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Rotation Numbers and Isometries

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Abstract. Using the notion of rotation set for homomorphisms of compact manifolds, we define the rotation homomorphism of a connected compact orientable Riemannian manifold and apply it to prove that the dimension of the isometry group of a connected compact orientable Riemannian 3-manifold without conjugate points is not greater than its first Betti number. In higher dimensions the same is true under the additional assumption that the fundamental cohomology class of the manifold is a cup product of integral one-dimensional classes.

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1. Introduction

A great amount of work in dynamical systems has been oriented towards the problem of finding conditions which guarantee the existence of periodic orbits for homeomorphisms and flows. The best known condition concerning orientation preserving homeomorphisms of the unit circle is the rationality of the Poincaré rotation number. A generalization to continuous flows on closed orientable 2-manifolds is given in [2]. An analogous result for 2-torus homeomorphisms is the Conley–Zehnder–Franks theorem which says that if a homeomorphism h of the 2-torus is homotopic to the identity, preserves a Borel probability measure ν that is positive on nonempty open sets and has a lift with ν-mean translation zero, then h has a fixed point (see [4] and [5]).

In this paper we clarify first in Section 3 the generalization of the notion of rotation number given in [8] using the asymptotic cycles of S. Schwartzman which are reviewed in Section 2. The ν-rotation number map of a homeomorphism h of a connected compact manifold M, which is homotopic to the identity, with respect to an h-invariant Borel probability measure ν can be viewed as an element of the group $H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})$, which is a torus with respect to the weak topology. In the case of the n-torus T^n, it is represented by the ν-mean translation vector of a lift of h, with respect to the natural basis of $H_1(T^n;\mathbf{R})$. In Section 4 we prove that the ν-rotation number map of an isometry of a connected compact metric space is independent of the invariant measure ν and derive useful corollaries. The connectedness is here essential. This allows us to define the rotation homomorphism for a connected compact orientable Riemannian manifold M, which is a continuous homomorphism $R\colon I(M) \to H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})$, where $I(M)$ is the path component of the identity of the group of isometries of M. The rotation homomorphism is a tool to finding relations between the isometry group of M and its homology, possibly under additional assumptions. One such relation is found in Section 5 where we prove that the kernel of the rotation homomorphism of a connected compact orientable Riemannian 3-manifold M without conjugate points is a finite group and therefore $\dim I(M) \leqslant \operatorname{rank} H_1(M;\mathbf{Z})$. In arbitrary dimensions the same is true under the additional assumption that the fundamental cohomology class of M is a cup product of integral one-dimensional classes.

2. Asymptotic Cycles

Let X be a compact metrizable space carrying a continuous flow $(\phi_t)_{t\in\mathbf{R}}$. Let tx denote the translation of the point $x \in X$ along its orbit in time $t \in \mathbf{R}$. For every continuous function $f\colon X \to S^1$ there is a continuous function $g\colon \mathbf{R}\times X \to \mathbf{R}$, called the *1-cocycle* of f such that $f(tx) = f(x)\exp(2\pi i g(t,x))$ and $g(t+s,x) = g(s,tx) + g(t,x)$ for every $x \in X$ and $t, s \in \mathbf{R}$. The Ergodic Theorem of Birkhoff implies that for every ϕ-invariant Borel probability measure μ on X the limit

$$g^*(x) = \lim_{t\to+\infty} \frac{g(t,x)}{t}$$

exists μ-almost for every $x \in X$ and $\int_X g^*\,\mathrm{d}\mu = \int_X g(1,.)\,\mathrm{d}\mu$. This integral describes the μ-average rotation of points moving along their orbits with respect to the projection f. Moreover, if $g^*(x)$ exists, then

$$g^*(x) = \frac{1}{t} \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(t, (\phi_t)^k(x))$$

for every $t > 0$. So, $\int_X g^*\,\mathrm{d}\mu = \frac{1}{t}\int_X g(t,.)\,\mathrm{d}\mu$.

The cocycle property of g implies that g^* is an μ-almost everywhere defined measurable flow invariant function, that is $g^*(tx) = g^*(x)$ for every $t \in \mathbf{R}$, whenever $g^*(x)$ is defined. So, if the measure μ is ergodic, then g^* is constant μ-almost everywhere. In case the flow is uniquely ergodic and μ is the unique ϕ-invariant Borel probability measure, then g^* is an everywhere defined constant, namely $\int_X g(1,.)\,\mathrm{d}\mu$.

