < \epsilon$. Fixing such a δ, by compactness, there exists $p_0 \in \exp_p(\partial B_p(0, \delta))$ such that

$$d(p_0, q) = \inf\{d(z, q) : z \in \exp_p(\partial B_p(0, \delta))\}.$$

Then, $p_0 = \exp_p(\delta v)$ for some $v \in T_pM$ with $\|v\| = 1$ and the unit speed geodesic

$$\gamma(t) = \exp_p(tv)$$

is defined on the entire real line $\mathbb{R}$, because we assume the the geodesic vector field is complete. It suffices to prove now that $d(\gamma(t), q) = r - t$ for every $\delta \leq t \leq r$, because then for $t = r$ we will get $\gamma(r) = q$ and $\gamma|_{[0,r]}$ will be minimizing.

Firstly, we have

$$r = d(p, q) \leq d(p, \gamma(t)) + d(\gamma(t), q) \leq t + d(\gamma(t), q)$$

and hence $d(\gamma(t), q) \geq r - t$ for every $0 \leq t \leq r$.

On the other hand we have

$$r \geq \inf\{d(p, z) + d(z, q) : z \in \exp_p(\partial B_p(0, \delta))\} = \delta + d(p_0, q)$$

and so $d(p_0, q) \leq r - \delta$. Hence $d(\gamma(\delta), q) = d(p_0, q) = r - \delta$. Let

$$T = \sup\{t \in [\delta, r] : d(\gamma(t), q) = r - t\}.$$

By continuity, $d(\gamma(T), q) = r - T$. Moreover, $d(\gamma(t), q) = r - t$ for all $\delta \leq t \leq T$, because

$$r - t \leq d(\gamma(t), q) \leq d(\gamma(t), \gamma(T)) + d(\gamma(T), q) \leq T - t + r - t = r - t.$$

It remains to prove that $T = r$. Suppose that $T < r$. We apply what we have already proved for p to $\gamma(T)$. Thus, there are some $\eta > 0$ and $p_0' \in \exp_{\gamma(T)}(\partial B_{\gamma(T)}(0, \eta))$ with

$$d(p_0', q) = \inf\{d(z, q) : z \in \exp_{\gamma(T)}(\partial B_{\gamma(T)}(0, \eta))\}$$

and $d(p_0', q) = d(\gamma(T), q) - \eta = r - T - \eta$. Therefore,

$$d(p, p_0') \geq d(p, q) - d(p_0', q) = r - (r - T - \eta) = T + \eta.$$

However the piecewise smooth parametrized curve, which is the concatenation of $\gamma|_{0,T]}$ and the unique geodesic in $\exp_{\gamma(T)}(\overline{B_{\gamma(T)}(0, \eta)})$ from $\gamma(T)$ to p_0' has length $T + \eta$, and from Proposition 5.5.6 its trace must be the trace of a geodesic path. Since part of this path coincides with $\gamma|_{0,T]}$, it follows from uniqueness of geodesics that this geodesic path is $\gamma|_{[0,T+\eta]}$. Hence $p_0' = \gamma(T + \eta)$ and $d(\gamma(T + \eta), q) = r - (T + \eta)$. This contradicts the definition of T. $\square$

A topological characterization of the completeness of the geodesic vector field is given by the following theorem also due to H. Hopf and W. Rinow.

Theorem 3.5.9. *For a connected Riemannian smooth manifold M with Riemannian distance d the following statements are equivalent:*
(i) The geodesic vector field of M is complete.
(ii) The metric space (M, d) is complete.

Proof. Suppose that the geodesic vector field of M is complete. In order to prove that (M, d) is a complete metric space, it suffices to show that every d-bounded set $C \subset M$ is contained in a compact set. Let $p \in M$. Since C is bounded, there exists $c > 0$ such that $d(p, q) < c$ for every $q \in C$. From Theorem 3.5.8, there exists some $v \in T_pM$ such that $q = \exp_p(v)$ and $\|v\| = d(p, q)$. This shows that $C \subset \exp_p(\overline{B_p(0, c)})$, and $\exp_p(\overline{B_p(0, c)})$ is compact, because $\exp_p$ is continuous.

Conversely, suppose that there exists a geodesic parametrized by arclength γ whose maximal interval of definition is an open interval (a, b) for some $a < b < +\infty$. Then, $d(\gamma(t), \gamma(s)) \leq |t - s|$ for every t, $s \in (a, b)$. If (M, d) is complete, then $p = \lim_{t \to b^-} \gamma(t)$ exists in M. From Proposition 3.4.8 there exists a uniformly normal open neighbourhood W of p, for which there exists some $\delta > 0$ such that $W \subset \exp_q(B_q(0, \delta))$ for every $q \in W$. There exists $b - \delta < T < b$ such that $\gamma(T) \in W$ and then the geodesic form $\gamma(T)$ with initial velocity $\dot{\gamma}(T)$ is defined at least on the interval $[0, \delta)$. By uniqueness of geodesics, this implies that γ is defined at least on $(a, T+\delta)$ and since $T+\delta > b$ this contradicts our assumption the $b < +\infty$. □

If any of the two equivalent conditions of the preceding theorem is satisfied, we shall call the Riemannian manifold M *complete*. As the proof shows, the following also holds.

Corollary 3.5.10. *A connected Riemannian smooth manifold M is complete if and only if there exists a point $p \in M$ such that $\exp_p$ is defined on the entire tangent space T_pM.* □

Corollary 3.5.11. *The geodesic vector field of a compact Riemannian smooth manifold is complete.* □

The fact that homogeneous Riemannian manifolds are complete is a consequence of the following.

Proposition 3.5.12. *Let (M, d) be a locally compact metric space. If it is homogeneous in the sense that for every x, $y \in M$ there exists a d-isometry $f : M \to M$ such that $f(x) = y$, then it is complete.*

Proof. Let $p \in M$. Since M is assumed to be locally compact, there exists some $r > 0$ such that $\overline{B(p, r)}$ is compact. The homogeneity implies now that $\overline{B(x, r)}$ is compact for every $x \in M$. If $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in M, there exists some $k_0 \in \mathbb{N}$ such that $d(x_{k_0}, x_k) < r$ for every $k \geq k_0$. Hence the sequence has a convergent subsequence, by compactness of $\overline{B(x_{k_0}, r)}$, which implies that it converges in M. □

Corollary 3.5.13. *A homogeneous connected Riemannian smooth manifold is complete.* □

The euclidean space, the spheres and the hyperbolic spaces are all complete Riemannian manifolds.

3.6 Geodesic convexity

Let M be a Riemannian smooth n-manifold and $p \in M$. By Proposition 3.4.8 and Proposition 3.5.3, there exists a uniformly normal open neighbourhood W of p for which there exists some $\delta > 0$ such that $W \subset \exp_q(B_q(0, \delta))$, for every $q \in W$, and for every q_1, $q_2 \in W$ there exists a unique minimizing geodesic path from q_1 to q_2 of length $< \delta$. However this geodesic path may not lie entirely in W.

Definition 3.6.1. A subset C of a Riemannian smooth manifold is said to be *strongly (geodesically) convex* if for every x, $y \in \overline{C}$ there exists a unique and minimizing geodesic path $\gamma : [a, b] \to \overline{C}$, for some a, $b \in \mathbb{R}$, $a < b$, from $x = \gamma(a)$ to $y = \gamma(b)$ such that $\gamma(t) \in C$ for $a < t < b$.

In this section we shall prove that sufficiently small geodesic balls with center any given point on a Riemennian smooth manifold are strongly convex (and of course uniformly normal). This result on the existence of strongly convex open neighbourhoods is due to J.H.C. Whitehead and is based on the following.

Lemma 3.6.2. *Let M be a Riemannian smooth n-manifold. For every $p \in M$ there exists some $\epsilon_0 > 0$ such that for $0 < \delta < \epsilon_0$ if $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a geodesic which is tangent to the geodesic sphere $\exp_p(\partial B_p(0, \delta))$ at the point $\gamma(t_0)$, for some $t_0 \in I$, then there exists some $\eta > 0$ such that*

$$\gamma((t_0 - \eta, t_0 + \eta) \setminus \{t_0\}) \subset M \setminus \exp_p(\overline{B_p(0, \delta)}).$$

Proof. There exists some $\epsilon > 0$ such that $\exp_p$ maps $B_p(0, \epsilon)$ diffeomorphically onto $U = \exp_p(B_p(0, \delta))$. Let $0 < \delta < \epsilon$. We choose an orthonormal basis $\{E_1, ..., E_n\}$ of T_pM and consider the normal chart (U, ϕ) at p, where $\phi = h \circ (\exp_p |_{B_p(0,\epsilon)})^{-1}$ and $h : T_pM \to \mathbb{R}^n$ is the linear isommetry with $h(E_i) = e_i$, $1 \le i \le n$. Let $\gamma : I \to U$ be a geodesic which is tangent to the geodesic sphere $\exp_p(\partial B_p(0, \delta))$ at the point $\gamma(t_0)$. Suppose that $\phi = (x^1, ..., x^n)$ and $\phi \circ \gamma = (\gamma^1, ..., \gamma^n)$. We consider the smooth function $f : I \to \mathbb{R}$ with

$$f(t) = \sum_{k=1}^{n} (\gamma^k(t))^2.$$

Since γ is tangent to $\exp_p(\partial B_p(0, \delta))$ at $\gamma(t_0)$, we have

$$f'(t_0) = 2\sum_{k=1}^{n} \gamma^k(t_0)(\gamma^k)'(t_0) = 0.$$

Since γ is a geodesic,

$$(\gamma^k)''(t) = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)(\gamma^j)'(t)$$

and substituting

$$f''(t) = 2\sum_{k=1}^{n} [((\gamma^k)'(t))^2 + (\gamma^k)(t)(\gamma^k)''(t)]$$

$$= \sum_{i,j=1}^{n} \left(2\delta_{ij} - 2\sum_{k=1}^{n} \Gamma_{ij}^k(\gamma(t))\gamma^k(t)\right)(\gamma^i)'(t)(\gamma^j)'(t)$$

for every $t \in I$. Since $\Gamma_{ij}^k(p) = 0$, $1 \le i, j, k \le n$, there exists some $0 < \epsilon_0 < \epsilon$ such that the quadratic form

$$\sum_{i,j=1}^{n} \left(\delta_{ij} - \sum_{k=1}^{n} \Gamma_{ij}^k(q) x^k(q) \right) v^i v^j$$

is positive definite for every $q \in \exp_p(B_p(0, \epsilon_0))$. Thus, if $0 < \delta < \epsilon_0$, then $f''(t_0) > 0$ and f has has a strict local minimum at t_0, which means that there exists $\eta > 0$ such that $f(t) > \delta^2$ for $t \in (t_0 - \eta, t_0 + \eta) \setminus \{t_0\}$. This proves the assertion. □

We shall also use the following remark. If $p \in M$, for every open neighbourhood U of p there exists an open neighbourhood V of $(p, 0)$ in TM such that $\exp_q(tv) \in U$ for every $0 \le t \le 1$ and $(q, v) \in V$. To see this, it suffices to consider the smooth map $g : [0, 1] \times E \to M$ with $g(t, q, v) = \exp_q(tv)$, where $E \subset TM$ is the domain of definition of the exponential map and note that $g(t, p, 0) = p$ for all $0 \le t \le 1$. By continuity, for every $t \in [0, 1]$ there exists an open neighbourhood $V_t \subset E$ of $(p, 0)$ and $\delta_t > 0$ such that $g((t - \delta_t, t + \delta_t) \times V_t) \subset U$. By compactness of $[0, 1]$, there exist t_1,..., $t_m \in [0, 1]$, for some $m \in \mathbb{N}$, such that

$$[0, 1] = \bigcup_{k=1}^{m} (t_k - \delta_{t_k}, t_k + \delta_{t_k}).$$

It suffices now to take $V = V_{t_1} \cap \cdots \cap V_{t_m}$.

Theorem 3.6.3. *If M is a Riemannian smooth n-manifold, then for every $p \in M$ there exists some $\epsilon > 0$ such that for every $0 < \delta < \epsilon$ the geodesic ball $\exp_p(B_p(0, \delta))$ is strongly convex.*

Proof. Let $\epsilon_0 > 0$ be as in the preceding Lemma 3.6.2 and let $F : E \to M \times M$ be the smooth map $F(q, v) = (q, \exp_q(v))$, where $E \subset TM$ is the domain of definition of the exponential map. As in the proof of Proposition 3.4.8, there exists an open neighbourhood $V \subset TM$ of $(p, 0)$ and some $0 < \epsilon < \epsilon_0$ such that F maps V diffeomorphically onto $\exp_p(B_p(0, \epsilon)) \times \exp_p(B_p(0, \epsilon))$ and $\exp_q(tv) \in \exp_p(B_p(0, \epsilon_0))$ for every $(q, v) \in V$ and $0 \le t \le 1$, form the above remark. Moreover, there exists some $\eta > 0$ such that $\exp_p(B_p(0, \epsilon)) \subset \exp_q(B_q(0, \eta))$ for every $q \in \exp_p(B_p(0, \epsilon))$.

We shall prove that $\exp_p(B_p(0, \delta))$ is strongly convex for every $0 < \delta < \epsilon$. Let q_1, $q_2 \in \overline{\exp_p(B_p(0, \delta))} = \exp_p(\overline{B_p(0, \delta)})$, Since $(q_1, q_2) \in F(V)$ there exists $v \in T_{q_1}M$ such that $q_1 = \exp_{q_1}(v)$ and $\gamma(t) = \exp_{q_1}(tv) \in \exp_p(B_p(0, \epsilon_0))$ for every $0 \le t \le 1$. By Proposition 3.5.3, γ is the unique and minimizing geodesic path from q_1 to q_2 in $\exp_{q_1}(B_{q_1}(0, \eta))$, hence in $\exp_p(B_p(0, \epsilon_0))$, and it suffices to show that $\gamma(t) \in \exp_p(B_p(0, \delta))$ for $0 < t < 1$. Let $(\gamma^1, ..., \gamma^n)$ be its local representation with respect to the normal chart on $\exp_p(B_p(0, \epsilon_0))$ and let again $f : [0, 1] \to \mathbb{R}$ be the smooth function

$$f(t) = \sum_{k=1}^{n} (\gamma^k(t))^2$$

as in the beginning of the proof of Lemma 3.6.2. If $\gamma((0,1))$ has points outside $\exp_p(B_p(0,\delta))$, then f takes its maximal value on $[0,1]$ at some $0 < t_0 < 1$ and

$$\delta^2 \leq f(t_0) < \epsilon_0^2$$

or equivalently $\gamma([0,1]) \in \exp_p(\overline{B_p(0,\sqrt{f(t_0)})})$. On the other hand, we must have

$$0 = f'(t_0) = 2\sum_{k=1}^{n}(\gamma^k)(t_0)(\gamma^k)'(t_0)$$

which means that the geodesic path $\gamma((0,1))$ is tangent to the geodesic sphere $\exp_p(\partial B_p(0,\sqrt{f(t_0)}))$. This contradicts Lemma 3.6.2. $\square$

Corollary 3.6.4. *If M is a Riemennian smooth manifold with Riemannian distance d, then for every $p \in M$ there exists some $\epsilon > 0$ such that for every $0 < \delta < \epsilon$ the open d-ball $B(p,\delta)$ is the geodesic ball with center p and radius δ and is uniformly normal and strongly convex.* $\square$

The existence of strongly convex geodesic balls can be applied to facilitate algebraic calculations on smooth manifolds involving de Rham and Čech cohomology.

3.7 Isometries

Let M be a Riemannian manifold with Riemannian distance d. Every Riemennian isometry $f : M \to M$ is a metric isometry of the metric space (M,d), that is f is surjective and $d(f(p),f(q)) = d(p,q)$ for every p, $q \in M$. The aim of this section is to prove that actually the converse also holds. This is a famous theorem first proved by S.B. Myers and N. Steenrod. The proof we present is due to R. Palais. As expected, the non-trivial part of the proof consists of the argument showing the differentiability of f. We shall need a preliminary fact.

Let (M,d) be a metric space. A continuous parametrized curve $\gamma : [a,b] \to M$, a, $b \in \mathbb{R}$ with $a < b$, is called *segment* if

$$d(\gamma(t_1),\gamma(t_2)) + d(\gamma(t_2),\gamma(t_3)) = d(\gamma(t_1),\gamma(t_3))$$

for all $a \leq t_1 \leq t_2 \leq t_3 \leq b$. Obviously, every unit speed minimizing geodesic in a Riemannian manifold is a segment.

Lemma 3.7.1. *Let M be a Riemannian manifold with Riemannian distance d. The image of every segment of the metric space (M,d) coincides with the image of a geodesic of M.*

Proof. Let $\gamma : [a,b] \to M$ be a segment and $p = \gamma(a)$. According to Corollary 3.6.4, there exists $\epsilon > 0$ such that for every $0 < \delta < \epsilon$ the open d-ball $B(p,\delta)$ is the geodesic ball with center p and radius δ and is uniformly normal and strongly convex. There exists $T > 0$ such that $\gamma([a,a+T]) \subset B(p,\epsilon)$. Let γ_0 be the unique and minimizing geodesic path in $B(p,\epsilon)$ from $p = \gamma(a)$ to $\gamma(a+T)$. Suppose that there exists some $a < t_0 < a + T$ such that $\gamma(t_0)$ does not belong to the image of γ_0. There is a unique and minimizing geodesic γ_1 in $B(p, \epsilon)$ from $p = \gamma(a)$ to $\gamma(t_0)$ and a unique and minimizing geodesic γ_2 in $B(p, \epsilon)$ from $\gamma(t_0)$ to $\gamma(a + T)$. Since the image of γ_0 does not coincide with the image of the concatenation $\gamma_1 * \gamma_2$ of γ_1 and γ_2, we have

$$d(\gamma(a), \gamma(a+T)) = L(\gamma_0) < L(\gamma_1 * \gamma_2) = L(\gamma_1) + L(\gamma_2)$$
$$= d(\gamma(a), \gamma(t_0)) + d(\gamma(t_0), \gamma(a+T)).$$

This contradicts our assumption that γ is a segment and shows that the image of $\gamma([a, a + T])$ coincides with the image of γ_0. If now

$$s = \sup\{a < t \leq b : \gamma([a,t]) \text{ coincides with the image of a geodesic path }\}$$

the same argument taking s in place of a shows that necessarily $s = b$. $\square$

Theorem 3.7.2. *Let M and M' be Riemannian n-manifolds with corresponding Riemannian distances d and d'. If $f : (M, d) \to (M, d')$ is a metric isometry, which means that f is surjective and $d'(f(p), f(q)) = d(p, q)$ for every p, $q \in M$, then f is a Riemannian isometry.*

Proof. Let $p \in M$, $p' = f(p)$ and let $\epsilon > 0$ be such that the open d'-ball $B(p', \epsilon)$ is the geodesic ball in M' with center p' and radius ϵ and is uniformly normal and strongly convex. We can choose $\epsilon > 0$ such that $f(B(p, \epsilon)) = B(p', \epsilon)$ and $B(p, \epsilon)$ is the geodesic ball in M with center p and radius ϵ and is uniformly normal and strongly convex. Let $v \in T_pM$ be such that $\|v\| = 1$ and $\gamma : [0, \frac{\epsilon}{2}] \to B(p, \delta)$ be the unique minimizing geodesic path with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then γ is a segment and so $f \circ \gamma$ is a segment in $B(p', \epsilon)$, since f is a metric isometry. By the preceding Lemma 3.7.1, the image of $f \circ \gamma$ coincides with the image of a geodesic path in $B(p', \epsilon)$ with initial point p', which is parametrized by arclength. Actually, $f \circ \gamma$ itself is a unit speed geodesic path, because

$$d'(f(\gamma(t)), f(\gamma(s))) = d(\gamma(t), \gamma(s)) = |t - s|$$

for every $t, s \in [0, \frac{\epsilon}{2}]$. If $u \in T_{p'}M'$ is the initial velocity of $f \circ \gamma$, we put $F(v) = u$. This defines a map F from the unit sphere in T_pM to the unit sphere in $T_{p'}M'$, which we extend to a map $F : T_pM \to T_{p'}M'$ putting $F(0) = 0$ and

$$F(w) = \|w\| \cdot F\left(\frac{1}{\|w\|} \cdot w\right)$$

for every non-zero $w \in T_pM$. Since f^{-1} is also a metric isometry, in a similar way follows that F is injective and surjective. Moreover, from the definition of F we have $\|F(w)\| = \|w\|$ for every $w \in T_pM$ and $f \circ \exp_p = \exp_{p'} \circ F$. Since $B(p, \epsilon)$ and $B(p', \epsilon)$ are uniformly normal neighbourhoods, it suffices to show that F is linear and preserves the inner products. It is obvious that $F(tw) = tF(w)$ for every $w \in T_pM$ and $t \geq 0$. If F preserves the inner products, then its linearity can be proved as follows. Let $\{v_1, ..., v_n\}$ be an orthonormal basis of T_pM. Then, $\{F(v_1), ..., F(v_n)\}$ is an orthonormal basis of $T_{p'}M'$. For v, $w \in T_pM$ we have

$$\langle F(v+w), F(E_j)\rangle = \langle v+w, E_j\rangle = \langle v, E_j\rangle + \langle w, E_j\rangle$$

$$= \langle F(v), F(E_j)\rangle + \langle F(w), F(E_j)\rangle = \langle F(v) + F(w), F(E_j)\rangle$$

for every $1 \le j \le n$. This implies that $F(v+w) - F(v) - F(w) = 0$ and hence F is linear.

Since $F(tv) = tF(v)$, for $t \ge 0$ and $v \in T_pM$, in order to prove that F preserves the inner products, it is sufficient to show that $\langle F(v), F(w)\rangle = \langle v, w\rangle$ for v, $w \in T_pM$ with $\|v\| = \|w\| = 1$. Then also $\|F(v)\| = \|F(w)\| = 1$. We put $\cos\theta = \langle v, w\rangle$ and $\cos\phi = \langle F(v), F(w)\rangle$. Let γ be the geodesic of M with $\gamma(0) = p$, $\dot\gamma(0) = v$, and σ be the geodesic with $\sigma(0) = p$, $\dot\sigma(0) = w$. Then, $f \circ \gamma$ and $f \circ \sigma$ are the geodesics in M' with initial point p' and initial velocities $F(v)$ and $F(w)$, respectively. It suffices now to prove that

$$\sin\frac{\theta}{2} = \lim_{t\to 0} \frac{1}{2t} d(\gamma(t), \sigma(t)),$$

$$\sin\frac{\phi}{2} = \lim_{t\to 0} \frac{1}{2t} d'(f(\gamma(t)), f(\sigma(t))),$$

because then

$$\langle v, w\rangle = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\phi}{2} = \langle F(v), F(w)\rangle,$$

since f is a metric isometry. We shall prove the first equality, the proof of the second being similar. On $B(p,\epsilon)$ we consider the euclidean Riemannian metric which makes the diffeomorphism $\exp_p : B_p(0,\epsilon) \to B(p,\epsilon)$ Riemannian isometry. Let ρ denote the corresponding Riemannian distance on $B(p,\epsilon)$. We proceed by contradiction. Suppose that

$$\limsup_{t\to 0} \frac{1}{2t} d(\gamma(t), \sigma(t)) > \sin\frac{\theta}{2}.$$

We choose some $c > 1$ such that

$$\limsup_{t\to 0} \frac{1}{2t} d(\gamma(t), \sigma(t)) > c\sin\frac{\theta}{2}.$$

As in the proof of Lemma 3.4.7, there exists $0 < \delta < \epsilon$ such that

$$\frac{1}{c}\Big(\sum_{k=1}^{n} w_k^2\Big)^{1/2} < \|w\| < c\Big(\sum_{k=1}^{n} w_k^2\Big)^{1/2}$$

for every $w \in T_qM$, $q \in B(p,\delta)$, where $(w_1, ..., w_n)$ are the normal coordinates of w (with respect to p). From the definition of d and ρ we have now

$$\frac{1}{c}\rho(q_1, q_2) < d(q_1, q_2) < c\rho(q_1, q_2)$$

for every q_1, $q_2 \in B(p,\delta)$. By continuity, there exists $\eta > 0$ such that

$$\frac{c}{2t}\rho(\gamma(t), \sigma(t)) > \frac{1}{2t} d(\gamma(t), \sigma(t)) > c\sin\frac{\theta}{2}.$$

But since ρ is the euclidean distance

$$\frac{1}{2t}\rho(\gamma(t), \sigma(t)) = \sin\frac{\theta}{2}.$$

This contradiction shows that

$$\limsup_{t\to 0}\frac{1}{2t}d(\gamma(t), \sigma(t)) \le \sin\frac{\theta}{2}.$$

In a similar way we can prove that

$$\liminf_{t\to 0}\frac{1}{2t}d(\gamma(t), \sigma(t)) \ge \sin\frac{\theta}{2}.$$

This concludes the proof. □

3.8 Exercises

1. Prove that the euclidean connection on $\mathbb{R}^n$ is the unique connection for which $\nabla_X Y = 0$ for every $X \in \mathcal{X}(\mathbb{R}^n)$ and every constant $Y \in \mathcal{X}(\mathbb{R}^n)$.

2. Let ∇ be a connection on a smooth n-manifold M. A smooth diffeomorphism $f : M \to M$ is called *affine*, if it preserves ∇, that is $f_*(\nabla_X Y) = \nabla_{f_*X} f_*Y$, for every X, $Y \in \mathcal{X}(M)$. The set of all affine diffeomorphisms of ∇ is a group. Prove that in case $M = \mathbb{R}^n$ and ∇ is the euclidean connection, for every affine diffeomorphism f there exist $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$ such that $f(x) = Ax + b$ for every $x \in \mathbb{R}^n$.

3. A smooth n-manifold M is said to be *affinely flat*, if there exists a smooth atlas $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ of M such that for every i, $j \in I$ with $U_i \cap U_j \neq \varnothing$ there exist $A_{ij} \in GL(n, \mathbb{R})$ and $b_{ij} \in \mathbb{R}^n$ such that

$$\phi_i \circ \phi_j^{-1}(x) = A_{ij}x + b_{ij}$$

for every $x \in \phi_j(U_i \cap U_j)$. Prove that then there exists a natural connection ∇ on M such that every $\phi_i : U_i \to \phi_i(U_i)$ transfers $\nabla|_U$ to the euclidean connection on $\phi_i(U_i) \subset \mathbb{R}^n$.

4. Let $A \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix and

$$M = \{x \in \mathbb{R}^n : \langle A^{-1}x, x\rangle = 1\}$$

be the $(n-1)$-dimensional ellipsoid with semi-axis the eigenvalues of A. Prove that a smooth parametrized curve $\gamma : \mathbb{R} \to M$ is a geodesic of M (with respect to the euclidean connection) if and only if

$$\gamma'' + \frac{\langle A^{-1}\gamma', \gamma'\rangle}{\|A^{-1}\gamma\|^2}A^{-1}\gamma = 0.$$

5. On $\mathbb{R}^2$ we consider the connection whose Christoffel symbols are $\Gamma_{11}^1 = x$, $\Gamma_{12}^1 = 1$, $\Gamma_{22}^2 = 2y$ and the rest vanish.
(a) Write down the system of differential equations of its geodesics.
(b) Let $\gamma : [0,1] \to \mathbb{R}^2$ be the smooth parametrized curve $\gamma(t) = (t, 0)$. Find the parallel translation of the vector $\left(\frac{\partial}{\partial y}\right)_{(0,0)}$ along γ on $(1,0)$ with respect to this connection.

6. Let M be a smooth manifold endowed with a connection ∇ and let $\rho : M \to \mathbb{R}$ be a smooth function. For every $X, Y \in \mathcal{X}(M)$ we put

$$\nabla_X^\rho Y = \nabla_X Y - Y(\rho)X - X(\rho)Y.$$

(a) Prove that ∇^ρ is a connection on M.
(b) Let $\epsilon > 0$ and $\gamma : (-\epsilon, \epsilon) \to M$ be a geodesic of ∇^ρ. If $h : (-\epsilon, \epsilon) \to \mathbb{R}$ is the smooth function with

$$h(t) = \int_0^t e^{2\rho(\gamma(s))} ds,$$

prove that $\gamma \circ h^{-1}$ is a geodesic of ∇. Thus, the two connections ∇ and ∇^ρ have the same non-parametrized geodesics.

7. On $\mathbb{R}^3$ we define $\nabla : \mathcal{X}(\mathbb{R}^3) \times \mathcal{X}(\mathbb{R}^3) \to \mathcal{X}(\mathbb{R}^3)$ by

$$\nabla_X Y = D_X Y + \frac{1}{2} X \times Y,$$

where $D_X Y$ is the directional derivetive of Y with respect to X and $X \times Y$ is the usual exterior product on $\mathbb{R}^3$.
(a) Prove that ∇ is a connection.
(b) Is ∇ symmetric?
(c) Is ∇ compatible with the euclidean Riemannian metric?

8. Let M, N be two connected Riemannian manifolds and let $f : M \to N$ be a smooth diffeomorphism. Assume that there exists some point $p \in M$ such that $f_{*p} : T_p M \to T_{f(p)} N$ is a linear isometry. Prove that f is an isometry if and only if it preserves the corresponding Levi-Civita connections.

9. Let M be a Riemannian smooth n-manifold and let $f : M \to \mathbb{R}$ be a smooth function. The gradient of f is the unique smooth vector field grad$f$ such that

$$f_{*p}(v) = \langle \mathrm{grad} f(p), v \rangle$$

for every $v \in T_p M$, $p \in M$.
(a) Prove that in the local coordinates $(x^1, ..., x^n)$ of a smooth chart of M the gradient of f is given by the formula

$$\mathrm{grad} f = (g_{ij})^{-1}_{1 \le i,j \le n} \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial x^n} \end{pmatrix}.$$

(b) If $\|\mathrm{grad} f\| = 1$ everywhere on M, prove that the integral curves of $\mathrm{grad} f$ are geodesics.

10. On $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ we consider the Riemannian metric

$$\langle v, w \rangle = \frac{4}{(1-|z|^2)^2} \cdot \mathrm{Re}(v\bar{w}), \qquad v, w \in T_z\mathbb{D}^2, \quad z \in \mathbb{D}^2.$$

(a) Prove that the map $C : \mathbb{D}^2 \to \mathbb{H}^2$ defined by

$$C(z) = -i\frac{z+i}{z-i}$$

is an isometry. C is called the Cayley transformation.
(b) Prove that if a, $b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$, then

$$h(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

is an isometry of $\mathbb{D}^2$.
(c) Describe the geodesics of $\mathbb{D}^2$.

11. Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be the smooth parametrized curve $\gamma(t) = (t, 1)$. Find the parallel vector field X along γ with $X(0) = \left(\frac{\partial}{\partial y}\right)_{\gamma(0)}$ and draw X on the interval $[-\frac{\pi}{2}, \pi]$.

12. Let M and N be two connected Riemannian manifolds.
(a) Let $p \in M$, $q \in N$ and $T : T_pM \to T_qN$ be a linear isometry. If there exists an isometry $h : M \to N$ such that $h(p) = q$ and $h_{*p} = T$, prove that there exist normal open neighbourhoods V of p and W of q such that $h(V) = W$ and

$$h|V = \exp_q \circ T \circ \exp_p^{-1}.$$

(b) Prove that if g, $h : M \to N$ are two isometries for which there exists some $p \in M$ such that $g(p) = h(p)$ and $g_{*p} = h_{*p}$, then $g = h$.

13. Let M ne a Riemannian smooth n-manifold and let G be a non-empty set of isometries of M. If $F = \{p \in M : g(p) = p \text{ for every } g \in G\}$, prove that F is a smooth submanifold of M.
(Hint: Consider for every $p \in F$ the vector subspace

$$V = \{v \in T_pM : g_{*p}(v) = v \text{ for every } g \in G\}$$

of T_pM and show that $\exp_p(U \cap V) = F \cap \exp_p(U)$ for a suitable open neighbourhood U of $0 \in T_pM$.)

14. Let M be a Riemannian smooth manifold with group of isometries $I(M)$. For a properly discontinuous subgroup G of $I(M)$, the orbit space M/G inherits a Riemannian metric, if it is a Hausdorff space, and the quotient map $p : M \to M/G$ is a local isometry. If M is complete, prove that M/G is complete as well. Describe

the geodesics of the flat 2-torus T^2 and the geodesics of $\mathbb{R}P^2$ with respect to the induced Riemannian metric from S^2.

