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# On the Ruelle rotation for torus diffeomorphisms

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## 1. Introduction

In this note we are concerned with the Ruelle rotation number, which is defined for a $C^1$ diffeomorphism $h$ of the 2-torus $T^2$ that is isotopic to the identity. Intuitively, the Ruelle rotation number is the mean value, with respect to an $h$-invariant Borel probability measure $\nu$, of the asymptotic rate at which the derivative of $h$ rotates the tangent planes. In order to measure it, we need to have fixed in advance a trivialization of the tangent bundle $TT^2$ of $T^2$. The Ruelle rotation number depends on the trivialization, as well as on the isotopy from $id$ to $h$. As was mentioned by Ruelle in [7], there is an independence on the isotopy if we take the number mod **Z**, i.e. consider the corresponding point on the unit circle $S^1$.

In Sect. 4 we clarify the dependence on the trivialization of $TT^2$. It turns out that the two points of $S^1$ taken starting with two different trivializations of $TT^2$ differ by an element of the group $\mathrm{Im}R_\nu(h)$, where $R_\nu(h) : H^1(T^2;\mathbf{Z}) \to S^1$ is the $\nu$-rotation number map of $h$. This may be seen as the discrete analogue of proposition 3.4 in [4]. It follows from this and the continuity of the Ruelle rotation number with respect to the $C^1$ topology [7], that the differential of the Ruelle rotation in group cohomology is a real bounded 2-cocycle on the "universal covering" group $\widetilde{Diff_0^1}(T^2,\nu)$ of the identity path component of the group of the $\nu$-preserving $C^1$ diffeomorphisms of $T^2$, which depends only on $\nu$ and not on the trivialization of $TT^2$. So it defines an element of the second bounded cohomology with real coefficients of $\widetilde{Diff_0^1}(T^2,\nu)$ made discrete, the Ruelle rotation class. This remark is perhaps the core of the present paper.

The Ruelle rotation class belongs to the kernel of the natural homomorphism $H_b^2(\widetilde{Diff_0^1(T^2,\nu)};\mathbf{R}) \to H^2(\widetilde{Diff_0^1(T^2,\nu)};\mathbf{R})$, which is known to be injective if $\widetilde{Diff_0^1(T^2,\nu)}$ is uniformly perfect [5]. Thus, it can be considered as an obstruction to the uniform perfectness of $\widetilde{Diff_0^1(T^2,\nu)}$. For instance, it follows from theorem 4.3 in [2] that if $\nu$ is the Haar measure, then the Ruelle rotation class is not trivial and so in this case $\widetilde{Diff_0^1(T^2,\nu)}$ is not uniformly perfect and not amenable. The same is true in the case of a Dirac point measure also.

## 2. Preliminaries on rotation vectors in homology

Let $X$ be a compact metrizable space carrying a continuous flow $(\phi_t)_{t\in\mathbf{R}}$. Let $tx$ denote the translation of the point $x \in X$ along its orbit in time $t \in \mathbf{R}$. For every continuous function $f : X \to S^1$ there is a continuous function $g : \mathbf{R} \times X \to \mathbf{R}$, called the *cocycle* of $f$ such that $f(tx) = f(x)\exp(2\pi i g(t,x))$ and $g(t+s,x) = g(s,tx) + g(t,x)$ for every $x \in X$ and $t,s \in \mathbf{R}$. The Ergodic Theorem of Birkhoff implies that for every $\phi$-invariant Borel probability measure $\mu$ on $X$ the limit

$$g^*(x) = \lim_{t\to+\infty} \frac{g(t,x)}{t}$$

exists $\mu$-almost for every $x \in X$ and $\int_X g^* d\mu = \int_X g(1,.)d\mu$. This integral describes the $\mu$-average rotation of points moving along their orbits with respect to the projection $f$. The cocycle property of $g$ implies that $g^*$ is an $\mu$-almost everywhere defined measurable flow invariant function, that is $g^*(tx) = g^*(x)$ for every $t \in \mathbf{R}$, whenever $g^*(x)$ is defined.

If $f_1$, $f_2 : X \to S^1$ are homotopic continuous functions with cocycles $g_1$, $g_2$, respectively, then $\int_X g_1^* d\mu = \int_X g_2^* d\mu$. Since the first integral $\check{C}$ech cohomology group $\check{H}^1(X;\mathbf{Z})$ is isomorphic to the group of homotopy classes of continuous functions of $X$ to $S^1$, there is a group homomorphism $A_\mu : \check{H}^1(X;\mathbf{Z}) \to \mathbf{R}$ defined by

$$A_\mu[f] = \int_X g(1,.)d\mu$$

where $g$ is the cocycle of $f : X \to S^1$ and $[f]$ the homotopy class of $f$.The homomorphism $A_\mu$ was defined by S.Schwartzman [8] and is called the *$\mu$-asymptotic cycle* of the flow. It describes how a $\mu$-average orbit winds around $X$. The image of $A_\mu$ is called the *$\mu$-winding numbers group* of the flow $\phi$ and is denoted by $W_\mu$. An exposition of the basic theory of asymptotic cycles with details is given in [1].

