ON SOME SECOND ORDER SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH NON-DEGENERATE INNER SOLUTIONS

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ABSTRACT. In this paper we study an inhomogeneous Allen-Cahn equation where the spatial reaction is positive but vanishes at a point. We also consider Fisher’s equation and the nonlinear Schrödinger equation with critical frequency.

1. Introduction

Throughout this paper we will denote by $c/C, C'$ a small/large positive generic constant independent of $\varepsilon$ whose value will change from line to line. Frequently we will not explicitly denote the obvious dependence of functions on $\varepsilon$.

2. Transition layer solutions in an Allen-Cahn equation with non-negative spatial reaction

In [9] the author studied the singularly perturbed boundary value problem

$$
\begin{align*}
\varepsilon^2 u'' + h(x)g(u) &= 0, \quad 0 < x < 1, \\
 u'(0) = 0 &= u'(1),
\end{align*}
$$

under the hypotheses:

- $h \in C[0,1]$ is strictly positive,
- $g \in C^2(\mathbb{R})$ and

$$
\begin{align*}
g &\text{ has precisely three zeros } a_- < 0 < a_+ \\
g_u(a_-) &< 0, \quad g_u(0) > 0, \quad g_u(a_+) < 0 \\
\int_{a_-}^{a_+} g(u)du &= 0 \\
g_u(u) &> g_u(u), \quad u \neq 0.
\end{align*}
$$

Theorem 2.1. [8, 9] Let $\{x_1, x_2, ..., x_m\}$ be an arbitrary subset of the set of local minimum points of $h$. Then there exists a stable solution of (2.1) which has one layer near each $x_k$ ($k = 1, \cdots, m$) and has no layer in the rest of the interval $(0,1)$.

The inner solution at each $x_k$ is $U \left( \frac{x-x_k}{\varepsilon} \right)$ where

$$
\begin{align*}
U'' + h(x_k)g(u) &= 0, \quad x \in \mathbb{R}, \\
U(x) &\to a_- \text{ as } x \to -\infty, \quad U(x) \to a_+ \text{ as } x \to \infty.
\end{align*}
$$

Partly supported by grant FONDECYT 3085026.
(the layer can also go in the opposite direction and is also stable). The above equation has a unique solution \( U \) (modulo translation) and Theorem 2.1 follows by constructing suitable upper and lower solutions.

Our interest is in the case when \( h \) is strictly positive and has a global minimum at \( x_1 \in (0,1) \) with \( h(x_1) = 0 \). A similar investigation for the nonlinear Schrödinger equation was carried out in [2] (see also Section 4 of the present paper).

We assume that there exists \( A > 0, \alpha > 0 \) such that
\[
h(x) = A|\alpha-x_1|^{\alpha} + O(|x-x_1|^{\alpha+\delta}), \quad x \in [0,1],
\]
for some \( \delta > 0 \) (\( \forall R > 1, |t| \leq R \) we have \( |O(t)| \leq C(R)|t| \)). We consider only odd functions \( g \) and more precisely \( g(u) = u - u^3 \) (i.e., \( a_1 = 1, a_+ = -1 \)) but our approach works for odd functions \( g \) with \( g_u(0) > 0 \), \( g(a_+) = 0 \), \( g_u(a_+) < 0 \) (we don’t need the last assumption in (2.2)).

Equation 2.1 models phase transitions between the two stable states \( a_- \), \( a_+ \) in the spatial domain \((0,1)\). By allowing \( h \) to take the value 0 we can include more general situations.

We will consider the following equation and boundary conditions
\[
\varepsilon^2u'' + h(x)(u-u^3) = 0, \quad x \in (0,1),
\]
\[
u'(0) = 0 = u'(1).
\]

2.1. The inner approximate solution \( u_{in,\varepsilon} \). For every \( u \in C^2[0,1] \), we have
\[
\varepsilon^2u'' + h(x)(u-u^3) = \varepsilon^2u'' + A|x-x_1|^{\alpha}(u-u^3) + O(|x-x_1|^{\alpha+\delta})(u-u^3).
\]
We will search the inner solution \( u_{in,\varepsilon} \) as a suitable solution of
\[
\varepsilon^2u'' + A|x-x_1|^{\alpha}(u-u^3) = 0, \quad x \in \mathbb{R}
\]
and then we will restrict its domain of definition to \([0,1]\). Using the transformation
\[
u(x) = U\left(\frac{x-x_1}{\varepsilon^{\frac{1}{\alpha}}}\right), \quad x \in \mathbb{R}
\]
we see that (2.7) becomes
\[
U''(x) + A|x|^{\alpha} U(x) - U^3(x), \quad x \in \mathbb{R}.
\]
Since \( u \) should be layered, the obvious boundary conditions for \( U \) are
\[
\lim_{x \to -\infty} U(x) = -1 \quad \text{and} \quad \lim_{x \to \infty} U(x) = 1.
\]

**Proposition 2.1.** There exists an odd solution \( U \) with \( U'(x) > 0, \ x \in \mathbb{R} \) of
\[
u'' = A|x|^{\alpha}(u^3 - u) \quad \text{in} \ \mathbb{R}
\]
satisfying
\[
u(x) \to -1 \quad \text{as} \ x \to -\infty, \quad u(x) \to 1 \quad \text{as} \ x \to +\infty.
\]