15. Prove that a connected isotropic and complete Riemannian manifold is homogeneous.

16. Let M be a connected, non-compact, complete Riemannian manifold with Riemannian distance d. Prove that for every $p \in M$ there exists a geodesic $\gamma : [0, +\infty) \to M$ with $\gamma(0) = p$ and $d(p, \gamma(t)) = t$ for every $t \geq 0$.

17. Let M and N be two Riemannian smooth manifolds and let $h : M \to N$ be a smooth diffeomorphism for which there exists $c > 0$ such hat $c\|h_{*p}(v)\| \leq \|v\|$ for every $v \in T_pM$ and $p \in M$. If N is complete, prove that M is also complete.

18. Let M be a Riemannian smooth manifold with Riemannian distance d. For every piecewise smooth parametrized curve $\gamma : [a, b] \to M$, where a, $b \in \mathbb{R}$, $a < b$, the non-negative real number

$$J(\gamma) = \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 dt$$

is called the energy of γ and is not invariant under reparametrizations.
(a) Prove that $(L(\gamma))^2 \leq 2(b - a)J(\gamma)$ and the equality holds if and only if $\|\dot{\gamma}\|$ is constant.

For every p, $q \in M$ we define

$$e(p, q) = \inf\{2J(\gamma) | \gamma : [0, 1] \to M \quad \text{piecewise smooth with} \quad \gamma(0) = p, \gamma(1) = q\}.$$

(b) Prove that $(d(p, q))^2 = e(p, q)$ for every p, $q \in M$.
(c) If p, $q \in M$ and γ is a piecewise smooth parametrized curve from p to q, prove that γ minimizes the energy, that is $2J(\gamma) = e(p, q)$, if and only if γ is a minimizing geodesic.

Chapter 4

Curvature

4.1 The Riemann curvature tensor

A first important step towards the classification of Riemannian manifolds is the answer to the following question: Are all Riemannian manifolds locally isometric? We shall see in this chapter that the answer is negative, by constructing local isometric invariants. All of them originate from the curvature tensor, which was introduced by B. Riemann in a purely geometric manner and generalizes the Gauss curvature of a surface in $\mathbb{R}^3$. Is should be noted that the local investigation of the Riemanian manifolds is a highly non-trivial task contrary to other geometric structures such as for instance the symplectic and contact structures which by Darboux's theorem are all locally isomorphic and thus have no local invariants.

Let ∇ be a connection on a smooth n-manifold M. The curvature of ∇ is an algebraic tool which describes how much ∇ locally deviates from the euclidean connection. For every $X \in \mathcal{X}(M)$ the linear map $\nabla_X : \mathcal{X}(M) \to \mathcal{X}(M)$ is a derivation and one can ask whether $\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z$ holds for every X, Y, $Z \in \mathcal{X}(M)$. This does not hold even for the euclidean connection on $\mathbb{R}^n$. More precisely, in this case we have

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

Thus, if ∇ is the Levi-Civita connection of a Riemannian metric on M and M is locally isometric to the euclidean n-space, we must necessarily have

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$$

for every X, Y, $Z \in \mathcal{X}(M)$. This leads to the following.

Definition 4.1.1. The *curvature tensor* of a connection ∇ on a smooth n-manifold M is the $C^\infty(M)$-multilinear map $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

An easy calculation shows that R is indeed $C^\infty(M)$-multilinear and therefore the value of $R(X,Y)Z$ at a point $p \in M$ depends only on the values X_p, Y_p and

Z_p. We can therefore write $R(X_p, Y_p)Z_p$. According to the definition, the curvature tensor of the euclidean connection is zero.

If M and $\tilde{M}$ are two smooth manifolds carrying connections ∇ and $\tilde{\nabla}$, respectively, and $f : M \to \tilde{M}$ is a smooth diffeomorphism such that $f_*(\nabla_X Y) = \tilde{\nabla}_{f_*X} f_*Y$ for every X, $Y \in \mathcal{X}(M)$ (such a diffeomorphism is called *conformal*), then from the definition and the behaviour of the Lie bracket under diffeomorphisms we have $f_*(R(X,Y)Z) = R(f_*X, f_*Y)f_*Z$ for every X, Y, $Z \in \mathcal{X}(M)$. In particular this holds in case ∇ and $\tilde{\nabla}$ are the Levi-Civita connections of Riemennian metrics on M and $\tilde{M}$, respectively, and $f : M \to \tilde{M}$ is an isometry.

Proposition 4.1.2. *The curvature tensor R of a symmetric connection ∇ on a smooth n-manifold M satisfies the following identities.*
(a) $R(X,Y)Z = -R(Y,X)Z$, and
(b) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ for every X, Y, $Z \in \mathcal{X}(M)$. This second identity is called the first (algebraic) identity of Bianchi.

Proof. The first identity is obvious from the definition of the curvature tensor. Since R is $C^\infty(M)$-multilinear, it suffices to check that (b) holds only in case $X = \dfrac{\partial}{\partial x^i}$, $Y = \dfrac{\partial}{\partial x^j}$ and $Z = \dfrac{\partial}{\partial x^k}$ are basic vector fields in some open set $U \subset M$ with respect to a chart $\phi = (x^1, ..., x^n) : U \to \mathbb{R}^n$. Now we have $[X,Y] = [Y,Z] = [Z,X] = 0$ and

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$$

$$= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X)$$

$$= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] = 0$$

since ∇ is assumed to be symmetric. $\square$

From now on we shall restrict ourselves in the case where ∇ is the Levi-Civita connection of a Riemannian manifold M with Riemannian metric $g = \langle .,. \rangle$. Since g is a non-degenerate, symmetric bilinear form on each tangent space T_pM, $p \in M$, the value of the curvature tensor $R(u,v)w$ for given u, v, $w \in T_pM$ is completely determined by the values of $\langle R(u,v)w, z\rangle$ for $z \in T_pM$. The $C^\infty(M)$-multilinear form defined by

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$$

is called the *covariant Riemann curvature tensor*.

Proposition 4.1.3. *The covariant Riemann curvature tensor has the following properties.*
(a) $Rm(W,X,Y,Z) = -Rm(X,W,Y,Z)$,
(b) $Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)$,
(c) $Rm(W,X,Y,Z) = Rm(Y,Z,W,X)$,
(d) $Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0$
for every W, X, Y, $Z \in \mathcal{X}(M)$.

Proof. Properties (a) and (d) are merely a restatement of Proposition 4.1.2. For (b) we prove first that $Rm(W, X, Y, Y) = 0$ for every $W,\ X,\ Y \in \mathcal{X}(M)$. Since ∇ is the Levi-Civita connection, we have

$$W(X(\|Y\|^2)) = W(2\langle\nabla_X Y, Y\rangle) = 2\langle\nabla_W\nabla_X Y, Y\rangle + 2\langle\nabla_X Y, \nabla_W Y\rangle,$$

$$X(W(\|Y\|^2)) = X(2\langle\nabla_W Y, Y\rangle) = 2\langle\nabla_X\nabla_W Y, Y\rangle + 2\langle\nabla_W Y, \nabla_X Y\rangle$$

and $[W, X](\|Y\|^2) = 2\langle\nabla_{[W,X]}Y, Y\rangle$. It follows that

$$2Rm(W, X, Y, Y) = 2\langle\nabla_W\nabla_X Y, Y\rangle - 2\langle\nabla_X\nabla_W Y, Y\rangle - 2\langle\nabla_{[W,X]}Y, Y\rangle$$

$$= W(X(\|Y\|^2)) - X(W(\|Y\|^2)) - [W, X](\|Y\|^2) = 0.$$

From this now we conclude that

$$Rm(W, X, Y, Z) + Rm(W, X, Z, Y) = Rm(W, X, Y + Z, Y + Z) = 0.$$

Property (c) follows from the rest noting first that from the first identity of Bianchi

$$Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0,$$

$$Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, Z) = 0,$$

$$Rm(Z, W, X, Y) + Rm(W, X, Z, Y) + Rm(X, Z, W, Y) = 0.$$

Summing up and using (a) and (b) we get

$$2Rm(Y, W, X, Z) - 2Rm(X, Z, Y, W) = 0. \quad \square$$

If (U, ϕ) is a chart of M and $\phi = (x^1, x^2, ..., x^n)$, there are uniquely determined smooth functions $R^l_{ijk} : U \to \mathbb{R}$, $1 \leq i, j, k, l \leq n$, such that

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = \sum_{l=1}^{n} R^l_{ijk}\frac{\partial}{\partial x^l},$$

which are called the local components of the curvature tensor with respect to the given chart. A straightforward calculation gives the following expression of the local components of the curvature tensor in terms of the Christoffel symbols

$$R^l_{ijk} = \sum_{m=1}^{n} \Gamma^m_{jk}\Gamma^l_{im} - \sum_{m=1}^{n} \Gamma^m_{ik}\Gamma^l_{jm} + \frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j}$$

due to C.F.Gauss.

The local components of the covariant Riemann curvature tensor are defined analogously by

$$R_{ijkl} = Rm(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) = \sum_{m=1}^{n} g_{ml}R^m_{ijk}.$$

Thus, the local expressions of the properties (a)-(d) of Proposition 4.1.3 are now (a) $R_{ijkl} = -Rjikl$, (b) $R_{ijkl} = -R_{ijlk}$, (c) $R_{ijkl} = R_{klij}$, (d) $R_{ijkl}+R_{jkil}+R_{kijl} = 0$.

The variation of the covariant Riemann curvature tensor has a property called the second (infinitesimal) identity of Bianchi. If

$$T : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M)$$

is a $C^\infty(M) - r-$multilinear form on a Riemannian manifold M with Levi-Civita connection ∇, the covariant derivative of T in the direction of the smooth vector field $X \in \mathcal{X}(M)$ is by definition the $C^\infty(M) - r-$multilinear form given by the formula

$$(\nabla_X T)(Y_1, ..., Y_r) = X(T(Y_1, ..., Y_r)) - \sum_{i=1}^{r} T(Y_1, ..., Y_{i-1}, \nabla_X Y_i, Y_{i+1}, ..., Y_r)$$

for every Y_1,...,$Y_r \in \mathcal{X}(M)$.

If $T : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{X}(M)$ is a $C^\infty(M) - r-$multilinear map, its covariant derivative in the direction of $X \in \mathcal{X}(M)$ is the $C^\infty(M) - r-$multilinear map defined by

$$(\nabla_X T)(Y_1, ..., Y_r) = \nabla_X(T(Y_1, ..., Y_r)) - \sum_{i=1}^{r} T(Y_1, ..., Y_{i-1}, \nabla_X Y_i, Y_{i+1}, ..., Y_r)$$

for every Y_1,...,$Y_r \in \mathcal{X}(M)$.

In both cases, T is called *parallel* if $\nabla_X T = 0$ for every $X \in \mathcal{X}(M)$. For example, in the case of the Riemannian metric g, which is $C^\infty(M)$-bilinear, we have

$$(\nabla_X g)(Y_1, Y_2) = X\langle Y_1, Y_2\rangle - \langle \nabla_X Y_1, Y_2\rangle - \langle Y_1, \nabla_X Y_2\rangle = 0$$

for every X, Y_1, $Y_2 \in \mathcal{X}(M)$.

Example 4.1.4. If M is a Riemannian n-manifold, $n \geq 2$, the map

$$T : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

defined by

$$T(X, Y, Z) = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

is $C^\infty(M)$-multilinear and parallel (with respect to the Levi-Civita connection), because

$$(\nabla_W T)(X, Y, Z) = \nabla_W(T(X, Y, Z)) - T(\nabla_W X, Y, Z) - T(X, \nabla_W Y, Z) - T(X, Y, \nabla_W Z)$$

$$= W(\langle Y, Z\rangle)X + \langle Y, Z\rangle \nabla_W X - W(\langle X, Z\rangle)Y - \langle X, Z\rangle \nabla_W Y - \langle Y, Z\rangle \nabla_W X$$

$$+\langle \nabla_W X, Z\rangle Y - \langle \nabla_W Y, Z\rangle X + \langle X, Z\rangle \nabla_W Y - \langle Y, \nabla_W Z\rangle X + \langle X, \nabla_W Z\rangle Y = 0.$$

Putting $(\nabla T)(X, Y_1, ..., Y_r) = (\nabla_X T)(Y_1, ..., Y_r)$ the second identity of Bianchi can be stated as follows.

Proposition 4.1.5. *The curvature tensor R of a Riemannian n-manifold M satisfies*

$$(\nabla R)(W, X, Y, Z) + (\nabla R)(X, Y, W, Z) + (\nabla R)(Y, W, X, Z) = 0$$

for every W, X, Y, $Z \in \mathcal{X}(M)$.

Proof. Since ∇R is $C^\infty(M)$-multilinear, it suffices to prove the identity at a point. Let $p \in M$ and let (U, ϕ) be a normal chart of M at p with $\phi = (x^1, ..., x^n)$. We need only consider the case

$$W = \frac{\partial}{\partial x^i}, \quad X = \frac{\partial}{\partial x^j}, \quad Y = \frac{\partial}{\partial x^k}, \quad Z = \frac{\partial}{\partial x^l}.$$

Since the chart is normal at p, the Christoffel symbols vanish at p and therefore

$$\nabla_{(\frac{\partial}{\partial x^i})_p} \frac{\partial}{\partial x^j} = 0$$

for all $1 \leq i, j \leq n$. It follows that

$$(\nabla R)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})(p) = \nabla_{(\frac{\partial}{\partial x^i})_p} R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) \frac{\partial}{\partial x^l}.$$

Since

$$\nabla_{\frac{\partial}{\partial x^i}} R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) \frac{\partial}{\partial x^l} + \nabla_{\frac{\partial}{\partial x^j}} R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}) \frac{\partial}{\partial x^l} + \nabla_{\frac{\partial}{\partial x^k}} R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^l}$$

$$= R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l}) + R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}) + R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i})(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}),$$

we get

$$(\nabla R)(W, X, Y, Z)(p) + (\nabla R)(X, Y, W, Z)(p) + (\nabla R)(Y, W, X, Z)(p)$$

$$= R((\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial x^j})_p)(\nabla_{(\frac{\partial}{\partial x^k})_p} \frac{\partial}{\partial x^l}) + R((\frac{\partial}{\partial x^j})_p, (\frac{\partial}{\partial x^k})_p)(\nabla_{(\frac{\partial}{\partial x^i})_p} \frac{\partial}{\partial x^l})$$

$$+ R((\frac{\partial}{\partial x^k})_p, (\frac{\partial}{\partial x^i})_p)(\nabla_{(\frac{\partial}{\partial x^j})_p} \frac{\partial}{\partial x^l}) = 0 + 0 + 0 = 0 \quad \square$$

If (U, ϕ) is a chart of M with $\phi = (x^1, ..., x^n)$, setting

$$\nabla_i R_{jklm} = \left(\nabla_{\frac{\partial}{\partial x^i}} Rm\right)\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m}\right)$$

$$= \frac{\partial R_{jklm}}{x^i} - \sum_{s=1}^n \Gamma_{ij}^s R_{sklm} - \sum_{s=1}^n \Gamma_{ik}^s R_{jslm} - \sum_{s=1}^n \Gamma_{il}^s R_{jksm} - \sum_{s=1}^n \Gamma_{im}^s R_{jkls}$$

the second identity of Bianchi locally takes the form

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.$$

4.2 Sectional curvature

The curvature tensor of a Riemannian manifold can be encoded through an arithmetical quantity, which is called the sectional curvature and had been originally introduced by C.F. Gauss in his differential geometry of surfaces in the euclidean 3-space.

Let M be a Riemannian n-manifold, $n \geq 2$, and let R be the curvature tensor of (the Levi-Civita connection of) M. Let $p \in M$ and u, $v \in T_pM$ be linearly independent tangent vectors spanning a 2-dimensional vector subspace S of T_pM. The real number

$$K_p(S) = \frac{\langle R(u,v)v,u\rangle}{\|u\|^2 \cdot \|v\|^2 - \langle u,v\rangle^2}$$

depends only on S and not on the choice of the particular basis $\{u, v\}$. Indeed, if $\{u_1, v_1\}$ is another basis of S, there are a, b, c, $d \in \mathbb{R}$ with $ad - bc \neq 0$ such that

$$u_1 = au + cv, \quad v_1 = bu + dv.$$

Then,

$$\|u_1\|^2 \cdot \|v_1\|^2 - \langle u_1, v_1\rangle^2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 \cdot (\|u\|^2 \cdot \|v\|^2 - \langle u,v\rangle^2)$$

and form Proposition 4.1.3

$$\langle R(u_1,v_1)v_1,u_1\rangle = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 \langle R(u,v)v,u\rangle.$$

Note that the denominator $\|u\|^2 \cdot \|v\|^2 - \langle u,v\rangle^2$ is the square of the area (with respect to the Riemannian inner product in T_pM) of the parallelogram with sides u, v.

Definition 4.2.1. If $p \in M$ and S is a 2-dimensional vector subspace of T_pM, the real number $K_p(S)$ is called the *sectional curvature* of M at p with respect to S.

It is obvious that the sectional curvature is invariant under local isometries. The complete determination of the curvature tensor by the sectional curvatures is a purely algebraic fact.

Lemma 4.2.2. *Let $(V, \langle .,. \rangle)$ be a real inner product vector space and let R_1, $R_2 : V \times V \times V \times V \to \mathbb{R}$ be two multilinear forms having the properties (a)-(d) of Proposition 4.1.3. If $R_1(u,v,v,u) = R_2(u,v,v,u)$ for every pair o linearly independent vectors u, $v \in V$, then $R_1 = R_2$.*

Proof. Putting $R = R_1 - R_2$, it suffices to show that if R has the properties (a)-(d) of Proposition 4.1.3 and $R(u,v,v,u) = 0$ for every u, $v \in V$, then $R = 0$. As a first step we have

$$0 = R(v+w,u,u,v+w) = 2R(v,u,u,w).$$

Thus,

$$0 = R(v,s+w,s+w,u) = R(v,s,w,u) + R(v,w,s,u)$$

for every s, v, w, $u \in V$. Finally,

$$0 = R(u,v,w,s) + R(v,w,u,s) + R(w,u,v,s)$$
$$= R(u,v,w,s) - R(v,u,w,s) - R(u,w,v,s) = 3R(u,v,w,s). \quad \square$$

Corollary 4.2.3. *Let M be a Riemannian n-manifold, $n \geq 2$, and $p \in M$. If there exists $c \in \mathbb{R}$ such that $K_p(S) = c$ for every 2-dimensional vector subspace S of T_pM, then*

$$R(u,v)w = c(\langle v,w\rangle u - \langle u,w\rangle v)$$

for every u, v, $w \in T_pM$. $\square$

By the definitions, the euclidean n-space $\mathbb{R}^n$, $n \geq 2$, has constant sectional curvature equal to zero.

Example 4.2.4. We shall compute the sectional curvature of the hyperbolic n-space $\mathbb{H}^n_R$ of radius $R > 0$. Recall from Example 3.3.5 that

$$\mathbb{H}^n_R = \{(p_1,...,p_n) \in \mathbb{R}^n : p_n > 0\}$$

and the Riemannian metric is

$$g_{ij}(p_1,...,p_n) = \frac{R^2}{p_n^2}\delta_{ij}, \quad 1 \leq 1, j \leq n.$$

First we shall calculate the Christoffel symbols which for each $1 \leq i,j \leq n$ are the solutions of the linear system

$$\sum_{k=1}^{n} \Gamma^k_{ij} g_{km} = \frac{1}{2}\left[\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right]$$

for every $1 \leq m \leq n$. Since

$$\frac{\partial g_{ij}}{\partial x^m} = -\delta_{ij}\delta_{mn}\frac{2R^2}{(x^n)^3}$$

substituting we find

$$\Gamma^m_{ij} = -\frac{1}{x^n}(\delta_{jm}\delta_{in} + \delta_{mi}\delta_{jn} - \delta_{ij}\delta_{mn})$$

for every $1 \leq i,j,m \leq n$. Now for every $1 \leq i,j \leq n-1$ we have

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \frac{\delta_{ij}}{x^n}\cdot\frac{\partial}{\partial x^n}, \quad \nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^j} = -\frac{1}{x^n}\cdot\frac{\partial}{\partial x^j}$$

and also

$$\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^n} = -\frac{1}{x^n}\cdot\frac{\partial}{\partial x^n}.$$

It follows that

$$\nabla_{\frac{\partial}{\partial x^1}}\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^n} = -\frac{1}{x^n}\nabla_{\frac{\partial}{\partial x^1}}\frac{\partial}{\partial x^n} = -\frac{1}{x^n}\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^1} = \frac{1}{(x^n)^2}\cdot\frac{\partial}{\partial x^1},$$

$$\nabla_{\frac{\partial}{\partial x^n}}\nabla_{\frac{\partial}{\partial x^1}}\frac{\partial}{\partial x^n} = -\frac{1}{x^n}\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^1} + \frac{1}{(x^n)^2}\frac{\partial}{\partial x^1} = \frac{2}{(x^n)^2}\frac{\partial}{\partial x^1}.$$

Therefore,

$$R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^n})\frac{\partial}{\partial x^n} = -\frac{1}{(x^n)^2}\frac{\partial}{\partial x^1}$$

and the sectional curvature of $\mathbb{H}^n_R$ at any point $p = (p_1, ..., p_n) \in \mathbb{H}^n_R$ with respect to the 2-dimensional vector subspace S of $T_p\mathbb{H}^n_R$ which is generated by $\{(\frac{\partial}{\partial x^1})_p, (\frac{\partial}{\partial x^n})_p\}$ is

$$K_p(S) = \frac{-\frac{1}{p_n^2}g_{11}(p)}{g_{11}(p)g_{nn}(p)} = -\frac{1}{R^2}.$$

Since for every $q \in \mathbb{H}^n_R$ and every pair of orthogonal vectors u, $v \in T_q\mathbb{H}^n_R$ with $\|u\| = \|v\| = 1$ there exists a hyperbolic isometry $h : \mathbb{H}^n_R \to \mathbb{H}^n_R$ such that $h(e_n) = q$ and

$$h_{*e_n}\left(\frac{\partial}{\partial x^1}\right)_{e_n} = Ru, \quad h_{*e_n}\left(\frac{\partial}{\partial x^n}\right)_{e_n} = Rv,$$

it follows that $\mathbb{H}^n_R$ has everywhere constant sectional curvature $-\dfrac{1}{R^2}$.

The following criterion which gives a condition that ensures everywhere constant sectional curvature is due to F. Schur.

Theorem 4.2.5. *Let M be a connected Riemannian n-manifold, $n \geq 3$. If there exists a function $f : M \to \mathbb{R}$ such that $K_p(S) = f(p)$ for every 2-dimensional vector subspace S of T_pM and every $p \in M$, then f is constant.*

Proof. Firstly we note that f is necessarily smooth, because if X and Y are two smooth local vector fields with $\|X\| = \|Y\| = 1$ and $\langle X, Y\rangle = 0$, then locally $f = \langle R(X,Y)Y, X\rangle$. Our assumption and Corollary 4.2.3 imply that $R = f \cdot T$, where T is the parallel $C^\infty(M)$-multilinear map of Example 4.1.4. Thus,

$$\nabla_X R = f \cdot \nabla_X T + Xf \cdot T = Xf \cdot T$$

for every local smooth vector field X. If now $\{X, Y, Z\}$ is a local orthonormal frame on M, that is $\{X_p, Y_p, Z_p\}$ is an orthonormal basis of T_pM for every p in an open subset of M, from Proposition 4.1.5 (the second identity of Bianchi) we have

$$0 = (\nabla_X R)(Y, Z, Z) + (\nabla_Y R)(Z, X, Z) + (\nabla_Z R)(X, Y, Z)$$

$$= Xf \cdot (\langle Z, Z\rangle Y - \langle Y, Z\rangle Z) + Yf \cdot (\langle X, Z\rangle Z - \langle Z, Z\rangle X) + Zf \cdot (\langle Y, Z\rangle X - \langle X, Z\rangle Y)$$

$$Xf \cdot Y - Yf \cdot X.$$

Hence $Xf = Yf = 0$. This shows that the derivative of f vanishes everywhere on M. Since M is connected, f must be constant. □

4.3 Submanifolds of the euclidean space

In this section we shall compute the curvature of a k-dimensional smooth submanifold M of the euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$, endowed with the Riemannian metric which is induced from the euclidean Riemannian metric on $\mathbb{R}^{n+1}$. We can identify the tangent space T_pM at a point $p \in M$ with a vector subspace of $\mathbb{R}^{n+1}$ and a tangent smooth vector field X of M with a smooth map $X : M \to \mathbb{R}^{n+1}$ such that $X_p \in T_pM$ for $p \in M$. The value $\nabla_{X_p}Y$ of the Levi-Civita connection ∇ of M for $X, Y \in \mathcal{X}(M)$, $p \in M$, is the tangent to M component of the directional derivative $DY(p)(X_p)$ of Y at p with respect to X_p. Let $B_p(X_p, Y) \in (T_pM)^{\perp}$ be the orthogonal to M component of $DY(p)(X_p)$. If $f \in C^{\infty}(M)$, the orthogonal to M component of $D(fY)(p)(X_p) = f(p)DY(p)(X_p) + X_p(f) \cdot Y(p)$ coincides with the orthogonal to M component of $f(p)DY(p)(X_p)$. This means that $B_p(X_p, fY) = f(p)B_p(X_p, Y)$, for every $f \in C^{\infty}(M)$ and $p \in M$, which implies that $B_p(X_p, Y)$ depends only on Y_p. So there is a well-defined bilinear map

$$B_p : T_pM \times T_pM \to (T_pM)^{\perp}$$

which evidently depends smoothly on p, meaning that $B_p(X_p, Y_p)$ is a smooth function of $p \in M$ for every $X, Y \in \mathcal{X}(M)$. By definition

$$DY(p)(X_p) = \nabla_{X_p}Y + B_p(X_p, Y_p)$$

for every $p \in M$ and $X, Y \in \mathcal{X}(M)$.

Lemma 4.3.1. *The bilinear map B_p is symmetric for every $p \in M$.*

Proof. Let (U, ϕ) be a chart of M around the point $p \in M$ with $\phi = (x^1, ..., x^k)$ coming from a M-straightening chart of $\mathbb{R}^{n+1}$. It suffices to show that

$$B_p((\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial x^j})_p) = B_p((\frac{\partial}{\partial x^j})_p, (\frac{\partial}{\partial x^i})_p)$$

for every $1 \leq i, j \leq k$. Recall that

$$\frac{\partial}{\partial x^i} = \frac{\partial \phi^{-1}}{\partial x^i} \circ \phi$$

on U, as a vector field in $\mathbb{R}^{n+1}$ along $U \subset M$. Since $\phi^{-1} : \phi(U) \to U \subset M \subset \mathbb{R}^{n+1}$ is smooth, we have

$$D(\frac{\partial}{\partial x^i})(\frac{\partial}{\partial x^j}) = \frac{\partial^2 \phi^{-1}}{\partial x^i \partial x^j} \circ \phi = \frac{\partial^2 \phi^{-1}}{\partial x^j \partial x^i} \circ \phi = D(\frac{\partial}{\partial x^j})(\frac{\partial}{\partial x^i}),$$

from which the symmetry of B_p follows. □

The curvature tensor of M can now be represented completely in terms of this bilinear form and the Riemannian metric.

Proposition 4.3.2. *The covariant Riemann curvature tensor of M is given by the formula*

$$\langle R(X,Y)Z,W\rangle = \langle B(Y,Z),B(X,W)\rangle - \langle B(X,Z),B(Y,W)\rangle$$

for X, Y, Z, $W \in \mathcal{X}(M)$.

Proof. Since both sides are $C^\infty(M)$-multilinear, it suffices to prove the formula for the basic local vector fields

$$X = \frac{\partial}{\partial x^i}, \quad Y = \frac{\partial}{\partial x^j}, \quad Z = \frac{\partial}{\partial x^l}$$

with respect to a chart $(U, x^1, ..., x^k)$ of M. Then,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z$$

and $DX(DY(Z)) = DY(DX(Z))$, as in the proof of the preceding Lemma 4.3.1. But

$$DX(DY(Z)) = DX(\nabla_Y Z) + DX(B(Y,Z)) = \nabla_X\nabla_Y Z + B(X,\nabla_Y Z) + DX(B(Y,Z))$$

and similarly

$$DY(DX(Z)) = \nabla_Y\nabla_X Z + B(Y,\nabla_X Z) + DY(B(X,Z)).$$

Subtracting we obtain

$$R(X,Y)Z = -B(X,\nabla_Y Z) + B(Y,\nabla_X Z) - DX(B(Y,Z)) + DY(B(X,Z))$$

and therefore

$$\langle R(X,Y)Z,W\rangle = -\langle DX(B(Y,Z)),W\rangle + \langle DY(B(X,Z)),W\rangle.$$

Also, differentiating the equation $\langle B(Y,Z),W\rangle = 0$ in the direction of X we get

$$0 = X\langle B(Y,Z),W\rangle = \langle DX(B(Y,Z)),W\rangle + \langle B(Y,Z),DX(W)\rangle$$

$$= \langle DX(B(Y,Z)),W\rangle + \langle B(Y,Z),B(X,W)\rangle$$

and similarly

$$\langle DY(B(X,Z)),W\rangle + \langle B(X,Z),B(Y,W)\rangle = 0.$$

Substituting now yields the formula. □

In the particular case of a hypersurface M in $\mathbb{R}^{n+1}$, that is $k = n$, if (U,ϕ) is a chart of M, there exists a unique up to sign (assuming that U is connected) unit normal vector field N along $U \subset M$. This is a smooth map $N : U \to S^n \subset \mathbb{R}^{n+1}$ such that $(T_pM)^\perp$ is generated by $N(p)$ for every $p \in U$, which is called the Gauss map on U. There is then a symmetric bilinear form $II_p : T_pM \times T_pM \to \mathbb{R}$ such that

$$B(u,v) = II_p(u,v)N$$

for every u, $v \in T_pM$, which is called *the second fundamental form* of M at the point $p \in M$ (with respect to N). The formula of the preceding Proposition 4.3.2 becomes

$$\langle R(u,v)w,s\rangle = II_p(v,w)II_p(u,s) - II_p(u,w)II_p(v,s)$$

for all u, v, w, $s \in T_pM$ and $p \in U$.

The sectional curvature of M at the point p with respect to the 2-dimensional vector subspace S of T_pM with basis $\{v,w\}$ is

$$K_p(S) = \frac{II_p(v,v)II_p(w,w) - II_p(v,w)^2}{\|v\|^2\|w\|^2 - \langle v,w\rangle^2}.$$

If M is a surface in $\mathbb{R}^3$, this is the Gauss curvature of M at p.

Note that the covariant Riemann curvature tensor and the sectional curvature of M do not depend on the choice of N.

Since $T_pM = T_{N(p)}S^n$ as vector subspaces of $\mathbb{R}^{n+1}$, if X and Y are local smooth vector fields tangent to M on U, the derivative $N_{*p} : T_pM \to T_{N(p)}S^n = T_pM$ at $p \in U$ of the Gauss map satisfies

$$\begin{aligned} 0 = X_p\langle Y,N\rangle &= \langle N_{*p}(X_p), Y_p\rangle + \langle N(p), DX(p)(Y(p))\rangle \\ &= \langle N_{*p}(X_p), Y_p\rangle + II_p(X_p,Y_p). \end{aligned}$$

Hence the second fundamental form of M at p is given by the formula

$$II_p(u,v) = -\langle N_{*p}(u), v\rangle, \quad u,v \in T_pM.$$

It follows from this and the symmetry of the second fundamental form that N_{*p} is self-adjoint and therefore has real eigenvalues, which are the *principal curvatures* of M at p. The corresponding eigenvectors define the *principal directions* of M at p.