The suspensions of homeomorphisms is a class of flows whose winding numbers groups can be computed in terms of initial data. Let $Y$ be a compact metrizable space and let $h : Y \to Y$ be a homeomorphism. On $[0,1]\times Y$ we consider the equivalence relation $(1,x) \sim (0,h(x))$, $x \in Y$. The quotient space $X = [0,1] \times Y/\sim$ is compact metrizable and is called the *mapping torus* of $h$. Let $[s,x]$ denote the class of $(s,x) \in [0,1] \times Y$. The flow on $X$ defined by

$$t[s,x] = [t+s-n, h^n(x)]$$

if $n \leq t+s < n+1$, is called the *suspension* of $h$.

If $\nu$ is an $h$-invariant Borel probability measure on $Y$ and $\lambda$ is the Lebesgue measure on $[0,1]$, then the product measure $\lambda \times \nu$ induces a Borel probability measure on $X$ which is invariant by the suspension of $h$. Every invariant by the suspension of $h$ Borel probability measure on $X$ is of this form.

Let $C(Y,\mathbf{Z})$ be the group of integer valued continuous functions on $Y$. Let $\gamma : C(Y,\mathbf{Z}) \to \check{H}^1(X;\mathbf{Z})$ be defined by $\gamma(\psi) = [f]$, where $f : X \to S^1$ is the continuous function defined by $f[t,x] = \exp(2\pi i t\psi(x))$ and $j^* : \check{H}^1(X;\mathbf{Z}) \to \check{H}^1(Y;\mathbf{Z})$ be the homomorphism induced by the inclusion $j : Y \to X$ with $j(x) = [0,x]$. Then one can easily verify that the following sequence is exact.

$$C(Y,\mathbf{Z}) \xrightarrow{h^*-id} C(Y,\mathbf{Z}) \xrightarrow{\gamma} \check{H}^1(X;\mathbf{Z}) \xrightarrow{j^*} \check{H}^1(Y;\mathbf{Z}) \xrightarrow{h^*-id} \check{H}^1(Y;\mathbf{Z})$$

If $\nu$ is an $h$-invariant Borel probability measure on $Y$ and $\mu$ is the corresponding invariant measure on $X$, then

$$W_\mu = \left\{ \int_Y \psi d\nu : \psi \in Log(Y,h) \right\}$$

where $Log(Y,h)$ is the set of continuous functions $\psi : Y \to \mathbf{R}$ satisfying $f \circ h = f \exp(2\pi i \psi)$ for some continuous function $f : Y \to S^1$ [1].

Let $Y$ be in addition connected and let $f : Y \to S^1$ be a continuous function such that $[f] \in \mathrm{Ker}(h^* - id)$. There is a continuous function $g : Y \to \mathbf{R}$ such that $f \circ h = f \exp(2\pi i g)$ and any two such functions differ by an integer. One can easily see that there is a well defined group homomorphism $R_\nu : \mathrm{Ker}(h^* - id) \to S^1$ with

$$R_\nu[f] = \exp\Big(2\pi i \int_Y g d\nu\Big)$$

called the *$\nu$-rotation number map* of $h$. The subgroup $\mathrm{Im}R_\nu$ of $S^1$ is called the *$\nu$-rotation numbers group* of $h$. We have now the following commutative diagram.

$$\begin{array}{ccc} \check{H}^1(X;\mathbf{Z}) & \xrightarrow{A_\mu} & \mathbf{R} \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle \exp} \\ \mathrm{Ker}(h^* - id) & \xrightarrow{R_\nu} & S^1 \end{array}$$

Thus we arrive at the following.

**2.1. Proposition.** *Let $Y$ be a connected, compact, metrizable space and $h : Y \to Y$ a homeomorphism. Let $\nu$ be an $h$-invariant Borel probability measure on $Y$. If $j^*$ has a right inverse, then $R_\nu \in \mathrm{Im}F$, where*

$$F : \mathrm{Hom}(\mathrm{Ker}(h^* - id), \mathbf{R}) \to \mathrm{Hom}(\mathrm{Ker}(h^* - id), S^1)$$

*is the homomorphism defined by $F(\alpha) = \exp(2\pi i\alpha)$.*

If $h : Y \to Y$ is homotopic to the identity and $\check{H}^1(Y;\mathbf{Z})$ is free, then $R_\nu$ can be considered an element of $\mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}), \mathbf{R}) / \mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}), \mathbf{Z})$, since $\mathrm{Ker}\, F = \mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}), \mathbf{Z})$, where $F$ is the homomorphism of Proposition 2.1.

If Y is a closed manifold, then $\mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}), \mathbf{R}) / \mathrm{Hom}(\check{H}^1(Y;\mathbf{Z}), \mathbf{Z})$ is isomorphic to $H_1(Y;\mathbf{R})/H_1(Y;\mathbf{Z})$, by the Universal Coefficient Theorems (see [9]). Thus for any $h$-invariant Borel probability measure $\nu$ on $Y$, the rotation number map $R_\nu$ can be considered an element of $H_1(Y;\mathbf{R})/H_1(Y;\mathbf{Z})$. A representative of $R_\nu$ in $H_1(Y;\mathbf{R})$ is $A_\mu \circ r$, where $r$ is a right inverse of $j^*$. The set $\rho(h) = \{R_\nu : \nu \in \mathcal{M}_h(Y)\}$ is the *ergodic rotation set* of $h$, where $\mathcal{M}_h(Y)$ denotes the set of $h$-invariant Borel probability measures on $Y$ and is weakly compact in $H_1(Y;\mathbf{R})/H_1(Y;\mathbf{Z})$ [6].

