# Stable and unstable minimal attractors

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## 0. Introduction

The study of compact invariant sets plays a central role in the qualitative theory of differential equations and dynamical systems. There are basic difficulties in this study.

1. The compact invariant sets are global objects and so one needs to develope global methods and tools for their study.

2. Their structure may be extremely complicated.

3. Even in the case of a simple compact invariant set, its structure may change dramatically under small perturbations of the system.

4. The structurally stable dynamical systems are not dense.

In practice, when studying a parametrized family of differential equations one has to handle all these four problems simultaneously.

We will be concerned with the study of the topology and dynamics in compact invariant sets of a continuous flow, and in particular compact minimal sets, in connection with the description of the dynamics around such a set. In particular, we want to examine how the complexity of a compact minimal set affects the behaviour of the flow around it. The most famous classical results are the Poincaré-Bendixson Theorem, about the structure of compact minimal sets for planar flows, and and the Lyapunov Stability Theorem, about the behavior of a flow near a compact invariant set.

## 1. Stable attractors

The simplest behavior occurs near an asymptotically stable compact invariant set. Let $(\phi_t)_{t\in\mathbb{R}}$ be of a continuous flow on a separable, locally compact, metrizable space $M$. The positive limit set of $x \in M$ is the closed, invariant set

$$L^+(x) = \{y \in M : \phi_{t_n}(x) \to y \quad \text{for some} \quad t_n \to +\infty\}.$$

Obviously, $L^+(\phi_t(x)) = L^+(x)$ for every $t \in \mathbb{R}$. Let $A \subset M$ be a compact invariant set. The invariant set

$$W^+(A) = \{x \in M : \varnothing \neq L^+(x) \subset A\}$$

is called the region of attraction of $A$. If $W^+(A)$ is an open neighbourhood of $A$, then $A$ is called an attractor. A compact invariant set $A$ is called (positively) Lyapunov stable if every neighbourhood of $A$ contains a positively invariant open neighbourhood of $A$. A Lyapunov stable attractor is usually called asymptotically stable.

If $A \subset M$ is an asymptotically stable compact invariant set, there exists a continuous Lyapunov function $f : M \to [0,1]$ such that

(i) $f^{-1}(0) = A$ and $f^{-1}(1) = M \setminus W^+(A)$, and

(ii) $f(\phi_t(x)) < f(x)$ for every $t > 0$ and $x \in W^+(A) \setminus A$.

One way to construct $f$ is the following. Let $\psi : M \to [0,1]$ be a continuous function such that $\psi^{-1}(0) = A$ and $\psi^{-1}(1) = M \setminus V$, where $V$ is a positively invariant, open neighbourhood of $A$ with $\overline{V} \subset W^+(A)$. If $g : M \to [0,1]$ is the continuous function defined by $g(x) = \sup\{\psi(\phi_t(x)) : t \geq 0\}$, then $g^{-1}(0) = A$, $M \setminus W^+(A) \subset g^{-1}(1)$ and $g(\phi_t(x)) \leq g(x)$ for $t \geq 0$ and $x \in M$. Moreover, $\lim_{t\to+\infty} g(\phi_t(x)) = 0$ for every $x \in W^+(A)$. It suffices now to define

$$f(x) = \int_0^{+\infty} e^{-t} g(\phi_t(x))dt.$$

If $0 < c < 1$, for every $x \in W^+(A) \setminus A$ there exists a unique $\tau(x) \in \mathbb{R}$ such that $f(\phi_{\tau(x)}(x)) = c$. Actually,

$$\tau(x) = \sup\{t \in \mathbb{R} : \phi_t(x) \in M \setminus f^{-1}([0, c])\}.$$

Obviously, $\tau(\phi_t(x)) = \tau(x) - t$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. We set $\tau(A) = -\infty$. Note that for every positively invariant, open neighbourhood $V$ of $A$ with compact closure there exists $0 < c < 1$ such that $f^{-1}([0, c]) \subset V$. Indeed, since $\partial V$ is compact, it suffices to take any $0 < c < \inf\{f(x) : x \in \partial V\}$.

**Lemma 1.1.** *If $0 < c < 1$ is such that $f^{-1}([0, c])$ is compact, then the so defined function $\tau : W^+(A) \to [-\infty, +\infty)$ is continuous.*

*Proof.* If $x \in W^+(A) \setminus A$ and $\epsilon > 0$, then $f(\phi_{\tau(x)+\epsilon}(x)) < c < f(\phi_{\tau(x)-\epsilon}(x))$ and from the continuity of $f$ and the flow, there exists an open neighbourhood $U$ of $x$ such that $f(\phi_{\tau(x)+\epsilon}(y)) < c < f(\phi_{\tau(x)-\epsilon}(y))$ for every $y \in U$. It follows that $\tau(x) - \epsilon < \tau(y) < \tau(x) + \epsilon$, which shows the continuity of $\tau$ on $W^+(A) \setminus A$. Let now $x \in A$ and $x_n \in W^+(A) \setminus A$, $n \in \mathbb{N}$, be such that there exists $a \in \mathbb{R}$ with $\tau(x_n) \geq a$, for every $n \in \mathbb{N}$ and $x_n \to x$. Then eventually $\tau(x_n) < 0$, and so $a < 0$. Thus, the sequence $(\tau(x_n))_{n\in\mathbb{N}}$ is bounded. Since $\phi_{\tau(x_n)}(x_n) \in f^{-1}(c)$ for every $n \in \mathbb{N}$, taking a subsequence if necessary, we may assume that there exist $a \leq t \leq 0$ and $z \in f^{-1}(c)$ such that $\tau(x_n) \to t$ and $\phi_{\tau(x_n)}(x_n) \to z$. But then $z = \phi_t(x) \in A$. This contradiction shows the continuity of $\tau$ at the points of $A$. □

We see immediately now that the map $h : W^+(A) \setminus A \to \mathbb{R} \times f^{-1}(c)$ defined by

$$h(x) = (-\tau(x), \phi_{\tau(x)}(x))$$

is a homeomorphism such that $h(\phi_t(x)) = (-\tau(x) + t, \phi_{\tau(x)}(x))$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. In other words, $h$ conjugates the restricted flow on $W^+(A) \setminus A$ with the parallel flow on $\mathbb{R} \times f^{-1}(c)$.

Note that $F : W^+(A) \to [0, +\infty)$ defined by

$$F(x) = \begin{cases} e^{\tau(x)}, & \text{if } x \in W^+(A) \setminus A \\ 0, & \text{if } x \in A \end{cases}$$

is also a continuous Lyapunov function for $A$ and $F(\phi_t(x)) = e^{-t}F(x)$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. Thus, $F^{-1}([0, a])$ is homeomorphic to $F^{-1}([0, b])$ for every $a$, $b > 0$, because

$$\phi_{\log(b/a)}(F^{-1}([0, b]) = F^{-1}([0, a]).$$

This implies that $F^{-1}([0,c])$ is compact for every $c > 0$. In the sequel we may replace $f$ with $F$.