Example 4.3.3. We shall apply the above in order to calculate the curvature tensor and the sectional curvature of the sphere S^n_R of radius $R > 0$ in $\mathbb{R}^{n+1}$. In this case there is a globally defined Gauss map $N : S^n_R \to S^n$ by

$$N(p) = \frac{1}{R}p$$

for every $p \in S^n_R$. The second fundamental form is thus

$$II_p(u,v) = -\frac{1}{R}\langle u,v\rangle$$

and the sectional curvature of S^n_R at the point p with respect to the 2-dimensional vector subspace S of $T_pS^n_R$ with orthonormal basis $\{u,v\}$ is

$$K_p(S) = II_p(u,u)II_p(v,v\_II_p(u,v)^2 = \big(-\frac{1}{R}\big)\big(-\frac{1}{R}\big) - 0 = \frac{1}{R^2}.$$

The covariant Riemann curvature tensor is given by the formula

$$\langle R(X,Y)Z,W\rangle = \frac{1}{R^2}\big(\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\big)$$

and the curvature tensor is

$$R(X,Y)Z = \frac{1}{R^2}\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big).$$

4.4 Riemannian submersions

Let M, N be two Riemannian manifolds and let $f : M \to N$ be a submersion onto N. From Corollary 1.3.5, for each $q \in N$ the level set $f^{-1}(q)$ is a smooth submanifold of M and $T_p f^{-1}(q) = \mathrm{Ker} f_{*p}$ for every $p \in f^{-1}(q)$. We shall use the notation $T_p^v M = T_p f^{-1}(q)$ and call this vector space the vertical subspace of T_pM (with respect to f), and $T_p^h M$ for the orthogonal complement of $T_p^v M$ in T_pM with respect to the Riemannian metric, which will be called the horizontal subspace of T_pM. Obviously, f_{*p} maps $T_p^h M$ isomorphically onto $T_{f(p)}M$.

Definition 4.4.1. A submersion $f : M \to N$ onto N is called *Riemannian submersion* if f_{*p} maps $T_p^h M$ isometrically onto $T_{f(p)}M$ for every $p \in M$.

If $f : M \to N$ is a Riemannian submersion and $X \in \mathcal{X}(N)$, there exists a unique $\tilde{X} \in \mathcal{X}(M)$ such that $\tilde{X}_p \in T_p^h M$ and $f_{*p}(\tilde{X}_p) = X_{f(p)}$ for every $p \in M$, which is called the *horizontal lift* of X.

Lemma 4.4.2. *If $f : M \to N$ is a Riemannian submersion, then*

$$\nabla_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^v$$

for every X, $Y \in \mathcal{X}(N)$, where $[\tilde{X}, \tilde{Y}]_p^v$ is the vertical component of $[\tilde{X}, \tilde{Y}]_p$ for $p \in M$ which depends only on $\tilde{X}_p$ and $\tilde{Y}_p$.

Proof. Let X, Y, $Z \in \mathcal{X}(N)$ and let $\tilde{X}$, $\tilde{Y}$, $\tilde{Z} \in \mathcal{X}(M)$ be their corresponding horizontal lifts. Let $q \in N$, $p \in f^{-1}(q)$ and $V_p \in T_p^v M$. There exists a (not unique) extension of V_p to some $V \in \mathcal{X}(M)$ with $V_x \in T_x^v M$ for every $x \in M$. Thus, V is orthogonal to $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$. Moreover, the smooth function $\langle \tilde{X}, \tilde{Y} \rangle$ takes the constant value $\langle X_q, Y_q \rangle$ on the level set $f^{-1}(q)$, because f is a Riemannian sumbersion, and therefore $V\langle \tilde{X}, \tilde{Y} \rangle = 0$, since V is vertical. Also, $\tilde{X}\langle \tilde{Y}, \tilde{Z} \rangle = X\langle Y, Z \rangle$, from the definition of the horizontal lifts and the chain rule. However,

$$\begin{aligned}
\tilde{X}\langle \tilde{Y}, \tilde{Z} \rangle &= \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \langle \tilde{Y}, \nabla_{\tilde{Z}} \tilde{X} \rangle + \langle \tilde{Y}, [\tilde{X}, \tilde{Z}] \rangle \\
&= \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \tilde{X}\langle \tilde{X}, \tilde{Y} \rangle - \langle \tilde{X}, \nabla_{\tilde{Z}} \tilde{Y} \rangle + \langle Y, [X, Z] \rangle \\
&= \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle - \langle \tilde{X}, \nabla_{\tilde{Z}} \tilde{Y} \rangle + Z\langle X, Y \rangle + \langle Y, [X, Z] \rangle
\end{aligned}$$

and similarly $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle - \langle X, \nabla_Z Y \rangle + Z\langle X, Y \rangle + \langle Y, [X, Z] \rangle$. Hence

$$\langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle - \langle \tilde{X}, \nabla_{\tilde{Z}} \tilde{Y} \rangle = \langle \nabla_X Y, Z \rangle - \langle X, \nabla_Z Y \rangle.$$

In the same way from the equality $\tilde{Y}\langle \tilde{X}, \tilde{Z} \rangle = Y\langle X, Z \rangle$ we get

$$\langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \langle \tilde{X}, \nabla_{\tilde{Z}} \tilde{Y} \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_Z Y \rangle.$$

Consequently, $\langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle = \langle \nabla_X Y, Z \rangle$.

On the other hand,

$$0 = V\langle \tilde{X}, \tilde{Y} \rangle = \langle \nabla_{\tilde{X}} V, \tilde{Y} \rangle + \langle [V, \tilde{X}], \tilde{Y} \rangle + \langle \nabla_V \tilde{Y}, \tilde{X} \rangle$$

$$= -\langle \nabla_{\tilde{X}} \tilde{Y}, V\rangle + 0 + \langle \nabla_{\tilde{Y}} V, \tilde{X}\rangle + \langle [V, \tilde{Y}], \tilde{X}\rangle = -\langle \nabla_{\tilde{X}} \tilde{Y}, V\rangle - \langle \nabla_{\tilde{Y}} \tilde{X}, V\rangle + 0$$

$$= -\langle \nabla_{\tilde{X}} \tilde{Y}, V\rangle - \langle \nabla_{\tilde{X}} \tilde{Y}, V\rangle - \langle [\tilde{Y}, \tilde{X}], V\rangle$$

because $[V, \tilde{X}]$ and $[V, \tilde{Y}]$ are vertical. This shows that $\nabla_{\tilde{X}_p} \tilde{Y} - \frac{1}{2}[\tilde{X}, \tilde{Y}]^v_p \in T^h_p M$ and from the above

$$\langle \nabla_{\tilde{X}_p} \tilde{Y} - \frac{1}{2}[\tilde{X}, \tilde{Y}]^v_p, \tilde{Z}_p\rangle = \langle \nabla_{\tilde{X}_p} \tilde{Y}, \tilde{Z}_p\rangle = \langle \nabla_{X_q} Y, Z_q\rangle = \langle (\widetilde{\nabla_X Y})_p, \tilde{Z}_p\rangle.$$

Finally, since $\langle V, [\tilde{X}, \tilde{Y}]\rangle = \langle V, \nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X}\rangle = -\langle \nabla_{\tilde{X}} V, \tilde{Y}\rangle + \langle \nabla_{\tilde{Y}} V, \tilde{X}\rangle$ it follows that $[\tilde{X}, \tilde{Y}]^v_p$ depends only on $\tilde{X}_p$ and $\tilde{Y}_p$. □

Corollary 4.4.3. *Let $f : M \to N$ be a Riemannian submersion, $\gamma : I \to N$ be a smooth parametrized curve and $\gamma^h : I \to M$ be a horizontal lift of γ, which means that $\dot{\gamma}^h(t) \in T^h_{\gamma^h(t)} M$ and $f \circ \gamma^h = \gamma$ for every $t \in I$. Then, γ is a geodesic of N if and only if γ^h is a geodesic of M.*

Proof. From the preceding Lemma 4.4.2 follows immediately that

$$\frac{D\dot{\gamma}^h}{dt} = \widetilde{\frac{D\dot{\gamma}}{dt}} + \frac{1}{2}[\dot{\gamma}^h(t), \dot{\gamma}^h(t)]^v = \widetilde{\frac{D\dot{\gamma}}{dt}}. \quad \square$$

The following formulas relating the curvature tensors and the sectional curvatures of the base space and the total space of a Riemannian submersion were found by B. O'Neil and are known in the literature with his name.

Theorem 4.4.4. *Let $f : M \to N$ be a Riemannian submersion, X, Y, Z, $W \in \mathcal{X}(N)$ and $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{W} \in \mathcal{X}(M)$ be their horizontal lifts, respectively. Then,*
(a) the covariant Riemann curvature tensor of N is given by the formula

$$\langle R(X, Y)Z, W\rangle = \langle R(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}\rangle - \frac{1}{4}\langle [\tilde{X}, \tilde{Z}]^v, [\tilde{Y}, \tilde{W}]^v\rangle + \frac{1}{4}\langle [\tilde{Y}, \tilde{Z}]^v, [\tilde{X}, \tilde{W}]^v\rangle$$

$$-\frac{1}{2}\langle [\tilde{Z}, \tilde{W}]^v, [\tilde{X}, \tilde{Y}]^v\rangle.$$

(b) If u, w is an orthonormal basis of a 2-dimensional subspace S of $T_q N$, $q \in N$, and $\tilde{S}$ is the horizontal lift of S at a point $p \in f^{-1}(q)$, then

$$K_q(S) = K_p(\tilde{S}) + \frac{3}{4}\|[\tilde{u}, \tilde{w}]^v\|^2 \geq K_p(\tilde{S}),$$

where $\tilde{u}$, $\tilde{w} \in T_p M$ are the horizontal lifts of u and w at p, respectively.

Proof. As in the proof of Lemma 4.4.2, if $V \in \mathcal{X}(M)$ is vertical we have

$$0 = \langle \nabla_V \tilde{Z}, \tilde{W}\rangle + \langle \tilde{Z}, \nabla_V \tilde{W}\rangle = \langle \nabla_V \tilde{Z}, \tilde{W}\rangle + \langle Z, \nabla_{\tilde{W}} V\rangle + \langle \tilde{Z}, [V, \tilde{W}]\rangle$$

$$= \langle \nabla_V \tilde{Z}, \tilde{W}\rangle - \langle V, \nabla_{\tilde{W}} \tilde{Z}\rangle + 0 = \langle \nabla_V \tilde{Z}, \tilde{W}\rangle - \langle V, \widetilde{\nabla_W Z}\rangle + \frac{1}{2}\langle V, [\tilde{Z}, \tilde{W}]^v\rangle$$

and therefore
$$\langle \nabla_V \tilde{Z}, \tilde{W}\rangle = -\frac{1}{2}\langle V, [\tilde{Z}, \tilde{W}]^v\rangle.$$
Again as in the proof of Lemma 4.4.2 we have $\tilde{X}\langle \nabla_{\tilde{Y}} \tilde{Z}, \tilde{W}\rangle = X\langle \nabla_Y Z, W\rangle$. But
$$\begin{aligned}\tilde{X}\langle \nabla_{\tilde{Y}} \tilde{Z}, \tilde{W}\rangle &= \langle \nabla_{\tilde{X}}\nabla_{\tilde{Y}} \tilde{Z}, \tilde{W}\rangle + \langle \nabla_{\tilde{Y}} \tilde{Z}, \nabla_{\tilde{X}} \tilde{W}\rangle \\ &= \langle \nabla_{\tilde{X}}\nabla_{\tilde{Y}} \tilde{Z}, \tilde{W}\rangle + \langle \widetilde{\nabla_Y Z}, \widetilde{\nabla_X W}\rangle + \frac{1}{4}\langle [\tilde{Y}, \tilde{Z}]^v, [\tilde{X}, \tilde{W}]^v\rangle \\ &= \langle \nabla_{\tilde{X}}\nabla_{\tilde{Y}} \tilde{Z}, \tilde{W}\rangle + \langle \nabla_Y Z, \nabla_X W\rangle + \frac{1}{4}\langle [\tilde{Y}, \tilde{Z}]^v, [\tilde{X}, \tilde{W}]^v\rangle\end{aligned}$$
from Lemma 4.4.2. Consequently,
$$\begin{aligned}\langle \nabla_{\tilde{X}}\nabla_{\tilde{Y}} \tilde{Z}, \tilde{W}\rangle &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \frac{1}{4}\langle [\tilde{Y}, \tilde{Z}]^v, [\tilde{X}, \tilde{W}]^v\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \frac{1}{4}\langle [\tilde{Y}, \tilde{Z}]^v, [\tilde{X}, \tilde{W}]^v\rangle\end{aligned}$$
and similarly
$$\langle \nabla_{\tilde{Y}}\nabla_{\tilde{X}} \tilde{Z}, \tilde{W}\rangle = \langle \nabla_Y \nabla_X Z, W\rangle - \frac{1}{4}\langle [\tilde{X}, \tilde{Z}]^v, [\tilde{Y}, \tilde{W}]^v\rangle.$$
Moreover, applying what we have proved in the beginning to the particular case $V = [\tilde{X}, \tilde{Y}]^v$ and Lemma 4.4.2 we obtain
$$\begin{aligned}\langle \nabla_{[\tilde{X},\tilde{Y}]} \tilde{Z}, \tilde{W}\rangle &= \langle \nabla_{[\tilde{X},\tilde{Y}]^h} \tilde{Z}, \tilde{W}\rangle + \langle \nabla_{[\tilde{X},\tilde{Y}]^v} \tilde{Z}, \tilde{W}\rangle \\ &= \langle \nabla_{[X,Y]} Z, W\rangle - \frac{1}{2}\langle [\tilde{X}, \tilde{Y}]^v, [\tilde{Z}, \tilde{W}]^v\rangle.\end{aligned}$$
Substituting we get the formula of assertion (a). The assertion for the sectional curvature is an immediate consequence of (a) taking $Z = Y$ and $W = X$. □

Example 4.4.5. We shall apply the above in order to calculate the sectional curvature and the covariant Riemann curvature tensor of the Fubini-Study metric on the complex projective space $\mathbb{C}P^n$, $n \geq 2$, which was defined in Example 3.3.6. By the definition of the Fubini-Study metric, the Hopf map $\pi : S^{2n+1} \to \mathbb{C}P^n$ becomes a Riemannian submersion, if on S^{2n+1} we consider the standard euclidean Riemannian metric of constant sectional curvature 1 according to Example 4.3.3. Recall from Example 3.3.6 that for each $z \in S^{2n+1}$ the vertical subspace $T_z^v S^{2n+1}$ is generated by the vector iz. The vertical smooth vector field $V \in \mathcal{X}(S^{2n+1})$ with $V_z = iz$ obviously extends to a smooth vector field on $\mathbb{C}^{n+1} \setminus \{0\}$ given by the same formula and
$$\nabla_{\tilde{X}} V = DV(\tilde{X}) = i\tilde{X}$$
for every horizontal $\tilde{X} \in \mathcal{X}(S^{2n+1})$. Thus, if $\tilde{X}$, $\tilde{Y} \in \mathcal{X}(S^{2n+1})$ are horizontal, we have
$$\begin{aligned}\langle [\tilde{X}, \tilde{Y}], V\rangle &= \langle \nabla_{\tilde{X}} \tilde{Y}, V\rangle - \langle \nabla_{\tilde{Y}} \tilde{X}, V\rangle = -\langle \tilde{Y}, \nabla_{\tilde{X}} V\rangle + \langle \tilde{X}, \nabla_{\tilde{Y}} V\rangle \\ &= -\langle \tilde{Y}, i\tilde{X}\rangle + \langle \tilde{X}, i\tilde{Y}\rangle = 2\langle \tilde{X}, i\tilde{Y}\rangle.\end{aligned}$$

Therefore, $[\tilde{X}, \tilde{Y}]^v = 2\langle \tilde{X}, i\tilde{Y}\rangle V$.

If now $\{u, w\}$ is an orthonormal basis of a 2-dimensional vector subspace S of $T_{\pi(z)}\mathbb{C}P^n$, according to Theorem 4.4.4 the sectional curvature at $\pi(z)$ with respect to S is

$$K_{\pi(z)}(S) = 1 + \frac{3}{4}\|2\langle \tilde{u}, i\tilde{w}\rangle V\| = 1 + 3|\langle \tilde{u}, i\tilde{w}\rangle|^2.$$

Note that if $\tilde{w} = i\tilde{u}$, then $K_{\pi(z)}(S) = 4$ and if $\langle \tilde{u}, i\tilde{w}\rangle = 0$, then $K_{\pi(z)}(S) = 1$. By continuity, this implies that the sectional curvatures at each point of $\mathbb{C}P^n$ cover the interval $[1, 4]$.

Finally the covariant Riemann curvature tensor is given by the formula

$$\langle R(X,Y)Z, W\rangle = \langle \tilde{Y}, \tilde{Z}\rangle\langle \tilde{X}, \tilde{W}\rangle - \langle \tilde{X}, \tilde{Z}\rangle\langle \tilde{Y}, \tilde{W}\rangle - \langle \tilde{X}, i\tilde{Z}\rangle\langle \tilde{X}, i\tilde{W}\rangle$$

$$+\langle \tilde{Y}, i\tilde{Z}\rangle\langle \tilde{X}, i\tilde{W}\rangle - 2\langle \tilde{X}, i\tilde{Y}\rangle\langle \tilde{Z}, i\tilde{W}\rangle$$

using the formula of the curvature tensor of S^{2n+1} found in Example 4.3.3.

4.5 The Ricci tensor and Einstein manifolds

Let M be a Riemannian n-manifold and $p \in M$. The bilinear form

$$Ric_p : T_pM \times T_pM \to \mathbb{R}$$

defined by $Ric_p(u, v) = \mathrm{Tr}R(., u)v$ is symmetric, because if $\{v_1, ..., v_n\}$ is an orthonormal basis of T_pM, then from Proposition 4.1.3 we have

$$Ric_p(u, v) = \sum_{j=1}^{n} Rm(v_j, u, v, v_j) = \sum_{j=1}^{n} Rm(v, v_j, v_j, u)$$

$$= \sum_{j=1}^{n} Rm(v_j, v, u, v_j) = Ric_p(v, u).$$

The $C^\infty(M)$-bilinear form $Ric : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$ defined in this way is called the *Ricci tensor* of M. If $v \in T_pM$ and $\|v\| = 1$, the real number $Ric_p(v, v)$ is called the *Ricci curvature* of M at p in the direction of v and it can be expressed in terms of sectional curvatures as follows. Let $\{v_1, ..., v_n\}$ with $v_n = v$ and let S_j be the 2-dimensional vector subspace of T_pM with basis $\{v_j, v\}$, $1 \leqq j \leq n-1$. Then,

$$Ric_p(v, v) = \sum_{j=1}^{n-1} Rm(v_j, v, v, v_j) = \sum_{j=1}^{n-1} K_p(S_j).$$

If (U, ϕ) is a chart of M with $\phi = (x^1, ..., x^n)$, the local components of the Ricci tensor on U with respect to ϕ are

$$R_{ij} = Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Since the matrix of $R\left(\cdot, \frac{\partial}{\partial x^i}\right)\frac{\partial}{\partial x^j}$ with respect to the ordered basis $\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\right]$ is $(R^k_{lij})_{1\leq k,l\leq n}$, it follows that

$$R_{ij} = \sum_{k=1}^{n} R^k_{kij}, \quad 1 \leq i,j \leq n.$$

The trace Sc of the Ricci tensor is called the *scalar curvature* of M. More precisely, for every $p \in M$ and $u \in T_pM$ there exists a unique $A(u) \in T_pM$ such that $Ric_p(u,v) = \langle A(u), v\rangle$ for all $v \in T_pM$. The so defined map $A_p : T_pM \to T_pM$ is obviously linear and self-adjoint, because Ric_p is symmetric, and by definition $Sc(p) = \mathrm{Tr}A_p$. If $\{v_1, ..., v_n\}$ is an orthonormal basis of T_pM, then

$$Sc(p) = \sum_{j=1}^{n} Ric_p(v_j, v_j) = \sum_{i,j=1}^{n} Rm(v_i, v_j, v_j, v_i).$$

In terms of the chart (U, ϕ), if $(a_{ij})_{1\leq i,j\leq n}$ is the matrix of A with respect to the ordered basis $\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\right]$, we have $R_{ij} = \sum_{k=1}^{n} g_{jk}a_{ki}$, $1 \leq i,j \leq n$, and therefore $a_{ki} = \sum_{l=1}^{n} g^{kl}R_{il}$, $1 \leq k,i \leq n$. Hence $Sc = \sum_{i,j=1}^{n} g^{ij}R_{ij}$, on U.

According to Schur's Theorem 4.2.5, if $n \geq 3$, the sectional curvature of M is constant if and only if at each point $p \in M$ the sectional curvature $K_p(S)$ does not depend on the 2-dimensional vector subspace S of T_pM. In analogy, suppose that the Ricci curvature at $p \in M$ in the direction of a unit tangent vector $v \in T_pM$ does not depend on v but only on p. In other words, suppose that there is a smooth function $f : M \to \mathbb{R}$ such that $Ric_p(v,v) = f(p)$ for every $v \in T_pM$ with $\|v\| = 1$ and $p \in M$, which is equivalent to saying $Ric = f \cdot g$, where as usual g denotes the Riemannian metric. Then, necessarily $Sc(p) = nf(p)$ for every $p \in M$.

Definition 4.5.1. If M is a Riemannian n-manifold, $n \geq 3$, the $C^\infty(M)$-bilinear form

$$Ric - \frac{Sc}{n}g : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$$

is called the *traceless Ricci tensor* of M.

Lemma 4.5.2. *If M is a connected Riemannian n-manifold, $n \geq 3$, with vanishing traceless Ricci tensor, then the scalar curvature of M is constant.*

Proof. Let $p \in M$ and let (U, ϕ) be a normal chart of M at p with $\phi = (x^1, ..., x^n)$. Then, $g_{ij}(p) = \delta_{ij}$, $\frac{\partial g_{ij}}{\partial x^m}(p) = 0$, hence also $g^{ij}(p) = \delta_{ij}$, $\frac{\partial g^{ij}}{\partial x^m}(p) = 0$ for every $1 \leq i,j,m \leq n$, and the Christoffel symbols vanish at p. Thus, at the point p the second identity of Bianchi becomes

$$\frac{\partial R_{jklm}}{\partial x^i}(p) + \frac{\partial R_{kilm}}{\partial x^j}(p) + \frac{\partial R_{ijlm}}{\partial x^k}(p) = 0$$

and for every $1 \le i \le n$ we have

$$\frac{\partial Sc}{\partial x^i}(p) = \left(\frac{\partial}{\partial x^i}\right)_p \left(\sum_{j,k=1}^n g^{jk} R_{jk}\right) = \sum_{j=1}^n \frac{\partial R_{jj}}{\partial x^i}(p) = \sum_{j,k=1}^n \frac{\partial R_{kjjk}}{\partial x^i}(p)$$

$$= -\sum_{j,k=1}^n \frac{\partial R_{jijk}}{\partial x^k}(p) - \sum_{j,k=1}^n \frac{\partial R_{ikjk}}{\partial x^j}(p) = \sum_{j,k=1}^n \frac{\partial R_{jkij}}{\partial x^k}(p) + \sum_{j,k=1}^n \frac{\partial R_{kjik}}{\partial x^j}(p)$$

$$= 2\sum_{j,k=1}^n \frac{\partial R_{jkij}}{\partial x^k}(p) = 2\sum_{j,k=1}^n \frac{\partial R^j_{jki}}{\partial x^k}(p) = 2\sum_{k=1}^n \frac{\partial R_{ki}}{\partial x^k}(p).$$

If the traceless Ricci tensor vanishes, then $R_{ij} = \dfrac{Sc}{n} g_{ij}$ on U and differentiating at the point p

$$\frac{\partial R_{ij}}{\partial x^k}(p) = \frac{1}{n} \cdot \frac{\partial Sc}{\partial x^k}(p) \cdot \delta_{ij}$$

for every $1 \le i, j, k \le n$. Substituting we get

$$\frac{\partial Sc}{\partial x^i}(p) = \frac{2}{n} \cdot \frac{\partial Sc}{\partial x^i}(p)$$

and therefore $\dfrac{\partial Sc}{\partial x^i}(p) = 0$ for every $1 \le i \le n$. Since M is connected, this implies that Sc is constant. $\square$

Definition 4.5.3. A connected Riemannian n-manifold M, $n \ge 3$, is called an *Einstein manifold* if its traceless Ricci tensor vanishes.

Thus, the Einstein manifolds are precisely the Riemannian manifolds with constant Ricci curvature. The following observation is due to J.A. Schouten and D.J. Struik.

Proposition 4.5.4. *A connected 3-dimensional Einstein manifold M has constant sectional curvature.*

Proof. Let $p \in M$ and let S be a 2-dimensional vector subspace of T_pM. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of T_pM such that $\{v_1, v_2\}$ generates S and let S_{ij} denote the 2-dimensional vector subspace of T_pM with basis $\{v_i, v_j\}$, $i \ne j$. Then, $S = S_{12} = S_{21}$ and

$$Ric_p(v_1, v_1) = K_p(S_{12}) + K_p(S_{13}),$$

$$Ric_p(v_2, v_2) = K_p(S_{21}) + K_p(S_{23}),$$

$$Ric_p(v_3, v_3) = K_p(S_{31}) + K_p(S_{32}).$$

Therefore,

$$Ric_p(v_1, v_1) + Ric_p(v_2, v_2) - Ric_p(v_3, v_3) = 2K_p(S).$$

Since M is a 3-dimensional Einstein manifold, we have $Sc(p) = 3Ric_p(v_j, v_j)$, for all $j = 1, 2, 3$. It follows that

$$K_p(S) = \frac{1}{6}Sc(p)$$

and by Schur's Theorem 4.2.5 the sectional curvature of M is constant. □

Example 4.5.5. The preceding Proposition 4.5.4 does not hold in dimensions greater then 3. We shall show that for $n \geq 2$ the complex projective space $\mathbb{C}P^n$ equipped with the Fubini-Study metric is an Einstein manifold. As we saw in Example 4.4.5, the sectional curvature of $\mathbb{C}P^n$ is not constant and takes all values in the interval $[1, 4]$. Let $p \in \mathbb{C}P^n$ and let $\{v_1, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be an orthonormal basis of $T_p\mathbb{C}P^n$ with horizontal lift $\{\tilde{v}_1, ..., \tilde{v}_n, \tilde{v}_{n+1}, ..., \tilde{v}_{2n}\}$ with respect to the Hopf map $\pi : S^{2n+1} \to \mathbb{C}P^n$ so that $\tilde{v}_{n+1} = i\tilde{v}_1$,..., $\tilde{v}_{2n} = i\tilde{v}_n$. From the formula of the covariant Riemann curvature tensor of $\mathbb{C}P^n$ of Example 4.4.5, for every u, $w \in T_p\mathbb{C}P^n$ we have

$$\langle R(v_j, u)w, v_j\rangle = \langle \tilde{u}, \tilde{w}\rangle - \langle \tilde{v}_j, \tilde{w}\rangle\langle \tilde{u}, \tilde{v}_j\rangle - \langle \tilde{v}_j, i\tilde{w}\rangle\langle \tilde{u}, i\tilde{v}_j\rangle$$

$$+\langle \tilde{u}, i\tilde{w}\rangle\langle \tilde{v}_j, i\tilde{v}_j\rangle - 2\langle \tilde{w}, i\tilde{v}_j\rangle\langle \tilde{v}_j, i\tilde{u}\rangle$$

$$= \langle u, w\rangle - \langle \tilde{v}_j, \tilde{w}\rangle\langle \tilde{u}, \tilde{v}_j\rangle + \langle \tilde{u}, i\tilde{v}_j\rangle\langle \tilde{w}, i\tilde{v}_j\rangle + 2\langle \tilde{u}, i\tilde{v}_j\rangle\langle \tilde{w}, i\tilde{v}_j\rangle.$$

Hence

$$Ric_p(u, w) = \sum_{j=1}^{2n} \langle R(v_j, u)w, v_j\rangle$$

$$= 2n\langle u, w\rangle - \sum_{j=1}^{2n} \langle \tilde{u}, \tilde{v}_j\rangle\langle \tilde{w}, \tilde{v}_j\rangle + \sum_{j=1}^{2n} \langle \tilde{u}, i\tilde{v}_j\rangle\langle \tilde{w}, i\tilde{v}_j\rangle + 2\sum_{j=1}^{2n} \langle \tilde{u}, i\tilde{v}_j\rangle\langle \tilde{w}, i\tilde{v}_j\rangle$$

$$= 2n\langle u, w\rangle + 2\langle u, w\rangle = (2n + 2)\langle u, w\rangle.$$

The traceless Ricci tensor should not be confused with the Einstein (gravitational) tensor

$$Ric - \frac{Sc}{2}g$$

which is important in General Relativity, as it occurs in Einstein's Equation

$$Ric - \frac{Sc}{2}g + \Lambda g = 8\pi T$$

in which $\Lambda \in \mathbb{R}$ is the cosmological constant and T is the energy momentum tensor describing the distribution of matter (in units where the gravitational constant and the velocity of light are equal to 1). The first part of the proof of Lemma 4.5.2 actually shows that the Einstein tensor is divergenceless.

4.6 Exercises

1. Let M be a parallelizable smooth n-manifold and X_1, X_2,..., $X_n \in \mathcal{X}(M)$ such that $\{X_1(p), X_2(p), ..., X_n(p)\}$ is a basis of T_pM for every $p \in M$. Prove that the formula

$$\nabla_X\Big(\sum_{k=1}^{n} f_k X_k\Big) = \sum_{k=1}^{n} X(f_k) \cdot X_k$$

for $X \in \mathcal{X}(M)$ and f_1, f_2,..., $f_k \in C^\infty(M)$ defines a connection on M with vanishing curvature tensor.

2. Let ∇ be a connection on a smooth manifold M. Let $p \in M$ and let (V, ϕ) be a normal chart at p. Let $E(p) \in T_pM$. For every $q \in V$ we consider the parallel translation $E(q) \in T_qM$ of $E(p)$ along the geodesic radius in V from p to q.
(a) Prove that E is a smooth vector field on V.
(b) If the curvature tensor of ∇ vanishes, prove that E is parallel that is $\nabla_X E = 0$ for every smooth vector field X on V.

3. Let ∇ be a connection on a connected smooth manifold M with the following property: For every p, $q \in M$ the parallel translation from p to q does not depend on choice of the smooth path from p to q. Prove that the curvature tensor of ∇ vanishes.

4. Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a real symmetric matrix and

$$M = \{(x, \frac{1}{2}\langle Ax, x\rangle) : x \in \mathbb{R}^n\}.$$

Find the second fundamental form of M at the point 0.

5. Prove that on a compact hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, there exists at least one point at which the second fundamental form is positive (or negative) definite.

6. If M and N are two Riemannian manifolds, express the Riemann curvature tensor of the Riemannian product $M \times N$ in terms of the Riemann curvature tensors of M and N.

7. Prove that $S^n \times S^n$, $n \geq 2$, with the product Riemannian metric, is an Einstein manifold.

8. Explain why the Ricci tensor of a 3-dimensional Riemannian manifold completely determines its Riemann curvature tensor.

9. Let N be a n-dimensional Riemannian manifold and $M \subset N$ a smooth $(n-1)$-dimensional submanifold. Let $U \subset N$ be an open set with $U \cap M \neq \varnothing$ on which there is an orthonormal frame e_1,...,e_{n-1}, e_n such that $\nu = e_n$ is always orthogonal to M and the rest are tangent to M. The bilinear form $h_p : T_pM \times T_pM \to \mathbb{R}$ defined by

$$h_p(X, Y) = \langle \nabla_X \nu, Y\rangle$$

is the second fundamental form of M at the point $p \in U \cap M$. The trace

$$H(p) = \sum_{i=1}^{n} h_p(e_i, e_i)$$

of h_p is called the mean curvature of M at p.
(a) Prove that h_p is symmetric.
(b) Let (U, ϕ) be a M-straightening smooth chart of N with $\phi = (x^1, ..., x^n)$ and $\phi(U \cap M) \subset \mathbb{R}^{n-1} \times \{0\}$. As usual let $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \rangle$, $1 \leq i, j \leq n$. Prove that

$$H(p) = \sum_{i,j=1}^{n-1} h_p\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) g^{ij}(p),$$

where $(g_{ij})^{-1}_{1 \leq i,j \leq n-1} = (g^{ij})_{1 \leq i,j \leq n-1}$.