**2.2. Example.** Let $h : T^n \to T^n$ be a homeomorphism of the $n$-torus homotopic to the identity. If $p_i : T^n \to S^1$ is the projection onto the $i$-th factor, $1 \le i \le n$, then the homotopy classes $[p_1], ..., [p_n]$ form a basis of $H^1(T^n;\mathbf{Z})$. Let $\nu$ be a $h$-invariant Borel probability measure and $\mu$ the corresponding measure that is invariant by the suspension of $h$. Then $R_\nu$ is represented by the vector

$$(A_\mu(r[p_1]), ..., A_\mu(r[p_n]))$$

with respect to the dual basis of $H_1(T^n;\mathbf{R})$.

Let $\tilde{h} : \mathbf{R}^n \to \mathbf{R}^n$ be a lift of $h$ and $\psi : T^n \to \mathbf{R}^n$ be the continuous function defined by $\psi(x) = \tilde{h}(y) - y$, where $y \in p^{-1}(x)$ and $p : \mathbf{R}^n \to T^n$ is the canonical covering projection. Then,

$$(A_\mu(r[p_1]), ..., A_\mu(r[p_n])) = \int_{T^n} \psi d\nu \ (\mathrm{mod}\ \mathbf{Z}^n).$$

That is, $R_\nu$ is represented by the $\nu$-mean translation vector of $h$. In case $\nu$ is ergodic, the Ergodic Theorem of Birkhoff implies that

$$\int_{T^n} \psi d\nu = \lim_{k\to+\infty} \frac{\tilde{h}^k(y)-y}{k}$$

$p^{-1}\nu$-almost for every $y \in [0,1)^n$.

*2.3. Remark.* Let $H$ be a homotopy from $id$ to $h$. We can extend it to $[0,+\infty)\times Y$ by setting inductively $H_t = H_{t-[t]} \circ H_{[t]}$. If $f : Y \to S^1$ is a continuous function, there exists a unique continuous function $g : [0,+\infty)\times Y \to \mathbf{R}$ such that $g(0,x) = 0$ and

$$\frac{f(H_t(x))}{f(x)} = \exp(2\pi i g(t,x)),$$

for every $t \geq 0$ and $x \in Y$. Then, $g(t,x) = g(t-[t], h^{[t]}(x)) + g([t],x)$ and inductively

$$g(n,x) = \sum_{k=0}^{n-1} g(1, h^k(x))$$

for every $n \in \mathbf{N}$. It follows from the Ergodic Theorem of Birkhoff that for every $\nu \in \mathcal{M}_h(Y)$ the limit

$$g^*(x) = \lim_{t\to+\infty} \frac{g(t,x)}{t} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(1, h^k(x))$$

exists $\nu$-almost for every $x \in Y$, is $\nu$-integrable and

$$R_\nu[f] = \exp\Big(2\pi i \int_Y g^* d\nu\Big).$$

*2.4. Remark.* Let $h_1$, $h_2$ be two homeomorphisms of the compact manifold $Y$, which are homotopic to the identity and preserve the Borel probability measure $\nu$. If $R_\nu(h_k)$ is the $\nu$-rotation number map of $h_k$, $k = 1,2$, then

$$R_\nu(h_1 \circ h_2) = R_\nu(h_1) R_\nu(h_2).$$

Indeed, if $f : Y \to S^1$ is a continuous function for which there are continuous functions $g_k : Y \to \mathbf{R}$ such that $f \circ h_k = f \exp(2\pi i g_k)$, $k = 1,2$, then

$$f \circ h_1 \circ h_2 = f \exp(2\pi i (g_1 \circ h_2 + g_2)).$$

Since $\nu$ is $h_2$-invariant, we have

$$R_\nu(h_1 \circ h_2)[f] = \exp(2\pi i \int_Y (g_1 \circ h_2 + g_2) d\nu) = R_\nu(h_1)[f] R_\nu(h_2)[f].$$

## 3. Rotation number maps and $SL(2, \mathrm{R})$

Every $A \in SL(2,\mathbf{R})$ has a unique decomposition $A = US$, where $U \in SO(2,\mathbf{R})$ and $S$ is a symmetric, positive definite matrix. This is the *polar decomposition* of $A$. In this way we get a $C^\infty$ diffeomorphism $S^1 \times (0,+\infty) \times \mathbf{R} \cong SL(2,\mathbf{R})$. It is easily verified [10] that

$$U = |\det(A + (A^t)^{-1})|^{-1/2}(A + (A^t)^{-1}).$$

If $p : \widetilde{SL(2,\mathbf{R})} \to SL(2,\mathbf{R})$ is the universal covering of $SL(2,\mathbf{R})$, then the preceeding diffeomorphism can be lifted to a $C^\infty$ diffeomorphism $\mathbf{R} \times (0,+\infty) \times \mathbf{R} \cong \widetilde{SL(2,\mathbf{R})}$. The group of the deck transformations is isomorphic to $\mathbf{Z}$. Let $d$ be a generator and $D = d(\tilde{I}_2)$, where $\tilde{I}_2$ is the unit element of $\widetilde{SL(2,\mathbf{R})}$. Since $p$ is a homomorphism, we have $P^{-1}d(P) \in \operatorname{Ker} p$ and $d(P)P^{-1} \in \operatorname{Ker} p$, for every $P \in \widetilde{SL(2,\mathbf{R})}$. The continuity of $d$, the connectedness of $\widetilde{SL(2,\mathbf{R})}$ and the discreteness of $\operatorname{Ker} p$ imply that $d(P) = PD = DP$ for every $P \in \widetilde{SL(2,\mathbf{R})}$.