**Remark 1.2.** In case $M$ is a smooth manifold and the flow is smooth, there is a smooth Lyapunov function $F$ for $A$. From the implicit function theorem follows that $F^{-1}([0,c])$ is a compact, smooth submanifold with boundary $\partial F^{-1}([0,c]) = F^{-1}(c)$. Moreover, $\tau$ is smooth on $W^+(A) \setminus A$.

If $c > 0$, the continuous map $r : W^+(A) \to F^{-1}([0,c])$ defined by

$$r(x) = \begin{cases} x, & \text{if } 0 \le F(x) \le c \\ \phi_{\tau(x)}(x), & \text{if } F(x) > c \end{cases}$$

is a retraction. The continuous map $H : [0,1] \times W^+(A) \to W^+(A)$ defined by

$$H(t,x) = \begin{cases} x, & \text{if } 0 \le F(x) \le c \\ \phi_{(1-t)\tau(x)}(x), & \text{if } F(x) > c \end{cases}$$

is a homotopy $H : i \circ r \simeq id$ rel $F^{-1}([0,c])$, where $i : F^{-1}([0,c]) \hookrightarrow W^+(A)$ is the inclusion. So, $F^{-1}([0,c])$ is a strong deformation retract of $W^+(A)$ and $i$ induces an isomorphism $i^* : \bar{H}^*(W^+(A);\mathbb{Z}) \cong \bar{H}^*(F^{-1}([0,c]);\mathbb{Z})$ in Alexander-Spanier cohomology. Similarly, the inclusion $F^{-1}([0,a]) \hookrightarrow F^{-1}([0,b])$ induces an isomorphism $\bar{H}^*(F^{-1}([0,b]);\mathbb{Z}) \cong \bar{H}^*(F^{-1}([0,a]);\mathbb{Z})$ for every $0 < a < b$. From the continuity property of the Alexander-Spanier cohomology follows that

$$\bar{H}^*(A;\mathbb{Z}) \cong \lim_{\to} \bar{H}^*(F^{-1}([0,c]);\mathbb{Z}) \cong \bar{H}^*(W^+(A);\mathbb{Z}),$$

since $A = \bigcap_{c>0} F^{-1}([0,c])$ and $F^{-1}([0,c])$ is compact for every $c > 0$.

**Proposition 1.3.** *Let $M$ be a locally compact* ANR *carrying a continuous flow $(\phi_t)_{t\in\mathbb{R}}$. If $A \subset M$ is an asymptotically stable, compact, invariant set, which contains no fixed point of the flow, then $\bar{H}^*(A;\mathbb{Z})$ is finitely generated and*

$$\sum_{q=0}^{\infty} (-1)^q \mathrm{rank} \bar{H}^q(A;\mathbb{Z}) = 0.$$

*Proof.* Since $M$ is a locally compact ANR, so is $W^+(A)$, and therefore $F^{-1}([0,c])$ is a positively invariant, compact ANR, for any $c > 0$, because it is a retract of $W^+(A)$. Hence $\bar{H}^*(A;\mathbb{Z}) \cong \bar{H}^*(F^{-1}([0,c]);\mathbb{Z})$ is finitely generated. The assumption that $A$ is compact and does not contains fixed points of the flow implies that there exists some $\epsilon > 0$ such that $\phi_t(x) \neq x$ for every $0 < t < \epsilon$ and $x \in A$, and so for every $x \in F^{-1}([0,c])$. If we choose any $0 < t < \epsilon$ and apply the Lefschetz fixed point theorem to $\phi_t$ on $F^{-1}([0,c])$, we get

$$\sum_{q=0}^{\infty} (-1)^q \mathrm{rank} \bar{H}^q(A;\mathbb{Z}) = \sum_{q=0}^{\infty} (-1)^q \mathrm{rank} H^q(F^{-1}([0,c]);\mathbb{Z}) = 0. \quad \square$$

**Remark 1.4.** One can show that actually $A$ and $F^{-1}([0,c])$ have the same shape. In case $M$ is a manifold, $F^{-1}([0,c])$ has the shape of a finite polyhedron, and therefore also $A$ [7], [9].

## 2. One-dimensional minimal sets

Of particular interest is the case of compact minimal sets. Recall that an invariant set is called minimal if it is nonempty, closed, invariant and has no proper subset with these properties. G. Allaud and E.S. Thomas have shown that an asymptotically stable, almost periodic, $k$-dimensional, compact minimal set of a flow on a manifold is homeomorphic to a $k$-torus (see Theorem 3.4 in [1]). The almost periodicity is a very restrictive internal property, which implies that the minimal set is an abelian compact topological group. This result is not true without the almost periodicity assumption if $k > 1$. As it is remarked in [11], there exists a smooth diffeomorphism of $\mathbb{R}^2$ having an asymptotically stable, 1-dimensional compact minimal set (the pseudocircle) and so its suspension gives a smooth flow on a smooth 3-manifold (the open solid torus) with an asymptotically stable, 2-dimensional compact minimal set.

We shall now be concerned with periodicity criteria of cohomological type for 1-dimensional compact minimal sets. Let $X$ be a 1-dimensional, compact metric space carrying a minimal flow $(\phi_t)_{t\in\mathbb{R}}$. Every point $x_0 \in X$ is contained in a some local section $S_0$ (see Lemma 1 in [8]). This means that there exists some $\epsilon > 0$ such that the flow maps $(-2\epsilon, 2\epsilon) \times S_0$ homeomorphically onto an open neighbourhood of $x_0$. It follows that $S_0$ is a 0-dimensional, locally compact, metric space and thus $x_0$ has an open-compact neighbourhood $S$ in $S_0$. Then $S$ is a compact local section at $x_0$ such that $(-2\epsilon, 2\epsilon) \times S$ is mapped by the flow onto an open neghbourhood of $x_0$. Let $f_S : X \to S^1$ be the continuous function defined by

$$f_S(x) = \begin{cases} e^{2\pi i t/\epsilon}, & \text{if } x \in \phi_t(S) \text{ and } 0 \leq t \leq \epsilon \\ 1, & \text{otherwise.} \end{cases}$$

The homotopy class of $f_S$ does not depend on $\epsilon$, but depends only on $S$. Recall that $\bar{H}^1(X;\mathbb{Z})$ is naturally isomorphic to the abelian group $[X; S^1]$ of homotopy classes of maps of $X$ into $S^1$ and is torsion free. Using this identification, the element $[f_S] \in \bar{H}^1(X;\mathbb{Z})$ is called the flow class of the local section $S$. The function $f_S$ is called the cosection map of $S$.

