Runge Kutta (3-σταδίων άμεσες) - Τάξη ακρίβειας.

$$
\begin{array}{ccc|c}
0 & 0 & 0 & \tau_1 \\
a_{21} & 0 & 0 & \tau_2 \\
a_{31} & a_{32} & 0 & \tau_3 \\
\hline
b_1 & b_2 & b_3 &
\end{array}
$$

$$\tau_1 = 0$$
$$\tau_2 = a_{21}$$
$$\tau_3 = a_{31} + a_{32}$$

$$t_{n,1} = t_n \quad , \quad t_{n,2} = t_n + a_{21} h \quad , \quad t_{n,3} = t_n + \tau_3 h$$

$$y_{n,1} = y_n$$
$$y_{n,2} = y_n + h a_{21} f(t_{n,1}, y_{n,1}) = y_n + h a_{21} f(t_n, y_n)$$
$$y_{n,3} = y_n + h\left(a_{31} f(t_n, y_n) + a_{32} f(t_{n,2}, y_{n,2})\right)$$

$$y_{n+1} = y_n + h\left(b_1 \underbrace{f(t_n, y_n)}_{k_1} + b_2 \underbrace{f(t_{n,2}, y_{n,2})}_{k_2} + b_3 \underbrace{f(t_{n,3}, y_{n,3})}_{k_3}\right)$$

$$\delta_n = y(t_{n+1}) - \left\{ y(t_n) + h\left(b_1 f(t_{n,1}, J_{n,1}) + b_2 f(t_{n,2}, J_{n,2}) + b_3 f(t_{n,3}, J_{n,3})\right) \right\}$$

$$J_{n,1} = y(t_n), \quad J_{n,2} = y(t_n) + h\, a_{21} f(t_n, y(t_n))$$

$$J_{n,3} = y(t_n) + h\left(a_{31} f(t_n, y(t_n)) + a_{32} f(t_{n,2}, J_{n,2})\right)$$

$$\tilde{f} = f(t_n, y(t_n)), \quad \tilde{f}_t = f_t(t_n, y(t_n)), \quad \text{u.s.w.}$$

$$f(t_{n,1}, J_{n,1}) = \tilde{f}$$

$$\begin{aligned}
f(t_{n,2}, J_{n,2}) &= f(t_n + a_{21}h,\ y(t_n) + h a_{21}\tilde{f}) \\
&= f(t_n, J_{n,2}) + a_{21} h\, f_t(t_n, J_{n,2}) + \frac{1}{2}(a_{21}h)^2 f_{tt}(t_n, J_{n,2}) + O(h^3) \\
&= \tilde{f} + h a_{21}\tilde{f}\tilde{f}_y + \frac{h^2}{2}(a_{21}\tilde{f})^2 \tilde{f}_{yy} + O(h^3) \\
&\quad + a_{21}h\left(\tilde{f}_t + h a_{21}\tilde{f}\tilde{f}_{ty} + O(h^2)\right) + \frac{1}{2}(a_{21}h)^2\left(\tilde{f}_{tt} + O(h)\right) \\
&= \tilde{f} + h a_{21}(\tilde{f}_t + \tilde{f}\tilde{f}_y) + \frac{h^2}{2} a_{21}^2\left(\tilde{f}_{tt} + \tilde{f}^2 \tilde{f}_{yy} + 2\tilde{f}\tilde{f}_{ty}\right) + O(h^3)
\end{aligned}$$

$$f(t_{n,3},y_{n,3}) = f(t_n,y_{n,3}) + c_3 h f_t(t_n,y_{n,3}) + c_3^2\frac{h^2}{2} f_{tt}(t_n,y_{n,3}) + O(h^3)$$

$$= \hat{f} + h(a_{31}\hat{f} + a_{32} f(t_{n,2},y_{n,2}))\hat{f}_y + \frac{h^2}{2}(a_{31}\hat{f} + a_{32} f(t_{n,2},y_{n,2}))^2\hat{f}_{yy} + O(h^3)$$

$$+ c_3 h\left(\hat{f}_t + h(a_{31}\hat{f} + a_{32} f(t_{n,2},y_{n,2}))\hat{f}_{ty} + O(h^2)\right)$$

$$+ c_3^2\frac{h^2}{2}\left(\hat{f}_{tt} + O(h)\right)$$

$$= \hat{f} + h\left(a_{31}\hat{f} + a_{32}\left[\hat{f} + h a_{21}(\hat{f}_t + \hat{f}\hat{f}_y) + O(h^2)\right]\right)\hat{f}_y$$

