Θεωρούμε Π.Α.Τ.

$$\begin{cases} \dfrac{dx}{dt} = -y(t) & x(0) = 1 \\ \dfrac{dy}{dt} = x(t) & y(0) = 0 \end{cases} \qquad t \geq 0$$

Γνωρίζουμε ότι $\underline{x^2(t) + y^2(t) = 1}$ $t \geq 0$

Ακριβώς $x(t) = \cos(t)$, $y(t) = \sin(t)$

Μέθοδο πεπλεγμένη Euler

$$x_n^2 + y_n^2 = ?$$

Πεπλεγμένη μέθοδος του Euler.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\underline{h>0}$, $t_n = nh$, $n = 0, 1, 2, \dots$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \approx \begin{pmatrix} x(t_n) \\ y(t_n) \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h \begin{pmatrix} -y_{n+1} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} x_n - h\,y_{n+1} \\ y_n + h\,x_{n+1} \end{pmatrix}$$

$$\left.\begin{aligned} x_{n+1} &= x_n - h y_{n+1} \\ y_{n+1} &= y_n + h x_{n+1} \end{aligned}\right\} \Rightarrow \quad x_{n+1} = x_n - h\left(y_n + h x_{n+1}\right) = x_n - h y_n - h^2 x_{n+1}$$

Άρα $$(1+h^2)\, x_{n+1} = x_n - h y_n$$

ή $$\boxed{x_{n+1} = \frac{1}{1+h^2}\left(x_n - h y_n\right)}$$

$$y_{n+1} = y_n + h\left(x_n - h y_{n+1}\right) = y_n + h x_n - h^2 y_{n+1}$$

ή $$\boxed{y_{n+1} = \frac{1}{1+h^2}\left(y_n + h x_n\right)}$$

$$x_{n+1}^2 + y_{n+1}^2 = \left(\frac{1}{(1+h^2)}\right)^2 \left[(x_n - hy_n)^2 + (y_n + hx_n)^2\right]$$

$$= \frac{1}{(1+h^2)^2}\left[x_n^2 + h^2 y_n^2 - \cancel{2hx_ny_n} + y_n^2 + h^2 x_n^2 + \cancel{2hx_ny_n}\right]$$

$$= \frac{1}{1+h^2}\left[x_n^2 + y_n^2\right]$$

$n=0$ $\quad x_0^2 + y_0^2 = 1$

$n=1$ $\quad x_1^2 + y_1^2 = \frac{1}{1+h^2}\left(x_0^2 + y_0^2\right) = \frac{1}{1+h^2}$

$\vdots$

Επαγωγικά $\quad x_n^2 + y_n^2 = \left(\frac{1}{1+h^2}\right)^n$ Παρατηρούμε $\quad x_n^2 + y_n^2 \xrightarrow{n\to\infty} 0$

Μέθοδος <u>τραπεζίου</u> $\qquad x_n^2 + y_n^2 = ?$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{h}{2}\left[\begin{pmatrix} -y_n \\ x_n \end{pmatrix} + \begin{pmatrix} -y_{n+1} \\ x_{n+1} \end{pmatrix}\right]$$

$$x_{n+1} = x_n - \frac{h}{2}\left(y_n + y_{n+1}\right)$$

$$x_{n+1} = x_n - \frac{h}{2}\left(y_n + y_n + \frac{h}{2}(x_n + x_{n+1})\right)$$

$$= x_n - h y_n - \frac{h^2}{2^2} x_n - \frac{h^2}{2^2} x_{n+1}$$

$$\boxed{\left(1 + \frac{h^2}{4}\right) x_{n+1} = \left(1 - \frac{h^2}{4}\right) x_n - h y_n}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(x_n + x_{n+1}\right)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(x_n + x_n - \frac{h}{2}(y_n + y_{n+1})\right)$$

$$= y_n + h x_n - \frac{h^2}{4} y_n - \frac{h^2}{4} y_{n+1}$$

$$\boxed{\left(1 + \frac{h^2}{4}\right) y_{n+1} = \left(1 - \frac{h^2}{4}\right) y_n + h x_n}$$

$$x_{n+1} = \frac{1}{1+h^2/4}\left(1-h^2/4\right)x_n - \frac{h}{1+h^2/4}y_n.$$

$$y_{n+1} = \frac{1}{1+h^2/4}\left(1-h^2/4\right)y_n + \frac{h}{1+h^2/4}x_n.$$

$x_0 = 1,\ y_0 = 0 \Rightarrow x_0^2 + y_0^2 = 1.$

$$x_1 = \frac{1-h^2/4}{1+h^2/4}x_0, \quad y_1 = \frac{h}{1+h^2/4}x_0$$

$$x_1^2 + y_1^2 = \left(\frac{1-h^2/4}{1+h^2/4}\right)^2 + \frac{h^2}{(1+h^2/4)^2} = \frac{1+\left(\frac{h^2}{4}\right)^2 - 2\frac{h^2}{4} + h^2}{\left(1+h^2/4\right)^2} = \frac{1+\left(\frac{h^2}{4}\right)^2 + 2\frac{h^2}{4}}{(1+h^2/4)^2}$$

$$= 1$$

$$x_{n+1}^2 + y_{n+1}^2 = \ ?$$

$$\left(1+\frac{h^2}{4}\right)^2 \left(x_{n+1}^2+y_{n+1}^2\right) = \left[\left(1-\frac{h^2}{4}\right)x_n - hy_n\right]^2 + \left[\left(1-\frac{h^2}{4}\right)y_n + hx_n\right]^2$$

$$= \left[\left(1-\frac{h^2}{4}\right)^2 x_n^2 + h^2 y_n^2 - \cancel{2h\left(1-\frac{h^2}{4}\right)} x_n y_n + \left(1-\frac{h^2}{4}\right)^2 y_n^2 + h^2 x_n^2 + \cancel{2h\left(1-\frac{h^2}{4}\right)} x_n y_n\right]$$

$$= \left(x_n^2+y_n^2\right)\left[\left(1-\frac{h^2}{4}\right)^2 + h^2\right] = \left(x_n^2+y_n^2\right)\left[1+\left(\frac{h^2}{4}\right)^2 - 2\frac{h^2}{4} + 4\frac{h^2}{4}\right]$$

$$= \left(x_n^2+y_n^2\right)\left(1+\frac{h^2}{4}\right)^2 = \dots = \left(1+\frac{h^2}{4}\right)^2\left(x_0^2+y_0^2\right) = \left(1+\frac{h^2}{4}\right)^2$$

$\Rightarrow \quad x_n^2+y_n^2 = 1 \quad$ για κάθε n.