**Proof.** We follow the idea of [4]. Equation (2.10) is equivalent to the non-autonomous system
\[
p' = q,
\]
\[
q' = A|x|^{\alpha}p(p+1)(p-1).
\]
Consider first the aspect of solution curves in \((p,q,x)\)-space with \( x \geq 0 \). Let
\[
W = \{(p,q,x)| \ 0 \leq p \leq 1, \ q \geq 0 \ \text{and} \ x \geq 0\}.
\]
First observe that any positive half orbit \( \{(p(x), q(x), x) \mid x \geq 0\} \) with \( p(0) = 0, q(0) > 0 \) which lies totally in \( W \) gives us a solution that we seek. Indeed, by odd reflection we see that \( p \) satisfies (2.10). Moreover \( p' = q > 0 \) (if \( q(x_0) = 0 \) then \( q \equiv 0 \)). It remains to show that \( p \) satisfies (2.11). \( p(x) \) must increase to some positive limit \( p_\infty \leq 1 \). Also since \( q' = -Ax|q|p(p + 1)(1 - \alpha) < 0, q \) must decrease to some nonnegative limit \( q_\infty \geq 0 \). Now if \( q_\infty > 0 \), then \( p_\infty \) would be \( +\infty \), thus \( q_\infty = 0 \). If \( p_\infty < 1 \), then \( \lim_{x \to +\infty} q'(x) = -\infty \) which implies that \( \lim_{x \to +\infty} q(x) = -\infty \), thus \( q_\infty = 0 \).

If an orbit leaves \( W \) it must do so across one of two faces, namely, \( \{p = 1, q > 0, x > 0\} \) or \( \{q = 0, 0 < p < 1, x > 0\} \). This follows from \( p' = q \geq 0 \) and \( q' = -Ax|q|p(p + 1)(1 - \alpha) \leq 0 \). Furthermore each exit point is a point of strict exit meaning (here) that orbits cross the faces non-tangentially; namely on the first face \( p' > 0 \) and on the second \( q' < 0 \) (note that the faces are “open”). Wazewski’s lemma implies that the map that assigns to each point of \( W \) its point of exit from \( W \) is continuous wherever it is defined. (This is easily proved directly in this simple case).

Consider the line \( S_0^Q = \{(0, q, 0) \mid \delta \leq q \leq Q\} \) where \( \delta, Q > 0 \).

**Claim 1:** If \( Q > 0 \) is sufficiently large, then the trajectory \( \{(p(x), q(x), x) \mid 0 \leq x < x_{\text{max}}\} \) starting at the upper endpoint \((0, Q, 0)\) of \( S_0^Q \) leaves \( W \) for the first time through the face \( \{p = 1, q > 0, x > 0\} \).

If not, then there exists a first \( 0 < x_0 < x_{\text{max}} \) such that
\[
q(x_0) = \frac{Q}{2} \quad \text{and} \quad (p(x), q(x), x) \in W, \ 0 \leq x \leq x_0.
\]

Making use of (2.12) we obtain that \( p(x) \geq \frac{Q}{2} x, x \in [0, x_0] \) and in particular \( x_0 \leq \frac{2}{Q} \).

Also \(-q'(x) \leq 2Ax^\alpha \) in \((0, x_0)\), and by integrating over \((0, x_0)\),
\[
\frac{Q}{2} \leq 2A \frac{x_0^{\alpha+1}}{\alpha+1} \leq 2A \frac{2^{\alpha+1}}{\alpha+1} Q^{\alpha+1}.
\]

This is impossible if \( Q \) is chosen sufficiently large.

**Claim 2:** If \( \delta > 0 \) is sufficiently small, then the trajectory \( \{(p(x), q(x), x) \mid 0 \leq x < x_{\text{max}}\} \) starting at the lower endpoint \((0, \delta, 0)\) of \( S_0^Q \) leaves \( W \) for the first time through the face \( \{q = 0, 0 < p < 1, x > 0\} \).

Indeed, \( p, q \) are \( O(\delta^3) \) close to \( \bar{p}, \bar{q} \) in compact intervals as \( \delta \to 0 \), where
\[
\bar{p}' = \bar{q}, \quad \bar{q}' = -Ax^\alpha \bar{p},
\]
\[
\bar{p}(0) = 0, \bar{q}(0) = \delta. \quad \text{Note that} \quad \bar{p}(x) = \delta v(x), \bar{q}(x) = \delta v'(x) \quad \text{where}
\]
\[
v'' + Ax^\alpha v = 0, \quad v(0) = 0, \quad v'(0) = 1.
\]

Since \( v \) is oscillatory, \( q \) becomes negative after an \( O(1) \) \( x \)-interval whereas \( 0 < p = O(\delta) \) in this interval.

It follows that there exists a point of \( S_0^Q \) whose positive half orbit is contained in \( W \) provided \( Q \) is chosen large and \( \delta > 0 \) small. If not, then the exit curve of \( S_0^Q \) (which would be continuous) would contain a point on the line \( \{(1, 0, x) \mid x > 0\} \) which is not possible.

The proof of the proposition is complete. \( \square \)
Remark 2.1. $U$ is the only odd solution of (2.10), (2.11). Indeed, let us suppose that there exists another odd solution $V$ of (2.10), (2.11). Then it is easy to see that $\theta V$, $0 \leq \theta \leq 1$ is a family of subsolutions of (2.10) in $(0, \infty)$ such that $\theta V(0) = U(0)$, $\theta V(\infty) < U(\infty)$, $0 \leq \theta < 1$. Moreover, $\theta V \leq U$ in $(0, \infty)$ when $\theta = 0$. By Serrin’s sweeping technique (see [10]) we get $V \leq U$ in $(0, \infty)$. Similarly, $U \leq V$ in $(0, \infty)$.

Proposition 2.2. 

$$0 < 1 - U(x) \leq C x^{-\frac{2}{\alpha}} e^{-\frac{2\sqrt{2}A}{\alpha+2} x^{\frac{\alpha+2}{\alpha+1}}}, \quad x > 0.$$  \hfill (2.13)

Proof. The function $\varphi = 1 - U$ is bounded in $(0, \infty)$ and satisfies 

$$-\varphi'' + q(x)\varphi = 0, \quad x > 0,$$

with $q(x) = A|\alpha|U(1 + U)$. The bound now follows from the asymptotic theory of linear equations (see [1]).