The map $r : SL(2,\mathbf{R}) \to SO(2,\mathbf{R})$ defined by $r(A) = U$ is a $C^\infty$ submersion and a strong deformation retraction that can be lifted to a $C^\infty$ submersion $\Theta : \widetilde{SL(2,\mathbf{R})} \to \mathbf{R}$, which has the following properties :

(i) If $P, Q$ correspond to elements of $\{0\} \times (0,+\infty) \times \mathbf{R}$, then $|\Theta(PQ)| < 1$, and

(ii) $|\Theta(PQ) - \Theta(P) - \Theta(Q)| < 1$ for every $P, Q \in \widetilde{SL(2,\mathbf{R})}$.

Of course $r$ is not a group homomorphism.

**3.1. Lemma.** *Under the usual multiplication of matrices $r$ is a homomorphism of $H$-groups.*

*Proof.* Let $A, B \in SL(2,\mathbf{R})$ have polar decompositions $A = r(A)s(A)$ and $B = r(B)s(B)$. Since $s(A)$ is symmetric and positive definite, $s(A)^\rho$ is defined for every $\rho \geq 0$. Note that since

$$AB = r(A)r(B)[r(B)^{-1}s(A)r(B)s(B)],$$

by the uniqueness of the polar decomposition we have

$$r(B)^{-1}r(A)^{-1}r(AB) = r(r(B)^{-1}s(A)r(B)s(B)).$$

The function $F : [0,1] \times SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \to SO(2,\mathbf{R})$ defined by

$$F(\rho, A, B) = r(A)r(B)r((r(B)^{-1}s(A)r(B))^{1-\rho}s(B)^{1-\rho})$$

is continuous and $F(0, A, B) = r(AB)$, $F(1, A, B) = r(A)r(B)$. Hence $r$ is a homomorphism of $H$-groups.

Let $Y$ be a compact metrizable space. The set $[Y; SL(2,\mathbf{R})]$ of homotopy classes of continuous functions of $Y$ into $SL(2,\mathbf{R})$ becomes a group in the obvious way, using the group structure of $SL(2,\mathbf{R})$. It follows from Lemma 3.1 that the induced map $r^* : [Y; SL(2,\mathbf{R})] \to \check{H}^1(Y;\mathbf{Z})$ is a group homomorphism. Since $r$ is a strong deformation retraction, $r^*$ is an isomorphism whose inverse is induced by the inclusion $SO(2,\mathbf{R}) \hookrightarrow SL(2,\mathbf{R})$.

Suppose that $Y$ is also connected and $h : Y \to Y$ is a homeomorphism homotopic to the identity. Let $f : Y \to SL(2,\mathbf{R})$ be a continuous function. If $H$ is a homotopy from $id$ to $h$, then $(f \circ H)f^{-1}$ is a homotopy from $I_2$ to $(f \circ h)f^{-1}$. We can extend $H$ to $[0,+\infty) \times Y$ inductively, by setting $H_t = H_{t-[t]} \circ H_{[t]}$. By the homotopy lifting property, there is a unique continuous map $G : [0,+\infty) \times Y \to \widetilde{SL(2,\mathbf{R})}$ such that $G(0,x) = \tilde{I}_2$ and

$$p(G(t,x)) = f(H_t(x))f(x)^{-1}$$

for every $t \geq 0$ and $x \in Y$. It follows that

$$f(H_t(x))f(x)^{-1} = p(G(t-[t], h^{[t]}(x))G([t],x).$$

Thus, there exists a function $k : [0,+\infty) \times Y \to \mathbf{Z}$ such that

$$G(t-[t], h^{[t]}(x))G([t],x)G(t,x)^{-1} = D^{k(t,x)}.$$

For each $x \in Y$, it is obvious that $k(.,x)$ is continuous on $[0,+\infty) \setminus \mathbf{N}$ and continuous from the right on $[0,+\infty)$. Since

$$\lim_{t \nearrow 1} G(t-[t], h^{[t]}(x))G([t],x)G(t,x)^{-1} = G(1,x)G(0,x)G(1,x)^{-1} = \tilde{I}_2,$$

it follows inductively that $k(.,x)$ is everywhere continuous and hence identically zero, by connectedness. Thus,

$$G(t,x) = G(t-[t], h^{[t]}(x))G([t],x)$$

for every $t \geq 0$ and $x \in Y$, and inductively

$$G(n,x) = G(1, h^{n-1}(x))...G(1,h(x))G(1,x)$$

for every $n \in \mathbf{N}$ and $x \in Y$. According to the Ergodic Theorem for $\widetilde{SL(2,\mathbf{R})}$ [7], which is a direct consequence of the almost subadditive ergodic theorem proved in [3], if $\nu$ is an $h$-invariant Borel probability measure on $Y$, the limit