If $f_1, f_2\colon X \to S^1$ are homotopic continuous functions with cocycles g_1, g_2, respectively, then $\int_X g_1^*\,\mathrm{d}\mu = \int_X g_2^*\,\mathrm{d}\mu$. Since the first Čech cohomology group with integer coefficients $\check{H}^1(X;\mathbf{Z})$ of X is isomorphic to the group of homotopy classes of continuous functions of X to S^1, there is a group homomorphism $A_\mu\colon \check{H}^1(X;\mathbf{Z}) \to \mathbf{R}$ defined by

$$A_\mu[f] = \int_X g(1,.)\,\mathrm{d}\mu,$$

where g is the 1-cocycle of $f: X \to S^1$ and $[f]$ the homotopy class of f.The homomorphism A_μ was defined by S. Schwartzman [9] and is called the *μ-asymptotic cycle* of the flow. It describes how a μ-average orbit winds around X. The image of A_μ is called the *μ-winding numbers group* of the flow ϕ and will be denoted by W_μ. An exposition of the basic theory of asymptotic cycles with details is given in [1].

EXAMPLES 2.1. (a) Let C be a periodic orbit in X of prime period $T > 0$ and let μ be the uniformly distributed Borel probability measure along C. Then,

$$A_\mu[f] = \frac{1}{T} \deg(f \mid C)$$

for every continuous function $f: X \to S^1$. Thus, $A_\mu = 0$, if C is null homologous in X. Suppose now that $X = S^1 \times S^1$ the 2-torus and the flow has no singular point. Then C is not null homologous. This means that if l is the longitude and m is the meridian, then C is homotopic to $l^a * m^b$ for some $a, b \in \mathbf{Z}$ not both zero. Let p and q be the projections onto the longitude and the meridian respectively. Every continuous function $f: S^1 \times S^1 \to S^1$ is homotopic to $p^\kappa q^\lambda$ for some $\kappa, \lambda \in \mathbf{Z}$ and we can compute

$$A_\mu[f] = \frac{1}{T} \deg(p^\kappa q^\lambda \mid l^a * m^b) = \frac{1}{T}(\kappa a + \lambda b).$$

Since a, b are not both zero, we have $W_\mu \cong \mathbf{Z}$.

(b) Each continuous one-parameter group $(\phi_t)_{t\in\mathbf{R}}$ of translations of the n-torus T^n has the form

$$\phi_t((x_1, ..., x_n)(\operatorname{mod} \mathbf{Z}^n)) = (x_1 + a_1 t, ..., x_n + a_n t)(\operatorname{mod} \mathbf{Z}^n),$$

for some $a_1, \ldots, a_n \in \mathbf{R}$ and preserves the Haar measure μ of T^n. It is uniquely ergodic if and only if $a_1, \ldots, a_n$ are linearly independent over $\mathbf{Q}$, by Kronecker's theorem. The winding numbers group with respect to μ is $W_\mu = a_1\mathbf{Z} + \cdots + a_n\mathbf{Z}$.

The following formula is often useful in the calculation of winding numbers and is proved using Fubini's theorem (see [1], Lemma 5.2).

LEMMA 2.2. *Let μ be a ϕ-invariant Borel probability measure on X, $f: X \to S^1$ a continuous function with 1-cocycle g and $A \subset X$ be a closed set with $\mu(A) = 0$. We make the following assumptions*

(a) *The time derivative $g'(0, x)$ exists for every $x \in X \setminus A$ and is continuous and bounded on $X \setminus A$.*
(b) *For every $x \in X$ the set of times $t \in \mathbf{R}$ such that $tx \in A$ is discrete.*

Then,

$$A_\mu[f] = \int_{X \setminus A} g'(0, x)\, \mathrm{d}\mu.$$

Let ξ be a smooth vector field on a connected compact smooth n-manifold X with flow $(\phi_t)_{t \in \mathbf{R}}$ and let μ be a ϕ-invariant Borel probability measure on X. By de Rham's theorem and the Universal Coefficient theorem, $H^1_{DR}(X) \cong H^1(X; \mathbf{Z}) \otimes \mathbf{R}$. So, there is a basis of $H^1_{DR}(X)$ every element of which can be represented by $f^*(\mathrm{d}\theta/2\pi)$, where $f\colon X \to S^1$ is a smooth function and $\mathrm{d}\theta/2\pi$ is the representative of the natural generator of $H^1_{DR}(S^1) \cong \mathbf{R}$. We can extend A_μ on $H^1_{DR}(X)$ linearly in the obvious way.