Chapter 5

Comparison Geometry

5.1 Variation of length

Let M be a Riemannian n-manifold. Let a, $b \in \mathbb{R}$, $a < b$, and $\gamma : [a, b] \to M$ be a piecewise smooth parametrized curve. A (piecewise smooth) *variation* of γ is a continuous map $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$, for some $\epsilon > 0$, such that there exists a partition $\{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$ for which $\Gamma|_{(-\epsilon,\epsilon)\times[t_{i-1},t_i]}$ is smooth for every $1 \leq i \leq m$ and $\Gamma(0, t) = \gamma(t)$ for every $a \leq t \leq b$. We say that Γ fixes endpoints if $\Gamma(s, a) = \gamma(a)$ and $\Gamma(s, b) = \gamma(b)$ for all $|s| < \epsilon$. The variation Γ is called smooth if it is a smooth map. The formula

$$V(t) = \frac{\partial \Gamma}{\partial s}(0, t) = \Gamma_{*(0,t)}\left(\frac{\partial}{\partial s}\right)_{(0,t)}, \quad t \in [a, b] \setminus \{t_0, t_1, ..., t_m\}$$

defines a piecewise smooth vector field along γ, which is smooth in case the variation Γ is smooth, that is called the *the variation field* of Γ.

Lemma 5.1.1. *Let $\gamma : [a, b] \to M$ be a smooth parametrized curve. Then, every $V \in \mathcal{X}(\gamma)$ is the variation field of some smooth variation of γ. The same holds in case γ is piecewise smooth and then the variation is only piecewise smooth. If $V(a) = 0$ and $V(b) = 0$, the variation fixes endpoints.*

Proof. By the compactness of $[a, b]$, there exists some $\delta > 0$ such that $\exp_{\gamma(t)}(w)$ is defined for all $w \in T_{\gamma(t)}M$ with $\|w\| < \delta$ and $t \in [a, b]$, form the existence of uniformly normal neighbourhoods. If $\epsilon > 0$ is such that $\max\{\|\epsilon V(t)\| : t \in [a, b]\} = \delta$, we define $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ by the formula

$$\Gamma(s, t) = \exp_{\gamma(t)}(sV(t)).$$

Obviously, Γ is a smooth variation of γ whose variation field is

$$\frac{\partial \Gamma}{\partial s}(0, t) = (\exp_{\gamma(t)})_{*0}(V(t)) = V(t),$$

as the proof of Proposition 3.2.4 shows. The rest is obvious. □

Let M be a Riemannian n-manifold, p, $q \in M$ and let $\gamma : [a,b] \to M$, a, $b \in \mathbb{R}$ with $a < b$, be a smooth parametrized by arclength curve from p to q. Let $\epsilon > 0$ and $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M$ be a smooth variation of γ. Since $\|\dot{\gamma}(t)\| = 1$ for every $t \in [a,b]$, taking a smaller ϵ if necessary, we may assume that $\dfrac{\partial\Gamma}{\partial t}(s,t) \neq 0$ for every $|s| < \epsilon$ and $t \in [a,b]$. The length of $\Gamma(s,.)$ is

$$L(s) = \int_a^b \left\| \frac{\partial\Gamma}{\partial t}(s,t) \right\| dt.$$

The so defined length function $L : (-\epsilon, \epsilon) \to \mathbb{R}$ of Γ is smooth and

$$L'(s) = \int_a^b \frac{\langle \frac{D}{ds}(\frac{\partial\Gamma}{\partial t})(s,t), \frac{\partial\Gamma}{\partial t}(s,t) \rangle}{\| \frac{\partial\Gamma}{\partial t}(s,t) \|} dt = \int_a^b \frac{\langle \frac{D}{dt}(\frac{\partial\Gamma}{\partial s})(s,t), \frac{\partial\Gamma}{\partial t}(s,t) \rangle}{\| \frac{\partial\Gamma}{\partial t}(s,t) \|} dt.$$

In particular,

$$\begin{aligned} L'(0) &= \int_a^b \langle \frac{D}{dt}\left(\frac{\partial\Gamma}{\partial s}\right)(0,t), \frac{\partial\Gamma}{\partial t}(0,t) \rangle dt \\ &= \int_a^b \left[\frac{d}{dt} \langle \frac{\partial\Gamma}{\partial s}(0,t), \frac{\partial\Gamma}{\partial t}(0,t) \rangle - \langle \frac{\partial\Gamma}{\partial s}(0,t), \frac{D}{dt}\left(\frac{\partial\Gamma}{\partial t}\right)(0,t) \rangle \right] dt \\ &= \langle \frac{\partial\Gamma}{\partial s}(0,b), \frac{\partial\Gamma}{\partial t}(0,b) \rangle - \langle \frac{\partial\Gamma}{\partial s}(0,a), \frac{\partial\Gamma}{\partial t}(0,a) \rangle - \int_a^b \langle \frac{\partial\Gamma}{\partial s}(0,t), \frac{D\dot{\gamma}}{dt}(t) \rangle dt. \end{aligned}$$

Thus, assuming that Γ fixes endpoints, we obtain the first variation formula

$$L'(0) = -\int_a^b \langle \frac{\partial\Gamma}{\partial s}(0,t), \frac{D\dot{\gamma}}{dt}(t) \rangle dt.$$

Proposition 5.1.2. *A smooth parametrized by arclength curve γ is a geodesic if and only if it is a critical point of the length function of every smooth variation of γ which fixes endpoints.*

Proof. We shall use the notation of the preceding discussion. If γ is a geodesic, then from the first variation formula we have $L'(0) = 0$. For the converse we consider any smooth function $g : [a,b] \to [0, +\infty)$ and the smooth vector field

$$V = g \frac{D\dot{\gamma}}{dt}$$

along γ. If moreover $g(t) > 0$ for $a < t < b$ and $g(a) = g(b) = 0$, then V is the variation field of a smooth variation Γ of γ which fixes endpoints, by Lemma 5.1.1. The first variation formula and our assumption give

$$0 = -\int_a^b \langle V(t), \frac{D\dot{\gamma}}{dt}(t) \rangle dt = -\int_a^b g(t) \left\| \frac{D\dot{\gamma}}{dt}(t) \right\|^2 dt.$$

Since this holds for any such g, this implies that $\left\| \dfrac{D\dot{\gamma}}{dt} \right\|^2 = 0$ on $[a,b]$, which means that γ is a geodesic. $\square$

In order to derive a second variation formula, that is compute $L''(0)$ for the length function L of a smooth variation of a geodesic path $\gamma : [a,b] \to M$, we shall need the following formula which is not at all unexpected if we recall the definition of the curvature tensor.

Lemma 5.1.3. *Let $A \subset \mathbb{R}^2$ be an open set and let $\Gamma : A \to M$ be a smooth map into a smooth n-manifold M carrying a connection ∇ with corresponding curvature tensor R. Let V be a smooth vector field along Γ, that is $V : A \to TM$ is a smooth map such that $V(s,t) \in T_{\Gamma(s,t)}M$ for every $(s,t) \in A$. Then,*

$$\frac{D}{ds}\left(\frac{DV}{dt}\right) - \frac{D}{dt}\left(\frac{DV}{ds}\right) = R\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right)V.$$

Proof. It is sufficient to prove the formula in the local coordinates of a chart $(U, x^1, ..., x^n)$ of M assuming that $\Gamma(A) \subset U$. There are smooth functions V_1,..., $V_n : A \to \mathbb{R}$ such that

$$V(s,t) = \sum_{i=1}^{n} V_i(s,t)\left(\frac{\partial}{\partial x_i}\right)_{\Gamma(s,t)}$$

for every $(s,t) \in A$. Then,

$$\frac{DV}{dt} = \sum_{i=1}^{n} \frac{\partial V_i}{\partial t} \cdot \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} V_i \cdot \frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right)$$

and

$$\frac{D}{ds}\left(\frac{DV}{dt}\right) = \sum_{i=1}^{n} \frac{\partial^2 V_i}{\partial s \partial t} \cdot \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} \frac{\partial V_i}{\partial t} \cdot \frac{D}{ds}\left(\frac{\partial}{\partial x^i}\right) + \sum_{i=1}^{n} \frac{\partial V_i}{\partial s} \cdot \frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right)$$
$$+ \sum_{i=1}^{n} V_i \frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right)\right).$$

A similar formula gives $\frac{D}{dt}\left(\frac{DV}{ds}\right)$ if we interchange s and t. Subtracting,

$$\frac{D}{ds}\left(\frac{DV}{dt}\right) - \frac{D}{dt}\left(\frac{DV}{ds}\right) = \sum_{i=1}^{n} V_i \left[\frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right)\right) - \frac{D}{dt}\left(\frac{D}{ds}\left(\frac{\partial}{\partial x^i}\right)\right)\right].$$

If $(\Gamma_1, ..., \Gamma_n)$ is the local representation of Γ with respect to the chart, we have

$$\frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right) = \nabla_{\frac{\partial \Gamma}{\partial t}} \frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial \Gamma_j}{\partial t} \cdot \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$$

and therefore

$$\frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right)\right) = \sum_{j=1}^{n} \frac{\partial^2 \Gamma_j}{\partial s \partial t} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} \frac{\partial \Gamma_j}{\partial t} \cdot \nabla_{\frac{\partial \Gamma}{\partial s}} (\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i})$$

$$= \sum_{j=1}^{n} \frac{\partial^2 \Gamma_j}{\partial s \partial t} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + \sum_{j,k=1}^{n} \frac{\partial \Gamma_j}{\partial t} \cdot \frac{\partial \Gamma_k}{\partial s} \cdot \nabla_{\frac{\partial}{\partial x^k}} (\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}).$$

Again $\frac{D}{dt}\left(\frac{D}{ds}\left(\frac{\partial}{\partial x^i}\right)\right)$ is given by a similar formula interchanging s and t. Subtracting,

$$\frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial}{\partial x^i}\right)\right) - \frac{D}{dt}\left(\frac{D}{ds}\left(\frac{\partial}{\partial x^i}\right)\right)$$

$$= \sum_{j,k=1}^{n} \frac{\partial \Gamma_j}{\partial t} \cdot \frac{\partial \Gamma_k}{\partial s} \cdot \nabla_{\frac{\partial}{\partial x^k}} (\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}) - \sum_{j,k=1}^{n} \frac{\partial \Gamma_j}{\partial s} \cdot \frac{\partial \Gamma_k}{\partial t} \cdot \nabla_{\frac{\partial}{\partial x^k}} (\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i})$$

$$= \sum_{j,k=1}^{n} \frac{\partial \Gamma_j}{\partial t} \cdot \frac{\partial \Gamma_k}{\partial s} \cdot \left(\nabla_{\frac{\partial}{\partial x^k}} (\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}) - \nabla_{\frac{\partial}{\partial x^j}} (\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i})\right)$$

$$= \sum_{j,k=1}^{n} \frac{\partial \Gamma_j}{\partial t} \cdot \frac{\partial \Gamma_k}{\partial s} \cdot R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^i} = R\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right) \frac{\partial}{\partial x^i}.$$

This completes the proof. □

We proceed now to compute the second derivative of the length function of a variation at the critical point 0, that is assuming that γ is a geodesic. We continue to use the same notations of the discussion preceding Proposition 5.1.2. Differentiating $L'(s)$ we find

$$L''(s) = \int_a^b \frac{d}{ds}\left(\frac{\langle \frac{D}{dt}(\frac{\partial \Gamma}{\partial s}), \frac{\partial \Gamma}{\partial t}\rangle}{\|\frac{\partial \Gamma}{\partial t}\|}\right) dt$$

$$= \int_a^b \frac{\langle \frac{D}{ds}\frac{D}{dt}(\frac{\partial \Gamma}{\partial s}), \frac{\partial \Gamma}{\partial t}\rangle}{\|\frac{\partial \Gamma}{\partial t}\|} dt + \int_a^b \frac{\langle \frac{D}{dt}(\frac{\partial \Gamma}{\partial s}), \frac{D}{ds}(\frac{\partial \Gamma}{\partial t})\rangle}{\|\frac{\partial \Gamma}{\partial t}\|} dt - \int_a^b \frac{\langle \frac{D}{dt}(\frac{\partial \Gamma}{\partial s}), \frac{\partial \Gamma}{\partial t}\rangle \langle \frac{D}{dt}(\frac{\partial \Gamma}{\partial s}), \frac{\partial \Gamma}{\partial t}\rangle}{\|\frac{\partial \Gamma}{\partial t}\|^3} dt.$$

In particular

$$L''(0) = \int_a^b \langle \frac{D}{ds}\frac{D}{dt}(\frac{\partial \Gamma}{\partial s})(0,t), \frac{\partial \Gamma}{\partial t}(0,t)\rangle dt + \int_a^b \left\|\frac{D}{dt}\left(\frac{\partial \Gamma}{\partial s}\right)(0,t)\right\|^2 dt$$

$$- \int_a^b \langle \frac{D}{dt}(\frac{\partial \Gamma}{\partial s})(0,t), \frac{\partial \Gamma}{\partial t}(0,t)\rangle^2 dt.$$

Let $V^\perp \in \mathcal{X}(\gamma)$ denote the orthogonal to γ component of the variation field, that is

$$V^\perp(t) = \frac{\partial \Gamma}{\partial s}(0,t) - \langle \frac{\partial \Gamma}{\partial s}(0,t), \frac{\partial \Gamma}{\partial t}(0,t)\rangle \frac{\partial \Gamma}{\partial t}(0,t)$$
$$= \frac{\partial \Gamma}{\partial s}(0,t) - \langle \frac{\partial \Gamma}{\partial s}(0,t), \dot\gamma(t)\rangle \dot\gamma(t)$$

for every $t \in [a,b]$. Since γ is a geodesic,

$$\langle \frac{DV^\perp}{dt}, \dot\gamma\rangle = \frac{d}{dt}\langle V^\perp, \dot\gamma\rangle = 0.$$

On the other hand

$$\left\|\frac{D}{dt}\left(\frac{\partial\Gamma}{\partial s}\right)(0,t)\right\|^2 = \left\|\frac{DV^\perp}{dt}(t)\right\|^2 + \langle\frac{D}{dt}(\frac{\partial\Gamma}{\partial s})(0,t),\dot{\gamma}(t)\rangle^2$$

and substituting we get

$$L''(0) = \int_a^b \langle\frac{D}{ds}\frac{D}{dt}(\frac{\partial\Gamma}{\partial s})(0,t),\dot{\gamma}(t)\rangle dt + \int_a^b \left\|\frac{DV^\perp}{dt}(t)\right\|^2 dt.$$

From Lemma 5.1.3,

$$\frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial\Gamma}{\partial s}\right)\right)(0,t) - \frac{D}{dt}\left(\frac{D}{ds}\left(\frac{\partial\Gamma}{\partial s}\right)\right)(0,t) = R\left(\frac{\partial\Gamma}{\partial s}(0,t),\dot{\gamma}(t)\right)\left(\frac{\partial\Gamma}{\partial s}\right)(0,t)$$

and

$$\langle R\left(\frac{\partial\Gamma}{\partial s}(0,t),\dot{\gamma}(t)\right)\left(\frac{\partial\Gamma}{\partial s}\right)(0,t),\dot{\gamma}(t)\rangle = -\langle R(V^\perp(t),\dot{\gamma}(t))\dot{\gamma}(t),V^\perp(t)\rangle.$$

Therefore,

$$\langle\frac{D}{ds}\frac{D}{dt}(\frac{\partial\Gamma}{\partial s})(0,t),\dot{\gamma}(t)\rangle = \langle\frac{D}{dt}\frac{D}{ds}(\frac{\partial\Gamma}{\partial s})(0,t),\dot{\gamma}(t)\rangle + \langle R\left(\frac{\partial\Gamma}{\partial s}(0,t),\dot{\gamma}(t)\right)\left(\frac{\partial\Gamma}{\partial s}\right)(0,t),\dot{\gamma}(t)\rangle$$

$$\frac{d}{dt}\langle\frac{D}{ds}\frac{\partial\Gamma}{\partial s})(0,t),\dot{\gamma}(t)\rangle - \langle R(V^\perp(t),\dot{\gamma}(t))\dot{\gamma}(t),V^\perp(t)\rangle.$$

Thus, we arrive at the formula

$$L''(0) = \int_a^b \left[\left\|\frac{DV^\perp}{dt}(t)\right\|^2 - \langle R(V^\perp(t),\dot{\gamma}(t))\dot{\gamma}(t),V^\perp(t)\rangle\right] dt$$

since the variation Γ fixes endpoints. The above calculation is due to J.L. Synge and is known as Synge's formula for the second variation of length. A second form is the following.

Theorem 5.1.4. *Let $\gamma : [a,b] \to M$ be a geodesic path parametrized by arclength and let Γ be a smooth variation of γ which fixes endpoints. If $V^\perp$ is the orthogonal to γ component of the variation field V of Γ and L is the corresponding length function, then*

$$L''(0) = -\int_a^b \langle V^\perp(t), \frac{D^2V^\perp}{dt^2}(t) + R(V^\perp(t),\dot{\gamma}(t))\dot{\gamma}(t)\rangle dt.$$

Proof. Since

$$\left\|\frac{DV^\perp}{dt}(t)\right\|^2 = \frac{d}{dt}\langle V^\perp, \frac{DV^\perp}{dt}\rangle - \langle V^\perp, \frac{D^2V^\perp}{dt^2}\rangle$$

and $V^\perp(a) = 0$, $V^\perp(b) = 0$, substituting to Synge's formula we arrive at the result.
□

A consequence of Synge's formula for the second variation of length is the following important theorem of S.B. Myers.

Theorem 5.1.5. *Let M be a connected, complete Riemannian n-manifold, $n \geq 2$. If there exists $r > 0$ such that*

$$Ric_p(v,v) \geq (n-1)\frac{1}{r^2}$$

for every $p \in M$ and $v \in T_pM$ with $\|v\| = 1$, then the following hold.
(a) $\operatorname{diam}(M) \leq \pi r$.
(b) M is compact.
(c) The fundamental group of M is finite.

Proof. (a) Let p, $q \in M$. By completeness, there exists a minimizing geodesic $\gamma : [0,\ell] \to M$ parametrized by arclength with $\gamma(0) = p$ and $\gamma(\ell) = q$, where $\ell = L(\gamma) = d(p,q)$. It is sufficient to show that $\ell \leq \pi r$. We proceed to prove the assertion by contradiction assuming that $\ell > \pi r$. Let E_1,..., E_{n-1}, E_n be a parallel orthonormal frame along γ such that $E_n = \dot{\gamma}$. For each $1 \leq j \leq n-1$ we consider the smooth vector field V_j along γ given by the formula

$$V_j(t) = \sin\left(\frac{\pi}{\ell}t\right) \cdot E_j(t), \quad t \in [0,\ell].$$

Since $V_j(0) = 0$ and $V_j(\ell) = 0$, each V_j is the variation field of a smooth variation of γ which fixes endpoints, by Lemma 5.1.1. Let L_j denote the corresponding length function. From Theorem 5.1.4 the second variation of the length function L_j is

$$\begin{aligned} L_j''(0) &= -\int_0^\ell \langle V_j(t), \frac{D^2V_j}{dt^2}(t) + R(V_j(t),\dot{\gamma}(t))\dot{\gamma}(t)\rangle dt \\ &= \int_0^\ell \sin^2\left(\frac{\pi}{\ell}t\right)\left[\frac{\pi^2}{\ell^2} - K_{\gamma(t)}(S_j(\gamma(t)))\right]dt, \end{aligned}$$

where $S_j(\gamma(t))$ is the 2-dimensional vector subspace of $T_{\gamma(t)}M$ with basis $\{E_j(t), \dot{\gamma}(t)\}$. Summing up

$$\begin{aligned} \sum_{j=1}^{n-1} L_j''(0) &= \int_0^\ell \sin^2\left(\frac{\pi}{\ell}t\right)\left[(n-1)\frac{\pi^2}{\ell^2} - Ric_{\gamma(t)}(\dot{\gamma}(t),\dot{\gamma}(t))\right]dt \\ &\leq (n-1)\int_0^\ell \sin^2\left(\frac{\pi}{\ell}t\right)\left[\frac{\pi^2}{\ell^2} - \frac{1}{r^2}\right]dt < 0 \end{aligned}$$

by our assumption. This implies that there exists at least one $1 \leq j \leq n-1$ such that $L_j''(0) < 0$. This means that the length function L_j has a strict local maximum at 0, which contradicts the fact that γ is minimizing.

Assertion (b) is an immediate consequence of (a) and the completeness of M, because $M = \exp_p(\overline{B_p(0,\pi r)})$ for any $p \in M$.

Assertion (c) follows from what we have already proved and some general considerations about covering spaces of Riemannian manifolds. If $\sigma : \hat{M} \to M$ is the universal covering of M, according to Example 3.3.3, the universal covering space $\hat{M}$ carries a Riemannian metric so that the universal covering map σ becomes a local isometry. By the path lifting property of covering spaces, each geodesic of $\hat{M}$ is a lifting of a geodesic of M. This implies that if M is geodesically complete, then so is $\hat{M}$. Thus, if M satisfies the assumptions of the theorem, then they are satisfied also by $\hat{M}$. From (a) and (b) the diameter of $\hat{M}$ is at most πr and $\hat{M}$ is compact. Since σ is a covering map of compact manifolds, its fibre is finite. But the cardinality of the fibre is equal to the cardinality of the fundamental group, because σ is the universal covering map. This concludes the proof. $\square$

The estimate $\operatorname{diam}(M) \leq \pi r$ is the best possible. For example, it is achieved in the case of the sphere of radius r. Also, if the sectional curvature K of M is everywhere positive but $\inf K = 0$, then the conclusion of the theorem may not hold. A simple counterexample is the paraboloid

$$M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$$

which is a connected, complete, non-compact, smooth surface in $\mathbb{R}^3$ having everywhere positive sectional curvature.

There have been several applications of Myers' theorem to General Relativity. For instance, T. Frankel has used Myers' theorem to obtain a bound for the size of a fluid mass in a stationary space-time universe and G.J. Galloway made use of Frankel's method to prove a closure theorem, which has as its conclusion the "finiteness" of the "spatial part" of a space-time obeying certain cosmological assumptions. Because of its importance, there are several generalizations of Myers' theorem, the most known being the ones by W. Ambrose and E. Calabi.

5.2 Jacobi fields

Let M be a Riemannian n-manifold, $n \geq 2$, and let $\gamma : [a, b] \to M$ be a geodesic path parametrized by arclength. The second variation formula derived in the previous section motivates the introduction of the symmetric bilinear form

$$I(X, Y) = \int_a^b \left[\langle \frac{DX}{dt}(t), \frac{DY}{dt}(t) \rangle - \langle R(X(t), \dot{\gamma}(t))\dot{\gamma}(t), Y(t) \rangle \right] dt$$

$$= -\int_a^b \langle Y(t), \frac{D^2X}{dt^2}(t) + R(X(t), \dot{\gamma}(t))\dot{\gamma}(t) \rangle dt$$

which is called the *index form* and is defined on the vector space $\mathcal{D}_0(\gamma)$ of continuous, piecewise smooth vector fields along γ which vanish at a and b, and are orthogonal to γ, because $L''(0) = I(X, X)$, if $X \in \mathcal{X}(\gamma)$ is orthogonal to γ and is the variation field of a smooth variation of γ with length function L. The vector space $\mathcal{D}_0(\gamma)$ carries the inner product

$$(X, Y) = \int_a^b \langle X(t), Y(t) \rangle dt.$$

Obviously, the linear operator $\mathcal{L} : \mathcal{X}(\gamma) \to \mathcal{X}(\gamma)$ defined by

$$\mathcal{L}(X) = -\frac{D^2 X}{dt^2} - R(X, \dot\gamma)\dot\gamma$$

satisfies $(\mathcal{L}(X), Y) = (X, \mathcal{L}(Y)) = I(X, Y)$ for every X, $Y \in \mathcal{X}(\gamma) \cap \mathcal{D}_0(\gamma)$ and is therefore self-adjoint on $\mathcal{X}(\gamma) \cap \mathcal{D}_0(\gamma)$.

Definition 5.2.1. A *Jacobi field* along a geodesic path $\gamma : [a, b] \to M$ is a solution of *Jacobi's differential equation*

$$\frac{D^2 X}{dt^2} + R(X, \dot\gamma)\dot\gamma = 0.$$

Thus, the Jacobi fields along γ which vanish at its endpoints are elements of the kernel of the self-adjoint operator $\mathcal{L}$. Another source of motivation for Jacobi's equation is the following. Let $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ be a smooth variation of γ by geodesics. This means that $\Gamma(s, .) : [a, b] \to M$ is a geodesic for all $|s| < \epsilon$. Then, the corresponding variation field V is a Jacobi field along γ. Indeed, in this case we have

$$\frac{D}{dt}\left(\frac{\partial \Gamma}{\partial t}\right) = 0$$

and from Lemma 3.5.1 and Lemma 5.1.3

$$\begin{aligned} 0 = \frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial \Gamma}{\partial t}\right)\right) &= \frac{D}{dt}\left(\frac{D}{ds}\left(\frac{\partial \Gamma}{\partial t}\right)\right) + R\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right)\frac{\partial \Gamma}{\partial t} \\ &= \frac{D}{dt}\left(\frac{D}{dt}\left(\frac{\partial \Gamma}{\partial s}\right)\right) + R\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right)\frac{\partial \Gamma}{\partial t}. \end{aligned}$$

Evaluating at $s = 0$ we obtain

$$\frac{D^2 V}{dt^2} + R(V, \dot\gamma)\dot\gamma = 0.$$

If E_1,..., E_n is a parallel orthonormal frame along the geodesic path γ, for every $V \in \mathcal{X}(\gamma)$ there are uniquely determined smooth functions, $V_k : [a, b] \to \mathbb{R}$, $1 \le k \le n$ such that $V = \sum_{k=1}^{n} V_k \cdot E_k$. Thus, V is a Jacobi field if and only if

$$\begin{aligned} 0 &= \sum_{k=1}^{n} V_k'' E_k + \sum_{k=1}^{n} V_k R(E_k, \dot\gamma)\dot\gamma \\ &= \sum_{k=1}^{n} V_k'' E_k + \sum_{k=1}^{n} V_k \sum_{j=1}^{n} \langle R(E_k, \dot\gamma)\dot\gamma, E_j\rangle E_j. \end{aligned}$$

Hence Jacobi's differential equation along γ is equivalent to the system of linear differential equations (with non-constant coefficients in general)

$$V_k'' + \sum_{j=1}^{n} \langle R(E_j, \dot\gamma)\dot\gamma, E_k\rangle V_j = 0, \quad 1 \le k \le n.$$

From the existence and uniqueness of solutions for linear differential equations, for every v, $w \in T_{\gamma(a)}M$ there exists a unique Jacobi field $V \in \mathcal{X}(\gamma)$ with initial conditions

$$V(a) = v, \quad \frac{DV}{dt}(a) = w.$$

Moreover, the set of all Jacobi fields along γ is a vector subspace of $\mathcal{X}(\gamma)$ of dimension $2n$.

Lemma 5.2.2. *Let $\ell > 0$ and $\gamma : [0,\ell] \to M$ be a geodesic path parametrized by arclength. If $J \in \mathcal{X}(\gamma)$ is a Jacobi field with $J(0) = 0$, then J is the variation field of a variation of γ by geodesics.*

Proof. Let $w = \dfrac{DJ}{dt}(0)$ and $v_0 = \dot\gamma(0)$. We think of w as an element of $T_{v_0}T_{\gamma(0)}M$ and consider any smooth curve $v : (-\epsilon, \epsilon) \to T_{\gamma(0)}M$ with $v(0) = v_0$, $\dot v(0) = w$, where $\epsilon > 0$ is so small that the smooth variation $\Gamma : (-\epsilon,\epsilon) \times [0,\ell] \to M$ of γ by geodesics with

$$\Gamma(s,t) = \exp_{\gamma(0)}(tv(s))$$

is defined. The variation field

$$V(t) = \frac{\partial\Gamma}{\partial s}(0,t) = t(\exp_{\gamma(0)})_{*tv_0}(w)$$

is a Jacobi field along γ and satisfies the initial conditions $V(0) = 0$ and $\dfrac{DV}{dt}(0) = (\exp_{\gamma(0)})_{*0}(w) = w$. By uniqueness with respect to initial conditions, $V = J$. $\square$

The velocity field $\dot\gamma$ of a geodesic path γ parametrized by arclength is trivially a Jacobi field along γ and is the variation field of the trivial variation $\Gamma(s,t) = \gamma(s+t)$. Non-trivial information for nearby geodesics of γ can be obtained from normal Jacobi fields. A Jacobi field J along a geodesic path $\gamma : [0,\ell] \to M$ parametrized by arclength is called *normal* if $\langle J(t), \dot\gamma(t)\rangle = 0$ for every $0 \le t \le \ell$.

Lemma 5.2.3. *Let $\gamma : [0,\ell] \to M$ be a geodesic path parametrized by arclength and $J \in \mathcal{X}(\gamma)$ be a Jacobi field.*
(a) J is normal if and only if $\langle J(0), \dot\gamma(0)\rangle = 0$ and $\langle \dfrac{DJ}{dt}(0), \dot\gamma(0)\rangle = 0$.
(b) If J is orthogonal to $\dot\gamma$ at two different times, then it is normal.

Proof. Since γ is a geodesic and J is a Jacobi field along γ, the second derivative of the smooth function $f : [0,\ell] \to \mathbb{R}$ defined by $f(t) = \langle J(t), \dot\gamma(t)\rangle$ is

$$f'' = \langle \frac{D^2J}{dt^2}, \dot\gamma\rangle = -\langle R(J,\dot\gamma)\dot\gamma, \dot\gamma\rangle = 0$$

from Proposition 4.1.3. This means that there are λ, $\mu \in \mathbb{R}$ such that $f(t) = \lambda t + \mu$ for every $0 \le t \le \ell$. Since

$$\lambda = f'(0) = \langle \frac{DJ}{dt}(0), \dot\gamma(0)\rangle,$$

it follows immediately that $\langle J(0), \dot\gamma(0)\rangle = 0$ and $\langle \frac{DJ}{dt}(0), \dot\gamma(0)\rangle = 0$ if and only if $f = 0$. The second assertion is obvious since f vanishes identically if and only if it vanishes at two different times. □.

Corollary 5.2.4. *The set of normal Jacobi fields along a geodesic path γ parametrized by arclength is a vector subspace of $\mathcal{X}(\gamma)$ of dimension $2n-2$.*

5.3 Conjugate points

Let M be a Riemannian n-manifold, $n \geq 2$, and $p \in M$. Let $\gamma : [0, \ell] \to M$ be a geodesic parametrized by arclength with $\gamma(0) = p$. If $\dot\gamma(0) = v$, then $\gamma(t) = \exp_p(tv)$. The point $\gamma(t_0)$ is said to be *conjugate* to p along γ if the derivative of the exponential map

$$(\exp_p)_{*t_0 v} : T_{t_0 v} T_p M \cong T_p M \to T_{\gamma(t_0)} M$$

at $t_0 v$ is not an isomorphism. The dimension of its kernel is called the multiplicity of $\gamma(t_0)$.

The set of points of M which are the first conjugate points to $p \in M$ along geodesics emanating from p is called the *conjugate locus* of p. By Sard's theorem, the conjugate locus of p has empty interior in M. The point p is called *pole* if its conjugate locus is empty.

Example 5.3.1. On the n-sphere S^n_R, $n \geq 2$, of radius $R > 0$, all geodesics emanating from a point p meet at its antipodal point $-p$, which lies at distance πR along any such geodesic. The exponential map $\exp_p$ maps $B_p(0, \pi R)$ diffeomorphically onto $S^n_R \setminus \{-p\}$ and $\exp_p(\partial B_p(0, \pi R)) = \{-p\}$. Thus, $-p$ is conjugate to p along any geodesic from p and since $\partial B_p(0, \pi R)$ is a smooth submanifold of $T_p S^n_R$ of dimension $n-1$, the multiplicity of $-p$ is equal to $n-1$. Of course the conjugate locus of p is $\{-p\}$.

The conjugate points can be characterized using Jacobi fields.