**Lemma 2.1.** *Let $P$, $Q$ be two disjoint, open-compact subsets of $S$. If $m$, $n \in \mathbb{Z}$ are such that $m \cdot n \neq 0$ and $f_P^m \cdot f_Q^n$ is homotopic to a constant, then $m \cdot n < 0$.*

*Proof.* Since $f_P^m \cdot f_Q^n$ is homotopic to a constant, there exists a continuous function $\alpha : X \to \mathbb{R}$ such that $f_P^m(x) \cdot f_Q^n(x) = e^{2\pi i \alpha(x)}$ for every $x \in X$. Let $\beta : X \to \mathbb{R}$ be the function defined by

$$\beta(x) = \begin{cases} mt/\epsilon, & \text{if } x \in \phi_t(P) \text{ and } 0 \leq t \leq \epsilon \\ nt/\epsilon, & \text{if } x \in \phi_t(Q) \text{ and } 0 \leq t \leq \epsilon \\ 0, & \text{otherwise.} \end{cases}$$

Then $e^{2\pi i\alpha(x)} = e^{2\pi i\beta(x)}$ for every $x \in X$ and $\beta$ is everywhere continuous except at the points of the set $\phi_\epsilon(P \cup Q)$. Let $\gamma = \alpha - \beta : X \to \mathbb{Z}$. By continuity of the restriction of $\gamma$ on the set $\bigcup_{|t|\leq\epsilon} \phi_t(P \cup Q)$, there exists a finite cover $\{W_1, W_2, ..., W_q\}$ of $P \cup Q$ consisting of mutually disjoint, open-compact subsets of $P \cup Q$ such that $\gamma$ takes a constant value $k_j \in \mathbb{Z}$ on $\bigcup_{|t|\leq\epsilon} \phi_t(W_j)$, $1 \leq j \leq q$. Taking a finer cover if necessary, we may assume that for each $1 \leq j \leq q$ we have $W_j \subset P$ or $W_j \subset Q$. If now $x \in \phi_\epsilon(W_j)$ and $(x_k)_{k\in\mathbb{N}}$ is a sequence of points of $X \setminus \bigcup_{0\leq t\leq\epsilon} \phi_t(S)$ converging to $x$, then $\lim_{k\to+\infty} \gamma(x_k) = \beta(x) + \gamma(x)$, and so the sequence $(\gamma(x_k))_{k\in\mathbb{N}}$ is eventually constant and equal to $m + k_j$, if $W_j \subset P$, or $n + k_j$, if $W_j \subset Q$. This implies that there exists some $0 < \delta < \epsilon$ such that if $x \in \bigcup_{-\epsilon\leq t\leq\epsilon+\delta} \phi_t(W_j)$, $1 \leq j \leq q$, then

$$\gamma(x) = \begin{cases} k_j, & \text{if } x \in \bigcup_{|t|\leq\epsilon} \phi_t(W_j) \\ m + k_j, & \text{if } x \in \bigcup_{\epsilon<t\leq\epsilon+\delta} \phi_t(W_j) \text{ and } W_j \subset P \\ n + k_j, & \text{if } x \in \bigcup_{\epsilon<t\leq\epsilon+\delta} \phi_t(W_j) \text{ and } W_j \subset Q. \end{cases}$$

Let $1 \leq j_0 \leq q$ be such that $x_0 \in W_{j_0}$. Since the flow is minimal, there are times $0 = t_0 < t_1 < ... < t_l$ such that $\phi_{t_l}(x_0) \in W_{j_0}$, $\phi_{t_j}(x_0) \in P \cup Q$ and $\phi_t(x_0) \in X \setminus (P \cup Q)$ for $t_j < t < t_{j+1}$, $1 \leq j < l$. Then, $\gamma(x_0) = \gamma(\phi_l(x_0))$ and $\gamma(\phi_{t_j}(x_0)) = n_j + \gamma(\phi_{t_{j-1}}(x_0))$, where $n_j = m$ or $n$, $1 \leq j \leq l$. This implies that $n_1 + n_2 + ... + n_l = 0$ and therefore $m$ and $n$ must have opposite signs. $\square$

**Corollary 2.2.** *If $X$ is a one-dimensional, compact metric space carrying a minimal flow, then $\bar{H}^1(X; \mathbb{Z}) \neq \{0\}$.*

*Proof.* Applying Lemma 2.1 for $P = S$, $Q = \varnothing$ and $m = n = 1$, we conclude that the cosection map $f_S$ is not homotopic to a constant, and therefore it defines a nonzero element of $\bar{H}^1(X; \mathbb{Z})$. $\square$

**Theorem 2.3.** *Let $X$ be a one-dimensional, compact metric space carrying a minimal flow. Then, $X$ is homeomorphic to the unit cricle $S^1$ if and only if $\bar{H}^1(X; \mathbb{Z}) \cong \mathbb{Z}$.*

*Proof.* Only the converse needs proof, and for this it suffices to prove that $x_0$ is an isolated point of $S$, using the same notation as above. Let $P$ be an open-compact, proper subset of $S$ and $Q = S \setminus P$. Then, $f_S = f_P \cdot f_Q$ and $f_Q$ is not homotopic to a constant, by Corollary 2.2. Thus, $f_S$ is not homotopic to $f_P$ and therefore $f_S^k$ is not homotopic to $f_P^k$ for any nonzero $k \in \mathbb{Z}$, since $\bar{H}^1(X; \mathbb{Z})$ is torsion free. The flow classes of $S$ and $P$ correspond to integers $n$ and $m$, respectively, through the assumed isomorphism $\bar{H}^1(X; \mathbb{Z}) \cong \mathbb{Z}$. Then, $m \cdot n \neq 0$ and $n \neq m$, by Corollary 2.2. Since $m \cdot n = n \cdot m$, $f_S^m$ is homotopic to $f_P^n$ and so $f_S^m \cdot f_P^{-n}$ is homotopic to a constant. However,

$$f_S^m \cdot f_P^{-n} = (f_S \cdot f_P^{-1})^m \cdot f_P^{m-n} = f_Q^m \cdot f_P^{m-n}$$

and from Lemma 2.1 we conclude that $m(m - n) < 0$ or equivalently $0 < |m| < |n|$. Suppose now that there exists a strictly decreasing, neighbourhood basis $\{S_k : k \in \mathbb{N}\}$ of $x_0$ in $S$ consisting of open-compact sets. The flow class of $S_k$ corresponds to a nonzero integer $n_k$ and what we have already proved shows that $|n_{k+1}| < |n_k|$ for every $k \in \mathbb{N}$, which is absurd. This contradiction proves that $x_0$ is an isolated point of $S$. $\square$

From Proposition 1.3 and Theorem 2.3 follows directly the Poincaré-Bendixson type result that if $A$ is an asymptotically stable, 1-dimensional, compact, minimal set of a flow on a locally compact ANR, then $A$ is a periodic orbit. Indeed, since $A$ is 1-dimensional, $\bar{H}^k(X;\mathbb{Z}) = \{0\}$, for $k > 1$. From Proposition 1.3 follows that $\bar{H}^1(A;\mathbb{Z})$ is a finitely generated, torsion free, abelian group of rank 1. Hence $\bar{H}^1(X;\mathbb{Z}) \cong \mathbb{Z}$ and $A$ must be a periodic orbit, from Theorem 2.3.