The 1-cocycle $g\colon \mathbf{R} \times X \to \mathbf{R}$ of a smooth function $f\colon X \to S^1$ is given by

$$g(t, x) = \int_0^t f^* \left(\frac{\mathrm{d}\theta}{2\pi} \right) (\xi(sx))\, \mathrm{d}s.$$

So, for every $x \in X$ we have

$$g'(0, x) = f^* \left(\frac{\mathrm{d}\theta}{2\pi} \right) (\xi(x)).$$

It follows from Lemma 2.2 that

$$A_\mu[f] = \int_X f^* \left(\frac{\mathrm{d}\theta}{2\pi} \right) (\xi)\, \mathrm{d}\mu.$$

Consequently, $A_\mu[\alpha] = \int_X (i_\xi \alpha)\, \mathrm{d}\mu$ for every closed 1-form α on X.

Since $H^1(X; \mathbf{Z})$ is a finitely generated free group, so is the μ-winding numbers group W_μ. If $W_\mu = 0$, then ξ is said to be *μ-homologically trivial*. If $W_\mu \cong \mathbf{Z}$, then $W_\mu = \lambda \mathbf{Z}$, for some $\lambda > 0$, and ξ is said to be *μ-homologically rational*. In any other case $W_\mu = \lambda(\mathbf{Z} + \alpha_1 \mathbf{Z} + \cdots + \alpha_k \mathbf{Z})$, for some $\lambda > 0$ and $0 < \alpha_i < 1$, $i = 1, 2, \ldots, k$, such that $1, \alpha_1, \ldots, \alpha_k$ are linearly independent over $\mathbf{Q}$ and $1 \leqslant k \leqslant \operatorname{rank} H^1(X; \mathbf{Z}) - 1$.

Suppose that X is oriented by a volume element ω and the flow of ξ preserves volume. Then, $\phi_t^* \omega = \omega$ for every $t \in \mathbf{R}$ and $i_\xi \omega$ is a closed $(n-1)$-form, called the *flux form*. Since $\alpha \wedge \omega = 0$, for every closed 1-form α, we have

$$0 = i_\xi(\alpha \wedge \omega) = (i_\xi \alpha)\omega - \alpha \wedge (i_\xi \omega)$$

and therefore

$$A_\omega[\alpha] = \int_X (i_\xi \alpha)\omega = \int_X \alpha \wedge (i_\xi \omega).$$

This means that the asymptotic cycle A_ω is the Poincare dual of the de Rham cohomology class represented by the flux form $i_\xi \omega$.

3. Rotation Number Maps and Rotation Sets of Homeomorphisms

The suspensions of homeomorphisms is a class of flows whose winding numbers groups can be computed in terms of initial data. Let Y be a compact metrizable space and let $h: Y \to Y$ be a homomorphism. On $[0, 1] \times Y$ we consider the equivalence relation $(1, x) \sim (0, h(x))$, $x \in Y$. The quotient space $X = [0, 1] \times Y/\sim$ is compact metrizable and is called the *mapping torus* of h. Let $[s, x]$ denote the class of $(s, x) \in [0, 1] \times Y$. The flow on X defined by

$$t[s, x] = [t + s - n, h^n(x)]$$

if $n \leqslant t + s < n + 1$, is called the *suspension* of h.

If ν is an h-invariant Borel probability measure on Y and λ is the Lebesgue measure on $[0, 1]$, then the product measure $\lambda \times \nu$ induces a Borel probability measure on X which is invariant by the suspension of h. It is easy to see that the converse is true. That is, every invariant by the suspension of h Borel probability measure on X is of this form.

In order to compute the winding numbers groups of the suspension of h we need to know the relation between the integral first Čech cohomology groups of Y and X. Let $C(Y, \mathbf{Z})$ be the group of integer valued continuous functions on Y. Let $\gamma: C(Y, \mathbf{Z}) \to \check{H}^1(X; \mathbf{Z})$ be defined by $\gamma(\psi) = [f]$, where $f: X \to S^1$ is the continuous function defined by $f[t, x] = \exp(2\pi i t \psi(x))$ and $j^*: \check{H}^1(X; \mathbf{Z}) \to \check{H}^1(Y; \mathbf{Z})$ be the homomorphism induced by the inclusion $j: Y \to X$ with $j(x) = [0, x]$. Then one can easily verify that the sequence

$$C(Y, \mathbf{Z}) \xrightarrow{h^* - \mathrm{id}} C(Y, \mathbf{Z}) \xrightarrow{\gamma} \check{H}^1(X; \mathbf{Z}) \xrightarrow{j^*} \check{H}^1(Y; \mathbf{Z}) \xrightarrow{h^* - \mathrm{id}} \check{H}^1(Y; \mathbf{Z})$$

is exact.

If now ν is an h-invariant Borel probability measure on Y and μ is the corresponding invariant measure on X, then

$$W_\mu = \left\{ \int_Y \psi \, \mathrm{d}\nu : \ \psi \in \mathrm{Log}(Y, h) \right\}$$

where $\mathrm{Log}(Y, h)$ is the set of continuous functions $\psi: Y \to \mathbf{R}$ satisfying $f \circ h = f \exp(2\pi i \psi)$ for some continuous function $f: Y \to S^1$ [1] or [7], Appendix.