Proposition 5.3.2. *Let $p \in M$, $\ell > 0$ and $\gamma : [0, \ell] \to M$ be a geodesic path parametrized by arclength with $\gamma(0) = p$ and $\dot\gamma(0) = v$. For $0 < t_0 \leq \ell$, the point $\gamma(t_0)$ is conjugate to p along γ with multiplicity k if and only if the exists a non-zero Jacobi field along γ which vanishes at 0 and t_0 and the vector space of all these Jacobi fields has dimension k.*

Proof. As in the proof of Lemma 5.2.2, if $w \in T_{t_0 v} T_p M \cong T_p M$, we consider any smooth curve $u : (-\epsilon, \epsilon) \to T_p M$ with $u(0) = v$, $\dot u(0) = w$, where $\epsilon > 0$ is so small that the smooth variation $\Gamma : (-\epsilon, \epsilon) \times [0, \ell] \to M$ of γ by geodesics with

$$\Gamma(s, t) = \exp_p(tu(s))$$

is defined. The variation field

$$J(t) = \frac{\partial \Gamma}{\partial s}(0, t) = t(\exp_p)_{*tv}(w)$$

is a Jacobi field along γ and on the interval $[0, t_0]$ satisfies the boundary conditions $J(0) = 0$ and $J(t_0) = t_0(\exp_p)_{*t_0 v}(w)$. The Jacobi field J is uniquely determined by w and so the vector space of all Jacobi fields along γ which vanish at 0 has dimension n. For such a Jacobi field J as above we have $J(t_0) = 0$ if and only if $(\exp_p)_{*t_0 v}(w) = 0$. $\Box$

By Lemma 5.2.3(b), a Jacobi field which vanishes at 0 and t_0 is necessarily normal to γ. The existence of conjugate points is the obstruction to the existence of a solution to the general boundary value problem for Jacobi's equation.

Proposition 5.3.3. *Let $\ell > 0$ and $\gamma : [0, \ell] \to M$ be a geodesic path parametrized by arclength. If $\gamma(0)$ and $\gamma(\ell)$ are not conjugate along γ, then for every $v \in T_{\gamma(0)}M$ and $w \in T_{\gamma(\ell)}M$ there exists a unique Jacobi field J along γ satisfying the boundary conditions $J(0) = v$ and $J(\ell) = w$.*

Proof. If J_1, J_2 are two solutions of the boundary value problem, then $J_1 - J_2$ is a Jacobi field which vanishes at 0 and ℓ. Thus, if $\gamma(0)$ and $\gamma(\ell)$ are not conjugate along γ, then $J_1 - J_2 = 0$. In order to prove existence we consider the n-dimensional vector space Λ of all Jacobi fields along γ which vanish at 0. The map $T : \Lambda \to T_{\gamma(\ell)}M$ with $T(J) = J(\ell)$ is a linear monomorphism, since $\gamma(\ell)$ is not conjugate to $\gamma(0)$ along γ. Hence T is a linear isomorphism and this means that for every $w \in T_{\gamma(\ell)}M$ there exists a unique Jacobi field J_1 along γ such that $J_1(0) = 0$ and $J_1(\ell) = w$. Similarly, for every $v \in T_{\gamma(0)}M$ there exists a unique Jacobi field J_2 along γ such that $J_2(0) = v$ and $J_1(0) = 0$. Thus, it is sufficient to take $J = J_1 + J_2$. $\Box$

An important feature of conjugate points to $p \in M$ along a geodesic emanating from p is that they occur after the first point at which the geodesic is no longer minimizing.

Theorem 5.3.4. *If $\ell > 0$ and $\gamma : [0, \ell] \to M$ is a minimizing geodesic path parametrized by arclength, then no point $\gamma(s_0)$ is conjugate to $\gamma(0)$ along γ for $0 < s_0 < \ell$.*

Proof. We proceed by contradiction assuming that $\gamma(s_0)$ is conjugate to $\gamma(0)$ along γ for some $0 < s_0 < \ell$. According to Proposition 5.3.2, there exists a non-zero normal Jacobi field J along $\gamma|_{[0,s_0]}$ with $J(0) = 0$ and $J(s_0) = 0$. Then $\langle \frac{DJ}{dt}(t), \dot\gamma(t)\rangle = 0$ for $0 < t < s_0$ and $\frac{DJ}{dt}(s_0) \neq 0$. Setting $J(t) = 0$ for $s_0 < t \leq \ell$ we obtain an element of $\mathcal{D}_0(\gamma)$. We perturb J as follows. Let $\psi : [0, \ell] \to [0, 1]$ be a smooth function such that $\psi(0) = \psi(\ell) = 0$ and $\psi(s_0) = 1$, and let Z be the parallel vector field along γ with $Z(s_0) = -\frac{DJ}{dt}(s_0)$. For every $\epsilon > 0$ we define $X_\epsilon = J + \epsilon\psi Z$. By Lemma 5.1.1, X_ϵ is the variation field of a piecewise smooth variation of γ which fixes endpoints. If L_ϵ is the corresponding length function, we have $L'_\epsilon(0) = 0$, by Proposition 5.1.2,

and

$$L_\epsilon''(0) = \int_0^\ell \left[\left\| \frac{DX_\epsilon}{dt}(t) \right\|^2 - \langle R(X_\epsilon(t), \dot\gamma(t))\dot\gamma(t), X_\epsilon(t)\rangle \right] dt$$

by Synge's formula. However,

$$\left\| \frac{DX_\epsilon}{dt} \right\|^2 = \left\| \frac{DJ}{dt} \right\|^2 + (\epsilon\psi'\|Z\|)^2 + 2\epsilon\langle \frac{DJ}{dt}, \frac{D}{dt}(\psi Z)\rangle$$

$$= \frac{d}{dt}\langle \frac{DJ}{dt}, J\rangle - \langle \frac{D^2J}{dt^2}, J\rangle + 2\epsilon\langle \frac{DJ}{dt}, \frac{D}{dt}(\psi Z)\rangle + (\epsilon\psi'\|Z\|)^2$$

and

$$\langle R(X_\epsilon, \dot\gamma)\dot\gamma, X_\epsilon\rangle = \langle R(J,\dot\gamma)\dot\gamma, J\rangle + 2\epsilon\langle R(J,\dot\gamma)\dot\gamma, \psi Z\rangle + \epsilon^2\langle R(\psi Z, \dot\gamma)\dot\gamma, \psi Z\rangle.$$

Substituting we find

$$L_\epsilon''(0) = \int_0^\ell \frac{d}{dt}\langle \frac{DJ}{dt}, J\rangle dt - \int_0^\ell \langle \frac{D^2J}{dt^2} + R(J,\dot\gamma)\dot\gamma, J\rangle dt$$

$$+2\epsilon \int_0^\ell \left[\langle \frac{DJ}{dt}, \frac{D}{dt}(\psi Z)\rangle - \langle R(J,\dot\gamma)\dot\gamma, \psi Z\rangle \right] dt + \lambda\epsilon^2$$

$$= 2\epsilon \int_0^\ell \left[\frac{d}{dt}\langle \frac{DJ}{dt}, \psi Z\rangle - \langle \frac{D^2J}{dt^2}, \psi Z\rangle - \langle R(J,\dot\gamma)\dot\gamma, \psi Z\rangle \right] dt + \lambda\epsilon^2$$

$$= 2\epsilon \int_0^{s_0} \frac{d}{dt}\langle \frac{DJ}{dt}, \psi Z\rangle dt + \lambda\epsilon^2 = 2\epsilon\langle \frac{DJ}{dt}(s_0), \psi(s_0)Z(s_0)\rangle + \lambda\epsilon^2 = -2\epsilon \left\| \frac{DJ}{dt}(s_0) \right\|^2 + \lambda\epsilon^2$$

where $\lambda \in \mathbb{R}$ is a constant. If $\lambda \le 0$, then $L_\epsilon''(0) < 0$ for every $\epsilon > 0$ and if $\lambda > 0$, then again $L_\epsilon''(0) < 0$ for $0 < \epsilon < \frac{2}{\lambda}\left\| \frac{DJ}{dt}(s_0) \right\|^2$. Thus, in any case there exists $\epsilon_0 > 0$ such that $L_\epsilon''(0) < 0$ for $0 < \epsilon < \epsilon_0$. This implies that γ is not minimizing. □

We shall conclude this section with a result due o M. Morse and I.J. Schönberg which gives an estimate on the distance of conjugate points along a geodesic under a curvature condition. It can be proved as an application of Wirtinger's analytical inequality.

Proposition 5.3.5. *Let $a > 0$ and $f : [0, a] \to \mathbb{R}$ be a C^1 function. If $f(0) = 0$ and $f(a) = 0$, then*

$$a^2 \int_0^a (f'(t))^2 dt \ge \pi^2 \int_0^a (f(t))^2 f dt.$$

Proof. It suffices to prove that there exists a continuous function $\psi : (0, a) \to \mathbb{R}$ such that the function $f\psi$ with $(f\psi)(t) = f(t)\psi(t)$ for $0 < t < a$ and $(f\psi)(0) = (f\psi)(a) = 0$ is continuous and

$$\int_0^a \left[(f'(t))^2 - \left(\frac{\pi}{a} f(t) \right)^2 \right] dt \ge \int_0^a [f'(t) - (f\psi)(t)]^2 dt.$$

We seek such a function ψ such that the equality holds. In order the equality to hold, it is sufficient ψ to satisfy the ordinary differential equation

$$\psi' + \psi^2 + \frac{\pi^2}{a^2} = 0.$$

Its general solution is

$$\psi(t) = -\frac{\pi}{a}\tan\left(\frac{\pi}{a}t + c\right)$$

where c is an arbitrary constant depending on initial conditions. For $c = -\dfrac{\pi}{2}$ we get the solution

$$\psi(t) = -\frac{\pi}{a}\tan\left(\frac{\pi}{a}t - \frac{\pi}{2}\right) = \frac{\pi}{a}\cot\left(\frac{\pi}{a}t\right), \quad 0 < t < a.$$

Applying L'Hospital's rule we have now

$$\lim_{t\to 0^+}(f(t))^2\psi(t) = \lim_{t\to 0^+}\frac{\frac{\pi}{a}(f(t))^2\cos(\frac{\pi}{a}t)}{\sin(\frac{\pi}{a}t)}$$

$$= \lim_{t\to 0^+}\frac{\pi}{a}\cdot\frac{2f(t)f'(t)\cos(\frac{\pi}{a}t) - (f(t))^2\frac{\pi}{a}\sin(\frac{\pi}{a}t)}{\frac{\pi}{a}\cos(\frac{\pi}{a})t} = 0,$$

since f is assumed to be C^1 on $[0,a]$. Similarly, $\lim\limits_{t\to a^-}(f(t))^2\psi(t) = 0$. This choice of ψ now satisfies

$$\int_0^a\left[2f(t)f'(t)\psi(t) - ((f\psi)(t))^2 - \frac{\pi^2}{a^2}(f(t))^2\right]dt$$

$$= \lim_{T\to 0^+}\int_T^{a-T}\left[2f(t)f'(t)\psi(t) - ((f\psi)(t))^2 - \frac{\pi^2}{a^2}(f(t))^2\right]dt$$

$$= \lim_{T\to 0^+}\Bigg[(f(a-T))^2\psi(a-T) - (f(T))^2\psi(T)$$

$$- \int_T^{a-T}(f(t))^2\left(\psi'(t) + (\psi(t))^2 + \frac{\pi^2}{a^2}\right)dt\Bigg] = 0. \quad \square$$

Let M be a Riemannian n-manifold and let $\gamma : [0,a] \to M$, $a > 0$, be a smooth parametrized curve. Let also $X \in \mathcal{X}(\gamma)$ be such that $X(0) = 0$ and $X(a) = 0$. Then,

$$\int_0^a\left\|\frac{DX}{dt}(t)\right\|^2 dt \geq \frac{\pi^2}{a^2}\int_0^a\|X(t)\|^2 dt.$$

Indeed, if $\{E_1, ..., E_n\}$ is a parallel orthonormal frame along γ, there are uniquely determined smooth functions $f_k : [0,a] \to \mathbb{R}$, $1 \leq k \leq n$, such that

$$X = \sum_{k=1}^n f_k E_k$$

and thus $f_k(0) = f_k(a) = 0$, $1 \leq k \leq n$. Wirtinger's inequality gives

$$\int_0^a \left\| \frac{DX}{dt}(t) \right\|^2 dt = \int_0^a \sum_{k=1}^n (f_k'(t))^2 dt \geq \frac{\pi^2}{a^2} \sum_{k=1}^n \int_0^a (f_k(t))^2 dt = \frac{\pi^2}{a^2} \int_0^a \|X(t)\|^2 dt.$$

Theorem 5.3.6. *Let M be a Riemannian n-manifold, $n \geq 2$, and let $\gamma : [0, \ell] \to M$, $\ell > 0$ be a geodesic path parametrized by arclength. We assume that there exists $r > 0$ such that $K_{\gamma(t)}(S) \leq \frac{1}{r^2}$ for every 2-dimensional vector subspace S of $T_{\gamma(t)}M$ and every $0 \leq t \leq \ell$. If $\gamma(\ell)$ is conjugate to $\gamma(0)$ along γ, then $\ell \geq \pi r$.*

Proof. Since $\gamma(\ell)$ is assumed to be conjugate to $\gamma(0)$ along γ, there exists a non-zero Jacobi field J along γ which vanishes at 0 and ℓ, by Proposition 5.3.2. The derivative of the smooth function $f : [0, \ell] \to \mathbb{R}$ defined by

$$f(t) = \langle J(t), \frac{DJ}{dt}(t) \rangle$$

is

$$f'(t) = \left\| \frac{DJ}{dt}(t) \right\|^2 + \langle J(t), \frac{D^2 J}{dt^2}(t) \rangle = \left\| \frac{DJ}{dt}(t) \right\|^2 - \langle R(J(t), \dot{\gamma}(t))\dot{\gamma}(t), J(t) \rangle$$

$$\geq \left\| \frac{DJ}{dt}(t) \right\|^2 - \frac{1}{r^2} \|J(t)\|^2$$

by our curvature condition. Integrating we find

$$0 = \int_0^\ell f'(t) dt \geq \int_0^\ell \left\| \frac{DJ}{dt}(t) \right\|^2 dt - \frac{1}{r^2} \int_0^\ell \|J(t)\|^2 dt.$$

Applying Wirtinger's inequality as in the above remark we get

$$\frac{1}{r^2} \int_0^\ell \|J(t)\|^2 dt \geq \int_0^\ell \left\| \frac{DJ}{dt}(t) \right\|^2 dt \geq \frac{\pi^2}{\ell^2} \int_0^\ell \|J(t)\|^2 dt.$$

Since J is non-zero, $\ell^2 \geq \pi^2 r^2$. $\square$

5.4 Manifolds without conjugate points

The Riemannian manifolds without conjugate points, that is Riemannian manifolds in which every point is a pole, is a distinguished class which contains the very important class of Riemannian manifolds of non-positive sectional curvature as we shall show now.

Proposition 5.4.1. *If M is a Riemannian n-manifold, $n \geq 2$, with non-positive sectional curvature, meaning that $K_p(S) \leq 0$ for every $p \in M$ and every 2-dimensional vector subspace S of T_pM, then there are no conjugate points on M.*

Proof. We proceed to prove the assertion by contradiction. Suppose that $\ell > 0$ and $\gamma : [0,\ell] \to M$ be a geodesic parametrized by arclength for which there exists a Jacobi field J along γ with $J(0) = 0$ and $J(\ell) = 0$. It is sufficient to show that $J = 0$. We consider the smooth function $f : [0,\ell] \to \mathbb{R}$ defined by

$$f(t) = \langle J(t), \frac{DJ}{dt}(t)\rangle$$

whose derivative is

$$\begin{aligned} f'(t) &= \left\| \frac{DJ}{dt}(t) \right\|^2 + \langle \frac{D^2J}{dt^2}(t), J(t)\rangle \\ &= \left\| \frac{DJ}{dt}(t) \right\|^2 - \langle R(J(t),\dot\gamma(t))\dot\gamma(t), J(t)\rangle \\ &= \left\| \frac{DJ}{dt}(t) \right\|^2 - K_{\gamma(t)}(S(t))(\|J(t)\|^2 \cdot \|\dot\gamma(t)\|^2 - \langle J(t),\dot\gamma(t)\rangle^2) \geq 0 \end{aligned}$$

where $S(t)$ is the 2-dimensional vector subspace of $T_{\gamma(t)}M$ generated by $\{J(t), \dot\gamma(t)\}$. Since $f(0) = f(\ell) = 0$, this implies $f = 0$. It follows that

$$\frac{d}{dt}\langle J, J\rangle = 2\langle \frac{DJ}{dt}, J\rangle = 2f = 0$$

and hence $\|J\|$ is constant. Therefore, $\|J(t)\| = \|J(0)\| = 0$ for every $0 \leq t \leq \ell$. □

Corollary 5.4.2. *In the euclidean and the hyperbolic spaces there are no conjugate points.* □

The topology of a manifold admitting a complete Riemannian metric without conjugate points is encoded in its fundamental group, because its higher homotopy groups are trivial. This follows from a theorem proved by S. Kobayashi according to which the universal covering space of a connected, complete Riemannian manifold without conjugate points is diffeomorphic to the euclidean space of the same dimension. In the topological literature, the topological n-manifolds whose universal covering space is homeomorphic to $\mathbb{R}^n$ are called aspherical. Its proof is based on the following.

Proposition 5.4.3. *Let M be a connected, complete Riemannian n-manifold, N a Riemannian manifold and $f : M \to N$ a smooth map with the following properties:*
(a) f is surjective and
(b) expanding, that is $\|f_{*p}(v)\| \geq \|v\|$ for every $v \in T_pM$ and $p \in M$.
Then, N is also n-dimensional and f is a covering map.

Proof. Since f is assumed to be expanding, its derivative $f_{*p} : T_pM \to T_{f(p)}N$ is a linear monomorphism and so the dimension of N is at least n. On the other hand, since f is assumed to be surjective, it follows from Sard's theorem that the dimension of N must be at most n. Hence N is n-dimensional.

On M we introduce a new Riemannian metric with corresponding norm on the tangent spaces denoted by $|.|$, putting $|v| = \|f_{*p}(v)\|$ for every $v \in T_pM$ and $p \in M$.

If d denotes the Riemannian distance on M and ρ the distance with respect to this new Riemannian metric, then $\rho \geq d$. Therefore, (M, ρ) is a complete metric space and so is N. Replacing the Riemannian metric of M with the new one, we may from the beginning assume that f is a local isometry.

Let $q \in N$ and $\epsilon > 0$ be such that $\exp_p : B_q(0, \epsilon) \to B(q, \epsilon)$ is a diffeomorphism. Since f is a local isometry, the level set $f^{-1}(q)$ is discrete, hence countable. For each $p \in f^{-1}(q)$ the following diagram commutes.

$$\begin{array}{ccc} B_p(0,\epsilon) & \xrightarrow{\exp_p} & B(p,\epsilon) \\ {\scriptstyle f_{*p}}\downarrow & & \downarrow{\scriptstyle f} \\ B_q(0,\epsilon) & \xrightarrow{\exp_q} & B(q,\epsilon) \end{array}$$

Consequently, $f(B(p, \epsilon)) = B(q, \epsilon)$ and $\exp_p |_{B_p(0,\epsilon)}$ is injective. Since M is complete, $\exp_p(B_p(0, \epsilon)) = B(p, \epsilon)$ and therefore $\exp_p |_{B_p(0,\epsilon)}$ is a diffeomorphism as well as $f|_{B(p,\epsilon)}$. Obviously,

$$\bigcup_{p \in f^{-1}(q)} B(p, \frac{\epsilon}{2}) \subset f^{-1}(B(q, \frac{\epsilon}{2})).$$

Conversely, let $z \in f^{-1}(B(q, \frac{\epsilon}{2}))$ and let $\gamma : [0, s] \to B(q, \frac{\epsilon}{2})$ be a minimizing geodesic parametrized by arclength from $f(z)$ to q, where $0 < s < \frac{\epsilon}{2}$. Let σ be the geodesic in M which is parametrized by arclength with initial conditions $\sigma(0) = z$ and $\dot\sigma(0) = f_{*z}^{-1}(\dot\gamma(0))$. Since f is a local isometry, $f \circ \sigma$ is a geodesic and therefore $f \circ \sigma = \gamma$. Hence $f(\sigma(s)) = q$. Also, $d(z, \sigma(s)) \leq s < \frac{\epsilon}{2}$, that is $z \in B(\sigma(s), \frac{\epsilon}{2}) \subset \bigcup_{p \in f^{-1}(q)} B(p, \frac{\epsilon}{2})$. This shows that

$$\bigcup_{p \in f^{-1}(q)} B(p, \frac{\epsilon}{2}) = f^{-1}(B(q, \frac{\epsilon}{2})).$$

Finally, if $p_1,\ p_2 \in f^{-1}(q)$ and $B(p_1, \frac{\epsilon}{2}) \cap B(p_2, \frac{\epsilon}{2}) \neq \varnothing$, then $p_1 \in B(p_2, \epsilon)$, contradiction, because $f|_{B(p_2,\epsilon)}$ is injective. □

Theorem 5.4.4. *If M is a connected, complete Riemannian n-manifold without conjugate points, then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.*

Proof. Let $p \in M$. Since M is assumed to be complete and p is a pole, the exponential map $\exp_p : T_pM \to M$ is a surjective local diffeomorphism. On T_pM we consider the Riemannian metric which makes $\exp_p$ a local isometry, according to Example 3.3.3. The straight lines through the origin in T_pM are mapped onto geodesics of M and are therefore geodesics in T_pM with this metric. Since M is complete, it follows that the geodesics in T_pM through the origin are defined on $\mathbb{R}$. By Corollary 3.5.10, T_pM is complete. The assertion follows now as immediate application of the preceding Proposition 5.4.3. □

Corollary 5.4.5. *If a connected smooth n-manifold M admits a complete Riemannian metric without conjugate points, then $\pi_k(M) = \{0\}$ for every integer $k \geq 2$.* □

Corollary 5.4.6. *If a connected smooth n-manifold M admits a complete Riemannian metric of non-positive sectional curvature, then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.* □

Historically, Theorem 5.4.4 was proved by S. Kobayashi as a generalization of Corollary 5.4.6, which had been proved much earlier by J. Hadamard in the case of surfaces with non-positive Gauss curvature and by E. Cartan in the case of Riemannian manifolds of non-positive sectional curvature.

5.5 The cut locus

Let M be a Riemannian n-manifold with corresponding Riemannian distance d. For each $p \in M$ and $v \in T_pM$ with $\|v\| = 1$ we shall denote by γ_v the unique geodesic with initial conditions $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. We call

$$c(v) = \sup\{t > 0 : \gamma_v(t) \text{ is defined and } d(p, \gamma_v(t)) = t\} \in (0, +\infty]$$

the distance of p from the cut point along γ_v.

Proposition 5.5.1. *If $0 < s < c(v)$, then $\gamma_v|_{[0,s]}$ is the unique minimizing geodesic path parametrized by arclength from p to $\gamma_v(s)$.*

Proof. Let $0 < s < c(v)$. It is evident from the definition of $c(v)$ that $\gamma_v|_{[0,s]}$ is minimizing. Suppose that there exists $w \in T_pM$ with $\|w\| = 1$ such that $\gamma_w(s) = \gamma_v(s)$ and $d(p, \gamma_w(s)) = s = L(\gamma_w|_{[0,s]})$. Then, the concatenation $(\gamma_w|_{[0,s]}) * (\gamma_v|_{[s,t]})$ has length t for every $s < t < c(v)$. Since $(\gamma_w|_{[0,s]}) * (\gamma_v|_{[s,t]})$ is minimizing, it is a geodesic, by Proposition 3.5.6. Necessarily now $\gamma_w|_{[0,s]} = \gamma_v|_{[0,s]}$. □

Lemma 5.5.2. *If M is complete and $c(v) < +\infty$, then one of the following holds:*
(i) The point $\gamma_v(c(v))$ is the first conjugate point to $p = \gamma_v(0)$ along γ_v or equivalently $c(v)$ is the distance from p of the first conjugate point to p along γ_v.
(ii) There exist at least two different minimizing geodesics from $p = \gamma_v(0)$ to $\gamma_v(c(v))$.

Proof. According to Theorem 5.3.4, no point $\gamma_v(t)$ is conjugate to p along γ_v for $0 < t < c(v)$. Thus, either $\gamma_v(c(v))$ is the first conjugate point to p along γ_v or there exists a conjugate point $\gamma_v(t)$ to p along γ_v for some $t > c(v)$, if any. Let $(t_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence converging to $c(v)$. Since M is assumed to be complete, from the Hopf-Rinow Theorem 3.5.8, for each $k \in \mathbb{N}$ there exists a minimizing geodesic γ_{v_k} parametrized by arclength with initial conditions $\gamma_{v_k}(0) = p$, $\dot{\gamma}_{v_k}(0) = v_k$ such that $\gamma_{v_k}(d_k) = \gamma_v(t_k)$, where $d_k = d(p, \gamma_v(t_k))$. By compactness of $\partial B_p(0,1)$, passing to a subsequence if necessary, we may assume that there exists $u \in T_pM$ with $\|u\| = 1$ such that $\lim\limits_{k\to+\infty} v_k = u$. If $u = v$, then the exponential map $\exp_p$ is not injective on any open neighbourhood of $c(v)v \in T_pM$ and therefore $\gamma_v(c(v)) = \exp_p(c(v)v)$ is conjugate to p along γ_v. In case $u \neq v$, $\lim\limits_{k\to+\infty} d_k = d(p, \gamma_v(c(v))) = c(v)$ and $\gamma_u(c(v)) = \gamma_v(c(v))$. In other words, γ_u and γ_v are two different minimizing geodesics from p to $\gamma_v(c(v))$. □

Theorem 5.5.3. *The function $c : T^1M \to (0, +\infty]$ is upper semicontinuous. If M is complete, then c is a continuous function.*

Proof. Let $v \in T^1M$ and let $(v_k)_{k\in\mathbb{N}}$ be a sequence in T^1M converging to v. For upper semicontinuity we need to show that $\limsup\limits_{k\to+\infty} c(v_k) \leq c(v)$. Let p, $p_k \in M$ are such that $v \in T_pM$, $v_k \in T_{p_k}M$, $k \in \mathbb{N}$. If the sequence $(c(v_k))_{k\in\mathbb{N}}$ is unbounded, there exists a diverging subsequence $(c(v_{k_m}))_{m\in\mathbb{N}}$. For every $t > 0$ we have eventually $c(v_{k_m}) > t$ and by continuity of the exponential map $\lim\limits_{m\to+\infty} \gamma_{v_{k_m}}(t) = \gamma_v(t)$. Hence

$$d(p, \gamma_v(t)) = \lim_{m\to+\infty} d(p_{k_m}, \gamma_{v_{k_m}}(t)) = t.$$

This implies that $c(v) = +\infty$. If the sequence $(c(v_k))_{k\in\mathbb{N}}$ is bounded, there exists a subsequence $(c(v_{k_m}))_{m\in\mathbb{N}}$ which converges to some $c \in \mathbb{R}$. For every $0 < \epsilon < c$ we have

$$d(p, \gamma_v(c - \epsilon)) = \lim_{m\to+\infty} d(p_{k_m}, \gamma_{v_{k_m}}(c(v_{k_m}) - \epsilon)) = \lim_{m\to+\infty} (c(v_{k_m}) - \epsilon) = c - \epsilon.$$

Hence $c(v) \geq c$. This shows the upper semicontinuity.

For the continuity assuming the completeness of M, we need to prove that $\liminf\limits_{k\to+\infty} c(v_k) \geq c(v)$. It is sufficient to assume that we have a sequence $(c(v_k))_{k\in\mathbb{N}}$ converging to some $c \in \mathbb{R}$ and prove that $\gamma_v|_{[0,t]}$ is not minimizing for $t > c(v)$. Passing to a subsequence if necessary, because of Lemma 5.5.2, we consider two cases.

Let $\gamma_{v_k}(c(v_k))$ be the first conjugate point to p along γ_{v_k} for every $k \in \mathbb{N}$. In this case, the point $\gamma_v(c)$ is conjugate to p along γ_v, by continuity, and hence $c(v) \leq c$, by Theorem 5.3.4.

In the second case, we may assume that for every $k \in \mathbb{N}$ there exists $w_k \in T_pM$ with $\|w_k\| = 1$, such that $v_k \neq w_k$ and $\gamma_{w_k}|_{[0,c(v_k)]}$ is minimizing with $\gamma_{w_k}(c(v_k)) = \gamma_{v_k}(c(v_k))$. Passing to a subsequence if necessary, we can further assume that the sequence $(w_k)_{k\in\mathbb{N}}$ converges to some $w \in T_pM$ with $\|w\| = 1$, by compactness. Then, $\gamma_w(c) = \gamma_v(c)$. If $w \neq v$, obviously $c(v) \leq c$. If $w = v$, then $\exp_p$ is not a diffeomorphism on any open neighbourhood of cv. Hence $\gamma_v(c) = \exp_p(cv)$ is conjugate to p along γ_v and again $c(v) \leq c$. □

The preceding Theorem 5.5.3 combined with Theorem 5.3.4 give the following compactness result of W. Ambrose.

Corollary 5.5.4. *Let M be a connected, complete Riemannian n-manifold, $n \geq 2$. If there exists a point $p \in M$ such that along every geodesic emanating from p there exists a conjugate to p point, then M is compact.*

Proof. Our assumptions and Theorem 5.3.4 imply that $c(v) < +\infty$ for every $v \in T_pM$ with $\|v\| = 1$. From Theorem 5.5.3 the function $c : T^1M \to (0, +\infty)$ is continuous and thus there exists $c > 0$ such that $0 < c(v) \leq c$ for all $v \in T_pM$ with $\|v\| = 1$. The Hopf-Rinow Theorem 3.5.8 now implies that $M = \exp_p(\overline{B_p(0, c)})$. Hence M is compact. $\square$

Let M be a connected, complete Riemannian n-manifold, $n \geq 2$. For every $p \in M$ the set

$$C(p) = \exp_p(\{c(v)v : v \in T_pM \text{ with } \|v\| = 1 \text{ and } c(v) < +\infty\})$$

is called the *cut locus* of M at p. The subset

$$\{tv : 0 \leq t < c(v), v \in T_pM, \|v\| = 1\}$$

of T_pM is the largest star-shaped on which the exponential map $\exp_p$ is a diffeomorphism and

$$\exp_p(\{tv : 0 \leq t < c(v), v \in T_pM, \|v\| = 1\}) = M \setminus C(p).$$

Note that $M \setminus C(p)$ is dense in M, by the Hopf-Rinow Theorem 3.5.8. The positive real number $\mathrm{inj}p = \inf\{c(v) : v \in T_pM, \|v\| = 1\}$ is called the injectivity radius at p and the non-negative real number $\mathrm{inj}M = \inf\{\mathrm{inj}p : p \in M\}$ is called the *injectivity radius* of M. If M is compact, then $\mathrm{inj}M > 0$, by Theorem 5.3.4.

Examples 5.5.5. (a) On the n-sphere S^n_R of radius $R > 0$ the cut locus of any point p is the singleton $\{-p\}$. Conversely, if M is a complete Riemannian n-manifold and there exists a point $p \in M$ such that $C(p)$ is a singleton, then M is homeomorphic to the n-sphere. Indeed, by Theorem 5.5.3, if $C(p) = \{q\}$, then $c(v) = d(p, q)$ for every $v \in T_pM$ with $\|v\| = 1$. Let $R = \dfrac{d(p,q)}{\pi}$. The map

$$(\exp_p|_{B_p(0,\pi R)})^{-1} : M \setminus \{q\} \to B_p(0, \pi R)$$

is a diffeomorphism which extends to a homeomorphism

$$h : M \to \overline{B_p(0, \pi R)}/\partial B_p(0, \pi R)$$

by putting $h(q) = [\partial B_p(0, \pi R)]$. But the quotient space $\overline{B_p(0, \pi R)}/\partial B_p(0, \pi R)$ which results in from $\overline{B_p(0, \pi R)}$ by identifying $\partial B_p(0, \pi R)$ to a point is homeomorphic to the n-sphere.
(b) If M is a circular cylinder in $\mathbb{R}^3$, for instance

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\}, \quad R > 0,$$

then for every $p \in M$, the cut locus $C(p)$ is the straight line opposite to p.
(c) If $p \in \mathbb{R}P^n$, the cut locus $C(p)$ is the copy of $\mathbb{R}P^{n-1}$ which is "perpendicular" to p. For $p = [0, ..., 0, 1]$ this is

$$C(p) = \{[t_0, ..., t_n, 0] \in \mathbb{R}P^n : (t_0, ..., t_n) \in S^{n-1}\}.$$

In classical n-dimensional Projective Geometry this is traditionally called the $(n-1)$-dimensional real projective space at infinity.