However, using another more elementary approach, one can show that this conclusion is true in general locally connected spaces [6].

**Theorem 2.4.** *Let $M$ be a locally compact, metric space carrying a continuous flow. Let $A \subset M$ be an asymptotically stable, one-dimensional, compact minimal set. If $M$ is locally connected at the points of $A$, then $A$ is a periodic orbit.* □

**Remarks 2.5.** (a) Theorem 2.3 says that if $X$ is a nonperiodic, one-dimensional, compact minimal set, then either rank$\bar{H}^1(A;\mathbb{Z}) \geq 2$, or rank$\bar{H}^1(A;\mathbb{Z}) = 1$ and $\bar{H}^1(X;\mathbb{Z})$ is not free abelian. An example for the first case is the unique minimal set of the Denjoy $C^1$ flow on the 2-torus, whose integral first Alexander-Spanier cohomology group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. On the other hand, the dyadic solenoid carries a minimal flow and is an example for the second case, since its integral first Alexander-Spanier cohomology group is isomorphic to the additive group of the dyadic rationals. One can prove that the integral first Alexander-Spanier cohomology group of a surface compact minimal set is always free abelian [3].

(b) The conclusion of Theorem 2.3 is true under the much more general assumption that $X$ is a one-dimensional continuum carrying a fixed point free flow (see Theorem 3.5 in [2]) and similarly for Theorem 2.4 in case the phase spase is a locally compact ANR (see Theorem 4.2 in [2]).

## 3. Isolated unstable attractors

Asymptotically stable compact invariant sets are very special cases of isolated invariant sets. A compact invariant set $A \subset M$ is called isolated if it has a compact neighbourhood $V$ such that $A$ is the maximal invariant set in $V$. Each such $V$ is called an isolating neighbourhood of $A$ and contains a smaller isolating neighbourhood $N$ of $A$ such that there are compact sets $N^+$, $N^- \subset \partial N$ with the following properties:

(i) $\partial N = N^+ \cup N^-$.

(ii) For every $x \in N^+$ there exists $\epsilon > 0$ such that $\phi_t(x) \in M \setminus N$ for $-\epsilon \leq t < 0$, and for every $y \in N^-$ there exists $\delta > 0$ such that $\phi_t(y) \in M \setminus N$. for $0 < t \leq \delta$.

(iii) For every $x \in \partial N \setminus N^+$ there exists $\epsilon > 0$ such that $\phi_t(x) \in \mathrm{int}N$ for $-\epsilon \leq t < 0$, and for every $y \in \partial N \setminus N^-$ there exists $\delta > 0$ such that $\phi_t(y) \in \mathrm{int}N$ for $0 < t \leq \delta$.

The triad $(N, N^+, N^-)$ is called an isolating block of $A$. The sets $A^{\pm} = \{x \in N : C^{\pm}(x) \subset N\}$ and $\alpha^{\pm} = \partial N \cap A^{\pm}$ are compact and $A = A^+ \cap A^-$. Moreover, $\varnothing \neq L^+(x) \subset A$ for every $x \in A^+$ and $\alpha^+ \subset N^+ \setminus N^-$.

If $A$ is asymptotically stable and $F : W \to \mathbb{R}^+$ is a Lyapunov function as before, then $(F^{-1}([0,c]), \partial F^{-1}([0,c]), \varnothing)$ is an isolating block for every $c > 0$.

If $M$ is a smooth $n$-manifold and the flow is smooth, then every neighbourhood of an isolated invariant set $A$ contains a smooth isolating block $(N, N^+, N^-)$ of $A$. This means that $N$ is a smooth compact $n$-manifold with boundary $\partial N = N^+ \cup N^-$, the sets

$N^+$ and $N^-$ are smooth compact $(n-1)$-manifolds with common boundary $N^+ \cap N^-$, which is a smooth compact $(n-2)$-manifold (without boundary) and on which the flow is externally tangent to $N$. Moreover, the flow is transverse to $N^+ \setminus N^-$ into $N$ and transverse to $N^- \setminus N^+$ out of $N$.

Let $A \subset M$ be an isolated compact invariant set and let $(N, N^+, N^-)$ be an isolating block of $A$. The final entrance time function $\tau : W^+(A) \to [-\infty, +\infty)$ defined by

$$\tau(x) = \sup\{t \in \mathbb{R} : \phi_t(x) \in M \setminus N\},$$

if $x \in W^+(A) \setminus A$ and $\tau(x) = -\infty$, if $x \in A$, is lower semicontinuous. This follows immediately from the definition and the continuity of the flow. Obviously, $\phi_{\tau(x)}(x) \in \alpha^+$ and $\tau(\phi_t(x)) = \tau(x) - t$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. The final entrance time function $\tau$ is discontinuous at $x \in W^+(A) \setminus A$ if and only if there are $x_n \to x$ such that $\tau(x_n) \to +\infty$ (see Lemma 3.1 [4]). These are the points of the set $A \cup \phi(\mathbb{R} \times \partial_{\partial N} a^+)$.

It is clear from the above that if $A$ is an isolated compact invariant set, then $A$ is not necessarily asymptotically stable with respect to the restricted flow in $W^+(A)$. However, it is possible to define a new topology in $W^+(A)$, which is finer than the subspace topology, with respect to which the flow remains continuous and $A$ becomes asymptotically stable. Roughly speaking, this new topology is obtained by cutting $W^+(A)$ along the discontinuity set of the final entrance time function with respect to any isolating block of $A$. It was originally defined in [13].

Let $(X_t, p_{st})_{s,t\in\mathbb{R}}$ be the following inverse system of compacta. For every $t \in \mathbb{R}$ we let $X_t = N/N^+$ and for $s \leq t$ the map $p_{st} : X_t \to X_s$ is defined by

$$p_{st}(x) = \begin{cases} \phi_{s-t}(x), & \text{if } \phi_r(x) \subset N \setminus N^+ \text{ for } s-t \leq r \leq 0, \\ [N^+], & \text{otherwise.} \end{cases}$$

Obviously, $p_{st}([N^+]) = [N^+]$ for every $s \leq t$.