We define the inner approximate solution of (2.4) by 

$$u_{in, \varepsilon}(x) = U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right), \quad x \in [0, 1].$$

Proposition 2.3. For small $\varepsilon > 0$, we have 

$$\|\varepsilon^2 u''_{in} + h(x)(u_{in} - u_{in}^3)\|_{L^\infty(0, 1)} \leq C\varepsilon^{\frac{2(\alpha + \delta)}{\alpha+2}}.$$ 

Proof. Note that, by its construction, $u_{in, \varepsilon}$ satisfies 

$$|\varepsilon^2 u''_{in} + h(x)(u_{in} - u_{in}^3)| \leq C|x - x_1|^{\alpha + \delta} \left[1 + U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right)\right] \left[1 - U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right)\right]$$

$$\leq C\varepsilon^{\frac{2(\alpha + \delta)}{\alpha+2}} \left|\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right|^{\alpha + \delta} e^{-c \varepsilon |x - x_1|^{\frac{1}{\alpha+2}}},$$

$x \in [0, 1]$. The proof is complete.
2.2. The approximate solution \( u_{ap,\varepsilon} \). The function \( U \left( \frac{x-x_1}{\varepsilon^{1/2}} \right) \) is certainly a good candidate for an approximate transition layer solution to the equation (2.4). However, it does not satisfy the boundary conditions. We, therefore, adjust the approximate solution to

\[
u_{ap,\varepsilon}(x) = \zeta_0(x-x_1)U \left( \frac{x-x_1}{\varepsilon^{1/2}} \right) + \zeta_+(x-x_1), \tag{2.14}\]

where \( \zeta_0, \zeta_+ \) are cutoff functions of class \( C^\infty(\mathbb{R}) \) with

\[
\zeta_0(x) = \begin{cases} 
1 & |x| \leq d \\
0 & |x| \geq 2d \\
0 \leq \zeta_0(x) \leq 1 & x \in \mathbb{R},
\end{cases}
\]

for some small fixed \( d > 0 \) (see also [6]).

**Proposition 2.4.** For small \( \varepsilon > 0 \), we have

\[
\|\varepsilon^2 u_{ap}'' + h(x)(u_{ap} - u_{ap}^3)\|_{L^\infty(0,1)} \leq C \varepsilon^{2(\alpha + \delta)}.
\]

**Proof.** In the interval \([x_1 - d, x_1 + d]\), \( u_{ap} = u_{in} \) and the bound follows from Proposition 2.3.

In \([0,1]\setminus(x_1 - d, x_1 + d)\) the bound holds as well. In the interval \([x_1 - 2d, x_1 - d]\) we have that \( 1 + u_{ap}, |u_{ap}'| \) are transcedentally small in \( \varepsilon \) (note that Proposition 2.2 implies that \( U'(z), U''(z) \) also converge to zero super-exponentially as \( |z| \to \infty \)). In \([x_1 + d, x_1 + 2d]\), \( 1 - u_{ap}, |u_{ap}'| \) are transcedentally small in \( \varepsilon \). In \([0, x_1 - 2d]\) we have \( u_{ap} = -1 \) and in \([x_1 + 2d, 1]\), \( u_{ap} = 1 \).

The proof of the proposition is complete. \( \square \)

2.3. Linear analysis: Asymptotic stability of \( U \). In Proposition 2.1 we showed that the limiting problem (2.10), (2.11) has a solution \( U \). In this subsection we show that \( U \) is non-degenerate and actually asymptotically stable.

The spectrum, in \( L^2(\mathbb{R}) \), of the linear operator

\[
M(\varphi) = -\varphi'' + A|x|^\alpha(3U^2 - 1)\varphi,
\tag{2.16}
\]

is discrete and consists of simple eigenvalues \( \mu_1 < \mu_2 < \cdots \) (recall that \( |x|^\alpha(3U^2 - 1) \to \infty \) as \( |x| \to \infty \)). We denote the corresponding \( L^2 \)-normalized eigenfunctions by \( \psi_1, \psi_2, \cdots \). Each \( \psi_i, i \geq 1 \) has exactly \( i - 1 \) zeroes and we may assume that \( \psi_1 > 0 \). Moreover each \( \psi_i \) satisfies the bound (??) (with constants depending on \( i \)). Without loss of generality we can assume that \( \psi_1 > 0 \) is an even function of \( x \).

**Proposition 2.5.** We have that \( \mu_1 > 0 \).

**Proof.** We have

\[
-\psi_1'' + A|x|^\alpha(3U^2 - 1)\psi_1 = \mu_1 \psi_1, \quad x \in \mathbb{R}.
\tag{2.17}
\]

Note that \( w = U' > 0 \) solves

\[
-w'' + A|x|^\alpha(3U^2 - 1)w = -\alpha A|x|^{-2}x(U^3 - U), \quad x \in \mathbb{R}.
\tag{2.18}
\]

By multiplying (2.17) with \( w \), (2.18) with \( \psi_1 \) and subtracting, we find that

\[
\mu_1 \int_{-\infty}^{\infty} \psi_1 wd = -\alpha A \int_{-\infty}^{\infty} |x|^{-2}x(U^3 - U)\psi_1 dx = -2\alpha A \int_{0}^{\infty} x^{\alpha - 1}(U^3 - U)\psi_1 dx > 0.
\]

The proof is complete. \( \square \)
2.4. Linear analysis: The linearized operator in a small neighborhood of the approximate solution. In this subsection we will study the linear operator

\[ L_{\varepsilon}(\varphi) = -\varepsilon^2 \varphi'' + h(x)(3u_{\varepsilon}^2 - 1)\varphi, \]

with \( \varphi \in H^2(0,1), \varphi'(0) = \varphi'(1) = 0 \) and \( u_{\varepsilon} \in C[0,1] \) satisfying

\[ \|u_{\varepsilon} - u_{app}\|_{L^{\infty}(0,1)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.19} \]

This operator arises when we linearize (2.4), (2.5) on the approximate solution \( u_{app} \).