5.6 Spaces of constant sectional curvature

The first step towards the answer to the question whether two given Riemannian manifolds are isometric is the local study of the problem. Contrary to other geometric structures this is a highly non-trivial task. A rather primitive approach would be the following. Let M and N be two Riemannian n-manifolds. Let $p \in M$ and $q \in N$. There exists a linear isometry $T : T_pM \to T_qN$. We seek for a Riemannian isometry f from some open neighbourhood U of p onto some open neighbourhood $f(U)$ of q such that $f(p) = q$ and $f_{*p} = T$. If such a local isometry f exists, shrinking U we may assume that U and $f(U)$ are (geodesic) open balls, necessarily of the same radius. Then, f commutes with the exponential maps, that is $f \circ \exp_p = \exp_q \circ T$. The question now arises under what conditions the diffeomorphism $\exp_q \circ T \circ \exp_p^{-1}$ is an isometry from a normal neighbourhood of p onto a normal neighbourhood of q. Such a sufficient condition has been found by E. Cartan.

Let U be a normal neighbourhood of p and W be a normal neighbourhood of q, so that $f = \exp_q \circ T \circ \exp_p^{-1}$ maps U diffeomorphically onto W. For every $x \in U \setminus \{p\}$ there exists a unique geodesic path $\gamma : [0, \ell] \to U$, $\ell > 0$, parametrized by arclength from p to x. The parallel translation $\tau_{x,p} : T_pM \to T_xM$ along γ is a linear isometry. Since T has been chosen to be a linear isometry, so is the map

$$F_x = \tau_{f(x),q} \circ T \circ \tau_{x,p}^{-1} : T_xM \to T_{f(x)}N.$$

We put $F_p = T$.

Theorem 5.6.1. *If for every $x \in U$ the equality*

$$\langle R(u, v)w, s\rangle = \langle R(F_x(u), F_x(v))F_x(w), F_x(s)\rangle$$

holds for all u, v, w, $s \in T_xM$, then $f = \exp_q \circ T \circ \exp_p^{-1}$ is an isometry and $f_{*x} = F_x$.

Proof. It is sufficient to prove that $\|f_{*x}(w)\| = \|w\|$ for every $w \in T_xM$ and $x \in U$. Let $\gamma : [0, \ell] \to U$ by the unique geodesic path parametrized by arclength from $\gamma(0) = p$ to $\gamma(\ell) = x$, where $\ell > 0$ and $x \neq p$. From Proposition 5.3.3, there exists a unique Jacobi field J along γ satisfying the boundary conditions $J(0) = 0$, $J(\ell) = w$. Let $\{E_1, ..., E_n\}$ be a parallel orthonormal frame along γ. If $J = \sum_{i=1}^{n} J_iE_i$, then

$$J_i''(t) + \sum_{j=1}^{n} \langle R(E_j(t), \dot\gamma(t))\dot\gamma(t), E_i(t)\rangle J_j(t) = 0$$

for every $0 \leq t \leq \ell$ and $1 \leq i \leq n$.

Let $\sigma : [0, \ell] \to N$ be the geodesic path with initial conditions $\sigma(0) = q$ and $\dot\sigma(0) = T(\dot\sigma(0))$. Obviously, $\sigma(\ell) = f(x)$. We define $V(t) = F_{\gamma(t)}(J(t))$ and $Z_i(t) = F_{\gamma(t)}(E_i(t))$, for all $0 \leq t \leq \ell$ and $1 \leq i \leq n$. Then, $\{Z_1, ..., Z_n\}$ is a parallel orthonormal frame along σ, from the construction of $F_{\gamma(t)}$, and $V(t) = \sum_{i=1}^{n} J_i(t)Z_i(t)$. By our assumption,

$$\langle R(E_j(t), \dot\gamma(t))\dot\gamma(t), E_i(t)\rangle = \langle R(Z_j(t), \dot\sigma(t))\dot\sigma(t), Z_i(t)\rangle$$

for every $0 \leq t \leq \ell$ and $1 \leq i, j \leq n$. This implies that V is a Jacobi field along σ. Moreover, $V(0) = 0$ and $\|V(\ell)\| = \|J(\ell)\| = \|w\|$. Thus, it suffices to show that $V(\ell) = f_{*x}(w)$. Note that

$$\frac{DV}{dt}(0) = T\Big(\frac{DJ}{dt}(0)\Big).$$

Since J and V are Jacobi fields which vanish at 0, they are given by the formulas

$$J(t) = (\exp_p)_{*t\dot\gamma(0)}\Big(t\frac{DJ}{dt}(0)\Big), \quad V(t) = (\exp_q)_{*t\dot\sigma(0)}\Big(t\frac{DV}{dt}(0)\Big),$$

as the proof of Lemma 5.2.2 shows. Consequently,

$$\begin{aligned} V(\ell) = (\exp_q)_{*\ell\dot\sigma(0)}\Big(\ell\frac{DV}{dt}(0)\Big) &= (\exp_q)_{*\ell\dot\sigma(0)}\Big(\ell T\Big(\frac{DJ}{dt}(0)\Big)\Big) \\ = (\exp_q)_{*t\dot\sigma(0)}(T((\exp_p)^{-1}_{*\ell\dot\gamma(0)}(J(\ell))) &= (\exp_q \circ T \circ \exp_p^{-1})_{*x}(J(\ell)) \\ &= f_{*x}(J(\ell)) = f_{*x}(w). \end{aligned}$$

Finally, $f_{*x}(w) = f_{*x}(J(\ell)) = V(\ell) = F_{\gamma(\ell)}(J(\ell)) = F_x(w)$. $\square$

Corollary 5.6.2. *If two Riemannian n-manifolds M and N, $n \geq 2$, have the same constant sectional curvature, then they are locally isometric.*

Proof. Suppose that M and N have the same constant sectional curvature $c \in \mathbb{R}$. From Corollary 4.2.3, the curvature tensor of both is given by the formula

$$R(u, v)w = c(\langle v, w\rangle u - \langle u, w\rangle v).$$

If $p \in M$, $q \in N$ and $T : T_pM \to T_qN$ is any choice of linear isometry, the hypothesis of Theorem 5.6.1 is satisfied. Hence there exists a Riemannian isometry from some normal neighbourhood of p onto some normal neighbourhood of q. $\square$

Theorem 5.6.3. *Let M be a connected, complete Riemannian n-manifold, $n \geq 2$. If M has constant sectional curvature K, then the universal covering space $\hat{M}$ of M is a simply connected complete Riemannian n-manifold of constant sectional curvature K and*
- *if $K < 0$, then $\hat{M}$ is isometric to the hyperbolic space $\mathbb{H}^n_{\frac{1}{\sqrt{-K}}}$,*
- *if $K = 0$, then $\hat{M}$ is isometric to the euclidean space $\mathbb{R}^n$,*
- *if $K > 0$, then $\hat{M}$ is isometric to the n-sphere $S^n_{\frac{1}{\sqrt{K}}}$.*

Proof. Let $\pi : \hat{M} \to M$ be the universal covering map. Since π is a local diffeomorphism, there is an induced Riemannian metric on $\hat{M}$ with respect to which π becomes a local isometry, according to Example 3.3.3. Hence $\hat{M}$ also has constant sectional curvature K. Every covering transformation $\alpha : \hat{M} \to \hat{M}$ is an isometry. Indeed, for every $x \in \hat{M}$ and v, $w \in T_x\hat{M}$ we have

$$\langle \alpha_{*x}(v), \alpha_{*x}(w)\rangle = \langle \pi_{*\alpha(x)}(\alpha_{*x}(v)), \pi_{*\alpha(x)}(\alpha_{*x}(w))\rangle = \langle (\pi \circ \alpha)_{*x}(v), (\pi \circ \alpha)_{*x}(w)\rangle$$

$$= \langle \pi_{*x}(v), \pi_{*x}(w) \rangle = \langle v, w \rangle.$$

In the case of the universal covering, the group of covering transformations is isomorphic to the fundamental group $\pi_1(M)$ of M which acts properly discontinuously on $\hat{M}$. The corresponding orbit space is precisely M and the quotient map is π.

Since M is complete, $\hat{M}$ is also complete, because every geodesic of $\hat{M}$ is a lifting of a geodesic of M, as we saw in the proof of Theorem 5.1.5(c). Thus, $\hat{M}$ is indeed a simply connected complete Riemannian manifold of constant sectional curvature K.

Note that in general if g is a Riemannian metric of sectional curvature K and $c \in \mathbb{R}$, then the sectional curvature of the Riemannian metric cg is $\dfrac{1}{c}K$. Thus, it suffices to proceed assuming that $K = -1$, 0 or 1.

If $K = -1$ or 0, then $\hat{M}$ is diffeomorphic to $\mathbb{R}^n$, by Corollary 5.4.6. We put $N = \mathbb{H}^n$ or $\mathbb{R}^n$, respectively. Actually, as the proof of Theorem 5.4.4. shows, if $x \in N$ and $y \in \hat{M}$, the corresponding exponential maps $\exp_x : T_xN \to N$ and $\exp_y : T_y\hat{M} \to \hat{M}$ are diffeomorphisms, since N and $\hat{M}$ are simply connected. Choosing any linear isometry $T : T_xN \to T_y\hat{M}$ we get a diffeomorphism

$$f = \exp_y \circ T \exp_x^{-1} : N \to \hat{M}$$

for which the hypothesis of Theorem 5.6.1 is satisfied. Hence f is an isometry.

Let now $K = 1$. In this case we put $N = S^n$ and using the same notations as above the exponential map $\exp_x : B_x(0, \pi) \to N \setminus \{-x\}$ is a diffeomorphism. Again the map $f = \exp_y \circ T \exp_x^{-1} : N \setminus \{-x\} \to \hat{M}$ is an isometric immersion onto an open subset of $\hat{M}$. We extend f on N as follows. Let $p \in N$, $p \neq x$, $-x$ and $q = f(p)$. The map $h = \exp_q \circ (f_{*p}) \circ \exp_p^{-1} : N \setminus \{-p\} \to \hat{M}$ is well defined and an isometric immersion onto an open subset of $\hat{M}$ such that $h(p) = q = f(p)$ and $h_{*p} = f_{*p}$. This implies that the coincidence set of f and h is non-empty, closed and open in $N \setminus \{-x, -p\}$ which is homeomorphic to $\mathbb{R} \setminus \{0\}$, hence connected. Therefore, $h = f$ on $N \setminus \{-x, -p\}$. Thus, we get a well defined map $\phi : N \to \hat{M}$ by

$$\phi(z) = \begin{cases} f(z), & \text{if } z \neq -x, \\ h(z), & \text{if } z \neq -p. \end{cases}$$

which is a local isometry of $N = S^n$ into $\hat{M}$. Since S^n is compact and $\hat{M}$ is connected, ϕ must necessarily be surjective. According to Proposition 5.4.3, ϕ is a covering map. Since S^n is simply connected, ϕ must be a diffeomorphism and an isometry. □

A connected, complete Riemannian manifold of constant sectional curvature is usually called *space form*. The preceding Theorem 5.6.3 implies that the problem of the isometric classification of space forms is essentially a group theoretical problem. More precisely, it is translated in the classification (modulo conjugacy) of the properly discontinuous subgroups of the isometry groups $I(\mathbb{H}^n)$, $I(\mathbb{R}^n)$ and $I(S^n)$. The study of hyperbolic space forms is a rich and very active field of contemporary research. The euclidean space forms have been thoroughly studied in dimensions ≤ 4. Especially, in dimension 3 the theory of compact euclidean space forms is essentially the theory of crystallographic groups. The spherical space forms have been found by J.A. Wolf. Here we shall present their description at even dimensions.

Proposition 5.6.4. *The isometry group of S^n, $n \geq 2$, is the orthogonal group $O(n+1,\mathbb{R})$.*

Proof. We already know that $O(n+1,\mathbb{R})$ is a subgroup of $I(S^n)$. In order to prove the reverse inclusion let $f \in I(S^n)$. If d denotes the spherical Riemannian distance, then $\cos d(p,q) = \langle p, q\rangle$ for every $p, q \in S^n$, where $\langle .,. \rangle$ on the right hand side denotes the euclidean inner product in $\mathbb{R}^{n+1}$. Thus, $\langle f(p), f(q)\rangle = \langle p, q\rangle$. In particular, p and q are orthogonal vectors in $\mathbb{R}^{n+1}$ if and only if $d(p,q) = \frac{\pi}{2}$, and so f maps the canonical basis $\{e_1, ..., e_{n+1}\}$ onto an orthonormal basis $\{f(e_1), ..., f(e_{n+1})\}$. By linear extension, there exists a unique $A \in O(n+1,\mathbb{R})$ such that $Ae_i = f(e_i)$ for every $1 \leq i \leq n+1$. If now $x \in S^n$, we have

$$\cos d(f(x), Ax) = \langle f(x), Ax\rangle = \sum_{i=1}^{n} \langle x, e_i\rangle \cdot \langle f(x), f(e_i)\rangle = \sum_{i=1}^{n} \langle x, e_i\rangle^2 = 1$$

and therefore $f(x) = Ax$. This concludes the proof. □

Theorem 5.6.5. *Let M be a connected, complete Riemannian n-manifold, $n \geq 2$, of constant sectional curvature* 1. *If n is even, then M is isometric to S^n or to $\mathbb{R}P^n$.*

Proof. According to what we have already proved in the present section, M is isometric to the orbit space of a subgroup G of $O(n+1,\mathbb{R})$ which acts properly discontinuously on S^n. Since S^n is compact, G must be a finite group. If $A \in G$, $A \neq I_{n+1}$ and $\det A = 1$, then 1 is an eigenvalue of A, since $n+1$ is odd. This contradicts the proper discontinuity of G. Thus, $\det A = -1$ and $A^2 = I_{n+1}$ for every $A \in G \setminus \{I_{n+1}\}$. If $\lambda \in \mathbb{C}$ is any root of the characteristic polynomial of $A \neq I_{n+1}$ in G, then λ^2 is an eigenvalue of A^2 and therefore $\lambda^2 = 1$. It follows that the characteristic polynomial of A has only one root in $\mathbb{C}$, namely -1. Consequently, $A = -I_{n+1}$, since A is orthogonal. This proves that either $G = \{I_{n+1}\}$ or $G = \{I_{n+1}, -I_{n+1}\}$. In the former case M is isometric to S^n itself and in the latter it is isometric to $\mathbb{R}P^n$. □

The preceding Theorem 5.6.5 does not hold in odd dimensions. An easy class of examples are the Lens spaces in dimension 3. Let $a > 1$ and b be relatively prime integers. The cyclic group of order a generated by the isometry

$$A(z_1, z_2) = (e^{2\pi i/a} z_1, e^{2\pi ib/a} z_2)$$

acts properly discontinuously on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. The corresponding orbit space $L(a,b)$ is a 3-dimensional spherical space form. If $a > 2$, certainly $L(a,b)$ is homeomorphic to neither S^3 nor $\mathbb{R}P^3$.

The Riemannian metric of a space form can be written down explicitly locally around a point in geodesic spherical coordinates. Since the space forms with the same curvature are locally isometric, it is sufficient to carry out the calculations only in the simply connected case.

In the euclidean space $\mathbb{R}^n$, $n \geq 2$, the map sending a non-zero vector x to $(x, \frac{1}{\|x\|}x)$ is a diffeomorphism of $\mathbb{R}^n \setminus \{0\}$ onto $(0,+\infty) \times S^{n-1}$. Thus, every smooth parametrized curve $\gamma : I \to \mathbb{R}^n \setminus \{0\}$, where $I \subset \mathbb{R}$ is an open interval, has the form $\gamma(t) = r(t)\xi(t)$ for suitable smooth maps $r : I \to (0,+\infty)$ and $\xi : I \to S^{n-1}$. Differentiating

$$\dot\gamma(t) = r'(t)\xi(t) + r(t)\dot\xi(t)$$

and $\|\dot\gamma(t)\|^2 = (r'(t))^2 + (r(t))^2\|\dot\xi(t)\|^2$. Thus, the euclidean Riemannian metric on $\mathbb{R}^n$ in local spherical coordinated around any point has the form

$$ds^2 = dr^2 + r^2\|d\xi\|^2$$

in traditional notation, since $\mathbb{R}^n$ is homogeneous.

The simply connected n dimensional space form, $n \geq 2$, of sectional curvature $\frac{1}{R^2}$, $R > 0$, is (isometric to) the n-sphere S^n_R of radius R in $\mathbb{R}^{n+1}$. Since S^n_R is homogeneous, it is sufficient to describe the geodesic spherical coordinates around the point Re_{n+1}. Every point $p \in S^n_R \setminus \{\pm Re_{n+1}\}$ can be written

$$p = \big(R\cos\frac{\rho}{R}\big)e_{n+1} + \big(R\sin\frac{\rho}{R}\big)\xi$$

for some $0 < \rho < \pi R$ and $\xi \in S^{n-1}$. Note that ρ is the length of the geodesic emanating from Re_{n+1} to p. A smooth parametrized curve $\gamma : I \to S^n_R \setminus \{\pm Re_{n+1}\}$, can be written

$$\gamma(t) = \big(R\cos\frac{\rho(t)}{R}\big)e_{n+1} + \big(R\sin\frac{\rho(t)}{R}\big)\xi(t)$$

for suitable smooth maps $\rho : I \to (0, \pi R)$ and $\xi : I \to S^{n-1}$. Differentiating

$$\dot\gamma(t) = \rho'(t)\big[\big(-\sin\frac{\rho(t)}{R}\big)e_{n+1} + \big(\cos\frac{\rho(t)}{R}\big)\xi(t)\big] + \big(R\sin\frac{\rho(t)}{R}\big)\dot\xi(t)$$

and therefore

$$\|\dot\gamma(t)\|^2 = (\rho'(t))^2 + \big(R^2\sin^2\frac{\rho(t)}{R}\big)\|\dot\xi(t)\|^2.$$

Thus, the standard Riemannian metric on S^n_R in local spherical coordinated around any point has the form

$$ds^2 = d\rho^2 + \big(R^2\sin^2\frac{\rho}{R}\big)\|d\xi\|^2$$

in traditional notation.

For the n-dimensional hyperbolic space of sectional curvature $-\frac{1}{R^2}$, $R > 0$, we shall use the unit ball model $\mathbb{D}^n_R$, $n \geq 2$. The traces of the hyperbolic geodesics through $0 \in \mathbb{D}^n_R$ are the euclidean diameters. Let $z \in \mathbb{D}^n_R \setminus \{0\}$. A parametrization of the geodesic path from 0 to z is $\gamma : [0,1] \to \mathbb{D}^n \setminus \{0\}$ with $\gamma(t) = tz$. The hyperbolic distance of z from 0 is

$$\rho = L(\gamma) = \int_0^1 \frac{2R\|z\|}{1-\|tz\|^2}dt = R\int_0^{\|z\|}\frac{2}{1-s^2}ds = R\log\frac{1+\|z\|}{1-\|z\|},$$

where $\|.\|$ is the euclidean norm. Thus,

$$\|z\| = \tanh \frac{\rho}{2R}.$$

Every smooth parametrized curve $\gamma : I \to \mathbb{R}^n \setminus \{0\}$, where $I \subset \mathbb{R}$ is an open interval, has the form $\gamma(t) = r(t)\xi(t)$ for suitable smooth maps $r : I \to (0,1)$ and $\xi : I \to S^{n-1}$. So,

$$\|\gamma(t)\| = r(t) = \tanh \frac{\rho(t)}{2R}$$

where $\rho(t) = L(\gamma|_[0,t])$ and

$$\|\dot\gamma(t)\|^2 = (r'(t))^2 + (r(t))^2\|\dot\xi(t)\|^2 = \frac{\rho'(t)}{4R^2\cosh^4 \frac{\rho(t)}{2R}} + \big(\tanh^2 \frac{\rho(t)}{2R}\big)\|\dot\xi(t)\|^2.$$

The square of the hyperbolic length of $\dot\gamma(t)$ is equal to

$$\frac{2R^2\|\dot\gamma(t)\|^2}{(1-\|\gamma(t)\|^2)^2} = (\rho'(t))^2 + \big(R^2\sinh^2 \frac{\rho(t)}{R}\big)\|\dot\xi(t)\|^2.$$

Thus, the hyperbolic Riemannian metric in geodesic spherical coordinates is

$$ds^2 = d\rho^2 + \big(R^2\sinh^2 \frac{\rho}{R}\big)\|d\xi\|^2$$

in traditional notation.

Summarizing, locally around a point of a space form of sectional curvature $K \in \mathbb{R}$ the Riemannian metric is

$$g_K = d\rho^2 + (S_K(\rho))^2 g_{S^{n-1}}$$

where $g_{S^{n-1}}$ is the usual Riemannian metric of S^{n-1}, ρ is the length of the geodesic radius and

$$S_K(\rho) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}\rho), & \text{if } K > 0, \\ \rho, & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}\rho), & \text{if } K < 0. \end{cases}$$

5.7 Infinitesimal isometries

Let M be a connected Riemannian n-manifold and $X \in \mathcal{X}(M)$ with flow $\phi : D \to M$. Recall that for every $t \in \mathbb{R}$ the set $D_t = \{p \in M : (t,p) \in D\}$ is open in M and $\phi_t = \phi(t,.) : D_t \to \phi_t(D_t) \subset M$ is a difeomorphism onto the open set $\phi_t(D_t)$. The vector field X is called an *infinitesimal isometry* or *Killing vector field* of M if ϕ_t is an isometric embedding for every $t \in \mathbb{R}$.

Proposition 5.7.1. *If M is complete, then every Killing vector field of M is complete.*

Proof. Since M is assumed to be connected and complete, from the Hopf-Rinow Theorem, $\overline{B(p,c)}$ is compact for every $p \in M$ and $c > 0$ and $M = \bigcup_{c>0} \overline{B(p,c)}$. Suppose that X is a non-complete Killing vector field of M. Then, there exists a point $p \in M$ such that the integral curve of X through p is defined on an open interval (a_p, b_p) and $b_p < +\infty$ (or $a_p > -\infty$). There exists some $\epsilon > 0$ such that $\overline{B(p,\epsilon)}$ is contained in a normal neighbourhood of p. There exists $T > 0$ such that $\phi_t(p) \in B(p,\epsilon)$ for $0 \leq t < T$ and $d(p, \phi_T(p)) = \epsilon$, where as usual d denotes the Riemannian distance. Dividing, for each $0 < t < b_p$ there exist $k \in \mathbb{Z}^+$ and $0 \leq s < T$ such that $t = kT + s$. From the triangle inequality,

$$d(\phi_t(p), p) \leq \sum_{j=1}^{k} d(\phi_{jT}(p), \phi_{(j-1)T}(p)) + d(\phi_{kT}(p), \phi_t(p))$$

$$= \sum_{j=1}^{k} d(\phi_T(p), p) + d(\phi_s(p), p) \leq (k+1)\epsilon < \left(\frac{b_p}{T} + 1\right)\epsilon.$$

This contradicts Lemma 2.2.4. $\square$

In the rest of this section we shall assume that M is a connected, complete Riemannian n-manifold, $n \geq 2$. A very useful characterization of Killing vector fields is the following.

Proposition 5.7.2. *If $X \in \mathcal{X}(M)$, then X is a Killing vector field if and only if it satisfies Killing's equation*

$$\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0$$

for every Y, $Z \in \mathcal{X}(M)$.

Proof. Let $(\phi_t)_{t\in\mathbb{R}}$ be the one-parameter group of diffeomorphisms generated by X. Then, X is a Killing vector field if and only if

$$\langle (\phi_t)_{*p}(Y_p), (\phi_t)_{*p}(Z_p) \rangle = \langle Y_p, Z_p \rangle$$

for every $p \in M$, $t \in \mathbb{R}$ and Y, $Z \in \mathcal{X}(M)$. Equivalenty,

$$0 = \lim_{t\to 0} \frac{1}{t} \Big[\langle (\phi_t)_{*p}(Y_p), (\phi_t)_{*p}(Z_p) \rangle - \langle Y_p, Z_p \rangle \Big]$$

$$\lim_{t\to 0} \frac{1}{t} \Big[\langle (\phi_t)_{*p}(Y_p), (\phi_t)_{*p}(Z_p) \rangle - \langle Y_{\phi_t(p)}, Z_{\phi_t(p)} \rangle + \langle Y_{\phi_t(p)}, Z_{\phi_t(p)} \rangle - \langle Y_p, Z_p \rangle \Big]$$

$$= X_p \langle Y, Z \rangle + \lim_{t\to 0} \frac{1}{t} \Big[\langle (\phi_t)_{*p}(Y_p), (\phi_t)_{*p}(Z_p) \rangle - \langle Y_{\phi_t(p)}, Z_{\phi_t(p)} \rangle \Big]$$

$$= X_p \langle Y, Z \rangle + \lim_{t\to 0} \frac{1}{t} \Big[\langle (\phi_t)_{*p}(Y_p), (\phi_t)_{*p}(Z_p) \rangle - \langle Y_{\phi_t(p)}, (\phi_t)_{*p}(Z_p)$$

$$+\langle Y_{\phi_t(p)}, (\phi_t)_{*p}(Z_p) - \langle Y_{\phi_t(p)}, Z_{\phi_t(p)}\rangle\Big]$$
$$= X_p\langle Y, Z\rangle - \langle [X,Y]_p, Z_p\rangle - \langle Y_p, [X,Z]_p\rangle$$
$$= X_p\langle Y, Z\rangle - \langle \nabla_X Y - \nabla_Y X, Z\rangle - \langle Y, \nabla_X Z - \nabla_Z X\rangle$$
$$= -\langle \nabla_Y X, Z\rangle - \langle Y, \nabla_Z X\rangle. \quad \square$$

Proposition 5.7.3. *$X \in \mathcal{X}(\mathcal{M})$ be a Killing vector field.*
(i) If $\ell > 0$ and $\gamma : [0, \ell] \to M$ is a geodesic path parametrized by arclength, then X is a Jacobi field when restricted along γ.
(ii) If there exists $p \in M$ such that $X_p = 0$ and $\nabla_u X = 0$ for every $u \in T_pM$, then $X = 0$.

Proof. (i) We consider the smooth variation $\Gamma : \mathbb{R} \times [0, \ell] \to M$ defined by $\Gamma(s,t) = \phi_s(\gamma(t))$, where $(\phi_t)_{t\in\mathbb{R}}$ is the one-parameter group of isometries of M generated by X. Thus, Γ is a variation by geodesics and the corresponding variation field is a Jacobi field along γ. However,

$$\frac{\partial \Gamma}{\partial s}(0,t) = \frac{\partial}{\partial s}\bigg|_{s=0} \phi_s(\gamma(t)) = X_{\gamma(t)}$$

for every $0 \le t \le \ell$.

(ii) The set $A = \{q \in M : X_q = 0 \text{ and } \nabla_u X = 0 \text{ for every } u \in T_pM\}$ is non-empty by assumption and obviously closed in M. Since M is connected, it is sufficient to prove that A is open. Let $q \in A$ and U be a normal open neighbourhood of q. For every $x \in U \setminus \{q\}$ there exists a unique geodesic path parametrized by arc length $\gamma : [0, \ell] \to U$ from $\gamma(0) = q$ to $\gamma(\ell) = x$, where $l = d(q,x)$. Since X is a Jacobi field when restricted along γ by (i) and satisfies the initial conditions $X_{\gamma(0)} = 0$ and $\dfrac{DX}{dt}(0) = \nabla_{\dot\gamma(0)} X = 0$, it follows that $X_{\gamma(t)} = 0$ for every $0 \le t \le \ell$, from uniqueness of solutions of Jacobi's linear differential equation. In particular, $X_x = 0$. This shows that X vanishes identically on U and therefore $U \subset A$. $\square$

We shall now investigate the square of the length function $h = \|X\|^2$ of a Killing vector field X of M.

Lemma 5.7.4. *If $X \in \mathcal{X}(M)$ is a Killing vector field and $p \in M$ is a critical point of $h = \|X\|^2$ such that $X_p \neq 0$, then the integral curve of X through p is a geodesic of M.*

Proof. From Proposition 5.7.2 we have

$$Y\langle X, X\rangle = 2\langle \nabla_Y X, X\rangle = -2\langle \nabla_X X, Y\rangle$$

for every $Y \in \mathcal{X}(M)$. Thus, if p is a critical point of $h = \|X\|^2$, we must have

$$0 = (\phi_t)_{*p}(-2\nabla_{X_p} X) = -2\nabla_{(\phi_t)_{*p}(X_p)}(\phi_t)_* X = -2\nabla_{X_{\phi_t(p)}} X$$

for every $t \in \mathbb{R}$, where $(\phi_t)_{t\in\mathbb{R}}$ is the one-parameter group of isometries of M generated by X. □

Lemma 5.7.5. *Let $X \in \mathcal{X}(M)$ be a Killing vector field and let $\gamma : \mathbb{R} \to M$ be a geodesic parametrized by arclength. If*

$$h_\gamma(t) = \frac{1}{2}\|X_{\gamma(t)}\|^2,$$

then

$$h''_\gamma(t) = \|\nabla_{\dot\gamma(t)}X\|^2 - \langle R(X_{\gamma(t)}, \dot\gamma(t))\dot\gamma(t), X_{\gamma(t)}\rangle, \quad t \in \mathbb{R}.$$

Proof. Applying the chain rule, the first derivative if h_γ is

$$h'_\gamma(t) = \frac{1}{2}\dot\gamma(t)\langle X, X\rangle = \langle \nabla_{\dot\gamma(t)}X, X_{\gamma(t)}\rangle$$

and differentiating once more

$$h''_\gamma(t) = \langle \frac{D^2}{dt^2}(t), X_{\gamma(t)}\rangle + \|\nabla_{\dot\gamma(t)}X\|^2 = \|\nabla_{\dot\gamma(t)}X\|^2 - \langle R(X_{\gamma(t)}, \dot\gamma(t))\dot\gamma(t), X_{\gamma(t)}\rangle$$

by Proposition 5.7.3(i). □

Lemma 5.7.6. *Let $X \in \mathcal{X}(M)$ be a Killing vector field with flow $\phi : \mathbb{R} \times M \to M$ and $p \in M$. We denote by γ the integral curve of X through p, that is $\gamma(t) = \phi_t(p)$, $t \in \mathbb{R}$. Suppose that $h = \|X\|^2$ has local minimum at p and $X_p \neq 0$. If $w \in T_pM$ and $Y(t) = (\phi_t)_{*p}(w)$, then*

$$\|\nabla_{Y(t)}X\|^2 \geq \langle R(X_{\gamma(t)}, Y(t))Y(t), X_{\gamma(t)}\rangle$$

for every $t \in \mathbb{R}$. If h has local maximum at p, the reverse inequality holds.

Proof. We consider the smooth variation by geodesics $\Gamma : \mathbb{R} \times \mathbb{R} \to M$ defined by $\Gamma(s,t) = \exp_{\gamma(s)}(tY(s))$. For each t, τ, $s \in \mathbb{R}$ we have

$$\phi_\tau(\Gamma(s,t)) = \exp_{\phi_{s+\tau}(p)}((\phi_{s+\tau})_{*p}(tw)) = \Gamma(s+\tau, t),$$

because ϕ_τ is an isometry. Consequently, $\dfrac{\partial\Gamma}{\partial s}(s,t) = X_{\Gamma(s,t)}$, and of course $\dfrac{\partial\Gamma}{\partial t}(s,0) = Y(s)$. Hence

$$\nabla_X \frac{\partial\Gamma}{\partial t} - \nabla_{\frac{\partial\Gamma}{\partial t}}X = \Gamma_{*(s,t)}\left(\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]_{(s,t)}\right) = 0$$

and in particular $\nabla_{Y(s)}X = \dfrac{DY}{ds}(s)$ for every $s \in \mathbb{R}$. According to the preceding Lemma 5.7.5 and using the same notations of its statement,

$$h''_{\Gamma(s,.)}(t) = \|\nabla_{\frac{\partial\Gamma}{\partial t}(s,t)}X\|^2 - \langle R(X_{\Gamma(s,t)}, \frac{\partial\Gamma}{\partial t}(s,t))\frac{\partial\Gamma}{\partial t}(s,t), X_{\Gamma(s,t)}\rangle$$

for every $(s,t) \in \mathbb{R} \times \mathbb{R}$. Since $(\phi_s)_{*p}(X_p) = X_{\phi_s(p)}$ and ϕ_s is an isometry, if h has local minimum at p and $X_p \neq 0$, it has local minimum at $\phi_s(p)$ and therefore

$$0 \leq h''_{\Gamma(s,.)}(0) = \|\nabla_{Y(s)} X\|^2 - \langle R(X_{\gamma(s)}, Y(s))Y(s), X_{\gamma(s)}\rangle. \quad \square$$

The preceding series of lemmas leads to the following famous vanishing theorem of S. Bochner.