$$+ \frac{h^2}{2}\left(a_{31}\hat{f} + a_{32}[\hat{f} + O(h)]\right)^2\hat{f}_{yy} + c_3 h\left(\hat{f}_t + h(a_{31}\hat{f} + a_{32}[\hat{f} + O(h)])\hat{f}_{ty}\right) + c_3^2\frac{h^2}{2}\hat{f}_{tt} + O(h^3)$$

$$= \hat{f} + h\left((a_{31}+a_{32})\hat{f}\hat{f}_y + c_3\hat{f}_t\right)$$

$$+ \frac{h^2}{2}\left[2a_{32}a_{21}(\hat{f}_t + \hat{f}\hat{f}_y)\hat{f}_y + (a_{31}+a_{32})^2\hat{f}^2\hat{f}_{yy} + 2(a_{31}+a_{32})^2\hat{f}\hat{f}_{ty} + c_3^2\hat{f}_{tt}\right] + O(h^3)$$

Για το $y(t_n) + h(b_1 \hat{f} + b_2 f(t_{n,2}, Y_{n,2}) + b_3 f(t_{n,3}, Y_{n,3}))$ για κάθε παράγοντα των όρων h^0, h^1, h^2, h^3

h^0 : $y(t_n)$

h^1 : $(b_1 + b_2 + b_3)\hat{f}$

h^2 : $b_2(a_{21}(\hat{f}_t + \hat{f}\hat{f}_y)) + b_3 c_3 (\hat{f}_t + \hat{f}\hat{f}_y)$

h^3 : $b_2 \frac{1}{2} a_{21}^2 (\hat{f}_{tt} + \hat{f}^2 f_{yy} + 2\hat{f}\hat{f}_{ty})$

$+ b_3 \frac{1}{2} c_3^2 (\hat{f}_{tt} + \hat{f}^2 \hat{f}_{yy} + 2\hat{f}\hat{f}_{ty}) + b_3 a_{32} a_{21} (\hat{f}_t + \hat{f}\hat{f}_y)\hat{f}_y$

Το ανάπτυγμα του $y(t_{n+1})$ θα είναι

$$y(t_{n+1}) = y(t_n) + h\, y'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(t_n) + O(h^4)$$

$$= y(t_n) + h\hat{f} + \frac{h^2}{2}\left(\hat{f}_t + \hat{f}\hat{f}_y\right) + \frac{h^3}{6}\left[\hat{f}_{tt} + \hat{f}^2\hat{f}_{yy} + 2\hat{f}\hat{f}_{ty} + (\hat{f}_t + \hat{f}\hat{f}_y)\hat{f}_y\right]$$

$$+ O(h^4)$$

Εξισώνουμε τους όρους με αντίστοιχες δυνάμεις του h και έχουμε ότι:

h^0 : $y(t_n) - y(t_n) = 0$

h^1 : $\hat{f} - (b_1 + b_2 + b_3)\hat{f}$

h^2 : $\frac{1}{2}(\hat{f}_t + \hat{f}\hat{f}_y) - (b_2 a_{21} + b_3 \tau_3)(\hat{f}_t + \hat{f}\hat{f}_y)$

h^3 : $\frac{1}{6}(\hat{f}_{tt} + \hat{f}^2\hat{f}_{yy} + 2\hat{f}\hat{f}_{ty}) - \left(\frac{1}{2} b_2 a_{21}^2 + \frac{1}{2} b_3 \tau_3^2\right)(\hat{f}_{tt} + \hat{f}^2\hat{f}_{yy} + 2\hat{f}\hat{f}_{ty})$

$+ \frac{1}{6}(\hat{f}_t + \hat{f}\hat{f}_y)\hat{f}_y - b_3 a_{32} a_{21}(\hat{f}_t + \hat{f}\hat{f}_y)\hat{f}_y$

Πρεπει να ισχυουν οι εξης εξισωσεις για να εχω ταξη ακριβειας τουλαχιστον $p=3$.

1. $$b_1+b_2+b_3=1$$

2. $$b_2a_{21}+b_3(a_{31}+a_{32})=\frac{1}{2}$$

3. $$\frac{1}{2}\left(b_2a_{21}^2+b_3(a_{31}+a_{32})^2\right)=\frac{1}{6}$$

4. $$b_3a_{32}a_{21}=\frac{1}{6}$$