Note that there exists a large number \( D > 0 \) such that

\[ 3u_{\varepsilon}^2(x) - 1 \geq 1, \quad |x - x_1| \geq D\varepsilon^{-\frac{2}{\alpha + 2}}, \tag{2.20} \]

if \( \varepsilon > 0 \) is sufficiently small. Hence,

\[ h(x)(3u_{\varepsilon}^2(x) - 1) \geq c|x - x_1|^{\alpha}, \quad |x - x_1| \geq D\varepsilon^{-\frac{2}{\alpha + 2}}, \tag{2.21} \]

for small \( \varepsilon \). Moreover, it is easy to see that

\[ h(x)(3u_{\varepsilon}^2(x) - 1) \geq -C\varepsilon^{-\frac{2\alpha}{\alpha + 2}}, \quad x \in [0,1]. \tag{2.22} \]

By (2.3), the definition of \( u_{app} \) and our assumption on \( u_{\varepsilon} \), we have

\[ h(x_1 + \varepsilon^{-\frac{2}{\alpha + 2}}x) \left( 3u_{\varepsilon}^2(x_1 + \varepsilon^{-\frac{2}{\alpha + 2}}x) - 1 \right) \to A|x|^{\alpha}(3U^2(x) - 1) \quad \text{in} \ C_{loc}(\mathbb{R}) \tag{2.23} \]

as \( \varepsilon \to 0. \)

**Proposition 2.6.** The exists \( \varepsilon_0 > 0, C > 0 \) such that, if \( 0 < \varepsilon < \varepsilon_0, f \in C[0,1] \) and \( \varphi \in C^2[0,1] \) satisfy

\[
\begin{cases}
L_{\varepsilon}(\varphi) = f & x \in (0,1) \\
\varphi'(0) = \varphi'(1) = 0,
\end{cases}
\]

then

\[ \|\varphi\|_{L^{\infty}(0,1)} \leq C\varepsilon^{-\frac{2\alpha}{\alpha + 2}}\|f\|_{L^{\infty}(0,1)}. \]

**Proof.** We will argue by contradiction. Assuming the opposite means that there are sequences \( \varepsilon_n \to 0, \varphi_n \in C^2[0,1], f_n \in C[0,1] \) such that

\[
\|\varphi_n\|_{L^{\infty}(0,1)} = 1, \quad n \geq 1, \quad \varepsilon_n^{-\frac{2\alpha}{\alpha + 2}}\|f_n\|_{L^{\infty}(0,1)} \to 0 \quad \text{as} \quad n \to \infty, \tag{2.24}
\]

\[
\begin{cases}
-\varepsilon_n^2\varphi''_n + h(x)(3u_{\varepsilon_n}^2 - 1)\varphi_n = f_n & \text{in} \ (0,1), \\
\varphi'_n(0) = \varphi'_n(1) = 0.
\end{cases} \tag{2.25}
\]

In view of the boundary conditions, without loss of generality, we assume that there exist \( \xi_n \in [0,1] \) such that

\[ \varphi_n(\xi_n) = 1, \quad \varphi'_n(\xi_n) = 0, \quad \varphi''_n(\xi_n) \leq 0, \quad n \geq 1. \]

We claim that

\[ \varepsilon_n^{-\frac{2}{\alpha + 2}}|\xi_n - x_1| \leq C, \quad n \geq 1. \tag{2.26} \]

Indeed, by (2.21), (2.25), we deduce that, for each \( n \geq 1, \)

\[ h(\xi_n)(3u_{\varepsilon_n}(\xi_n) - 1) \leq \|f_n\|_{L^{\infty}(0,1)}. \]

Relation (2.26) now follows immediately from (2.21), (2.24).

Let

\[ \tilde{\varphi}_n(x) = \varphi_n(x_1 + \varepsilon_n^{-\frac{2}{\alpha + 2}}x), \quad \tilde{f}_n(x) = f_n(x_1 + \varepsilon_n^{-\frac{2}{\alpha + 2}}x), \quad x \in I_n = \left[ \frac{x_1}{\varepsilon_n^{-\frac{2}{\alpha + 2}}}, \frac{1 - x_1}{\varepsilon_n^{-\frac{2}{\alpha + 2}}} \right]. \]
We see that
\[
\|\tilde{\varphi}_n\|_{L^\infty(I_n)} = 1, \ n \geq 1, \ \varrho_n \frac{2\varepsilon_n}{n^{\frac{2}{n+2}}} \|\tilde{f}_n\|_{L^\infty(I_n)} \to 0 \quad \text{as} \ n \to \infty, \tag{2.27}
\]
\[
\begin{cases}
-\varrho_n'' + \varepsilon_n \frac{2\varepsilon_n}{n^{\frac{2}{n+2}}} h(x + \varepsilon_n \frac{2\varepsilon_n}{n^{\frac{2}{n+2}}} x) \left(3u_n^2(x + \varepsilon_n \frac{2\varepsilon_n}{n^{\frac{2}{n+2}}} x) - 1\right) \tilde{\varphi}_n = \varepsilon_n \frac{2\varepsilon_n}{n^{\frac{2}{n+2}}} \tilde{f}_n \\
\varphi_n'(0) = \varphi'(1) = 0.
\end{cases}
\tag{2.28}
\]