If Y is in addition connected, then W_μ can also be described through the rotation number map of h with respect to ν. Let $f: Y \to S^1$ be a continuous function such that $[f] \in \mathrm{Ker}(h^* - \mathrm{id})$. There is a continuous function $g: Y \to \mathbf{R}$ such that $f \circ h = f \exp(2\pi i g)$ and any two such functions differ by an integer. One can easily see that there is a well defined group homomorphism $R_\nu\colon \mathrm{Ker}(h^* - \mathrm{id}) \to S^1$ with

$$R_\nu[f] = \exp\Big(2\pi i \int_Y g \,\mathrm{d}\nu\Big)$$

called the *ν-rotation number map* of h. The subgroup $\mathrm{Im}\, R_\nu$ of S^1 is called the *ν-rotation numbers group* of h. It is clear that we have a commutative diagram

$$\begin{array}{ccc} \check{H}^1(X;\mathbf{Z}) & \xrightarrow{A_\mu} & \mathbf{R} \\ \Big\downarrow{\scriptstyle j^*} & & \Big\downarrow{\scriptstyle \exp} \\ \mathrm{Ker}(h^* - \mathrm{id}) & \xrightarrow{R_\nu} & S^1 \end{array}$$

and therefore $W_\mu = \exp^{-1}(\mathrm{Im}\, R_\nu)$. Thus we arrive at the following.

PROPOSITION 3.1. *Let Y be a connected, compact, metrizable space and $h\colon Y \to Y$ a homeomorphism. Let μ be an invariant by the suspension of h Borel probability measure on the mapping torus X of h corresponding to the h-invariant Borel probability measure ν on Y.*

(i) *If the ν-rotation numbers group of h is trivial or is a finite cyclic subgroup of S^1, then $W_\mu \cong \mathbf{Z}$.*
(ii) *If j^* has a right inverse, then $R_\nu \in \mathrm{Im}\, F$, where*

$$F\colon \mathrm{Hom}(\mathrm{Ker}(h^* - \mathrm{id}), \mathbf{R}) \to \mathrm{Hom}(\mathrm{Ker}(h^* - id), S^1)$$

is the homomorphism defined by $F(\alpha) = \exp(2\pi i\alpha)$.

So, if the homeomorphism $h\colon Y \to Y$ is homotopic to the identity and $\check{H}^1(Y;\mathbf{Z})$ is free, then R_ν can be considered an element of $\mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}),\mathbf{R})/\mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}),\mathbf{Z})$, since $\mathrm{Ker} F = \mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}),\mathbf{Z})$, where F is the homomorphism of Proposition 3.1.

If Y is a connected compact manifold, then $\mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}),\mathbf{R})/\mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}),\mathbf{Z})$ is isomorphic to $H_1(Y;\mathbf{R})/H_1(Y;\mathbf{Z})$, by the Universal Coefficient Theorems (see [11]). Thus for any h-invariant Borel probability measure ν on Y, the rotation number map R_ν can be considered as an element of $H_1(Y;\mathbf{R})/H_1(Y;\mathbf{Z})$. A representative of R_ν in $H_1(Y;\mathbf{R})$ is $A_\mu \circ r$, where r is a right inverse of j^*. The set $\rho(h) = \{R_\nu\colon \nu \in \mathcal{M}_h(Y)\}$ is called the *rotation set* of h, where $\mathcal{M}_h(Y)$ denotes the set of h-invariant Borel probability measures on Y and is weakly compact in $H_1(Y;\mathbf{R})/H_1(Y;\mathbf{Z})$.

EXAMPLE 3.2. Let $h: T^n \to T^n$ be a homeomorphism of the n-torus homotopic to the identity. If $p_i: T^n \to S^1$ is the projection onto the ith factor, $1 \leqslant i \leqslant n$, then the homotopy classes $[p_1], \ldots, [p_n]$ form a basis of $H^1(T^n; \mathbf{Z})$. Let ν be a h-invariant Borel probability measure and μ the corresponding measure that is invariant by the suspension of h. Then R_ν is represented by the vector

$$(A_\mu(r[p_1]), \ldots, A_\mu(r[p_n]))$$

with respect to the dual basis of $H_1(T^n; \mathbf{R})$.