Let $X_N = \varprojlim(X_t, p_{st})$ and let $*$ denote the point of $X$ all of whose coordinates are equal to $[N^+]$. Clearly, $X_N \setminus \{*\}$ is a locally compact, separable, metrizable space. If $(x_t)_{t\in\mathbb{R}} \in X_N \setminus \{*\}$ and $x_{t_0} = [N^+]$, then $x_t = [N^+]$, for every $t \leq t_0$. Moreover, there exists $s \in \mathbb{R}$ such that $x_s \neq [N^+]$ and so $x_t \in N \setminus N^+$ and $x_s = \phi_{s-t}(x_t)$ for every $t \geq s$. Let $h_N : X_N \setminus \{*\} \to M$ be defined by $h_N((x_t)_{t\in\mathbb{R}}) = \phi_{(-s)}(x_s)$, where $s \in \mathbb{R}$ is any such that $x_s \neq [N^+]$. Since $\phi_{t-s}(x_s) = x_t \in N \setminus N^+$ for every $t \geq s$, we have $C^+(x_s) \subset N$, and so $x_s \in W^+(A)$, because $N$ is an isolating neighbourhood of $A$. Thus, $h_N(X_N \setminus \{*\}) \subset W^+(A)$. Conversely, if $x \in W^+(A)$, we let

$$x_t = \begin{cases} [N^+], & \text{if } t \leq \tau(x), \\ \phi_t(x), & \text{if } t > \tau(x). \end{cases}$$

Then, $h_N((x_t)_{t\in\mathbb{R}}) = x$. This shows that $h_N(X_N \setminus \{*\}) = W^+(A)$, and it is easy to see that $h_N$ is also injective and continuous. Note that

$$(h_N)^{-1}(A) = \{(\phi_t(x))_{t\in\mathbb{R}} : x \in A\} = \varprojlim(A, p_{st}|A),$$

which is homeomorphic to $A$, since $(\phi_s|A) \circ (p_{st}|A) = \phi_t|A$, for every $s \leq t$, hence compact. So, $h_N$ maps $(h_N)^{-1}(A)$ homeomorphically onto $A$. Similarly, $h_N$ maps $(h_N)^{-1}(\alpha^+)$ homeomorphically onto $\alpha^+$.

It is immediate from the above formula giving $(h_N)^{-1}(x)$ that $(h_N)^{-1}$ is discontinuous at $x \in W^+(A)$ if and only if the final entrance time function $\tau$ is discontinuous at $x$.

**Lemma 3.1.** *If $(N, N^+, N^-)$ and $(\Lambda, \Lambda^+, \Lambda^-)$ are two isolating blocks of $A$, then $(h_N)^{-1} \circ h_\Lambda : X_\Lambda \setminus \{*\} \to X_N \setminus \{*\}$ is a homeomorphism.*

*Proof.* It suffices to show that $(h_N)^{-1} \circ h_\Lambda$ is continuous. First we observe that if $x \in A$ then $((h_N)^{-1} \circ h_\Lambda)((\phi_t(x))_{t\in\mathbb{R}}) = (\phi_t(x))_{t\in\mathbb{R}}$. If now $(x_t)_{t\in\mathbb{R}} \in (X_\Lambda \setminus \{*\}) \setminus (h_\Lambda)^{-1}(A)$, there exists $x \in W^+(A) \setminus A$ and some $t_0 \in \mathbb{R}$ such that $x_t = [N^+]$ for $t \leq t_0$ and $x_t = \phi_t(x)$ for $t > t_0$. Suppose that $(y_t)_{t\in\mathbb{R}} = ((h_N)^{-1} \circ h_\Lambda)((x_t)_{t\in\mathbb{R}})$. There exists some $s_0 \in \mathbb{R}$ such that $y_t = [N^+]$ for $t \leq s_0$ and $y_t = \phi_t(x)$ for $t > s_0$. The pair $(x, t_0)$ is uniquely determined by $(x_t)_{t\in\mathbb{R}}$ and similarly $(x, s_0)$ by $(y_t)_{t\in\mathbb{R}}$, and $\phi_{t_0}(x) \in \alpha^+_\Lambda$, $\phi_{s_0}(x) \in \alpha^+_N$. There is a continuous function $\lambda : \alpha^+_\Lambda \to \mathbb{R}$ such that $\lambda(z)$ is the unique real number with $\phi_{\lambda(z)}(z) \in \alpha^+_N$ for every $z \in \alpha^+_\Lambda$. Then, $\lambda(\phi_{t_0}(x)) = s_0 - t_0$. It follows that each coordinate $y_t$ is a continuous function of $(x_t)_{t\in\mathbb{R}}$. $\square$

It follows now that there is a topology on $W^+(A)$, which is finer than the subspace topology inherited from $M$, and which makes $h_N$ a homeomorphism and does not depend on the chosen isolating block $(N, N^+, N^-)$. It is called the intrinsic topology of the region of attraction of $A$ and we denote by $W^+_i(A)$ the region of attraction of $A$ equiped with this topology.

**Lemma 3.2.** *The final entrance time function $\tau : W^+_i(A) \to [-\infty, +\infty)$ is continuous for any isolating block $(N, N^+, N^-)$ of $A$.*

*Proof.* We have to prove the continuity of $g = f \circ h_N : X_N \setminus \{*\} \to [-\infty, +\infty)$, defined by $g((x_t)_{t\in\mathbb{R}}) = \inf\{t \in \mathbb{R} : x_t \neq [N^+]\}$. Suppose that $g((x_t)_{t\in\mathbb{R}}) = t_0$ and $a < t_0 < b$ for some $a$, $b \in \mathbb{R}$. There exists $x \in \alpha^+$ such that $x_t = \phi_{t-t_0}(x)$ for every $t > t_0$. Since $\alpha^+$ is a compact subset of $N^+ \setminus N^-$, there is an open neighbourhood $V$ of $x$ such that $V \cap N^- = \varnothing$, and $\phi_{-\epsilon}(V) \subset M \setminus N$ and $\phi_\epsilon(V) \subset \mathrm{int}N$, for some $\epsilon > 0$ such that $a < t_0 - \epsilon < t_0 < t_0 + \epsilon < b$. The set

$$C = \big(\phi_\epsilon(V) \times \prod_{t \neq t_0+\epsilon} N/N^+\big) \cap (X_N \setminus \{*\})$$

is an open neighbourhood of $(x_t)_{t\in\mathbb{R}}$ in $X_N \setminus \{*\}$. If $(y_t)_{t\in\mathbb{R}} \in C$, then $y_{t_0+\epsilon} \in \phi_\epsilon(V) \subset \mathrm{int}N$ and so $g((y_t)_{t\in\mathbb{R}}) < t_0 + \epsilon < b$. On the other hand, there exists a unique $-2\epsilon < s \leq 0$ such that $\phi_s(y_{t_0+\epsilon}) \in \alpha^+$, because $\phi_{-\epsilon}(V) \subset M \setminus N$, and so $g((y_t)_{t\in\mathbb{R}}) = t_0 + \epsilon + s > t_0 - \epsilon > a$. This shows the continuity in case $t_0 \in \mathbb{R}$. If $t_0 = -\infty$, there exists some $x \in A$ such that $x_t = \phi_t(x)$ for every $t \in \mathbb{R}$ and there is an open neighbourhood $V$ of $x$ such that $\phi_\epsilon(V) \subset \mathrm{int}N$ for some $\epsilon > 0$. In this case it suffices to take

$$C = \big(\phi_\epsilon(V) \times \prod_{t \neq \epsilon} N/N^+\big) \cap (X_N \setminus \{*\}). \quad \square$$