The above two relations and (2.23) imply that for any $L > 0$ we have
\[
\|\tilde{\varphi}_n\|_{C^2[-L,L]} \leq C(L) \quad \text{if} \ n \geq N(L).
\]
Using the Arzela-Ascoli theorem, (2.23), (2.28) and the standard diagonal argument we obtain that, for a subsequence,
\[
\tilde{\varphi}_n \to \bar{\varphi} \quad \text{as} \ n \to \infty,
\]
where $\bar{\varphi}$ is bounded in $\mathbb{R}$ and satisfies
\[
-\bar{\varphi}'' + A|x|^\alpha \left(3U^2(x) - 1\right) \bar{\varphi} = 0 \quad \text{in} \ \mathbb{R}.
\]
Since $\bar{\varphi} \in L^\infty(\mathbb{R})$ and $A|x|^\alpha (3U^2 - 1) \to \infty$ as $|x| \to \infty$, it is easy to see that $\bar{\varphi} \to 0$ as $|x| \to \infty$ super-exponentially. Hence, by Proposition 2.5,
\[
\bar{\varphi} \equiv 0.
\]

On the other hand, since
\[
\tilde{\varphi}_n \left(\frac{x_n - x}{\varepsilon_n}\right) = 1, \ n \geq 1,
\]
we obtain, via (2.26) and the $C^2_{loc}$ convergence of $\tilde{\varphi}_n$, that $\bar{\varphi}(\bar{x}) = 1$ for some $\bar{x}$; a contradiction and the proof is complete. \hfill \Box

**Proposition 2.7.** The exists $\varepsilon_0 > 0$, $C > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, $f \in C[0,1]$ and $\varphi \in C^2[0,1]$ satisfy
\[
\begin{cases}
L_\varepsilon(\varphi) = hf & x \in (0,1) \\
\varphi'(0) = \varphi'(1) = 0,
\end{cases}
\]
then
\[
\|\varphi\|_{L^\infty(0,1)} \leq C\|f\|_{L^\infty(0,1)}.
\]

**Proof.** We will argue by contradiction. Assuming the opposite means that there are sequences $\varepsilon_n \to 0$, $\varphi_n \in C^2[0,1]$, $f_n \in C[0,1]$ such that
\[
\|\varphi_n\|_{L^\infty(0,1)} = 1, \ n \geq 1, \ \|f_n\|_{L^\infty(0,1)} \to 0 \quad \text{as} \ n \to \infty, \tag{2.29}
\]
\[
\begin{cases}
-\varepsilon_n^2 \varphi_n'' + h(x)(3u_2^2 - 1)\varphi_n = h(x)f_n \quad \text{in} \ (0,1), \\
\varphi_n'(0) = \varphi_n'(1) = 0.
\end{cases}
\tag{2.30}
\]

In view of the boundary conditions, without loss of generality, we assume that there exist $\xi_n \in [0,1]$ such that
\[
\varphi_n(\xi_n) = 1, \ \varphi_n'(\xi_n) = 0, \ \varphi_n''(\xi_n) \leq 0, \ n \geq 1.
\]

We claim that
\[
\varepsilon_n \frac{2\varepsilon_n}{n^{\frac{2}{n+2}}} |\xi_n - x_1| \leq C, \ n \geq 1. \tag{2.31}
\]
Indeed, by (2.30) we deduce that, for each $n \geq 1$,
\[
h(\xi_n) \left(3u_{\varepsilon_n}^2(\xi_n) - 1\right) \leq h(\xi_n)\|f_n\|_{L^\infty(0,1)}.
\]
Relation (2.31) now follows immediately from (2.20).

The rest of the argument is similar to the one in the proof of Proposition 2.6.

\[\square\]

Similarly we can show (see [13])

Proposition 2.8. For any fixed integer $m \geq 1$ the first $m$ eigenvalues $\lambda_{1,\varepsilon} < \cdots \lambda_{m,\varepsilon}$ and the corresponding $L^2$-normalized eigenfunctions $\varphi_{i,\varepsilon}$ of $L_\varepsilon$ satisfy
\[
e^{-\frac{2\alpha}{\varepsilon^2}}\lambda_{i,\varepsilon} \rightarrow \mu_i
\]
\[
\|\varphi_{i,\varepsilon}(x) - \varepsilon^{-\frac{1}{\alpha+2}}\psi_i \left(\frac{x-x_1}{\varepsilon^{\alpha+2}}\right)\|_{L^2(0,1)} \rightarrow 0
\]
as $\varepsilon \rightarrow 0$, $i = 1, \ldots, k$, where $\mu_i$, $\psi_i$ are as in Section 2.3.

2.5. Existence and stability for the nonlinear problem.

Theorem 2.2. If $\varepsilon > 0$ is sufficiently small, there exists a solution $u_\varepsilon$ of (2.4), (2.5) such that
\[
\|u_\varepsilon - u_{ap,\varepsilon}\|_{L^\infty(0,1)} \leq C\varepsilon^{\frac{2\alpha}{\alpha+2}}.
\]
Moreover, the smallest eigenvalue $\lambda_{1,\varepsilon}$ of the eigenvalue problem
\[
\begin{cases}
\varepsilon^{-\frac{2\alpha}{\varepsilon^2}}\lambda_{1,\varepsilon} \rightarrow \mu_1 \\
\varepsilon^{-\frac{1}{\alpha+2}}\lambda_{i,\varepsilon} \rightarrow \mu_i \\
\|\varphi_{i,\varepsilon}(x) - \varepsilon^{-\frac{1}{\alpha+2}}\psi_i \left(\frac{x-x_1}{\varepsilon^{\alpha+2}}\right)\|_{L^2(0,1)} \rightarrow 0
\end{cases}
\]
as $\varepsilon \rightarrow 0$, $i = 1, \ldots, k$, where $\mu_i$, $\psi_i$ are as in Section 2.3.