$$G^*(x) = \lim_{n\to+\infty} \frac{1}{n}\Theta(G(n,x))$$

exists $\nu$-almost for every $x \in Y$, is $h$-invariant, $\nu$-integrable and

$$\int_Y G^* d\nu = \lim_{n\to+\infty} \frac{1}{n} \int_Y \Theta(G(n,x)) d\nu.$$

Since for every $t > 1$ we have

$$|\Theta(G(t,x)) - \Theta(G(t-[t], h^{[t]}(x)) - \Theta(G([t],x))| < 1$$

and by compactness and uniform continuity

$$\lim_{t\to+\infty} \frac{1}{t} \Theta(G(t-[t], h^{[t]}(x)) = 0,$$

we conclude that

$$G^*(x) = \lim_{t\to+\infty} \frac{1}{t} \Theta(G(t,x)).$$

**3.2. Lemma.** *The limit* $G^*$ (mod $\mathbf{Z}$) *does not depend on the choice of the homotopy* $H$ *from* $id$ *to* $h$.

*Proof.* If $\phi = (f \circ h) f^{-1}$, then $G(1,.)$ is a lift of $\phi$ and it is sufficient to prove that if $\psi : Y \to \widetilde{SL}(2,\mathbf{R})$ is another lift of $\phi$ and

$$G'(n,x) = \psi(h^{n-1}(x))...\psi(h(x))\psi(x),$$

then

$$\lim_{n\to+\infty} \frac{1}{n} \Theta(G'(n,x)) = \lim_{n\to+\infty} \frac{1}{n} \Theta(G(n,x)).$$

Indeed, there exists $m \in \mathbf{Z}$ such that $\psi(x) = \widetilde{D^m G}(1,x)$ for every $x \in Y$, and since $D$ commutes with every element of $\widetilde{SL}(2,\mathbf{R})$, we have $G'(n,x) = D^{mn} G(n,x)$. Note that $\Theta(D^{mn}) = mn$, since $D \in \operatorname{Ker} p$. Thus, we have $|\Theta(G'(n,x)) - \Theta(G(n,x)) - mn| < 1$ and the conclusion follows.

**3.3. Proposition.** *If* $R_\nu$ *is the rotation number map of* $h$ *with respect to the* $h$*-invariant Borel probability measure* $\nu$*, then using the above notation,*

$$(R_\nu \circ r^*)[f] = \exp\Big(2\pi i \int_Y G^* d\nu\Big).$$

*Proof.* There exists a unique continuous function $g : [0,+\infty) \times Y \to \mathbf{R}$ such that $g(0,x) = 0$ and

$$\frac{r(f(H_t(x)))}{r(f(x))} = \exp(2\pi i g(t,x))$$

for every $t \geq 0$ and $x \in Y$. Choose any $Q(x) \in \widetilde{SL(2,\mathbf{R})}$ such that $p(Q(x)) = f(x)$. Then, we have

$$\frac{r(f(H_t(x)))}{r(f(x))} = \frac{1}{r(f(x))} r(p(G(t,x)Q(x)))$$

and so there exists $k(t,x) \in \mathbf{Z}$ such that

$$g(t,x) = \Theta(G(t,x)Q(x)) - \Theta(Q(x)) + k(t,x).$$

By continuity with respect to $t$ and connectedness, we get $k(t,x) = k(0,x) = 0$. Hence

$$g^*(x) = \lim_{t \to +\infty} \frac{g(t,x)}{t} = \lim_{t \to +\infty} \frac{1}{t}[\Theta(G(t,x)Q(x)) - \Theta(Q(x))]$$

$\nu$-almost for every $x \in Y$ and by property (ii) of $\Theta$, we get $g^*(x) = G^*(x)$. This proves the proposition.

## 4. The Ruelle rotation

Let $h : T^2 \to T^2$ be a $C^1$ diffeomorphism isotopic to the identity. Let $H$ be an isotopy from $id$ to $h$. We need not assume that $H$ is $C^1$, but merely that $H : [0,1] \times T^2 \to T^2$ is a continuous map such that $H_0 = id$, $H_1 = h$ and $H_t : T^2 \to T^2$, $t \in [0,1]$, is a continuous path of $C^1$ diffeomorphisms in the $C^1$ topology. As usual we extend it to $[0,+\infty) \times T^2$ by setting inductively $H_t = H_{t-[t]} \circ H_{[t]}$. Let $\tau : TT^2 \to T^2 \times \mathbf{R}^2$ be a trivialization of the tangent bundle of $T^2$, compatible with its usual orientation. Let $\tau_x : T_xT^2 \to \{x\} \times \mathbf{R}^2$ denote the restriction to the fiber. The continuous function $F : [0,+\infty) \times T^2 \to SL(2,\mathbf{R})$ defined by

$$F(t,x) = \frac{\tau_{H(t,x)} \circ DH_t(x) \circ \tau_x^{-1}}{(\det(\tau_{H(t,x)} \circ DH_t(x) \circ \tau_x^{-1}))^{1/2}}$$

can be lifted to a unique continuous function $G : [0,+\infty) \times T^2 \to \widetilde{SL(2,\mathbf{R})}$ such that $G(0,x) = \tilde{I}_2$ and $p(G(t,x)) = F(t,x)$, for every $t \geq 0$ and $x \in T^2$. From the chain rule and a same argument used in an analogous situation of Sect. 3 we have $G(t,x) = G(t-[t], h^{[t]}(x))G([t],x)$. So, inductively, $G(n,x) = G(1,h^{n-1}(x))...G(1,h(x))G(1,x)$. According to the Ergodic Theorem for $\widetilde{SL(2,\mathbf{R})}$ [7], if $\nu$ is an $h$-invariant Borel probability measure, the limit

$$G^*(x) = \lim_{t \to +\infty} \frac{1}{t}\Theta(G(t,x)) = \lim_{n \to +\infty} \frac{1}{n}\Theta(G(n,x))$$

exists $\nu$-almost for every $x \in T^2$, is $h$-invariant, $\nu$-integrable and

$$\int_{T^2} G^* d\nu = \lim_{n\to+\infty} \frac{1}{n} \int_{T^2} \Theta(G(n,x))d\nu.$$