We observe that for every $x \in W^+(A)$ and $s \in \mathbb{R}$, if $(h_N)^{-1}(sx) = (y_t)_{t\in\mathbb{R}}$, then

$$y_t = \begin{cases} [N^+], & \text{if } s + t \leq f(x), \\ tx, & \text{if } s + t > f(x). \end{cases}$$

Thus, $(h_N)^{-1}$ transforms the restricted flow on $W^+(A)$ to the left shift on $X_N \setminus \{*\}$, which is a continuous flow. This implies that the flow remains continuous with respect to the intrinsic topology and $A$ remains a compact invariant set in $W_i^+(A)$.

The function $F : W_i^+(A) \to [0, +\infty)$ defined by

$$F(x) = \begin{cases} e^{\tau(x)}, & \text{if } x \in W_i^+(A) \setminus A, \\ 0, & \text{if } x \in A \end{cases}$$

where $\tau$ is the final entrance time function with respect to any isolating block $(N, N^+, N^-)$ of $A$. From Lemma 3.2 we have that $F$ is continuous. It is also immediate from the definition that $A = F^{-1}(0)$ and $F(\phi_t(x)) = e^{-t}F(x)$ for every $t \in \mathbb{R}$ and $x \in W_i^+(A) \setminus A$. This shows that $A$ is globally asymptotically stable in $W_i^+(A)$. Moreover, the restricted flow on $W_i^+(A) \setminus A$ is parallelizable and each level set $F^{-1}(c)$, $c > 0$, is a compact global section. In particular the set $\alpha^+ = F^{-1}(1)$ is a global section to the flow on $W_i^+(A) \setminus A$ and thus $W_i^+(A) \setminus A$ is homeomorphic to $\mathbb{R} \times \alpha^+$.

Note that $F : W^+(A) \to [0, +\infty)$ is a lower semicontinuous Lyapunov function for $A$, which by no means implies that $A$ is stable with respect to the restricted flow in $W^+(A)$. The identity map $id : W^+(A) \to W_i^+(A)$ is continuous that a point $x \in W^+(A)$ if and only if $F$ is continuous at $x$.

**Lemma 3.3.** *Let $X$ be a locally compact, metric space and $h : X \to Y$ be an injective, continuous map onto a metric space $Y$. Let $D$ be the set of points of $X$ such that $h^{-1}$ is not continuous at $h(x)$. Then, $h(D)$ is closed in $Y$ and $D$ is closed in $X$.*

*Proof.* Suppose that $h(D)$ is not closed in $Y$. There exists $x \in X$ such that $h(x) \in (Y \setminus h(D)) \cap \overline{h(D)}$. Since $x \in X \setminus D$, $h^{-1}$ is continuous at $h(x)$, and if $V$ is a compact neighbourhood of $x$, there exists an open neighbourhood $U$ of $h(x)$ such that $h^{-1}(U) \subset V$ or equivalently $U \subset h(V)$. Let $z \in X$ be such that $h(z) \in U \cap h(D) \subset h(V \cap D)$, that is $z \in V \cap D$. There exists a sequence $(z_n)_{n\in\mathbb{N}}$, which does not converge to $z$, such that the sequence $(h(z_n))_{n\in\mathbb{N}}$ converges to $h(z)$. Passing to a subsequence if necessary, we may assume that $z$ is not a limit point of $(z_n)_{n\in\mathbb{N}}$. Eventually $h(z_n) \in U$ and so $z_n \in V$. Since $V$ is compact, the sequence $(z_n)_{n\in\mathbb{N}}$ has a limit point $y \in V$. It follows that $h(y) = h(z)$ and therefore $y = z$, contradiction. ☐

Since $W_i^+(A)$ is locally compact, $id : W^+(A) \to W_i^+(A)$ is continuous on an invariant, open subset of $W^+(A)$ by Lemma 3.3. In general, $W^+(A)$ may not be locally compact, but is only an $F_\sigma$-set in $M$. If $W^+(A)$ is not locally compact, then $id : W^+(A) \to W_i^+(A)$ may be nowhere continuous.

An alternative description of the intrinsic topology is given in [15] and is stated in the following.

**Proposition 3.4.** *The intrinsic topology is the smallest topology $\mathcal{T}$ which contains the subspace topology of $W^+(A)$ and the sets*

$$W^+(A) \cap \bigcap_{t \geq 0} \phi_{-t}(V),$$

*where $V$ runs over the open neighbourhoods of $A$ in $M$.*

*Proof.* We shall prove first that if $V$ is an open neighbourhood of $A$ in $M$ then $W^+(A) \cap \bigcap_{t\geq 0} \phi_{-t}(V)$ is intrinsically open. In the special case where $V = \text{int}N$ and $(N, N^+, N^-)$ is an isolating block, this set is equal to $\tau^{-1}[-\infty, 0)$, where $\tau$ is the corresponding final entrance time function, and is intrinsically open. In any case, every open neighbourhood $V$ of $A$ contains such an $N$. If $x \in W^+(A) \cap \bigcap_{t\geq 0} \phi_{-t}(V)$, there is some $s > \tau(x)$ such that $\phi_s(x) \in \text{int}N$ and so $x \in W^+(A) \cap \phi_{-s}(\bigcap_{t\geq 0} \phi_{-t}(V))$, which is intrinsically open. There is also an open neighbourhood $U$ of $x$ such that $\phi_r(y) \in V$ for $0 \leq r \leq s$ and $y \in U$, by compactness. Therefore

$$x \in U \cap W^+(A) \cap \phi_{-s}(\bigcap_{t\geq 0} \phi_{-t}(V)) \subset W^+(A) \cap \bigcap_{t\geq 0} \phi_{-t}(V).$$

This shows that $W^+(A) \cap \bigcap_{t\geq 0} \phi_{-t}(V)$ is intrinsically open.

To prove the converse it suffices to show that given an isolating block $(N, N^+, N^-)$ of $A$, the map

$$h_N^{-1} : (W^+(A), \mathcal{T}) \to X_N \setminus \{*\} \subset \prod_{t\in\mathbb{R}} X_t$$

is continuous. Let $t \in \mathbb{R}$ and $x \in W^+(A)$. In order to prove that the $t$-coordinate of $h_N^{-1}$ is continuous at $x$, we consider two cases.