Proof. We look for a solution to (2.4), (2.5) in the form
\[
u = u_{ap,\varepsilon} + \varphi,
\]
where $\varphi \in C^2[0,1]$ with $\varphi'(0) = \varphi'(1) = 0$ is a small perturbation. Thus the equation for $u$ is equivalent to
\[
\begin{cases}
L_\varepsilon(\varphi) = h(x)N_\varepsilon(\varphi) + E_\varepsilon \quad \text{in} \quad (0,1), \\
\varphi'(0) = \varphi'(1) = 0,
\end{cases}
\]
(2.34)

where
\[
L_\varepsilon(\varphi) = -\varepsilon^2\varphi'' + h(x)(3u_{ap}^2 - 1)\varphi,
\]
\[
N_\varepsilon(\varphi) = -\varphi^3 - 3u_{ap}\varphi^2,
\]
\[
E_\varepsilon = \varepsilon^2u_{ap}' - h(x)(u_{ap}^3 - u_{ap}).
\]

Note that $L_\varepsilon$ satisfies the hypotheses of Section 2.4 (see (2.19)).

In view of Proposition 2.6, for small $\varepsilon$, we can define a mapping $T_\varepsilon : C[0,1] \rightarrow C[0,1]$ by the relation
\[
\begin{cases}
L_\varepsilon(T_\varepsilon(\varphi)) = h(x)N_\varepsilon(\varphi) + E_\varepsilon \quad \text{in} \quad (0,1), \\
(T_\varepsilon(\varphi))'(0) = (T_\varepsilon(\varphi))'(1) = 0.
\end{cases}
\]
(2.35)
We will show that we can choose a large $M$ (indep. of $\varepsilon$) such that, for small $\varepsilon$, the nonlinear operator $T_\varepsilon$ maps

$$B_{\varepsilon,M} := \{ \varphi \in C[0,1] : \| \varphi \|_{L^{\infty}(0,1)} \leq M \varepsilon^{\frac{2\alpha}{\alpha+2}} \}$$

into itself and is a contraction with respect to the $L^\infty$ norm. If $\varphi \in B_{\varepsilon,M}$, by Propositions 2.4, 2.6, 2.7 we get

$$\| T_\varepsilon(\varphi) \|_{L^{\infty}(0,1)} \leq C \| N_\varepsilon(\varphi) \|_{L^{\infty}(0,1)} + C \varepsilon^{-\frac{2\alpha}{\alpha+2}} \| E_\varepsilon \|_{L^{\infty}(0,1)}$$

provided $\varepsilon$ is small (independently of $\varphi, M$). (The constant $C$ in the above relation is independent of $\varphi, M$). By choosing $M = 2C$ and $\varepsilon_0$ such that $4C^2 \varepsilon_0^{\frac{2\alpha}{\alpha+2}} < 1$ we deduce that $T_\varepsilon(\varphi) \in B_{\varepsilon,M}$ if $0 < \varepsilon < \varepsilon_0$.

If $\varphi_1, \varphi_2 \in B_\varepsilon$ (we dropped the subscript $M$ since we fixed it), we have

$$\| T_\varepsilon(\varphi_1) - T_\varepsilon(\varphi_2) \|_{L^{\infty}(0,1)} \leq C \| N_\varepsilon(\varphi_1) - N_\varepsilon(\varphi_2) \|_{L^{\infty}(0,1)}$$

$$\leq C \varepsilon^{\frac{2\alpha}{\alpha+2}} \| \varphi_1 - \varphi_2 \|_{L^{\infty}(0,1)}.$$

It follows that, if $\varepsilon$ is sufficiently small, $T_\varepsilon : B_\varepsilon \to B_\varepsilon$ is a contraction. By the Banach fixed point theorem, there exists a unique

$$\varphi_\varepsilon \in B_\varepsilon$$

such that

$$T_\varepsilon(\varphi_\varepsilon) = \varphi_\varepsilon.$$

The function

$$u_\varepsilon := u_{ap,\varepsilon} + \varphi_\varepsilon$$

is a solution of (2.4), (2.5) satisfying estimate (2.32). The principal eigenvalue of the linearized problem (3.3) satisfies the assertion of Proposition 2.8 because $u_\varepsilon$ satisfies (2.19). The proof of the theorem is complete. \[\square\]

### 3. Layered Solutions in a Problem Arising in Population Genetics

We consider the singularly perturbed boundary value problem

$$\begin{cases}
\varepsilon^2 \Delta u + h(x) g(u) = 0, & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega,
\end{cases}$$

(3.1)

where $\Omega$ is a smooth domain in $\mathbb{R}^N$, $h \in C(\Omega)$ and $g \in C^2(\mathbb{R})$ satisfies

$$\begin{cases}
g(0) = 0, & g(1) = 0, \\
g_u(0) > 0, & g_u(1) < 0, \\
g_u > 0, & \forall u \in (0,1).
\end{cases}$$

Note that (3.1) includes the important Fisher’s equation (with $g(u) = u(1-u)$).
This problem was treated in [7] in the context of population genetics. The existence of a family of global minimizers \( u_\varepsilon \) of the corresponding energy functional which develop inner transition layers with a single interface given by

\[
S = \{ x \in \Omega : h(x) = 0 \}
\]

was proved if \( h \) changes sign at \( S \). More precisely, the author assumed that \( h \in L^\infty(\Omega) \), \( h \neq 0 \) a.e., and that the set

\[
P = \{ x \in \Omega : h(x) > 0 \ \text{a.e.} \}
\]

has finite capacity. He showed that any global minimizer \( u_\varepsilon \) satisfies

\[
u_\varepsilon \to \chi_P \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^q(\Omega) \quad \forall q < \infty,
\]

(see [7] Chapter 10, Exercise 3).

In the case where the interface \( S \) is smooth, one can apply the arguments of the previous sections and give a perturbation proof which yields fine estimates (the one dimensional profile turns out to be asymptotic stable). For simplicity we focus on the one dimensional case and we assume that \( h \) changes sign only once. We will assume that \( h \in C[0,1] \) satisfies

\[
h(x) < 0, \quad x \in [0,x_1), \quad h(x) > 0, \quad x \in (x_1,1],
\]
 \[h(x) = \text{Sign}(x-x_1)|x-x_1|^\alpha + O(|x-x_1|^\alpha + \delta) \quad \text{as} \quad x \to x_1,
\]

for some \( x_1 \in (0,1) \) and \( A, \alpha, \delta > 0 \).