Let $\tilde{h}: \mathbf{R}^n \to \mathbf{R}^n$ be a lift of h and $\psi: T^n \to \mathbf{R}^n$ be the continuous function defined by $\psi(x) = \tilde{h}(y) - y$, where $y \in p^{-1}(x)$ and $p: \mathbf{R}^n \to T^n$ is the canonical covering projection. Then,

$$(A_\mu(r[p_1]), \ldots, A_\mu(r[p_n])) = \int_{T^n} \psi \, \mathrm{d}\nu \ (\mathrm{mod}\, \mathbf{Z}^n).$$

That is, R_ν is represented by the ν-mean translation vector of h. In case ν is ergodic, the Ergodic Theorem of Birkhoff implies that

$$\int_{T^n} \psi \, \mathrm{d}\nu = \lim_{k \to +\infty} \frac{\tilde{h}^k(y) - y}{k}$$

$p^{-1}\nu$-almost for every $y \in [0, 1)^n$.

EXAMPLE 3.3. Let $(\phi_t)_{t \in \mathbf{R}}$ be a continuous flow on a compact metrizable space X and μ a ϕ-invariant Borel probability measure on X. If R_μ^t denotes the rotation number map of the homeomorphism ϕ_t, then $R_\mu^t = \exp(2\pi i t A_\mu)$, for every $t > 0$. Indeed, for every continuous function $f: X \to S^1$ we have $f \circ \phi_t = f \exp(2\pi i g(t, .))$, where g is the 1-cocycle of f with respect to the flow. Thus,

$$\begin{aligned} R_\mu^t[f] &= \exp\left(2\pi i \int_X g(t, .) \, \mathrm{d}\mu\right) = \exp\left(2\pi i t \int_X g^* \, \mathrm{d}\mu\right) \\ &= \exp(2\pi i t A_\mu[f]). \end{aligned}$$

4. Isometric Systems

Let X be a compact metrizable space and $h: X \to X$ be a homeomorphism. From any compatible metric d of X, one can define a new metric d^* by

$$d^*(x, y) = \sup\{d(h^n(x), h^n(y)) : n \in \mathbf{Z}\}$$

for $x, y \in X$, which defines a finer topology on X. Clearly h is a d^*-isometry. The metric d^* is compatible with the topology of X if and only if $\{h^n : n \in \mathbf{Z}\}$ is a d-equicontinuous family. Similar considerations hold for continuous flows.

A homeomorphism $h\colon X \to X$ is called *isometric* if there is a compatible metric d of X such that h is a d-isometry. Analogously, a continuous flow $(\phi_t)_{t\in\mathbf{R}}$ on X is called *isometric* if there is a compatible metric d of X such that ϕ_t becomes a d-isometry for every $t \in \mathbf{R}$.

The isometric homeomorphisms (and flows) satisfy a strong form of the Ergodic Theorem of Birkhoff. More precisely, if $h\colon X \to X$ is an isometric homeomorphism, then for every continuous function $g\colon X \to \mathbf{R}$ there is a continuous function $g^*\colon X \to \mathbf{R}$ such that

$$g^* = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} g \circ h^k$$

uniformly on X (see [6], Theorem 2.6).

THEOREM 4.1. *Let X be a compact metrizable space and $h\colon X \to X$ be an isometric homeomorphism. If $f\colon X \to S^1$ is a continuous function for which there is a continuous function $g\colon X \to \mathbf{R}$ such that $f \circ h = f \exp(2\pi i g)$, then g^* is a locally constant function.*

Proof. Let d be a compatible metric of X such that h is a d-isometry and let $0 < \varepsilon < 1/3$. By uniform continuity, there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{3} \quad \text{and} \quad |g(x) - g(y)| < \frac{\varepsilon}{3},$$

whenever $x, y \in X$ and $d(x, y) < \delta$. So for every $n \in \mathbf{N}$ we have $|g(h^n(x)) - g(h^n(y))| < \varepsilon/3$ and

$$\left|\frac{f(x)}{f(y)} - \exp\left(2\pi i \sum_{k=0}^{n-1} g(h^k(y)) - g(h^k(x))\right)\right| = |f(h^n(x)) - f(h^n(y))| < \frac{\varepsilon}{3}.$$

It follows that

$$\left|1 - \exp\left(2\pi i \sum_{k=0}^{n-1} g(h^k(x)) - g(h^k(y))\right)\right| < \frac{2\varepsilon}{3}$$

and consequently there are $\lambda_n \in \mathbf{Z}$ such that

$$\left|\lambda_n - \sum_{k=0}^{n-1} g(h^k(x)) - g(h^k(y))\right| < \frac{4\varepsilon}{3}$$

for every $n \in \mathbf{N}$. From this we conclude that

$$\begin{aligned}\lambda_{n+1} - \frac{4\varepsilon}{3} &< \sum_{k=0}^{n} g(h^k(x)) - g(h^k(y)) \\ &< \frac{4\varepsilon}{3} + \lambda_n + g(h^n(x)) - g(h^n(y)) < \lambda_n + \frac{5\varepsilon}{3},\end{aligned}$$

that is $\lambda_{n+1} - \lambda_n < 3\varepsilon$ and similarly $\lambda_n - \lambda_{n+1} < 3\varepsilon$. Hence $\lambda_n = \lambda_1$ for every $n \in \mathbf{N}$. Dividing now by n and taking the limit for $n \to +\infty$ we have $|g^*(x) - g^*(y)| = 0$ for every $x, y \in X$ with $d(x, y) < \delta$. This shows that g^* is locally constant.