Let first $\tau(x) < t$. Then the $t$-coordinate of $h_N^{-1}(x)$ is $\phi_t(x) \in N/N^+$ and $\phi_r(x) \in \text{int}N$ for $r \geq t$. Let $U \subset \text{int}N$ be an open neighbourhood of $\phi_t(x)$. By continuity, there is an open neighbourhood $G$ of $x$ such that $\phi_t(G) \subset U$. For every

$$y \in G \cap W^+(A) \cap \phi_{-t}(\bigcap_{r\geq 0} \phi_{-r}(\text{int}N))$$

we have $\tau(y) > t$ and the $t$-coordinate of $h_N^{-1}(y)$ is $\phi_t(y) \in U$. This proves the continuity of $t$-coordinate of $h_N^{-1}$ at $x$ in this case.

Let now $\tau(x) \geq t$. The $t$-coordinate of $h_N^{-1}(x)$ is $[N^+]$. Let $U$ be an open neighbourhood of $N^+$ in $M$. By continuity, there exist $\epsilon > 0$ such that $\phi_r(x) \in U$ for $|\tau(x) - r| \leq \epsilon$ and $\phi_{\tau(x)-\epsilon}(x) \notin N$, and an open neighbourhood $G$ of $x$ such that $\phi_r(y) \in U$ for $|\tau(x) - r| \leq \epsilon$, $y \in G$ and $\phi_{\tau(x)-\epsilon}(G) \cap N = \varnothing$. If now

$$y \in G \cap W^+(A) \cap \phi_{\tau(x)+\epsilon}(\bigcap_{r\geq 0} \phi_{-r}(\text{int}N)),$$

then either $\tau(y) \geq t$, and so $t$-coordinate of $h_N^{-1}(y)$ is $[N^+]$, or $\tau(y) < t$ and it is $\phi_t(y)$. In the latter, we have $\tau(x) - \epsilon < \tau(y) < t \leq \tau(x)$ and therefore $\phi_t(y) \in U$. This completes the proof. □

A compact invariant set $A \subset M$ is called an isolated unstable attractor if it is an isolated invariant set such that $W^+(A)$ is an open neighbourhood of $A$, but $A$ is not (positively) Lyapunov stable. Having in mind Theorem 2.4, the question arises whether a 1-dimensional compact minimal set $A$ which is an isolated unstable attractor of a flow on a locally compact ANR $M$ must be a periodic orbit. This is true in case $A$ is almost periodic and $M$ is a 3-manifold under the weaker assumption that $A$ is only isolated and not necessarily an attractor [1], [17]. The observations of the present section and Theorem 2.4 show that it is also true if $W_i^+(A)$ is locally connected at the points of $A$. So we are led to the following topological problem: Let $M$ be a separable, locally compact, ANR (for instance a manifold) and $X$ be a connected, locally compact, metrizable space. If there is a continuous, one-to-one, onto map $h : X \to M$, such that $h^{-1} : M \to X$ is continuous on an open, dense subset of $M$, under what conditions is $X$ locally connected or even an ANR?

In the sequel we will make some remarks about the complexity of the flow in $W^+(A)$ (see [5]).

**Proposition 3.5.** *If $A$ is an isolated compact invariant set and $W^+(A)$ is locally compact, then*

*(a) $A$ is asymptotically stable with respect to the restricted flow in $W^+(A)$ if and only if the identity $id : W^+(A) \to W_i^+(A)$ is continuous at every point of $A$.*

*(b) The identity $id : W^+(A) \to W_i^+(A)$ is continuous on an invariant, dense, open subset of $W^+(A) \setminus A$.*

*Proof.* (a) Suppose that $id : W^+(A) \to W_i^+(A)$ is continuous at every point of $A$. Then the final entrance time function $\tau : W^+(A) \to [-\infty, +\infty)$, with respect to any isolating block of $A$, is continuous at every point of $A$. If $A$ is not stable in $W^+(A)$, then, since $W^+(A)$ is locally compact, there exist points $x \in A$, $y \in W^+(A) \setminus A$, $x_n \in W^+(A) \setminus A$, $n \in \mathbb{N}$, and times $t_n \to +\infty$ such that $x_n \to x$ and $\phi_{t_n}(x_n) \to y$. Since $\tau$ is continuous at $x$, we have $\tau(x_n) \to -\infty$, and therefore $\tau(\phi_{t_n}(x_n)) = \tau(x_n) - t_n \to -\infty$. But since $\tau$ is lower semicontinuous, $-\infty < \tau(y) \leq \liminf_{n\to+\infty} \tau(\phi_{t_n}(x_n)) = -\infty$. The converse is trivial.

(b) The identity from $W^+(A)$ to $W_i^+(A)$ is discontinuous at a point $x \in W^+(A) \setminus A$ if and only if the final entrance time function $\tau : W^+(A) \setminus A \to \mathbb{R}$ is discontinuous at $x$. Since $\tau$ is lower semicontinuous and $W^+(A) \setminus A$ is locally compact, the set of points of $W^+(A) \setminus A$ at which $\tau$ is continuous, is dense in $W^+(A) \setminus A$ from the Baire Category Theorem. □

Let now $A \subset M$ be an isolated unstable attractor. The identity from $W^+(A)$ to $W_i^+(A)$ is continuous on an open, invariant subset $G_0$ of $W^+(A) \setminus A$. Since $W^+(A)$ is locally compact, $G_0$ is also dense in $W^+(A) \setminus A$, and the invariant set $W_1^+(A) = W^+(A) \setminus G_0$ is locally compact. If $\alpha$ is an ordinal, and $W_\alpha^+(A) \subset W^+(A)$ has been defined, then the identity from $W_\alpha^+(A)$ to $W_{\alpha i}^+(A)$, which is the same set with the intrinsic topology, is continuous on an open, invariant subset $G_\alpha$ of $W_\alpha^+(A) \setminus A$, and $G_\alpha$ is dense in $W_\alpha^+(A) \setminus A$, beause $W_\alpha^+(A)$ is locally compact. We define then $W_{\alpha+1}^+(A) = W_\alpha^+(A) \setminus G_\alpha$. If $\alpha$ is a limit ordinal, we put

$$W_\alpha^+(A) = \bigcap_{\beta<\alpha} W_\beta^+(A).$$

Since $M$ has a countable basis, there exists an ordinal $\delta$ smaller than the first uncountable ordinal such that $W_\alpha^+(A) = W_\delta^+(A)$, for all $\alpha > \delta$. We call the least such $\delta$ the instability depth of $W^+(A)$. It is a measure of the complexity of the flow in $W^+(A)$, and measures how far $A$ is from being asymptotically stable, with respect to the restricted flow in $W^+(A)$. Note that at every step $W^+_\alpha(A)$ is locally compact, and $G_\alpha \neq \varnothing$, since it is dense in $W^+_\alpha(A)$. So the instability depth is $\delta$ if and only if $\delta$ is the least ordinal such that $W^+_\delta(A) = A$.

