Εφαρμογή μεθόδου RK

$$\begin{array}{cc|c} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ \hline 0 & 1 & \end{array}$$

Βελτιωμένη Euler

$$y_{n,1} = y_n$$

$$y_{n,2} = y_n + h\tfrac{1}{2} f(t_n + 0h, y_{n,1})$$

$$= y_n + \frac{h}{2} f(t_n, y_n)$$

$$y_{n+1} = y_n + h f(t_{n,2}, y_{n,2})$$

$$= y_n + h\, f\left(t_n + \frac{h}{2},\ y_n + \frac{h}{2} f(t_n, y_n)\right)$$

$$\begin{cases} y'(t) = (1-2t)\,y(t) & t>0 \\ y(0) = 1 \end{cases}$$

Για $h = 0.1$

$$y_{0,1} = y_0\ , \quad y_{0,2} = y_0 + \frac{h}{2} f(t_0, y_0)$$

$$y_1 = y_0 + h f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(t_0, y_0)\right)$$

$t_0 = 0\ ,\ y_0 = 1$

$$y_{0,1} = 1\ , \quad y_{0_2} = 1 + \frac{0.1}{2}(1-2\cdot 0)\cdot 1 = 1.05$$

$$y_1 = y_0 + h f\left(\frac{h}{2},\ y_0 + \frac{h}{2} f(0, y_0)\right) = 1 + 0.1\, f(0.05,\ 1.05)$$

$$= 1 + 0.1\,(1 - 2\cdot 0.05)\cdot 1.05 = \ldots = 1.0945$$

Κανόνας πρόλεψης μέχρι το $t = 1{,}2$

Σφάλμα RK : 0.00721

Για άλλες μεθόδους

Άμεση Euler : 0.07344 $(p=1)$

Πεπ. Euler : 0.06510 $(p=1)$

Τραπεζίου : $0{,}00789$ $(p=2)$

Πρόγραμμα Runge Kutta.

$$
\begin{array}{cc|c}
1/4 & 1/4-\mu & 1/2-\mu \\
1/4+\mu & 1/4 & 1/2+\mu \\
\hline
1/2 & 1/2 &
\end{array}
\qquad \mu = \frac{\sqrt{3}}{6}
$$

$$
\begin{cases} y'(t) = \lambda y(t) \\ y(0) = 1 \end{cases}, \quad \lambda \in \mathbb{R}.
$$

$$t_{n,1} = t_n + \left(\tfrac{1}{2} - \mu\right) h, \quad t_{n,2} = t_n + \left(\tfrac{1}{2} + \mu\right) h$$

$$y_{n,1} = y_n + h\left(\tfrac{1}{4} f(t_{n,1}, y_{n,1}) + \left(\tfrac{1}{4} - \mu\right) f(t_{n,2}, y_{n,2})\right)$$

$$y_{n,2} = y_n + h\left(\left(\tfrac{1}{4} + \mu\right) f(t_{n,1}, y_{n,1}) + \left(\tfrac{1}{4}\right) f(t_{n,2}, y_{n,2})\right)$$

Γραμμικό σύστημα για $f(t,y)=\lambda y$.

$$y_{n,1} = y_n + \frac{1}{4} h\lambda y_{n,1} + \left(\tfrac{1}{4} - \mu\right) h\lambda y_{n,2}$$

$$y_{n,2} = y_n + \left(\frac{1}{4} + \mu\right) h\lambda y_{n,1} + \left(\tfrac{1}{4}\right) h\lambda y_{n,2}$$

$$\begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} = \underbrace{\begin{bmatrix} \tfrac{1}{4} h\lambda & \left(\tfrac{1}{4} - \mu\right) h\lambda \\ \left(\tfrac{1}{4} + \mu\right) h\lambda & \tfrac{1}{4} h\lambda \end{bmatrix}}\begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} + \begin{pmatrix} y_n \\ y_n \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} 1 - \tfrac{1}{4} h\lambda & -\left(\tfrac{1}{4} - \mu\right) h\lambda \\ -\left(\tfrac{1}{4} + \mu\right) h\lambda & 1 - \tfrac{1}{4} h\lambda \end{bmatrix}}_{B}\underbrace{\begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix}} = \underbrace{\begin{pmatrix} y_n \\ y_n \end{pmatrix}}$$

$$B = \underset{\sim}{L} \cdot \underset{\sim}{U}$$

κάτω τριγωνικός   άνω τριγωνικός

$$L = \begin{bmatrix} 1 & 0 \\ -\dfrac{(1/4\, f'(u))h_j}{1 - 1/4\, h_j} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 - 1/4\, h_j & -(1/4\, f'(u))h_j \\ 0 & \dfrac{1 - \frac{h_j}{2} + f'(u)^2 (h_j)^2}{1 - h_j/4} \end{bmatrix}$$

$$LU \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} = \begin{pmatrix} y_n \\ y_n \end{pmatrix}$$

1) $Lw = \begin{pmatrix} y_n \\ y_n \end{pmatrix}$ 2) $U \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

1) $Lw = \begin{pmatrix} y_n \\ y_n \end{pmatrix} \Rightarrow \quad w_1 = y_n, \quad w_2 = y_n \dfrac{1 + \mu h\lambda}{1 - h\lambda/4}$

2) $U \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y_{n,2} = y_n \dfrac{1 + \mu h\lambda}{1 - \frac{h\lambda}{2} + \mu^2 (h\lambda)^2}$

$$y_{n,1} = y_n \frac{1 - \mu h\lambda}{1 - \frac{h\lambda}{2} + \mu^2 (h\lambda)^2}$$

$$y_{n+1} = y_n + \frac{1}{2} h \Big(f(t_{n,1}, y_{n,1}) + f(t_{n,2}, y_{n,2}) \Big)$$

$$= y_n + \frac{h\lambda}{2} (y_{n,1} + y_{n,2}) = \dots = y_n \cdot \frac{1 + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{12}}{1 - \frac{h\lambda}{2} + \frac{(h\lambda)^2}{12}} \qquad \left(\mu = \frac{\sqrt{3}}{6}\right)$$