COROLLARY 4.2. *Let X be a connected compact metrizable space and $h: X \to X$ be an isometric homeomorphism. If $[f] \in \mathrm{Ker}(h^* - \mathrm{id})$ and $g: X \to \mathbf{R}$ is a continuous function such that $f \circ h = f \exp(2\pi i g)$, then $R_\nu[f] = \exp(2\pi i g^*)$ for every $\nu \in \mathcal{M}_h(X)$. Thus, R_ν is independent of ν. In particular, the rotation set $\rho(h)$ is a point, in case h is homotopic to the identity.*

COROLLARY 4.3. *Let X be a connected compact metrizable space and $h: X \to X$ be an isometric homeomorphism. If h has a periodic point of period q, the rotation numbers group of h is a subgroup of the qth roots of unity.*

Proof. Let x_0 be a periodic point of h of period q. Note that the rotation number map of h^q is R^q, where R is the rotation number map of h. If $f: X \to S^1$ and $g: X \to \mathbf{R}$ are continuous functions such that $f \circ h^q = f \exp(2\pi i g)$, then $g(x_0) \in \mathbf{Z}$. Obviously, $g^* = g(x_0)$, since h^q fixes x_0. So we have $(R[f])^q = \exp(2\pi i g^*) = 1$.

Similar results hold for continuous flows.

THEOREM 4.4. *Let X be a connected compact metrizable space and $(\phi_t)_{t\in\mathbf{R}}$ be a continuous flow on X. If the flow is isometric, all its asymptotic cycles coincide.*

Proof. Let μ be a ϕ-invariant Borel probability measure on X and let $f: X \to S^1$ be a continuous function with 1-cocycle g. Then

$$g^*(x) = \lim_{n\to+\infty} \frac{g(t,x)}{t} = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(1, (\phi_1)^k(x)),$$

uniformly on X. Since ϕ_1 is an isometric homeomorphism and $f \circ \phi_1 = f \exp(2\pi i g(1, .))$, by Theorem 4.1, g^* is constant. Hence $A_\mu[f] = g^*$.

COROLLARY 4.5. *Let X be a connected compact metrizable space and $(\phi_t)_{t\in\mathbf{R}}$ be an isometric flow on X. If there is a nonnull homologous periodic orbit of period T, the winding numbers group is a nontrivial subgroup of $\frac{1}{T}\mathbf{Z}$.*

One case where the converse of Corollary 4.5 holds is the class of Killing vector fields on connected compact orientable Riemannian 3-manifolds.

PROPOSITION 4.6. *Let ξ be a Killing vector field on a connected compact orientable Riemannian 3-manifold. If ξ is homologically rational, it has at least one periodic orbit.*

Proof. Since ξ is assumed to be homologically rational it is everywhere nonzero. So, its orbits are the leaves of a Riemannian foliation in the sense of [3]. Suppose that ξ has no periodic orbit. According to the classification Theorem III.A.1

and Corollary III.B.4 of [3], the flow of ξ is topologically equivalent to one of the following:

(i) A uniquely ergodic one-parameter group of tranlations of the 3-torus T^3.
(ii) A nonergodic one-parameter group of translations of T^3 with slopes a_1, a_2, 0, where a_1, a_2 are linearly independent over $\mathbf{Q}$.

In both cases we do not have homologically rational vector fields.

In arbitrary dimensions we have the following.