The instability depth $\delta$ of an isolated unstable attractor $A$ is a successor ordinal. To see this, assume on the contrary that it is a limit ordinal and let $(N, N^+, N^-)$ be an isolating block of $A$ such that $N \subset W^+(A)$. For every $\alpha < \delta$ the set $\partial N \cap W^+_\alpha(A)$ is compact and so $(\partial N \cap W^+_\alpha(A))_{\alpha<\delta}$ is a decreasing family of non-empty, compact subsets of $\partial N$ such that

$$\bigcap_{\alpha<\delta} \partial N \cap W^+_\alpha(A) = \partial N \cap W^+_\delta(A) = \partial N \cap A = \varnothing$$

which contradicts the compactness of $\partial N$.

Isolated unstable atractors whose region of attraction have instability depth 1 have been studied in [12] and [14].

**Example 3.6.** Consider the smooth flow on $\mathbb{R}^2$ defined by the system of differential equations (in polar coordinates)

$$r' = r(1-r), \quad \theta' = \sin^2(\frac{\theta}{2}).$$

Then, $\{(1,0)\}$ is an isolated unstable attractor with $W^+(1,0) = \mathbb{R}^2 \setminus \{(0,0)\}$. The closed disc of radius $1/2$ centered at $(1,0)$ is an isolating block $N$ and $\alpha^+$ is the southern semicircle on $\partial N$, hence homeomorphic to the closed interval $[0,1]$. The final entrance time function is discontinuous at $(s,0)$ or equivalently the identity $id : W^+(1,0) \to W^+_i(1,0)$ is not continuous at $(s,0)$. Now $W^+_i(1,0) \setminus \{(1,0)\}$ is homeomorphic to $\mathbb{R} \times [0,1]$ and $W^+_i(1,0)$ is homeomorphic to $\mathbb{R} \times \mathbb{R}^+$. Here the instability depth is 2, because $W^+_1(1,0) = (0,+\infty) \times \{0\}$ and $W^+_2(1,0) = \{(1,0)\}$. It can be proved that if a fixed point of a flow on $\mathbb{R}^2$ or $S^2$ is an isolated unstable attractor, then the instability depth is at most 2. This is not true for flows on the 2-torus. However, in the case of a fixed point of a flow on an orientable, closed 2-manifold, which is an isolated unstable attractor, the instability depth is always finite [4].

Let $M$ be a connected smooth manifold carrying a smooth flow $(\phi_t)_{t\in\mathbb{R}}$ and let $A \subset M$ be an isolated unstable attractor. If $(N, N^+, N^-)$ is a smooth isolating block of $A$ with $N \subset W^+(A)$, then the interior of $a^+$ in $\partial N$ is not empty, because otherwise the corresponding final entrance time function $\tau : W^+(A) \to [-\infty, +\infty)$ would be discontinuous at every point of $W^+(A) \setminus A$. However $\tau$ is lower semicontinuous and its points of continuity form a dense subset of $W^+(A)$, since the latter is locally compact. If now $S \subset \text{int}_{\partial N} a^+$ is open in $\partial N$, then it is a local section to the flow, because the flow finally enters $N$ through $a^+$ intersecting transversally $\partial N$. It follows that $\phi(I \times S)$ is an open subset of $M$ for every open interval $I \subset \mathbb{R}$. Now any open neighbourhood $V \subset W^+(A)$ of $A$ contains such a smooth isolating block of $A$ and for every open set $U \subset W^+(A)$ we have

$$U \cap \phi((0,+\infty) \times \text{int}_{\partial N} a^+) \subset U \cap \bigcap_{t\geq 0} \phi_{-t}(V).$$

This and the above observation show that each open subset of $W^+_i(A)$ contains an open subset of $W^+(A)$. Stated otherwise, the identity $id : W^+_i(A) \to W^+(A)$ sends open sets to sets with non-empty interior, although it is not open. In particular, the open dense subset of $W^+(A) \setminus A$ at the points of which $id : W^+(A) \to W_i^+(A)$ is continuous (see Proposition 3.5) is also open and dense in $W_i^+(A) \setminus A$.

Since we are interested in 1-dimensional compact minimal sets and such sets for flows on 2-manifolds are never isolated, the following shows that the flow is sufficiently complex around it.

**Theorem 3.7.** *Let $\xi$ be a smooth vector field on a connected, smooth $n$-manifold $M$ and $A \subset M$ be an invariant continuum of dimension at most $n-2$. If $A$ is an isolated unstable attractor, then the instability depth of $W^+(A)$ is at least 2.*

*Proof.* Suppose that the instability depth of $W^+(A)$ is 1. This means that the identity maps $W_i^+(A) \setminus A$ homeomorphically onto $W^+(A) \setminus A$. If $(N, N^+, N^-)$ is a smooth isolating block of $A$, then the flow on $W^+(A) \setminus A$ is parallelizable with section $\alpha^+$. It follows that $\alpha^+$ is a union of connected components of $\partial N$, thus being a compact, $(n-1)$-dimensional, smooth submanifold of $M$ without boundary. Since the dimension of $A$ is at most $n-2$ and $W^+(A)$ is connected and open, $W^+(A) \setminus A$ is connected. It follows now from Theorem 3.4 in [4] that $M$ is compact and $M = W^+(A)$. Moreover, $A$ is an isolated unstable attractor with respect to $-\xi$, whose region of attraction (with respect to $-\xi$) has instability depth 1. This implies that $\partial N = \alpha^+ \cup \alpha^-$ and

$$N \setminus A = \bigcup_{t \geq 0} \phi_t(\alpha^+) \cup \bigcup_{t \leq 0} \phi_t(\alpha^-),$$

where these two sets are nonempty, disjoint and open in $N \setminus A$. This contradicts our assumption that $A$ has dimension at most $n-2$, since $A \subset \text{int}N$. □

We note that although there is an example of a $C^1$ vector field on $S^3$ having an isolated, nonperiodic, 1-dimensional compact minimal set, constructed by P. Schweitzer in [16], we do not have a smooth example. The question on the existence of such a smooth vector field on a 3-manifold is Problem 3.112 on page 177 in R. Kirby's List of Problems on Low-dimensional Topology. The minimal set in Schweitzer's example is not an isolated unstable attractor and we do not know such an example. In any case, an isolated, 1-dimensional, compact minimal set of a $C^1$ vector field on a 3-manifold is a surface minimal set [10], and so its integral first Alexander-Spanier cohomology group is free abelian [3]. Concluding, we arrive at the following question: Let $A$ be a one-dimensional compact minimal set of a $C^1$ vector field on an orientable 3-manifold $M$. If $A$ is an isolated unstable attractor, is then $A$ necessarily a periodic orbit? If no, can the instability depth be infinite?

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