**Theorem 3.1.** If \( \varepsilon \) is sufficiently small, there exists a layered solution \( u_\varepsilon \) of

\[
\begin{align*}
\varepsilon^2 u'' + h(x)g(u) &= 0, \quad x \in (0,1), \\
u'(0) &= u'(1) = 0,
\end{align*}
\]

such that

\[
|u_\varepsilon(x) - U\left(\frac{x-x_1}{\varepsilon}\right)| \leq C\varepsilon^{\frac{\alpha}{\alpha+2}}, \quad x \in [0,1],
\]

where \( U \) is the unique monotone solution of

\[
\begin{align*}
\phi'' + \text{Sign}(x)|x|^\alpha g(u) &= 0, \quad x \in \mathbb{R}, \\
u(x) &\to 0 \quad \text{as} \quad x \to -\infty, \quad u(x) \to 1 \quad \text{as} \quad x \to \infty,
\end{align*}
\]

(see [5]). Moreover, the smallest eigenvalue \( \lambda_{1,\varepsilon} \) of the eigenvalue problem

\[
\begin{align*}
-\varepsilon^2 \phi'' + h(x)g_u(u_\varepsilon)\phi &= \lambda \phi \quad \text{in} \quad (0,1), \\
\phi'(0) &= \phi'(1) = 0,
\end{align*}
\]

is positive and satisfies

\[
\varepsilon^{-\frac{\alpha}{\alpha+2}} \lambda_{1,\varepsilon} \to \mu_1 \quad \text{as} \quad \varepsilon \to 0,
\]

where \( \mu_1 > 0 \) is the principal eigenvalue of

\[
-\psi'' - \text{Sign}(x)|x|^\alpha g_u(U)\psi = \mu \psi, \quad \psi \in L^2(\mathbb{R}).
\]

We refer to [5] for the asymptotic stability of \( U \) (we note that \( U' > 0 \)). The proof follows almost word by word the arguments of Sections 2.2, 2.4, 2.5 and, thus, is omitted.
4. Standing waves with critical frequency for the nonlinear Schrödinger equation

In [2] the authors studied the problem

\[
\begin{align*}
\varepsilon^2 \Delta u - V(x)u + u^p &= 0, \quad x \in \mathbb{R}^N, \\
u(x) &\to 0, \quad |x| \to \infty,
\end{align*}
\]

(4.1)

under the hypotheses:

\[
\begin{cases}
V \in C(\mathbb{R}^N) \\
\liminf_{|x| \to \infty} V(x) > 0 \\
\inf_{x \in \mathbb{R}^N} V(x) = 0 \\
1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3, \quad p > 1 \text{ if } N = 1, 2.
\end{cases}
\]

(4.2)

In the case when \(V\) has an isolated minimum at \(x_1 \in \mathbb{R}^N\) such that

\[V(x) = A|x - x_1|^\alpha + o(|x - x_1|^\alpha) \quad \text{as } x \to x_1,\]

for some constants \(A, \alpha > 0\), their general result says that (4.1) has a positive solution \(u_\varepsilon\) such that

\[\varepsilon^{-\frac{2\alpha}{(p-1)(\alpha+2)}} u_\varepsilon(x_1 + \varepsilon^{\frac{2}{\alpha+2}} x) \to U(x) \text{ uniformly in } \mathbb{R}^N \text{ as } \varepsilon \to 0,\]

(4.3)

where \(U\) is the unique positive solution of

\[
\begin{cases}
\Delta u - A|x|^\alpha u + u^p = 0, \quad x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0,
\end{cases}
\]

(see [3]). Their approach is variational (equation (4.1) under (4.2) has mountain-pass structure).

From [3] we know that \(U\) is non-degenerate, i.e.,

\[
\begin{cases}
\Delta \varphi - A|x|^\alpha \varphi + pU^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} \varphi(x) = 0,
\end{cases}
\]

has only the trivial solution \(\varphi \equiv 0\).

Here we assume that \(V, p \geq 2\) satisfy (4.2),

\[V(x) = A|x - x_1|^\alpha + O(|x - x_1|^\alpha+\delta) \quad \text{as } x \to x_1,\]

for some \(A, \alpha, \delta > 0\) and \(V(x) > 0, x \neq x_1\).

We set

\[u_{ap,\varepsilon}(x) = \varepsilon^{-\frac{2\alpha}{(p-1)(\alpha+2)}} \zeta \left( \frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}} \right) \eta(x - x_1), \quad x \in \mathbb{R}^N,\]

(4.4)

where \(\zeta \in C_0(\mathbb{R}^N)\) is a smooth cut-off (independent of \(\varepsilon\)) such that

\[
\zeta(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2. \end{cases}
\]
We introduce the weighted norm
\[ \|u\|_w = \|\exp\left\{\frac{|x-x_1|}{\varepsilon^{\frac{1}{2}\alpha}}\right\} u\|_{L^\infty(\mathbb{R}^N)}, \]
and the Banach space
\[ W = \{u \in C(\mathbb{R}^N) : \|u\|_w < \infty\} \]
equipped with this norm.