PROPOSITION 4.7. *Let M be a connected compact orientable Riemannian n-manifold such that its fundamental cohomology class is a cup product of integral one-dimensional classes. Then, every homologically rational Killing vector field of M has periodic flow.*

Proof. Let ξ be a homologically rational Killing vector field on M and let $(\phi_t)_{t\in\mathbf{R}}$ be its flow. The closure G of $\{\phi_t: t \in \mathbf{R}\}$ in the isometry group of M is a k-torus, $k \geqslant 1$, acting effectively and smoothly on M. Moreover, $(\phi_t)_{t\in\mathbf{R}}$ is a uniquely ergodic continuous one-parameter group of translations of G. The orbit closures of the points of M under the flow of ξ are the orbits of G. So, $k = 1$ if and only if the flow of ξ is periodic. Let $x \in M$ and let $f_x: G \to M$ be the natural map with $f_x(g) = gx$. Let ν be the Haar measure on G and $\mu = (f_x)_*\nu$ be the induced invariant measure on M under the flow of ξ. Then, $A_\mu = A_\nu \circ (f_x)^*$, where A_ν is the asymptotic cycle of the one-parameter group of translations $(\phi_t)_{t\in\mathbf{R}}$ of G. Our cohomological assumption implies that $(f_x)^*: H^1(M; \mathbf{Z}) \to H^1(G; \mathbf{Z})$ is an epimorphism, according to Theorem 4 in [10]. Hence, $\mathbf{Z} \cong W_\mu = W_\nu \cong \mathbf{Z}^k$ and we must necessarily have $k = 1$.

We close this section with a small generalization of the well known fact that if a parallel vector field of a connected compact orientable Riemannian manifold vanishes at some point, it vanishes everywhere.

PROPOSITION 4.8. *Let M be a connected compact orientable Riemannian manifold. Then, any homologically trivial parallel vector field vanishes identically on M.*

Proof. If ξ is a parallel vector field on M, then it is Killing and its dual 1-form α with respect to the Riemannian metric is closed. The value of the asymptotic cycle A of ξ on $[\alpha]$ is

$$A[\alpha] = \int_M (\|\xi\|^2)\omega = \|\xi\|^2,$$

where ω is the normalized Riemannian volume element. The conclusion is now obvious.

5. The Rotation Homomorphism

Let M be a connected compact orientable Riemannian manifold and let $I(M)$ denote the path component of the group of isometries of M, which contains the identity map, endowed with the compact-open topology. For each $h \in I(M)$ the rotation number map $R(h)$ of h can be considered as an element of $H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})$. The map

$$R\colon I(M) \to H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})$$

defined in this way is a homomorphism and is called the *rotation homomorphism.* To see that R is indeed a homomorphism let $h_1, h_2 \in I(M)$ and note that they both preserve the Borel probability measure ν on M defined by the normalized Riemannian volume. If $f\colon M \to S^1$ is a continuous function for which there are continuous functions $g_k\colon M \to \mathbf{R}$ such that $f \circ h_k = f \exp(2\pi i g_k)$, $k = 1, 2$, then

$$f \circ h_1 \circ h_2 = f \exp(2\pi i (g_1 \circ h_2 + g_2)).$$

Since ν is h_2-invariant, it follows that

$$\begin{aligned} R(h_1 \circ h_2)[f] &= \exp\Big(2\pi i \int_M (g_1 \circ h_2 + g_2)\, \mathrm{d}\nu\Big) \\ &= R(h_1)[f]R(h_2)[f], \end{aligned}$$

LEMMA 5.1. *The rotation homomorphism is continuous with respect to the quotient weak topology on $H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})$.*

Proof. Let $h_n, h \in I(M)$, $n \in \mathbf{N}$, be such that $h_n \to h$ uniformly on M. Let $f\colon M \to S^1$ be a continuous function and let $g_n, g\colon M \to \mathbf{R}$, $n \in \mathbf{N}$, be continuous functions such that $f \circ h_n = f \exp(2\pi i g_n)$, $n \in \mathbf{N}$, and $f \circ h = f \exp(2\pi i g)$. It suffices to show that

$$\exp\Big(2\pi i \int_M g_n\, \mathrm{d}\nu\Big) \to \exp\Big(2\pi i \int_M g\, \mathrm{d}\nu\Big),$$

where ν is the Borel probability measure on M defined by the normalized Riemannian volume. Let $\varepsilon > 0$ and choose $0 < \delta < 1/4$ such that $|\exp(2\pi i t) - \exp(2\pi i s)| < \varepsilon$, whenever $|t - s| < \delta$. Since $f \circ h_n \to f \circ h$ uniformly on M, there is some $n_0 \in \mathbf{N}$ such that $\|1 - \exp(2\pi i (g_n - g))\| < \delta/2$ for $n \geqslant n_0$. Thus, for every $x \in M$ there is some $k_n(x) \in \mathbf{Z}$ such that $|g_n(x) - g(x) - k_n(x)| < \delta$. Since M is connected and $g_n - g$ continuous, $k_n(x)$ is a constant k_n. Integrating we get

$$\Big|\int_M (g_n - g)\, \mathrm{d}\nu - k_n\Big| < \delta$$

and therefore

$$\left|\exp\left(2\pi i \int_M g_n \, \mathrm{d}\nu\right) - \exp\left(2\pi i \int_M g \, \mathrm{d}\nu\right)\right| < \varepsilon$$

for every $n \geqslant n_0$. This proves the Lemma.