The method of proof of Lemma 3.2 applies here also to show that $G^*$ (mod $\mathbf{Z}$) is independent on the isotopy $H$. It does depend however on the trivialization $\tau$ of the tangent bundle of $T^2$. Let

$$\rho_\nu^\tau(H) = \int_{T^2} G^* d\nu,$$

and $P_\nu^\tau(h) = \exp(2\pi i \rho_\nu^\tau(H))$.

Let now $\tau_1$ and $\tau_2$ be two trivializations of $TT^2$ compatible with the usual orientation of $T^2$ and $f : T^2 \to SL(2,\mathbf{R})$ be the continuous function defined by

$$f(x) = \frac{\tau_{1x} \circ \tau_{2x}^{-1}}{(\det(\tau_{1x} \circ \tau_{2x}^{-1}))^{1/2}}.$$

**4.1. Proposition.** *If $R_\nu(h)$ is the $\nu$-rotation number map of $h$, then*

$$R_\nu(h)(r^*[f]) = P_\nu^{\tau_1}(h) P_\nu^{\tau_2}(h)^{-1}.$$

*Proof.* Let $F_j$, $G_j$ be the functions used in the definition of $\rho_\nu^{\tau_j}$ as above, $j = 1,2$ and $G : [0,+\infty) \times T^2 \to \widetilde{SL(2,\mathbf{R})}$ be the unique continuous map such that $G(0,x) = \tilde{I}_2$ and $p(G(t,x)) = f(H_t(x))f(x)^{-1}$ for every $t \geq 0$ and $x \in T^2$. It is easy to see that since

$$F_1(t,x) = (f(H_t(x))f(x)^{-1})(f(x)F_2(t,x)f(x)^{-1}),$$

we have $G_1(t,x) = G(t,x)Q(x)G_2(t,x)Q(x)^{-1}$, where $p(Q(x)) = f(x)$. This imlpies that $G_1^* = G^* + G_2^*$, $\nu$-almost everywhere and the conclusion follows from Proposition 3.3.

The preceeding Proposition shows that $\rho_\nu(h) = P_\nu^\tau(h)\mathrm{Im}R_\nu(h)$ is a well defined element of the quotient group $S^1/\mathrm{Im}R_\nu(h)$, independent of the trivialization $\tau$, and is called the *Ruelle rotation* of $h$ with respect to the invariant measure $\nu$.

*4.2. Remark.* For every $x \in T^2$ and $n \in \mathbf{N}$ we have

$$|\frac{1}{n}\Theta(G(n,x)) - \frac{1}{n}\sum_{k=0}^{n-1} \Theta(G(1,h^k(x)))| < 1.$$

From the Ergodic Theorem of Birkhoff the limit

$$\tilde{G}(t,x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Theta(G(1,h^k(x)))$$

exists $\nu$-almost for every $x \in T^2$, is $h$-invariant, $\nu$-integrable and

$$\int_{T^2} \tilde{G}(1,x)d\nu = \int_{T^2} \Theta(G(1,x))d\nu.$$

Thus, taking limits and integrating, we get

$$|\rho_\nu^\tau(H) - \int_{T^2} \Theta(G(1,x))d\nu| \leq 1.$$

Let now $\nu$ be a Borel probability measure on $T^2$ and $Diff_0^1(T^2,\nu)$ be the path component of the identity in the $C^1$ topology of the group of $C^1$ diffeomorphisms of $T^2$ which preserve $\nu$. In the space of continuous paths in $Diff_0^1(T^2,\nu)$ with initial point $id$ endowed with the compact-open topology the relation of homotopy with fixed endpoints is an equivalence relation. Let $\widetilde{Diff_0^1}(T^2,\nu)$ be the corresponding quotient space and $q : \widetilde{Diff_0^1}(T^2,\nu) \to Diff_0^1(T^2,\nu)$ be the continuous map $q([H]) = H_1$. If $Diff_0^1(T^2,\nu)$ is locally contractible, then $q$ is its universal covering projection. The space $\widetilde{Diff_0^1}(T^2,\nu)$ has a topological group structure, if we define the composition of paths by $(H \circ H')_t = H_t \circ H'_t$, $t \in [0,1]$, and then $q$ becomes a group homomorphism.