Theorem 5.7.7. *Let M be a connected, complete Riemannian n-manifold, $n \geq 2$, whose Ricci curvature is everywhere negative. If $X \in \mathcal{X}(M)$ is a Killing vector field and the function $\|X\|$ takes a maximum value at some point of M, then $X = 0$.*

Proof. Suppose that $X \neq 0$ and $\|X\|$ takes a maximum value at a point $p \in M$. Then, $X_p \neq 0$. Let $w \in T_pM$ with $\|w\| = 1$ be orthogonal to X_p and let S be the 2-dimensional vector subspace of T_pM generated by $\{w, X_p\}$. Then Lemma 5.7.6 implies

$$K_p(S) = \frac{1}{\|X_p\|^2}\langle R(X_p, w)w, X_p\rangle \geq \frac{\|\nabla_w X\|^2}{\|X\|^2} \geq 0.$$

It follows from this that $Ric_p(X_p, X_p) \geq 0$. $\square$

Corollary 5.7.8. *Let M be a compact Riemannian n-manifold, $n \geq 2$. If the Ricci curvature (or the sectional curvature) is everywhere negative on M, then every Killing vector field vanishes identically on M.* $\square$

The isometry group $I(M)$ of a connected complete Riemannian manifold M is a Lie group endowed with the compact-open topology, which in this case coincides with the topology of pointwise convergence, and acts on M smoothly as a Lie transformation group. Its Lie algebra is precisely the Lie algebra of all Killing vector fields of M. This justifies the term infinitesimal isometry. If M is compact, then $I(M)$ is also compact. It follows from the last Corollary 5.7.8 that the isometry group of a compact Riemannian manifold of negative Ricci (or sectional) curvature is finite. Thus, the compact Riemannian manifolds of negative curvature virtually have no symmetries. This is a deeper reason which explains why their classification is not an easy task.

5.8 Exercises

1. Let M be a Riemannian n-manifold, $n \geq 2$, and $\gamma : [0,l] \to M$ be a geodesic path parametrized by arclength. Let $X \in \mathcal{X}(M)$ and $\phi : D \to M$ be its flow.
(a) Prove that there exists $T > 0$ such that $[-T,T] \times \gamma([0,l]) \subset D$.
(b) Let $\Gamma : [0,T] \times [0,l] \to M$ be the smooth variation of γ with $\Gamma(s,t) = \phi_s(\gamma(t))$. If $\Gamma_T = \Gamma(T,.)$ and $L(\gamma)$, $L(\Gamma_T)$ are the lengths of γ, Γ_T, respectively, prove that

$$|L(\Gamma_T) - L(\gamma)| \leq \int_0^T \int_0^l \|\nabla_{\frac{\partial\Gamma}{\partial t}} X\| ds dt.$$

2. Let M be a Riemannian manifold with Riemannian distance function d and $p \in M$.

(a) Prove that the real function $f = d(p,.)$ is smooth on $D(p) \setminus \{p\}$, where $D(p) = M \setminus C(p)$.
(b) If $q \in D(p) \setminus \{p\}$ and γ is the unit speed geodesic in $D(p)$ from p to q, prove that $f_{*q}(v) = \langle v, \dot{\gamma}(d(p,q))\rangle$ for every $v \in T_pM$.

3. Let M be a Riemannian n-manifold, $n \geq 2$, and $\gamma : [0,\ell] \to M$ be a geodesic path parametrized by arclength, where $\ell > 0$. If there exists $r > 0$ and a smooth function $f : [0,\ell] \to \mathbb{R}$ such that

$$Ric_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \geq (n-1)\big(\frac{1}{r^2} + f'(t)\big)$$

for every $0 \leq t \leq \ell$, prove that

$$\ell \leq \pi r^2\big(\|f\| + \sqrt{\|f\|^2 + \frac{1}{r^2}}\big)$$

where $\|f\| = \sup\{|f(t)| : 0 \leq t \leq \ell\}$.

4. Let M be a Riemannian 2-manifold and $\gamma : [0,l] \to M$ be a geodesic path parametrized by arclength. Let $X \in \mathcal{X}(\gamma)$ with $\langle X, \dot{\gamma}\rangle = 0$ and $\|X\| = 1$ on $[0,l]$.
(a) Prove that X is parallel along γ.
(b) Let $f : [0,l] \to \mathbb{R}$ be a smooth function. Prove that fX is a Jacobi field along γ if and only if $f''(t) + K(\gamma(t))f(t) = 0$ for every $0 \leq t \leq l$, where K is the sectional curvature of M.

5. On $\mathbb{R}^2 \setminus \{(0,0)\}$ we consider the Riemannian metric g

$$g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 1, \quad g(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}) = 0, \quad g(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) = (f(r,\phi))^2$$

in polar coordinates, where $f : \mathbb{R}^2 \setminus \{(0,0)\} \to (0,+\infty)$ is a smooth function.
(a) Find the differential equation of geodesics and prove that γ_ϕ, for constant ϕ, is a geodesic.
(b) If $X(r)$ is a parallel vector field along γ_ϕ, which is orthogonal to γ_ϕ, prove that $Y(r) = f(r,\phi)X(r)$ is a Jacobi field along γ_ϕ.
(a) Prove that the sectional curvature is given by the formula

$$K(r,\phi) = -\frac{1}{f(r,\phi)}\frac{\partial^2 f}{\partial r^2}(r,\phi).$$

6. Let $r > 0$, $\ell > 0$ and $\gamma : [0,\ell] \to S^n_r$, $n \geq 2$, be a smooth curve parametrized by arclength.
(a) If $E \in \mathcal{X}(\gamma)$ is parallel along γ, prove that

$$E'(t) = -\frac{\langle E(t), \dot{\gamma}(t)\rangle}{r^2}\gamma(t)$$

for every $0 \leq t \leq \ell$. In case γ is a geodesic, prove that

$$E'(t) = -\frac{\langle E(0), \dot{\gamma}(0)\rangle}{r^2}\gamma(t)$$

for every $0 \leq t \leq \ell$ and thus if $E(0)$ is orthogonal to $\dot{\gamma}(0)$, then E is constant.
(b) Let $p = (0, ..., 0, r) \in S^n_r$ and u, $v \in T_p S^n_r$ be orthogonal with $\|u\| = \|v\| = 1$. Let $\Gamma : (-\pi, \pi) \times [0, \pi r] \to S^n_r$ be the smooth variation

$$\Gamma(s,t) = \Big(\cos\frac{t}{r}\Big)p + \Big(\sin\frac{t}{r}\Big)r\big((\cos s)u + (\sin s)v\big).$$

Prove that the variation field V of Γ is given by the formula

$$V(t) = \Big(\sin\frac{t}{r}\Big)r \cdot E(t), \quad 0 \leq t \leq \pi R,$$

where E is the parallel vector field along $\Gamma(0, .)$ with initial condition $E(0) = v$. Compute then $R(v,u)u = \dfrac{1}{r^2}v$ and so $\langle R(v,u)u, v\rangle = \dfrac{1}{r^2}$.

7. Let $z \in S^{2n+1}$, $u \in T_{[z]}\mathbb{C}P^n$ with $\|u\| = 1$ and γ_u denote the geodesic in $\mathbb{C}P^n$ with $\gamma_u(0) = [z]$ and $\dot{\gamma}_u(0) = u$. Let $v \in T_{[z]}\mathbb{C}P^n$ be orthogonal to u, and let $\tilde{u}$, $\tilde{v}$ be horizontal lifts of u, v, respectively, with respect to the Hopf map $\pi : S^{2n+1} \to \mathbb{C}P^n$. The variation by geodesics

$$\Gamma(s,t) = (\cos t)z + (\sin t)\big((\cos s)\tilde{u} + (\sin s)\tilde{v}\big)$$

project to a variation $\pi \circ \Gamma$ of γ_u by geodesics.
(a) Prove that if $\tilde{v}$ is orthogonal to $i\tilde{u}$, the corresponding Jacobi field is $V(t) = (\sin t)E(t)$, where E is the parallel vector field along γ_u with $E(0) = v$.
(b) If $\tilde{v} = i\tilde{u}$, prove that the corresponding Jacobi field is given by the formula $V(t) = (\sin t \cdot \cos t) \cdot i\dot{\gamma}_u(t)$.

8. Let M be a Riemannian manifold with curvature tensor R. Let $p \in M$, $v \in T_pM$ with $\|v\| = 1$ and γ be the geodesic with initial conditions $\gamma(0) = p$, $\dot{\gamma}(0) = v$. Let u, $w \in T_pM$ and Y, Z be the Jacobi fields along γ with $Y(0) = Z(0) = 0$ and

$$\frac{DY}{dt}(0) = u, \qquad \frac{DZ}{dt}(0) = w.$$

Prove that for t sufficiently close to 0, we have

$$\langle Y(t), Z(t)\rangle = t^2\langle u, w\rangle - \frac{1}{3}\langle R(u,v)v, w\rangle t^4 + o(t^5)$$

where $\lim\limits_{t\to 0}\dfrac{o(t^5)}{t^4} = 0$.
(Hint : Apply Taylor's formula to the function $f(t) = \langle Y(t), Z(t)\rangle$. Then $f(0) = f'(0) = 0$, $f''(0) = 2\langle u, w\rangle$, $f^{(3)}(0) = 0$ and $f^{(4)}(0) = 8\langle R(u,v)v, w\rangle$. For the latter we need to show that

$$\frac{DR(Y,\dot{\gamma})\dot{\gamma}}{dt}(0) = R(\frac{DY}{dt}(0), \dot{\gamma}(0))\dot{\gamma}(0) = R(u,v)v.)$$

9. Let M be a Riemannian n-manifold, $n \geq 2$, and K be its sectional curvature. Let $p \in M$, u, $v \in T_pM$ with $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 0$. if γ is the geodesic

with initial conditions $\gamma(0) = p$, $\dot{\gamma}(0) = v$, prove that there exists $\epsilon > 0$ such that for $|t| < \epsilon$ the Jacobi field Y along γ with initial conditions

$$Y(0) = 0, \qquad \frac{DY}{dt}(0) = u$$

satisfies
(a)$\|Y(t)\|^2 = t^2 - \frac{1}{3}K_p(S)t^4 + o(t^5)$, and
(b)$\|Y(t)\| = t - \frac{1}{6}K_p(S)t^3 + o(t^4)$,
where S is the 2-dimensional vector subspace of T_pM generated by $\{v, u\}$.

10. Let M be a complete Riemannian n-manifold, $n \geq 2$, with vanishing curvature tensor. Prove that for every $p \in M$ there exists $\epsilon > 0$ such that $\exp_p : B_p(0, \epsilon) \to B(p, \epsilon)$ is an isometry.

11. Let M be a connected, complete Riemannian n-manifold, $n \geq 2$, of constant sectional curvature $c \in \mathbb{R}$ and let $\gamma : \mathbb{R} \to M$ be a geodesic parametrized by arclength. Find explicit formulas for the normal Jacobi fields along γ in terms of c. Deduce that if $c \leq 0$ there are no conjugate points, while if $c > 0$, the first conjugate point to $\gamma(0)$ along γ in positive time is $\gamma(\frac{\pi}{\sqrt{c}})$.

12. Prove that the sectional curvature of the paraboloid

$$M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$$

is given by the formula

$$K(x, y, z) = \frac{4}{(1 + 4x^2 + 4y^2)^2}$$

and therefore $\inf\{K(x, y, z) : (x, y, z) \in M\} = 0$. Prove also that $(0, 0, 0)$ is a pole.

13. Prove that for every point $p \in \mathbb{C}P^n$, $n \geq 1$, the cut point along any unit speed geodesic emanating from p, with respect to the Fubini-Study metric, occurs at distance $\frac{\pi}{2}$ from p.

14. Let X, $Y \in \mathcal{X}(M)$ be two Killing vector fields of the Riemannian manifold M. Prove that their Lie bracket $[X, Y]$ is also a Killing vector field.
(Hint: It is sufficient to prove that $\langle \nabla_w X, Y, w \rangle = 0$ for every $w \in T_pM$ and $p \in M$. Consider the geodesic $\gamma(t) = \exp_p(tw)$ and prove that $\left.\frac{d}{dt}\right|_{t=0} \langle [X, Y]_{\gamma(t)}, \dot{\gamma}(t) \rangle = 0$.)

Chapter 6

Riemannian volume

6.1 Geodesic spherical coordinates

Let M be a n-dimensional Riemannian manifold and let $\exp : E \to M$ be its exponential map. Let $p \in M$ and $S_p = \{v \in T_pM : \|v\| = 1\}$. To each smooth map $\xi : U \to S_p$ defined on some open set $U \subset \mathbb{R}^{n-1}$ corresponds the smooth map

$$\psi(t, u) = \exp(t\xi(u))$$

which is defined on $\{(t, u) \in (0, +\infty) \times U : t\xi(u) \in E\}$. In case M is complete, ψ is defined on $(0, +\infty) \times U$. Obviously,

$$\frac{\partial \psi}{\partial t}(s, u) = \psi_{*(s,u)}\big(\frac{\partial}{\partial t}\big)_{(s,u)} = \dot{\gamma}_{\xi(u)}(s),$$

where $\gamma_{\xi(u)}$ is the unique geodesic with $\gamma_{\xi(u)}(0) = p$ and $\dot{\gamma}_{\xi(u)}(0) = \xi(u)$. Also, for every $u \in U$ and $1 \leq j \leq n-1$ the smooth vector field

$$Y_j(., u) = \frac{\partial \psi}{\partial u^j}(., u)$$

along $\gamma_{\xi(u)}$ is the Jabobi field with initial conditions $Y_j(0, u) = 0$ and

$$\frac{DY_j}{dt}(0, u) = \frac{D}{dt}\frac{\partial \psi}{\partial u^j}(0, u) = \frac{D}{\partial u^j}\frac{\partial \psi}{\partial t}(0, u) = \frac{D\dot{\gamma}_{\xi(u)}}{\partial u^j}(0) = \frac{\partial \xi}{\partial u^j}(u).$$

Finally, $\left\|\frac{\partial \psi}{\partial t}\right\| = 1$ and $\langle \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial u^j} \rangle = 0$, by the Gauss' Lemma 3.5.2. Thus, in order to have a description of the Riemennian metric along the geodesic $\gamma_{\xi(u)}$ we need to compute $\langle Y_i(t, u), Y_j(t, u)\rangle$, $1 \leq i, j \leq n-1$.

For every $\xi \in S_p$, $v \in T_pM$ and $t > 0$ we put

$$R(t)v = \tau_t^{-1} R(\tau_t(v), \dot{\gamma}_\xi(t))\dot{\gamma}_\xi(t)$$

where τ_t denotes the parallel translation along the geodesic γ_ξ from $p = \gamma_\xi(0)$ to $\gamma_\xi(t)$. By Proposition 4.3.1 $R(t)$ is self-adjoint and since $R(t)\xi = 0$, we have $R(t) \in \mathrm{Hom}(\xi^\perp, \xi^\perp)$. Let $A(t, \xi)$ denote the unique solution of the linear ordinary differential equation

$$A'' + R(t)A = 0$$

defined on $\mathrm{Hom}(\xi^\perp, \xi^\perp)$ satisfying the initial conditions $A(0,\xi) = 0$ and $A'(0,\xi) = id_{\xi^\perp}$.

Proposition 6.1.1. *The operators $A(t,\xi)^* A'(t,\xi)$ and $A'(t,\xi)A(t,\xi)^{-1}$ are self-adjoint.*

Proof. Indeed, if we put $W(t) = A'(t,\xi)^* A(t,\xi) - A(t,\xi)^* A'(t,\xi)$, then

$$W'(t) = A''(t,\xi)^* A(t,\xi) + A'(t,\xi)^* A'(t,\xi) - A'(t,\xi)^* A'(t,\xi) - A(t,\xi)^* A''(t,\xi)$$
$$= (-R(t)A(t,\xi))^* A(t,\xi) - A(t,\xi)^*(-R(t)A(t,\xi)) = 0,$$

because $R(t)$ is self-adjoint. Since $W(0) = 0$, we must have $W = 0$.

Also, if $U(t) = A'(t,\xi)A(t,\xi)^{-1}$, we have

$$U(t)^* - U(t) = (A(t,\xi)^{-1})^*[A'(t,\xi)^* A(t,\xi) - A(t,\xi)^* A'(t,\xi)]A(t,\xi)^{-1}$$
$$= (A(t,\xi)^{-1})^* W(t) A(t,\xi)^{-1} = 0. \quad \square$$

For every $v \in \xi^\perp$ the smooth vector field $Y(t) = \tau_t A(t,\xi)v$ along γ_ξ is the Jacobi field with initial conditions $Y(0) = 0$ and $\dfrac{DY}{dt}(0) = v$.

The above now become

$$\frac{\partial \psi}{\partial u^j}(t,u) = Y_j(t,u) = \tau_t A(t,\xi(u))\frac{\partial \xi}{\partial u^j},$$

since $\dfrac{\partial \xi}{\partial u^j} \in \xi(u)^\perp$, because $\|\xi(u)\| = 1$ for every $u \in U$. It follows that

$$\langle Y_i(t,u), Y_j(t,u)\rangle = \langle A(t,\xi(u))\frac{\partial \xi}{\partial u^i}, A(t,\xi(u))\frac{\partial \xi}{\partial u^j}\rangle.$$

Using the traditional notation, the Riemannian metric on the image of ψ can now be written

$$ds^2 = dt^2 + (A(t,\xi))^2 \|d\xi\|^2.$$

In the special case of a space form Jacobi's differential equation along a unit speed geodesic is particularly simple. Suppose that M has constant sectional curvature $K \in \mathbb{R}$ and let γ be a unit speed geodesic. Let Y be a normal Jacobi field along γ. If E_1,..., E_{n-1}, $E_n = \dot\gamma$ is a parallel orthonormal frame along γ and

$$Y = \sum_{j=1}^{n-1} f_j E_j$$

then

$$f'' + K f_j = 0, \quad j = 1, ..., n-1.$$

It follows that there are parallel vector fields A, $B \in \mathcal{X}(\gamma)$, which are combinations of the elements of the frame, such that Y is given by the formula

$$Y(t) = C_K(t)A(t) + S_K(t)B(t),$$

where

$$C_K(t) = \begin{cases} \cos(\sqrt{K}t), & \text{if } K > 0, \\ 1, & \text{if } K = 0, \\ \cosh(\sqrt{-K}t), & \text{if } K < 0. \end{cases}$$

and

$$S_K(t) = \begin{cases} \frac{1}{\sqrt{K}} \sinh(\sqrt{K}t), & \text{if } K > 0, \\ t, & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}t), & \text{if } K < 0. \end{cases}$$

If $Y(0) = 0$, then $Y(t) = S_K(t)B(t)$, which shows that

$$Y_j(t,u) = S_K(t)\tau_t(\frac{\partial \xi}{\partial u^j}), \quad j = 1, ..., n-1$$

with the previous notations. Moreover, $A(t,\xi) = S_K(t)I$. Thus, we recover the formulas for the Riemannian metric in local geodesic spherical coordinates of the end of section 5.6.

As a final remark we note that the conjugate locus of a point $p \in M$ is the image under the exponential map $\exp_p$ of the set of all points $t\xi \in T_pM$ for $t > 0$, $\xi \in S_p$, such that $A(t,\xi)$ is not an isomorphism.

6.2 Riemannian measure

On a n-dimensional Riemannian manifold M there is a globally defined natural measure. Let $\phi : U \to \mathbb{R}^n$ and $\psi : W \to \mathbb{R}^n$ be two smooth charts of M with $\phi = (x^1, ..., x^n)$ and $\psi = (y^1, ..., y^n)$. As usual, we denote $g^\phi = (g^\phi_{ij})_{1 \le i,j \le n}$ and $g^\psi = (g^\psi_{ij})_{1 \le i,j \le n}$ where

$$g^\phi_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \quad \text{and} \quad g^\psi_{ij} = \langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle.$$

If $U \cap W \neq \varnothing$ and $S = D(\phi \circ \psi^{-1}) = (S_{ij})_{1 \le i,j \le n}$, then

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^n S_{ij} \frac{\partial}{\partial y^i}$$

and $g^\phi = S^T g^\psi S$. Therefore, $\sqrt{\det g^\phi} = \sqrt{\det g^\psi} \cdot |\det S|$. From the change of variables formula follows now that for every continuous function $f : \phi(U) \to \mathbb{R}$ the Riemann integral

$$\int_{\phi(U)} (f \cdot \sqrt{\det g^\phi}) \circ \phi^{-1}$$

depends only on f and U and not on the choice of the chart $\phi : U \to \mathbb{R}^n$. Thus, if we choose a smooth atlas $\mathcal{A}$ of M and a subordinated smooth partition of unity $\{f_{(U,\phi)} : (U,\phi) \in \mathcal{A}\}$, then for every continuous function with compact support $f : M \to \mathbb{R}$ the quantity

$$\int_M f d\mathrm{Vol} = \sum_{(U,\phi) \in \mathcal{A}} \int_{\phi(U)} (f_{(U,\phi)} f \cdot \sqrt{\det g^\phi}) \circ \phi^{-1}$$

depends only on f and not on the choice of the smooth atlas $\mathcal{A}$ and the subordinated smooth partition of unity. By the Riesz Representation Theorem, there is a well defined σ-finite Borel measure dVol on M, which is called the *Riemannian measure* of M. A function $f : M \to \mathbb{R}$ is measurable with respect to dVol if and only if $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is Lebesgue measurable for every smooth chart $\phi : U \to \mathbb{R}^n$.

It is obvious that the above definition of the Riemannian measure is not suitable for efficient calculations. A simple idea to overcome this difficulty is to remove from M a set of measure zero such that on the rest of M there are some kind of coordinates.

Let M be a complete, connected, n-dimensional Riemannian manifold and $p \in M$. If $C(p)$ denotes the cut locus of p, as it was defined in section 5.5, then $C(p)$ has measure zero. Indeed, it follows from Theorem 5.5.3 that the set

$$C_p = \{c(v)v : v \in T_pM, \|v\| = 1, c(v) < +\infty\}$$

has Lebesgue measure zero in T_pM. Since $\exp_p : T_pM \to M$ is smooth, $C(p) = \exp_p(C_p)$ must have measure zero in M. On $D(p) = M \setminus C(p)$ there are geodesic spherical coordinates since $\exp_p$ maps $D_p = \{tv : 0 \leq t < c(v), v \in T_pM, \|v\| = 1\}$ diffeomorphically onto $D(p)$. The Riemannian measure on $D(p) \setminus \{p\}$ has the form $g(t,\xi)dtd\xi$, where $d\xi$ is the spherical Lebesgue measure on $S_p = \{\xi \in T_pM : \|\xi\| = 1\}$ induced by the Lebesgue measure on T_pM. So for every integrable function $f : M \to \mathbb{R}$ we have

$$\int_M f d\mathrm{Vol} = \int_{S_p} \left(\int_0^{c(\xi)} f(\exp_p(t\xi)) g(t,\xi) dt \right) d\xi.$$

We shall show that $g(t,\xi) = \det A(t,\xi)$ for every $\xi \in S_p$ and $0 < t < c(\xi)$, that is $t\xi \in D_p \setminus \{0\}$. Let $u = (u^1, ..., u^n)$ be a smooth chart of S_p. We consider the smooth chart $\phi = (x^1, ..., x^n)$ on $D_p \setminus \{p\}$ with

$$\phi = \left(u \circ \left(\frac{(\exp_p|_{D_p})^{-1}}{\|(\exp_p|_{D_p})^{-1}\|} \right), \|(\exp_p|_{D_p})^{-1}\| \right).$$

Note that $\dfrac{\partial}{\partial x^n}$ is $\dfrac{\partial \psi}{\partial t}$ and $\dfrac{\partial}{\partial x^i}$ is $\dfrac{\partial \psi}{\partial u^i}$, $i = 1, ..., n-1$, in the notation of the preceding section 6.1. If as usual $g_{ij} = \langle \dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j} \rangle$, $1 \leq i,j \leq n$, from the definition of the Riemannian measure we have $g = \sqrt{\det(g_{ij})_{1 \leq i,j \leq n}}$. Putting $h_{ij} = \langle \dfrac{\partial}{\partial u^i}, \dfrac{\partial}{\partial u^j} \rangle$, $1 \leq i,j \leq n-1$, we have $g_{nn} = 1$, $g_{nj} = 0$, $j = 1, ..., n-1$, from the Gauss' Lemma 3.5.2, and

$$g_{ij} = \langle A(t,\xi)\frac{\partial}{\partial u^i}, A(t,\xi)\frac{\partial}{\partial u^j} \rangle, \quad 1 \leq i,j \leq n-1,$$

where $\xi = u^{-1}$. Therefore, $\det(g_{ij})_{1 \leq i,j \leq n} = \det(h_{ij})_{1 \leq i,j \leq n-1} \cdot (\det A(t,\xi))^2$. This proves our assertion, because $d\xi = \sqrt{\det(h_{ij})_{1 \leq i,j \leq n-1}} du^1 \cdots du^{n-1}$ is the local expression of the spherical Riemannian measure on S_p.

Example 6.2.1. In case M has constant sectional curvature $K \in \mathbb{R}$, we have $A(t,\xi) = S_K(t)I$ and therefore $\sqrt{\det(g_{ij})_{1 \leq i,j \leq n}} = (S_K(t))^{n-1}$. In the particular case of the euclidean space $\mathbb{R}^n$ we have $K = 0$, $S_0(t) = t$ and thus the Riemannian measure is $t^{n-1}dtd\xi$ in spherical coordinates, where $d\xi$ is the Riemannian measure of the unit sphere S^{n-1}. The volume

$$c_{n-1} = \int_{S^{n-1}} d\xi$$

of S^{n-1} can be computed as follows. We observe that

$$\left(\int_{-\infty}^{+\infty} e^{-t^2}dt\right)^n = \int_{\mathbb{R}^n} e^{-\|x\|^2}d\mathrm{Vol} = \int_{S^{n-1}} \left(\int_0^{+\infty} t^{n-1}e^{-t^2}dt\right)d\xi$$
$$= c_{n-1}\int_0^{+\infty} t^{n-1}e^{-t^2}dt = c_{n-1}\int_0^{+\infty} \frac{1}{2}e^{-s}s^{\frac{n}{2}-1}ds = c_{n-1}\frac{\Gamma(\frac{n}{2})}{2}.$$

Since $\int_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{\pi}$, we conclude that

$$c_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

The volume of the unit n-ball $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is

$$\mathrm{Vol}(D^n) = \int_{D^n} t^{n-1}dtd\xi = c_{n-1}\int_0^1 t^{n-1}dt = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Let M be a complete, n-dimensional Riemannian manifold and $p \in M$. For $r > 0$ we put $D_p(r) = \{\xi \in S_p : r\xi \in D_p\} = \{\xi \in S_p : 0 < r < c(\xi)\}$ and

$$E(p,r) = \int_{D_p(r)} \det A(r,\xi)d\xi.$$

If $0 < r < \mathrm{inj}p$, then $E(p,r)$ is the $(n-1)$-dimensional volume of $\partial B(p,r)$, which is a smooth $(n-1)$-dimensional submanifold of M. In any case, the volume of the ball $B(p,r) = \exp(B_p(0,r))$ is

$$V(p,r) = \int_{B(p,r)} d\mathrm{Vol} = \int_{D_p \cap B(p,r)} \det A(t,\xi)dtd\xi$$
$$= \int_0^r \left(\int_{D_p(r)} \det A(t,\xi)d\xi\right)dt = \int_0^r E(p,t)dt.$$

Thus, the function $E(p,.)$ is integrable, while $V(p,.)$ is obviously continuous. Since

$$D_p(r+\epsilon) = \frac{1}{r+\epsilon}(D_p \cap B_p(0,r+\epsilon)) \subset \frac{1}{r}(D_p \cap B_p(0,r)) = D_p(r)$$

for every $\epsilon > 0$, we have

$$\frac{V(p,r+\epsilon) - V(p,r)}{\epsilon} = \frac{1}{\epsilon}\int_r^{r+\epsilon} E(p,t)dt = \frac{1}{\epsilon}\int_r^{r+\epsilon} \left(\int_{D_p(t)} \det A(t,\xi)d\xi\right)dt$$

$$\leq \frac{1}{\epsilon}\int_r^{r+\epsilon}\left(\int_{D_p(r)} \det A(t,\xi)d\xi\right)dt = \int_{D_p(r)}\left(\frac{1}{\epsilon}\int_r^{r+\epsilon} \det A(t,\xi)dt\right)d\xi.$$

Therefore,

$$\limsup_{\epsilon\to 0}\frac{V(p,r+\epsilon)-V(p,r)}{\epsilon} \leq \int_{D_p(r)}\left(\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_r^{r+\epsilon} \det A(t,\xi)dt\right)d\xi$$

$$= \int_{D_p(r)} \det A(r,\xi)d\xi = E(p,r),$$

by Lebesgue dominated convergence.

If M is the simply connected space form of sectional curvature $K \in \mathbb{R}$, the volume of a ball of radius $r > 0$ is

$$V_K(r) = c_{n-1}\int_0^r (S_K(t))^{n-1}dt$$

and of a sphere of radius r is $E_K(r) = c_{n-1}(S_K(r))^{n-1}$.

6.3 Volume comparison theorems

In this section we shall present results comparing the Riemannian volumes of balls in a complete Riemannian manifold with the volumes of balls in space forms of the same dimension under conditions concerning the sectional or the Ricci curvature. We shall need the following comparison theorem which is due to H.E. Rauch.

Theorem 6.3.1. *Let M be a complete, n-dimensional Riemannian manifold, $K \in \mathbb{R}$ and let $\gamma : [0,a] \to M$ be a geodesic path parametrized by arclength such that $K_{\gamma(t)}(S) \leq K$ for every 2-dimensional vector subspace S of $T_{\gamma(t)}M$ and for every $0 \leq t \leq a$. Let Y be a non-zero normal Jacobi field along γ.*
(i) Then for every $0 \leq t \leq a$ we have

$$\frac{d^2}{dt^2}\|Y(t)\| + K\|Y(t)\| \geq 0.$$

(ii) If $\psi : [0,a] \to \mathbb{R}$ is the solution of the linear differential equation $x'' + Kx = 0$ on $[0,a]$ satisfying the initial conditions $\psi(0) = \|Y(0)\|$, $\psi'(0) = \left.\frac{d}{dt}\right|_{t=0}(\|Y\|)$ and ψ does not vanish at any point of the open interval $(0,a)$, then on this interval

$$\frac{d}{dt}\left(\frac{\|Y\|}{\psi}\right) \geq 0, \quad \|Y\| \geq \psi.$$

(iii) Moreover, if $0 < t_0 < a$, then $\left.\frac{d}{dt}\right|_{t=t_0}\left(\frac{\|Y\|}{\psi}\right) = 0$ if and only if $K_{\gamma(t)}(S(t)) = K$ for every $0 \leq t \leq t_0$, where $S(t)$ is the 2-dimensional vector subspace of $T_{\gamma(t)}M$ generated by $\{\dot{\gamma}(t), Y(t)\}$, and there exists a parallel vector field E along $\gamma|_{(0,t_0)}$ such that $\|E\| = 1$ and $Y(t) = \psi(t)E(t)$ for every $0 < t < t_0$.

Proof. In the beginning we estimate

$$\frac{d^2}{dt^2}(\|Y\|) = \frac{d}{dt}\left(\frac{\langle Y, \frac{DY}{dt}\rangle}{\|Y\|}\right) = \frac{1}{\|Y\|}\cdot\frac{d}{dt}\langle Y, \frac{DY}{dt}\rangle + \langle Y, \frac{DY}{dt}\rangle\cdot\frac{d}{dt}\left(\frac{1}{\|Y\|}\right)$$

$$= \frac{1}{\|Y\|}\left(\left\|\frac{DY}{dt}\right\|^2 + \langle Y, \frac{D^2Y}{dt^2}\rangle\right) - \frac{\langle Y, \frac{DY}{dt}\rangle^2}{\|Y\|^3} = \frac{1}{\|Y\|}\left(\left\|\frac{DY}{dt}\right\|^2 - \langle Y, R(Y,\dot\gamma)\dot\gamma\rangle\right) - \frac{\langle Y, \frac{DY}{dt}\rangle^2}{\|Y\|^3}$$

$$\geq \frac{1}{\|Y\|}\left(\left\|\frac{DY}{dt}\right\|^2 - K\|Y\|^2\right) - \frac{\langle Y, \frac{DY}{dt}\rangle^2}{\|Y\|^3} = -K\|Y\| + \frac{1}{\|Y\|^3}\left(\left\|\frac{DY}{dt}\right\|^2\|Y\|^2 - \langle Y, \frac{DY}{dt}\rangle^2\right) \geq -K\|Y\|,$$

using our assumption and the Cauchy-Schwartz inequality.