Remark 5.2. It is not hard to see that the rotation homomorphism R of a connected compact orientable Riemannian manifold M can be lifted to a continuous homomorphism $\tilde{R}\colon \widetilde{I(M)} \to H_1(M; \mathbf{R})$, where $\widetilde{I(M)}$ is the universal covering space of $I(M)$. $\tilde{R}$ is called the *isometric flow homomorphism.*

THEOREM 5.3. *If M is a connected compact orientable Riemannian 3-manifold without conjugate points, then the kernel of the rotation homomorphism is a finite group consisting of periodic isometries. Hence* $\dim I(M) \leqslant \operatorname{rank} H_1(M; \mathbf{Z})$.

Proof. If $\phi \in I(M)$ is sufficiently close to the identity, there is a Killing vector field ξ on M such that $\phi = \phi_1$, where $(\phi_t)_{t\in\mathbf{R}}$ is the flow of ξ. Since M has no conjugate points, ξ is parallel [12]. If $\phi \in \operatorname{Ker} R$, then ξ is homologically trivial or rational, more precisely its winding numbers group is a subgroup of $\mathbf{Z}$. In the first case we have $\phi = \mathrm{id}$ from Proposition 4.8. In the later ξ has a periodic orbit from Proposition 4.6. However, if G denotes the closure in $I(M)$ of the group $\{\phi_t : t \in \mathbf{R}\}$, then G is a torus acting effectively and smoothly on M. The orbits of G are exactly the orbit closures of the flow of ξ. The Lie algebra of the isotropy group of a point $x \in M$ consists of parallel vector fields that vanish at x, hence vanish everywhere. It follows that every isotropy group is finite and every orbit of G is a torus of the same dimension as G. Since ξ has a periodic orbit, we conclude that $G \cong S^1$ and there is a $T > 0$ such that $\phi_T = \mathrm{id}$. Moreover, $T \in \mathbf{Q}$, since the winding numbers group of ξ is a nontrivial subgroup of $\mathbf{Z} \cap \frac{1}{T}\mathbf{Z}$. Hence ϕ is periodic. So we have shown that the elements of $\operatorname{Ker} R$ which are sufficiently close to the identity are periodic. Since by [12] $I(M)$ is a torus, every element of $\operatorname{Ker} R$ is periodic. In particular, $\operatorname{Ker} R$ is a totally disconnected subgroup of $I(M)$. Since R is continuous by Lemma 5.1, it is also closed, hence a compact Lie group. It follows that $\operatorname{Ker} R$ is finite.

Using Proposition 4.7 instead of Proposition 4.6 in the proof of Theorem 5.3 we have the following.

THEOREM 5.4. *Let M be a connected compact orientable Riemannian n-manifold without conjugate points such that its fundamental cohomology class is a cup product of integral one-dimensional classes. Then, the kernel of the rotation homomorphism is finite and hence* $\dim I(M) \leqslant \operatorname{rank} H_1(M; \mathbf{Z})$.

References

1. Athanassopoulos, K.: Some aspects of the theory of asymptotic cycles, *Expositiones Math.* **13** (1995), 321–336.

2. Athanassopoulos, K.: Flows with cyclic winding numbers groups, *J. reine angew. Math.* **481** (1996), 207–215.
3. Carriere, Y.: Flots riemanniens, *Asterisque* **116** (1984), 31–52.
4. Franks, J.: Recurrence and fixed points of surface homomorphisms, *Ergodic Theory Dynam. Systems* **8*** (1988), 99–107.
5. Flucher, M.: Fixed points of measure preserving torus homomorphisms, *Manuscripta Math.* **68** (1990), 271–293.
6. Krengel, U.: *Ergodic Theorems,* Walter de Gruyter, Berlin, 1985.
7. Packer, J. A.: K-theoretic invariants for C^*-algebras associated to transformations and induced flows, *J. Funct. Anal.* **67** (1986), 25–59.
8. Pollicott, M.: Rotation sets for homomorphisms and homology, *Trans. Amer. Math. Soc.* **331** (1992), 881–894.
9. Schwartzman, S.: Asymptotic cycles, *Ann. of Math.* **66** (1957), 270–284.
10. Schwartzman, S.: A split action associated with a compact transformation group, *Proc. Amer. Math. Soc.* **83** (1981), 817–824.
11. Spanier, E. H.: *Algebraic Topology,* McGraw-Hill, New York, 1966.
12. O'Sullivan, J. J.: Manifolds without conjugate points, *Math. Ann.* **210** (1974), 295–311.