**4.3. Lemma.** *If $H$ and $H'$ are two homotopic paths with fixed endpoints the initial point being $id$ in $Diff_0^1(T^2,\nu)$, then $\rho_\nu^\tau(H) = \rho_\nu^\tau(H')$ for every trivialization $\tau$ of $TT^2$.*

*Proof.* Let $(H^s)_{s\in[0,1]}$ be a homotopy with fixed endpoints such that $H^0 = H$ and $H^1 = H'$. Let $F^s$ and $G^s$ be the functions as above in the definition of the Ruelle rotation corresponding to $H^s$. Then, $F^s(n,x) = F^0(n,x)$, for every $x \in T^2$ and $n \in \mathbf{N}$, and $G^s(1,.)$ is a lift of $F^0(1,.)$ for every $s \in [0,1]$. Thus, there exist $m(s) \in \mathbf{Z}$ such that $G^s(1,.) = G^0(1,x)D^{m(s)}$, for every $x \in T^2$ and $s \in [0,1]$. By continuity and connectedness, we must have $m(s) = m(0) = 0$. So, $G^s(1,.) = G^0(1,.)$ for every $s \in [0,1]$ and the conclusion follows.

So we have a well defined function $\rho_\nu^\tau : \widetilde{Diff_0^1}(T^2,\nu) \to \mathbf{R}$ and the following diagramm commutes.

$$\begin{array}{ccc} \widetilde{Diff_0^1}(T^2,\nu) & \xrightarrow{\rho_\nu^\tau} & \mathbf{R} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \exp} \\ Diff_0^1(T^2,\nu) & \xrightarrow{P_\nu^\tau} & S^1 \end{array}$$

The continuous case of the Ergodic Theorem for $\widetilde{SL}(2,\mathbf{R})$ implies also that $\rho_\nu^\tau$ is continuous with respect to the $C^1$ topology in $\widetilde{Diff_0^1}(T^2,\nu)$ (see [7]).

It is immediate from the definition that $\rho_\nu^\tau(H^m) = m\rho_\nu^\tau(H)$ for every $m \in \mathbf{Z}$. If $H$ and $H'$ are two paths in $Diff_0^1(T^2,\nu)$ with initial point $id$, then one can easily verify using Remark 4.2 that

$$|\rho_\nu^\tau(H \circ H') - \rho_\nu^\tau(H) - \rho_\nu^\tau(H')| \le 4.$$

Thus, $\rho_\nu^\tau$ is a homogeneous quasi-morphism. Using this one can show that $P_\nu^\tau$ is constant on conjugancy classes in $Diff_0^1(T^2,\nu)$ (see the remark following proposition 2.8 in [4]).

We recall now the notion of bounded cohomology with real coefficients of a discrete group $G$. Let $B^n(G)$ be the Banach space of all bounded real functions on $G^n$, $n \ge 1$. If $d_n : B^n(G) \to B^{n+1}(G)$ is the continuous operator defined by

$$(d_n f)(g_0, ..., g_n) = f(g_1, ..., g_n) + \sum_{k=1}^{n} (-1)^k f(g_0, ..., g_{k-1}g_k, ..., g_n)$$
$$+(-1)^{n+1} f(g_0, ..., g_{n-1}),$$

then $d_{n+1} \circ d_n = 0$. The bounded cohomology of $G$ with real coefficients $H_b^*(G;\mathbf{R})$ is the cohomology of the cochain complex of Banach spaces $(B^*(G), d_*)$. It is always true that $H_b^1(G;\mathbf{R}) = 0$.

It follows from the above that $d\rho_\nu^\tau : \widetilde{Diff_0^1}(T^2,\nu) \times \widetilde{Diff_0^1}(T^2,\nu) \to \mathbf{R}$ defined by $(d\rho_\nu^\tau)(H,H') = \rho_\nu^\tau(H \circ H') - \rho_\nu^\tau(H) - \rho_\nu^\tau(H')$ is a bounded 2-cocycle. If $\tau_1$, $\tau_2$ are two trivializations of $TT^2$, it follows from Remark 2.4 and Proposition 4.1 that $d\rho_\nu^{\tau_1} - d\rho_\nu^{\tau_2}$ takes on integer values. Since $\widetilde{Diff_0^1}(T^2,\nu)$ is connected, and $\rho_\nu^{\tau_k}$, $k = 1, 2$, are continuous, both taking the value zero at $id$, we conclude that $d\rho_\nu^{\tau_1} = d\rho_\nu^{\tau_2}$. Thus, in the sequel we shall simply write $d\rho_\nu$. We shall call the element $[d\rho_\nu]_b$ of the second bounded cohomology with real coefficients of the group $\widetilde{Diff_0^1}(T^2,\nu)$ made discrete, the *Ruelle rotation class*. It belongs to the kernel of the natural homomorphism $H_b^2(\widetilde{Diff_0^1}(T^2,\nu);\mathbf{R}) \to H^2(\widetilde{Diff_0^1}(T^2,\nu);\mathbf{R})$.

**4.4. Lemma.** *If $\nu$ is a Borel probability measure on $T^2$, then $[d\rho_\nu]_b = 0$ if and only if $\rho_\nu^\tau : \widetilde{Diff_0^1}(T^2, \nu) \to \mathbf{R}$ is a group homomorphism for any trivialization $\tau$ of $TT^2$.*

*Proof.* Suppose that there exists $\sigma \in B^1(\widetilde{Diff_0^1}(T^2, \nu))$ such that $d_1\sigma = d\rho_\nu^\tau$. Then, $\sigma(id) = \rho_\nu^\tau(id) = 0$. Moreover, it follows inductively that $\sigma$ is homogeneous, since $\rho_\nu^\tau$ is. But then we must have $\sigma = 0$, because $\sigma$ is bounded. Hence $\rho_\nu^\tau$ is a group homomorphism. The converse is trivial.