Since

$$\frac{d}{dt}\left(\frac{\|Y\|}{\psi}\right) = \frac{1}{\psi^2}\left(\frac{d}{dt}(\|Y\|)\cdot\psi - \|Y\|\frac{d\psi}{dt}\right),$$

we consider the function

$$f = \frac{d}{dt}(\|Y\|)\cdot\psi - \|Y\|\frac{d\psi}{dt}$$

for which we have $f(0) = 0$ by our choice of ψ and

$$f' = \psi\frac{d^2}{dt^2}(\|Y\|) - \|Y\|\psi'' \geq \psi(-K\|Y\|) - (-K\psi)\|Y\| = 0.$$

Therefore, $f(t) \geq f(0) = 0$ for $0 < t < a$ and

$$\frac{d}{dt}\left(\frac{\|Y\|}{\psi}\right) \geq 0$$

on the interval $(0, a)$. It follows that

$$\frac{\|Y(t)\|}{\psi(t)} \geq \frac{\|Y(0)\|}{\psi(0)} = 1$$

for every $0 < t < a$. This proves assertion (ii). In order to prove (iii), we observe that $f(t_0) = 0$, and thus $f(t) = 0$ for all $0 \leq t \leq t_0$, by monotonicity. It follows that $\|Y\| = \psi$ and thus

$$\frac{d^2}{dt^2}(\|Y\|) + K\|Y\| = 0$$

on the interval $[0, t_0]$. This means that the inequalities appearing in our initial estimate are equalities. In particular,

$$\left\|\frac{DY}{dt}\right\|^2\|Y\|^2 - \langle Y, \frac{DY}{dt}\rangle^2 = 0$$

and hence $\frac{DY}{dt}$, Y must be linearly dependent on $[0, t_0]$. If now $Y = \psi E$, where $\|E\| = 1$, then

$$\frac{DY}{dt} = \psi' E + \psi\frac{DE}{dt},$$

and E is perpendicular to $\dfrac{DE}{dt}$. Thus, necessarily $\dfrac{DE}{dt} = 0$ on $(0, t_0)$. $\square$

Let M be a connected, complete, n-dimensional Riemennian manifold, $K \in \mathbb{R}$ and let $a = \dfrac{\pi}{\sqrt{K}}$, where $a = +\infty$ in case $K \leq 0$. Let $\gamma : [0, a] \to M$ be a geodesic path parametrized by arclength such that $K_{\gamma(t)}(S) \leq K$ for every 2-dimensional vector subspace of $T_{\gamma(t)}M$ and every $0 \leq t \leq a$. Let Y be a normal Jacobi field along γ. Applying Rauch's Theorem 6.3.1 for $\left\| \dfrac{DY}{dt}(0) \right\|^{-1} Y$, then $\psi(t) = S_K(t)$ and $\|Y\| \geq \left\| \dfrac{DY}{dt}(0) \right\| S_K$. Moreover,

$$\frac{\frac{d}{dt}(\|Y\|)}{\|Y\|} \geq \frac{\frac{dS_K}{dt}}{S_K}$$

and the equality holds at some point $0 < t_0 < a$ if and only if there exists a non-zero parallel vector field along γ such that $Y = S_K \cdot E$ on $(0, t_0]$ and

$$K = \langle R(t)E(t), E(t) \rangle, \quad 0 < t \leq t_0,$$

where $R(t)$ was defined in section 6.1. Thus, if $\xi = \dot{\gamma}(0)$, that is $\gamma = \gamma_\xi$, then

$$\tau_t A(t, \xi) \left(\frac{DY}{dt}(0) \right) = Y(t) = S_K(t) \cdot E(t), \quad 0 < t \leq t_0.$$

Since $\dfrac{DY}{dt}(0) = \dfrac{dS_K}{dt}(0) \cdot E(0) = E(0)$, it follows that

$$A(t, \xi) \left(\frac{DY}{dt}(0) \right) = S_K(t) \cdot \tau_t^{-1} E(t) = S_K(t) \cdot \frac{DY}{dt}(0).$$

This shows that $A(t, \xi) = S_K(t) I$ for $0 < t \leq t_0$ in case we have equality in the above inequality at t_0, because Y was any non-zero normal Jacobi vector field along γ. Since $A''(t, \xi) + R(t)A(t, \xi) = 0$, we also have $R(t) = K \cdot I$, $0 < t \leq t_0$.

The following theorem is due to P. Günter and R.L. Bishop. We shall need two elementary facts.

Lemma 6.3.2. *Let $a > 0$ and f, $g : [0, a] \to [0, +\infty)$ be two C^1 functions such that $f(0) = g(0) = 0$, $f(t) > 0$, $g(t) > 0$ for $0 < t \leq a$ and $f'(0) = g'(0) \neq 0$. If $\dfrac{f'}{f} \geq \dfrac{g'}{g}$ on $(0, a]$, then $f \geq g$ on $[0, a]$.*

Proof. Our assumption implies that

$$\frac{f'g - fg'}{g^2} = \frac{f}{g} \left(\frac{f'}{f} - \frac{g'}{g} \right) \geq 0.$$

Therefore,

$$\frac{f(t)}{g(t)} \geq \lim_{s \to 0^+} \frac{f(s)}{g(s)} = \frac{f'(0)}{g'(0)} = 1$$

for every $0 < t \leq a$. $\square$.

If $G : (-\epsilon, \epsilon) \to \mathbb{R}^{n\times n}$ is a smooth map for some $\epsilon > 0$ with $G(0) = I_n$, then from Taylor's formula we have

$$G(t) = I_n + tG'(0) + O(t^2)$$

and therefore

$$\det G(t) = 1 + t\mathrm{Tr}G'(0) + O(t^2).$$

This implies that $(\det G)'(0) = \mathrm{Tr}G'(0)$. Applying this to $G(t) = B(t)B(0)^{-1}$ we obtain

$$\frac{(\det B)'(0)}{\det B(0)} = \mathrm{Tr}(B'(0)B(0)^{-1})$$

for any smooth $B : (-\epsilon, \epsilon) \to \mathbb{R}^{n\times n}$.

Theorem 6.3.3. *Let M be a connected, complete, Riemannian n-manifold, $p \in M$ and $\xi \in S_p$. We assume that there exists $K \in \mathbb{R}$ such that $K_{\gamma_\xi(t)}(S) \leq K$ for every 2-dimensional vector subspace S of $T_{\gamma_\xi(t)}M$ and every $t \in \mathbb{R}$. Then*

$$\frac{\frac{d}{dt}(\det A(t,\xi))}{\det A(t,\xi)} \geq (n-1)\frac{S_K'(t)}{S_K(t)}, \quad \det A(t,\xi) \geq (S_K(t))^{n-1}$$

for every $0 < t \leq \dfrac{\pi}{\sqrt{K}}$. The equality holds at some $0 < t_0 < \dfrac{\pi}{\sqrt{K}}$ if and only if $A(t,\xi) = S_K(t)\cdot I$ and $R(t) = K\cdot I$ for $0 < t \leq t_0$.

Proof. Putting $B(t) = A(t,\xi)^*A(t,\xi)$ we have

$$\frac{\frac{d}{dt}(\det A(t,\xi))}{\det A(t,\xi)} = \frac{1}{2}\cdot\frac{(\det B)'(t)}{\det B(t)}.$$

Let $0 < s < \dfrac{\pi}{\sqrt{K}}$. Since $B(s)$ is self-adjoint, there exists an orthonormal basis $\{v_1, ..., v_{n-1}\}$ of $\xi^\perp$ consisting of eigenvectors of $B(s)$. Let η_1,..., η_{n-1} be the solutions of Jacobi's differential equation

$$\eta'' + R(t)\eta = 0$$

on $\xi^\perp$ satisfying the initial conditions $\eta_j(0) = 0$, $\eta_j'(0) = v_j$, $1 \leq j \leq n-1$. As we have seen in section 6.1, $\eta_j(t) = A(t,\xi)v_j$, $1 \leq j \leq n-1$. By Rauch's Theorem 6.3.1 and the subsequent comments, and Proposition 6.1.1,

$$\frac{1}{2}\cdot\frac{(\det B)'(s)}{\det B(s)} = \frac{1}{2}\mathrm{Tr}(B'(s)B(s)^{-1}) = \mathrm{Tr}(A(s,\xi)^*A'(s,\xi)B^{-1}(s))$$

$$= \sum_{j=1}^{n-1}\langle A(s,\xi)^*A'(s,\xi)B^{-1}(s)v_j, v_j\rangle = \sum_{j=1}^{n-1}\langle A(s,\xi)^*A'(s,\xi)\Big(\frac{1}{\langle Bv_j, v_j\rangle}v_j\Big), v_j\rangle$$

$$= \sum_{j=1}^{n-1} \frac{\langle A(s,\xi)^* A'(s,\xi)v_j, v_j\rangle}{\langle A(s,\xi)^* A(s,\xi)v_j, v_j\rangle} = \sum_{j=1}^{n-1} \frac{\langle A'(s,\xi)v_j, A(s,\xi)v_j\rangle}{\langle A(s,\xi)v_j, A(s,\xi)v_j\rangle}$$

$$= \sum_{j=1}^{n-1} \frac{\langle \eta_j'(s), \eta_j(s)\rangle}{\langle \eta_j(s), \eta_j(s)\rangle} = \sum_{j=1}^{n-1} \frac{\frac{d}{dt}\big|_{t=s}\|\eta_j\|}{\|\eta_j(s)\|} \geq (n-1)\frac{S_K'(s)}{S_K(s)}.$$

The second inequality follows from the preceding elementary Lemma 6.3.2. Finally, suppose that there exists some $0 < t_0 \leq \dfrac{\pi}{\sqrt{K}}$ such that

$$\frac{\frac{d}{dt}\big|_{t=t_0}(\det A(t,\xi))}{\det A(t_0,\xi)} = (n-1)\frac{S_K'(t_0)}{S_K(t_0)}.$$

Since

$$\frac{\frac{d}{dt}\big|_{t=t_0}\|\eta_j\|}{\|\eta_j(s)\|} \geq \frac{S_K'(s)}{S_K(s)}$$

for all $1 \leq j \leq n-1$, necessarily we have equality. From the above $A(t,\xi) = S_K(t)I$ and $R(t) = K \cdot I$ for every $0 < t \leq t_0$. □

Corollary 6.3.4. *Let M be a complete, n-dimensional Riemannian manifold such that there exists $K \in \mathbb{R}$ with $K_p(S) \leq K$ for every $p \in M$ and every 2-dimensional vector subspace S of T_pM. Then, $V(p,r) \leq V_K(r)$ for every $p \in M$ and every $0 < r \leq \min\{\mathrm{inj}p, \dfrac{\pi}{\sqrt{K}}\}$. The equality holds for $r = r_0$ if and only if the ball $B(p,r_0)$ in M is isometric to a ball of radius r_0 in the simply connected space form M_K of sectional curvature K.* □

There is an analogous to the preceding Theorem 6.3.3 due to R.L. Bishop with the assumption that the Ricci curvature has a lower bound.

Theorem 6.3.5. *Let M be a complete, n-dimensional Riemannian manifold, $p \in M$, $\xi \in S_p$ and let γ_ξ be the geodesic with $\gamma_\xi(0) = p$ and $\dot\gamma_\xi(0) = \xi$. We denote*

$$\mathrm{conj}\xi = \inf\{t > 0 : \gamma_\xi(t) \text{ is conjugate to } p \text{ along } \gamma_\xi\}$$

$$= \min\{t > 0 : \det A(t,\xi) = 0\} > 0.$$

If there exists $K \in \mathbb{R}$ such that $Ric_{\gamma_\xi(t)}(\dot\gamma_\xi(t), \dot\gamma_\xi(t)) \geq (n-1)K$ for every $0 < t \leq \mathrm{conj}\xi$, then

$$\frac{\frac{d}{dt}(\det A(t,\xi))}{\det A(t,\xi)} \leq (n-1)\frac{S_K'(t)}{S_K(t)}, \quad \det A(t,\xi) \leq (S_K(t))^{n-1}$$

for every $0 < t \leq \mathrm{conj}\xi$. The equality holds at some $0 < t_0 < \mathrm{conj}\xi$ if and only if $A(t,\xi) = S_K(t) \cdot I$ and $R(t) = K \cdot I$ for $0 < t \leq t_0$.

Proof. Recall that $Ric_{\gamma_\xi(t)}(\dot\gamma_\xi(t), \dot\gamma_\xi(t)) = \mathrm{Tr}R(t)$ and

$$\frac{\frac{d}{dt}(\det A(t,\xi))}{\det A(t,\xi)} = \mathrm{Tr}\frac{d}{dt}A(t,\xi)A(t,\xi)^{-1}.$$

For the sake of brevity we shall use the notation $\mathrm{Ct}_K(t) = \dfrac{S_K'(t)}{S_K(t)}$ and denote by arcCt_K its inverse. If $\psi = (n-1)\mathrm{Ct}_K$, then ψ satisfies Ricatti's differential equation

$$\psi' + \frac{1}{n-1}\psi^2 + (n-1)K = 0.$$

Also ψ is strictly decreasing and in case $K < 0$ we have $\lim_{t\to+\infty} \psi(t) = (n-1)\sqrt{-K}$. By Proposition 6.1.1, the operator $U(t) = A'(t,\xi)A(t,\xi)^{-1}$ is self-adjoint. Also it satisfies Ricatti's differential equation for operators

$$U' + U^2 + R = 0.$$

This is easily verified by differentiating the equation $A(t,\xi)A(t,\xi)^{-1} = I$, which gives $(A(t,\xi)^{-1})' + A(t,\xi)^{-1}A'(t,\xi)A(t,\xi)^{-1} = 0$, and substituting

$$\begin{aligned} &U'(t) + U(t)^2 + R(t) \\ &= A''(t,\xi)A(t,\xi)^{-1} + A'(t,\xi)(A(t,\xi)^{-1})' + A'(t,\xi)A(t,\xi)^{-1}A'(t,\xi)A(t,\xi)^{-1} + R(t) \\ &= -R(t)A(t,\xi)A(t,\xi)^{-1} + A'(t,\xi)\big((A(t,\xi)^{-1})' + A(t,\xi)^{-1}A'(t,\xi)A(t,\xi)^{-1}\big) + R(t) = 0. \end{aligned}$$

Thus, $\mathrm{Tr}U' + \mathrm{Tr}U^2 + \mathrm{Tr}R = 0$ and by the Cauchy-Schwartz inequality, since $U(t)$ is diagonalizable,

$$(\mathrm{Tr}U)^2 \le (n-1)\mathrm{Tr}U^2.$$

If now

$$\phi(t) = \mathrm{Tr}U(t) = \mathrm{Tr}(A'(t,\xi)A(t,\xi)^{-1}) = \frac{d}{dt}\log\det A(t,\xi),$$

then the above and our assumption imply

$$\begin{aligned} \phi' + \frac{1}{n-1}\phi^2 + (n-1)K &= \mathrm{Tr}U' + \frac{1}{n-1}(\mathrm{Tr}U)^2 + (n-1)K \\ &\le \mathrm{Tr}U' + \frac{1}{n-1}(n-1)\mathrm{Tr}U^2 + \mathrm{Tr}R = 0. \end{aligned}$$

So, we want to compare the solution ψ of the above differential equation of Ricatti with the solution ϕ of the last differential inequality.

Since

$$\psi'(t) = \begin{cases} \frac{-(n-1)K}{\sin^2(\sqrt{K}t)}, & \text{if } K > 0, \\ -1, & \text{if } K = 0, \\ \frac{-(n-1)(-K)}{\sinh^2(\sqrt{-K}t)}, & \text{if } K < 0, \end{cases}$$

we have

$$\frac{1}{n-1}(\psi(t))^2 + (n-1)K > 0$$

for $0 < t < \dfrac{\pi}{\sqrt{K}}$.

We observe that $\lim_{t\to 0^+} \phi(t) = +\infty$. Indeed, for any $0 < a < b$ we have

$$\int_a^b \phi(s) = \log\frac{\det A(b,\xi)}{\det A(a,\xi)}.$$

Since $A(0,\xi)=0$, it follows that $\int_0^b \phi(s)ds=+\infty$. From the mean value theorem of integral calculus, there exist $s_m \to 0$ such that $\lim_{m\to+\infty} \phi(s_m)=+\infty$. Let now $c>0$. There exists $m_0 \in \mathbb{N}$ such that $0<s_m<1$ and $\phi(s_m)>c+(n-1)|K|$ for every $m \geq m_0$. Since

$$\phi' + (n-1)K \leq -\frac{\phi^2}{n-1} \leq 0,$$

integrating for every $0<t<s_m$ we obtain

$$\phi(s_m)-\phi(t)+(n-1)K(s_m-t) \leq 0$$

and therefore

$$\phi(t) \geq \phi(s_m)+(n-1)K(s_m-t) > c+(n-1)[K(s_m-t)+|K|] \geq c$$

for every $0<t<s_{m_0}$.

The above imply that there exists $0<\epsilon<\mathrm{conj}\xi$ such that

$$\frac{(\phi(t))^2}{n-1}+(n-1)K>0$$

for $0<t<\epsilon$ and hence

$$\frac{-\phi'(t)}{\frac{(\phi(t))^2}{n-1}+(n-1)K} \geq 1$$

for $0<t\leq\epsilon$. Integrating,

$$t \leq \lim_{a\to 0^+} \int_a^t \frac{-\phi'(s)}{\frac{(\phi(s))^2}{n-1}+(n-1)K}ds = \lim_{a\to 0^+} \int_a^t -\frac{\frac{\phi'(s)}{n-1}}{\frac{(\phi(s))^2}{(n-1)^2}+K}ds$$

$$\lim_{a\to 0^+} \int_{\phi(a)/n-1}^{\phi(t)/n-1} \frac{-1}{s^2+K}ds = \mathrm{arcCt}_K\left(\frac{\phi(t)}{n-1}\right)$$

for every $0<t\leq\epsilon$. Consequently, $\psi(t)=(n-1)\mathrm{Ct}_K(t)\geq\phi(t)$ or equivalently

$$\frac{\frac{d}{dt}(\det A(t,\xi))}{\det A(t,\xi)} \leq (n-1)\frac{S_K'(t)}{S_K(t)}$$

and $\det A(t,\xi) \leq (S_K(t))^{n-1}$ for every $0<t\leq\epsilon$, by Lemma 6.3.2.

Suppose now that there exists some $0<t_0\leq\epsilon$ such that these inequalities are equalities for $t=t_0$. Then, $\phi(t_0)=\psi(t_0)$ and

$$\phi'(t)+\frac{(\phi(t))^2}{n-1}+(n-1)K=0,$$

hence also $(\mathrm{Tr}U(t))^2=(n-1)\mathrm{Tr}(U(t)^2)$ for all $0<t\leq t_0$. By the uniqueness of solutions of Ricatti's differential equation, $\phi(t)=\psi(t)$ and $\mathrm{Tr}R(t)=(n-1)K$ for $0<t\leq t_0$. Moreover, since the Cauchy-Schwartz inequality we have used above is an equality, $U(t)$ must be a multiple of the identity operator I for $0<t\leq t_0$.

It follows that $R(t)$ must also be a multiple of the identity operator and therefore $R(t) = K \cdot I$ for $0 < t \leq t_0$.

Returning to $U(t)$, since this is a multiple of I and has trace

$$\phi(t) = \psi(t) = (n-1)\frac{S'_K(t)}{S_K(t)},$$

necessarily

$$A'(t,\xi)A(t,\xi)^{-1} = \frac{S'_K(t)}{S_K(t)} \cdot I$$

for every $0 < t \leq t_0$. Therefore, $\frac{1}{S_K}A(.,\xi)$ is constant on the interval $(0, t_0]$. Taking the limit for $t \to 0^+$ we find that this constant must be I, because $A'(0,\xi) = I$. In other words $A(t,\xi) = S_K(t) \cdot I$ for every $0 < t \leq t_0$.

It remains to show that the above hold not only on the interval $(0, \epsilon]$ but also on $(0, \mathrm{conj}\xi]$. We proceed by contradiction assuming that

$$T = \sup\{\epsilon \in (0, \mathrm{conj}\xi] : \phi(t) \leq \psi(t) \quad \text{for} \quad 0 < t \leq \epsilon\} < \mathrm{conj}\xi.$$

By continuity, $\phi(T) = \psi(T)$ and therefore

$$\frac{(\phi(T))^2}{n-1} + (n-1)K > 0.$$

There exists $\epsilon_1 > 0$ such that

$$\frac{(\phi(t))^2}{n-1} + (n-1)K > 0$$

and thus

$$\frac{-\phi'(t)}{\frac{(\phi(t))^2}{n-1} + (n-1)K} \geq 1$$

for $T \leq t < T + \epsilon_1$. As before, this implies that $\phi(t) \leq \psi(t)$ for $T \leq t < T + \epsilon_1$, which contradicts the choice of T.

Finally, suppose that there exists some $0 < t_0 < \mathrm{conj}\xi$ such that $\phi(t_0) = \psi(t_0)$, but this equality does not hold on the whole interval $(0, t_0]$. Then, there exist $0 < t_1 < t_2 \leq t_0$ such that $\phi(t) < \psi(t)$ for $t_1 < t < t_2$ and $\phi(t_2) = \psi(t_2)$. Hence

$$\frac{(\phi(t_2))^2}{n-1} + (n-1)K > 0$$

and by continuity there exists $\delta > 0$ such that

$$\frac{(\phi(t))^2}{n-1} + (n-1)K > 0$$

for $|t - t_2| < \delta$. Thus,

$$\frac{-\phi'(t)}{\frac{(\phi(t))^2}{n-1} + (n-1)K} \geq 1$$

for $|t-t_2|<\delta$. If $t_2-\delta<t<t_2$, integrating from t to t_2 we find

$$t_2 - \operatorname{arcCt}_K\left(\frac{\phi(t)}{n-1}\right) = \operatorname{arcCt}_K\left(\frac{\phi(t_2)}{n-1}\right) - \operatorname{arcCt}_K\left(\frac{\phi(t)}{n-1}\right)$$

$$= \int_t^{t_2} \frac{-\phi'(s)}{\frac{(\phi(s))^2}{n-1} + (n-1)K} ds \geq t_2 - t.$$

Hence $\operatorname{arcCt}_K\left(\frac{\phi(t)}{n-1}\right) \leq t$. This shows that $\phi(t) \geq \psi(t)$ for every $t_2-\delta<t<t_2$. This concludes the proof. □

Theorem 6.3.6. *Let M be a complete, connected, Riemannian n-manifold for which there exists $K \in \mathbb{R}$ such that $Ric_p(v,v) \geq (n-1)K$ for every $v \in T_pM$ with $\|v\|=1$ and $p \in M$. Then, $V(p,r) \leq V_K(r)$ for every $r>0$. Moreover, $V(p,r)=V_K(r)$ for some $r>0$ if and only if $B(p,r)$ is isometric to the open ball of radius r in the simply connected space form of sectional curvature K.*

Proof. For every $r>0$ Theorem 6.3.5 implies that

$$V(p,r) = \int_{S_p}\left(\int_0^{\min\{c(\xi),r\}} \det A(t,\xi)dt\right)d\xi \leq \int_{S_p}\left(\int_0^{\min\{c(\xi),r\}} (S_K(t))^{n-1}dt\right)d\xi$$

$$\leq \int_{S_p}\left(\int_0^{r} (S_K(t))^{n-1}dt\right)d\xi = V_K(r).$$

The case of the equality is obvious. □

Corollary 6.3.7. *For every complete, connected, Riemannian n-manifold M and $p \in M$ the function $V(p,.)$ is locally uniformly Lipschitz.*

Proof. Let $\rho>0$ and $K_\rho = \inf\{Ric_q(v,v) : v \in S_q, q \in B(p,\rho)\}$. For every $0<s<r<\rho$ applying Bishop's Theorem 6.3.5 we have

$$\frac{V(p,r)-V(p,s)}{r-s} = \frac{1}{r-s}\int_{D_p(s)}\left(\int_s^{\min\{c(\xi),r\}} \det A(t,\xi)dt\right)d\xi$$

$$\leq c_{n-1}\cdot \max\{(S_{K_\rho}(t))^{n-1} : 0 \leq t \leq \rho\}. \quad \square$$

In the rest of this section we shall study the monotonicity of the function $\frac{V(p,.)}{V_K(.)}$ under the assumptions of Theorem 6.3.6.

Proposition 6.3.8. *Let M be a complete, connected, Riemannian n-manifold for which there exists $K \in \mathbb{R}$ such that $Ric_p(v,v) \geq (n-1)K$ for every $v \in T_pM$ with $\|v\|=1$ and $p \in M$. Then, the function $\frac{E(p,.)}{E_K(.)}$ is decreasing for all $p \in M$.*

Proof. Let $0 < r < s$, so that $D_p(s) \subset D_p(r)$. By Bishop's Theorem 6.3.6, for every $\xi \in S_p$ the function

$$\frac{\det A(.,\xi)}{(S_K(.))^{n-1}}$$

is decreasing. Thus,

$$\frac{E(p,r)}{E_K(r)} = \frac{1}{c_{n-1}} \int_{D_p(r)} \frac{\det A(r,\xi)}{(S_K(r))^{n-1}} d\xi \geq \frac{1}{c_{n-1}} \int_{D_p(s)} \frac{\det A(r,\xi)}{(S_K(r))^{n-1}} d\xi$$

$$\geq \frac{1}{c_{n-1}} \int_{D_p(s)} \frac{\det A(s,\xi)}{(S_K(s))^{n-1}} d\xi = \frac{E(p,s)}{E_K(s)}. \quad \square$$

The monotonicity of the function $\dfrac{V(p,.)}{V_K(.)}$ will now be a direct consequence of the following lemma of M. Gromov.

Lemma 6.3.9. *Let f, $g : \mathbb{R} \to (0,+\infty)$ be two integrable functions. If the function $\dfrac{f}{g}$ is decreasing, then the function*

$$\frac{\int_0^t f(s)ds}{\int_0^t g(s)ds}$$

is decreasing as well.

Proof. Let $r < s$. Since

$$\left(\int_0^r f(t)dt\right)\left(\int_0^s g(t)dt\right) = \left(\int_0^r f(t)dt\right)\left(\int_0^r g(t)dt\right) + \left(\int_0^r f(t)dt\right)\left(\int_r^s g(t)dt\right)$$

and

$$\left(\int_0^s f(t)dt\right)\left(\int_0^r g(t)dt\right) = \left(\int_0^r f(t)dt\right)\left(\int_0^r g(t)dt\right) + \left(\int_r^s f(t)dt\right)\left(\int_0^r g(t)dt\right).$$

Let $h = \dfrac{f}{g}$, which is assumed to be decreasing. Then,

$$\left(\int_0^r f(t)dt\right)\left(\int_r^s g(t)dt\right) = \left(\int_0^r g(t)h(t)dt\right)\left(\int_r^s g(t)dt\right) \geq h(r)\left(\int_0^r g(t)dt\right)\left(\int_r^s g(t)dt\right)$$

$$\geq \left(\int_0^r g(t)dt\right)\left(\int_r^s g(t)h(t)dt\right) = \left(\int_0^r g(t)dt\right)\left(\int_r^s f(t)dt\right). \quad \square$$

Combining now Proposition 6.3.8 and Lemma 6.3.9 we obtain the following result of M. Gromov.

Theorem 6.3.10. *Let M be a complete, connected, Riemannian n-manifold for which there exists $K \in \mathbb{R}$ such that $Ric_p(v,v) \geq (n-1)K$ for every $v \in T_pM$ with*

$\|v\| = 1$ *and* $p \in M$. *Then, the function* $\dfrac{V(p,.)}{V_K(.)}$ *is decreasing for all* $p \in M$. $\square$

Recall that Myers' Theorem 5.1.5 states that if M is a complete, connected, Riemannian n-manifold for which there exists $r > 0$ such that $Ric_p(v,v) \geq (n-1)\dfrac{1}{r^2}$ for every $v \in T_pM$ with $\|v\| = 1$ and $p \in M$, then $\operatorname{diam}(M) \leq \pi r$, M is compact and it has finite fundamental group. Using the above results of R.L. Bishop and M. Gromov we can examine what happens in case $\operatorname{diam}(M) = \pi r$. The corresponding result is originally due to V.A. Toponogov for the case of sectional curvature and S.Y. Cheng for the case of Ricci curvature. The proof we present here was given later by K. Shiohama.

Theorem 6.3.11. *Let* M *be a complete, connected, Riemannian n-manifold for which there exists* $r > 0$ *such that* $Ric_p(v,v) \geq (n-1)\dfrac{1}{r^2}$ *for every* $v \in T_pM$ *with* $\|v\| = 1$ *and* $p \in M$. *If* $\operatorname{diam}(M) = \pi r$, *then* M *is isometric to the n-sphere* S^n_r *of radius* r.

Proof. By Bishop's Theorem 6.3.6, it suffices to prove that $\operatorname{Vol}(M) = \operatorname{Vol}(S^n_r)$. By Myers' Theorem 5.1.5, M is compact and there exist $p, q \in M$ such that $d(p,q) = \pi r$, where as usual d denotes the Riemannian distance. By Gromov's Theorem 6.3.10,

$$\frac{V(p,\frac{\pi r}{2})}{\frac{1}{2}\operatorname{Vol}(S^n_r)} \geq \frac{V(p,\pi r)}{\operatorname{Vol}(S^n_r)} = \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^n_r)}$$

and therefore $V(p,\dfrac{\pi r}{2}) \geq \dfrac{1}{2}\operatorname{Vol}(M)$. Similarly, we have $V(q,\dfrac{\pi r}{2}) \geq \dfrac{1}{2}\operatorname{Vol}(M)$, and since $B(p,\dfrac{\pi r}{2}) \cap B(q,\dfrac{\pi r}{2}) = \varnothing$, because $d(p,q) = \pi r$, it follows that

$$V(p,\frac{\pi r}{2}) = V(q,\frac{\pi r}{2}) = \frac{1}{2}\operatorname{Vol}(M).$$

Moreover, the function $\dfrac{V(p,.)}{V_{1/r^2}(.)}$ is constant on the interval $[\dfrac{\pi r}{2}, \pi r]$. By monotonicity, the function $\dfrac{E(p,.)}{E_{1/r^2}(.)}$ is also constant on the interval $[\dfrac{\pi r}{2}, \pi r]$ as the proof of Gromov's Lemma 6.3.6 shows. By the proof of Proposition 6.3.8, the function

$$\frac{\det A(t,\xi)}{(S_{1/r^2}(t))^{n-1}}$$

is constant for $\dfrac{\pi r}{2} \leq t \leq \pi r$ for every $\xi \in S_p$. The conclusion follows now from Bishop's Theorem 6.3.5 by taking $t_0 = \pi r$. $\square$

6.4 Exercises

1. Let M be a Riemannian manifold. Explain why the property that a set $A \subset M$ is measurable does not depend on the choice of the Riemannian metric and the

same holds for subsets of M of measure zero.

2. Prove that the Riemannian volume of the complex projective space $\mathbb{C}P^n$, $n \geq 1$, with respect to the Fubini-Study metric is equal to $\dfrac{\pi^n}{n!}$.

3. Let M be a n-dimensional Riemannian manifold and $p \in M$. If $0 < t < \mathrm{inj}p$, prove that the quantity

$$\frac{\frac{d}{dt}(\det A(t,\xi))}{\det A(t,\xi)}$$

is the mean curvature of the geodesic sphere $\partial B(p,t)$ at the point $\gamma_\xi(t)$, where γ_ξ denotes the geodesic with $\gamma_\xi(0) = p$ and $\dot{\gamma}_\xi(0) = \xi \in S_p$.

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