Runge-Kutta - 2 σταδίων (άμεσοι)

$$\begin{array}{cc|c} 0 & 0 & 0 \\ a & 0 & a \\ \hline b_1 & b_2 & \end{array}$$

$$t_{n,1} = t_n \qquad t_{n,2} = t_n + ah$$

$$y_{n,1} = y_n \quad , \quad y_{n,2} = y_n + a h f(t_{n,1}, y_{n,1}) = y_n + ah f(t_n, y_n)$$

$$y_{n+1} = y_n + h\left(b_1 f(t_n, y_n) + b_2 f(t_n + ah, y_n + ah f(t_n, y_n))\right)$$

$$\delta_n = y(t_{n+1}) - \left\{ y(t_n) + h\left(b_1 f(t_{n,1}, J_{n,1}) + b_2 f(t_{n,2}, J_{n,2})\right)\right\}$$

$$J_{n,1} = y(t_n) \quad , \quad J_{n,2} = y(t_n) + ah f(t_n, y(t_n))$$

Αναπτυγμα Taylor της $y(t_{n+1})$ με κεντρο το $y(t_n)$

$$y(t_{n+1}) = y(t_n) + h\,y'(t_n) + \frac{h^2}{2}\,y''(t_n) + \frac{h^3}{3!}\,y'''(t_n) + O(h^4)$$

$$y'(t_n) = f(t_n, y(t_n))$$

$$y''(t_n) = \frac{d}{dt}\Big(f(t_n, y(t_n))\Big) = f_t(t_n, y(t_n)) + f(t_n, y(t_n)) \cdot f_y(t_n, y(t_n))$$

Αναπτυγμα της f με κεντρο το (t, y)

$$f(t+\alpha h, y+\beta h) = f(t, y+\beta h) + \alpha h \cdot f_t(t, y+\beta h) + O(h^2)$$

$$= f(t,y) + \beta h\, f_y(t,y) + \alpha h\Big(f_t(t,y) + \beta h f_{yy}(t,y) + O(h^2)\Big) + O(h^2)$$

$$= f(t,y) + h\big(\alpha f_t(t,y) + \beta f_y(t,y)\big) + O(h^2)$$

Θα χρησιμοποιήσουμε τον συμβολισμό: $\tilde{f} = f(t_n, y(t_n))$, $\tilde{f}_t = f_t(t_n, y(t_n))$

$\tilde{f}_y = f_y(t_n, y(t_n))$ κ.ο.κ

$$f(t_n + ah,\, y(t_n) + \beta h) = \tilde{f} + h(a\tilde{f}_t + a\tilde{f}\tilde{f}_y) + O(h^2)$$

$\beta = a\tilde{f}$

Άρα

$$\delta_n = y(t_n) + h\tilde{f} + \frac{h^2}{2}(\tilde{f}_t + \tilde{f}\tilde{f}_y) + O(h^3)$$

$$- \left\{ y(t_n) + h\left(b_1\tilde{f} + b_2\left(\tilde{f} + ha(\tilde{f}_t + \tilde{f}\tilde{f}_y)\right)\right) \right\}$$

$$= h\left(1 - (b_1 + b_2)\right)\tilde{f} + \frac{h^2}{2}(1 - 2b_2 \cdot a)(\tilde{f}_t + \tilde{f}\tilde{f}_y) + O(h^3)$$

$b_1 + b_2 = 1$ & $b_2 a = 1/2$ $\Rightarrow$ $\delta_n = O(h^3)$ $\Rightarrow$ τάξη ακρίβειας τουλάχιστον 2.

$$y'''(t) = \frac{d^2}{dt^2}\left(f(t,y(t)\right) = \frac{d}{dt}\left(f_t(t,y(t)) + f(t,y(t))f_y(t,y(t))\right)$$

$$= f_{tt}(t,y(t)) + f_{ty}(t,y(t))\,y'(t) + \left(f_t(t,y(t)) + f(t,y(t))\,f_y(t,y(t))\right)f_y(t,y(t))$$

$$+ f(t,y(t))\cdot\left(f_{ty}(t,y(t)) + f_{yy}(t,y(t))\,y'(t)\right)$$

$$y'''(t_n) = \tilde{f}_{tt} + \tilde{f}_{ty}\hat{f} + (\hat{f}_t + \tilde{f}\hat{f}_y)\tilde{f}_y + \hat{f}(\hat{f}_{ty} + \tilde{f}_{yy}\hat{f})$$

$$= \tilde{f}_{tt} + 2\hat{f}_{ty}\hat{f} + \hat{f}_t\hat{f}_y + \hat{f}^2\hat{f}_{yy} + \hat{f}\hat{f}_y^{\,2}$$

$$f(t+\alpha h, y+\beta h) = f(t, y+\beta h) + \alpha h f_t(t, y+\beta h) + \frac{(\alpha h)^2}{2} f_{tt}(t, y+\beta h) + O(h^3)$$

$$= f(t,y) + \beta h f_y(t,y) + \frac{(\beta h)^2}{2} f_{yy}(t,y) + O(h^3)$$

$$+ \alpha h f_t(t,y) + \alpha h \beta h f_{ty}(t,y) + \frac{(\alpha h)^2}{2} f_{tt}(t,y)$$

$$= f(t,y) + \alpha h f_t(t,y) + \beta h f_y(t,y)$$

$$+ \frac{(\alpha h)^2}{2} f_{tt}(t,y) + \frac{(\beta h)^2}{2} f_{yy}(t,y) + \alpha\beta h^2 f_{ty}(t,y) + O(h^3)$$

$$f(t_n+\alpha h, y(t_n)+\alpha h\tilde{f}) = \tilde{f} + \alpha h(\tilde{f}_t + \hat{f}\hat{f}_y) + \frac{h^2}{2}\left(\alpha^2 \tilde{f}_{tt} + \alpha^2 \tilde{f}^2 \tilde{f}_{yy} + 2\alpha^2 \hat{f}\tilde{f}_{ty}\right) + O(h^3)$$

$$\delta_n = y(t_{n+1}) - \left\{ y(t_n) + h\left(b_1 \tilde{f} + b_2 f(t_n + ah, y(t_n) + ah\tilde{f})\right)\right\}$$

$$= h\left(1 - (b_1 + b_2)\right)\tilde{f} + \frac{h^2}{2}(1 - 2ab_2)(\tilde{f}_t + \tilde{f}\tilde{f}_y)$$

$$+ \frac{h^3}{3!}\left(\tilde{f}_{tt} + 2\tilde{f}_{ty}\tilde{f} + \tilde{f}_t\tilde{f}_y + \tilde{f}^2\tilde{f}_{yy} + \tilde{f}\tilde{f}_y^2\right)$$

$$- \frac{h^3}{2} b_2 a^2 \left(\tilde{f}_{tt} + \tilde{f}^2\tilde{f}_{yy} + 2\tilde{f}\tilde{f}_{ty}\right) + O(h^4)$$

$$= h\left(1 - (b_1 + b_2)\right)\hat{f} + \frac{h^2}{2}(1 - 2ab_2)(\hat{f}_t + \hat{f}\hat{f}_y) + \frac{h^3}{2}\left(\frac{1}{3} - a^2 b_2\right)\left(\hat{f}_{tt} + 2\hat{f}\hat{f}_{ty} + \hat{f}^2\hat{f}_{yy}\right)$$

$$+ \frac{h^3}{3!}\left(\hat{f}_t\hat{f}_y + \hat{f}\hat{f}_y^2\right) + O(h^4)$$

Τάξη ακρίβειας $p = 2$