**4.5. Theorem.** *The Ruelle rotation class with respect to the Haar measure on $T^2$ is not trivial.*

*Proof.* Let $D^2$ be the closed unit disc, $\lambda$ be the normalized Lebesgue measure on $D^2$ and $K$ be a collar closed neighbourhood of $\partial D^2$. The group $Diff^\infty(D^2, K, \lambda)$ of $\lambda$-preserving diffeomorphisms of $D^2$, which are the identity on $K$, is contractible in the $C^\infty$ topology. There is a continuous monomorphism $l : Diff^\infty(D^2, K, \lambda) \to Diff_0^\infty(T^2, \nu)$, where $\nu$ is the Haar measure on $T^2$. The latter is a locally contractible group in the $C^\infty$ topology. By contractibility, $l$ is lifted to a continuous homomorphism $\tilde{l} : Diff^\infty(D^2, K, \lambda) \to \widetilde{Diff_0^\infty}(T^2, \nu)$. Let $\tau$ be the trivialization of $TT^2$ defined by the derivative of the universal covering projection $p : \mathbf{R}^2 \to T^2$. According to theorem 4.3 in [2], $\rho_\nu^\tau \circ \tilde{l}$ is not a group homomorphism. Hence $\rho_\nu^\tau$ is not a group homomorphism and $[d\rho_\nu]_b$ is not trivial, by Lemma 4.4.

From corollary 2.11 in [5], we have :

**4.6. Corollary.** *The group $\widetilde{Diff_0^1}(T^2, \nu)$ is not uniformly perfect for the Haar measure $\nu$.*

Finally, we shall be concerned with the Ruelle rotation class of a Dirac measure $\nu$ at a point $x \in T^2$. In this case $Diff_0^1(T^2, \nu)$ is the path component of the identity in the $C^1$ topology of the group of $C^1$ diffeomorphisms that fix $x$.

If $h : T^2 \to T^2$ is a $C^1$ diffeomorphism isotopic to the identity that fixes $x$, then $R_\nu(h) = 1$ and $\rho(x, h) = P_\nu^\tau(h)$ is independent of the trivialization $\tau$ of $TT^2$, by Proposition 4.1. $\rho(x, h)$ is the *infinitesimal rotation number* of the fixed point $x$ of $h$. It measures the rate at which the fixed point $x$ is rotating infinitesimally.

If $H$ is an isotopy from $id$ to $h$ and $F$, $G$ as above, then

$$F(n, x) = \frac{(\tau_x \circ Dh(x) \circ \tau_x^{-1})^n}{(\det(\tau_x \circ Dh(x) \circ \tau_x^{-1}))^{n/2}}$$

for every $n \in \mathbf{N}$ and

$$\rho_\nu^\tau(H) = \lim_{n\to+\infty} \frac{1}{n}\Theta(G(n,x)).$$

Let $P \in \widetilde{SL(2,\mathbf{R})}$. Applying the Ergodic Theorem for $\widetilde{SL(2,\mathbf{R})}$ for the trivial one point probability space, we conclude that the limit

$$\Omega(P) = \lim_{n\to+\infty} \frac{1}{n}\Theta(P^n)$$

exists. Since $\Theta$ is continuous, $\Omega : \widetilde{SL(2,\mathbf{R})} \to \mathbf{R}$ is a continuous function, which is also constant on conjugancy classes, by property (ii) of $\Theta$. Moreover, we have $|\Theta((D^kP)^n) - \Theta(P^n) - kn| < 1$, from which follows that $\Omega(D^kP) = \Omega(P) + k$. Thus, $\Omega$ descends to a continuous function $\omega : SL(2,\mathbf{R}) \to S^1$ defined by $\omega(A) = \exp(2\pi i\Omega(P))$, where $P$ is any lift of $A$, which is constant on conjugancy classes.

If $A$ is hyperbolic with positive eigenvalues, then $\omega(A) = 1$. If $A$ has negative eigenvalues, then $\omega(A) = -1$. If $A$ is parabolic, then $\omega(A) = 0$. If $A$ is elliptic with eigenvalues $\cos 2\pi\phi \pm i\sin 2\pi\phi$, then $\omega(A) = e^{2\pi i\phi}$.

Let $A$ be a rotation by an angle $2\pi\phi$, with $0 < \phi < 1/4$ and $B$ be diagonal hyperbolic with eigenvalues $-4$ and $-1/4$. Then $AB$ is hyperbolic with negative eigenvalues and according to the above we have $\omega(AB) = \omega(B) = -1$ and $\omega(A) = e^{2\pi i\phi}$. Hence $\omega$ is not a group homomorphism.

**4.7. Theorem.** *The Ruelle rotation class with respect to a Dirac point measure is not trivial.*

*Proof.* With the above notation we have

$$\rho(x,h) = \omega(Dh(x)/(\det Dh(x))^{1/2}),$$

for every $h \in Diff_0^1(T^2,\nu)$, where $\nu$ is the Dirac measure at $x$. The above remarks imply that $\rho(x,.) : Diff_0^1(T^2,\nu) \to S^1$ is not a group homomorphism and the result follows.

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