**Proposition 4.1.** If \( \varepsilon > 0 \) is sufficiently small,
\[ \|\varepsilon^2 \Delta u_{ap} - V(x)u_{ap} + u_{ap}^p\|_w \leq C\varepsilon^{2\alpha p + 2\delta(p-1)}\]

**Proof.** By the definition of \( U \) and its super-exponential decay, we have
\[ |\varepsilon^2 \Delta u_{ap} - A|x-x_1|^\alpha u_{ap} + u_{ap}^p| \leq Ce^{-\frac{2}{\alpha}e^{-\frac{|x-x_1|}{\varepsilon}}} \]
in \( \mathbb{R}^N \). So,
\[ |\varepsilon^2 \Delta u_{ap} - V(x)u_{ap} + u_{ap}^p| \leq \|(A|x-x_1|^\alpha - V(x))u_{ap}\| + Ce^{-\frac{2}{\alpha}e^{-\frac{|x-x_1|}{\varepsilon}}} \]
\[ \leq C\varepsilon^{2\alpha p + 2\delta(p-1)}\frac{|x-x_1|}{\varepsilon^{\frac{1}{2}\alpha}} \|U\left(\frac{x-x_1}{\varepsilon^{\frac{1}{2}\alpha}}\right)\| + Ce^{-\frac{2}{\alpha}e^{-\frac{|x-x_1|}{\varepsilon}}} \]
\[ \leq C\varepsilon^{2\alpha p + 2\delta(p-1)} e^{-\frac{|x-x_1|}{\varepsilon^{\frac{1}{2}\alpha}}}, \]
in \( \mathbb{R}^N \), and the desired bound follows immediately. \( \square \)

**Remark 4.1.** We used the cut-off in case \( V \) increases exponentially fast at as \( |x| \to \infty \).

**Remark 4.2.** The estimate \( 0 < U(x) \leq Ce^{-2|x|} \) was all that we used.

Let
\[ L_\varepsilon(\varphi) = -\varepsilon^2 \Delta \varphi + (V(x) - C\varphi_{ap}^{\nu-1}) \varphi. \]
Note that for some fixed \( D > 1 \) independent of \( \varepsilon \),
\[ V(x) - C\varphi_{ap}^{\nu-1} \geq \begin{cases} c|x-x_1|^\alpha - \frac{\varepsilon^{2\alpha}}{2} & D\varepsilon^{\frac{2}{\alpha}+\delta} \leq |x-x_1| \leq 1 \\ c & |x-x_1| \geq 1, \end{cases} \]
\[ \varepsilon^{-\frac{2}{\alpha}+\frac{\alpha}{2}} V(x_1 + \varepsilon^{\frac{2}{\alpha}+\delta} x) \to A|x|^\alpha \text{ in } C_{\text{loc}}(\mathbb{R}^N) \]
as \( \varepsilon \to 0 \) (these are the analogs of (2.21), (2.23)).

As in Proposition 2.6, we have

**Proposition 4.2.** There exist constants \( \varepsilon_0, C > 0 \) such that if \( \varepsilon \in (0, \varepsilon_0), \varphi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) satisfy
\[ L_\varepsilon(\varphi) = f \text{ in } \mathbb{R}^N, \]
then
\[ \|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq C\varepsilon^{-\frac{\alpha}{\alpha+2}} \|f\|_{L^\infty(\mathbb{R}^N)}. \]
Proposition 4.3. There exist constants $\varepsilon_0$, $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, and $f \in W$, then there exists a unique $\varphi \in W \cap C^2(\mathbb{R}^N)$ such that

$$L_\varepsilon(\varphi) = f \quad \text{in } \mathbb{R}^N.$$ 

Moreover,

$$\|\varphi\|_W \leq C\varepsilon^{-\frac{2\alpha}{\pi^2}} \|f\|_W.$$ 

Proof. The existence and uniqueness of $\varphi$ can be deduced from Proposition 4.3 and a standard barrier argument. We already know that

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq C\varepsilon^{-\frac{2\alpha}{\pi^2}} \|f\|_W. \quad (4.8)$$ 

So,

$$\exp\left\{\left[\frac{|x - x_1|}{\varepsilon^\frac{2\alpha}{\pi^2}}\right]\right\} |\varphi(x)| \leq C\varepsilon^{-\frac{2\alpha}{\pi^2}} \|f\|_W \quad \text{if } |x - x_1| \leq D\varepsilon^\frac{2\alpha}{\pi^2},$$ 

(D is as in (4.7)). Let

$$\psi_M(x) = M\varepsilon^{-\frac{2\alpha}{\pi^2}} \|f\|_W \exp\left\{\left[-\frac{|x - x_1|}{\varepsilon^\frac{2\alpha}{\pi^2}}\right]\right\}, \quad |x - x_1| \geq D\varepsilon^\frac{2\alpha}{\pi^2}.$$ 

In view of (4.7), (4.8) it is easy to check that

$$L_\varepsilon(\psi_M) \geq f \quad |x - x_1| \geq D\varepsilon^\frac{2\alpha}{\pi^2}$$

$$\psi_M = \varphi \quad |x - x_1| = D\varepsilon^\frac{2\alpha}{\pi^2},$$

provide $\varepsilon$ is small and $M$ is large (indep. of $\varepsilon$). By the maximum principle (recall also that $\varphi, \psi_M$ vanish at infinity) we deduce that

$$\varphi(x) \leq M\varepsilon^{-\frac{2\alpha}{\pi^2}} \|f\|_W \exp\left\{\left[-\frac{|x - x_1|}{\varepsilon^\frac{2\alpha}{\pi^2}}\right]\right\}.$$ 

Doing the same for $-\varphi$ yields

$$\exp\left\{\left[\frac{|x - x_1|}{\varepsilon^\frac{2\alpha}{\pi^2}}\right]\right\} \|\varphi(x)| \leq C\varepsilon^{-\frac{2\alpha}{\pi^2}} \|f\|_W,$$

and the proof is complete. \qed

Remark 4.3. As in Section 2.4, the above two propositions also hold true when we linearize on functions $u_\varepsilon$ satisfying $\varepsilon^{-\frac{2\alpha}{\pi^2}} \|u_\varepsilon - u_{ap}\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Theorem 4.1. If $\varepsilon > 0$ is sufficiently small, there exists a positive solution $u_\varepsilon > 0$ of (4.1) such that

$$\|u_\varepsilon - u_{ap, \varepsilon}\|_W \leq C\varepsilon^{\frac{2\alpha + 2(1 - p)}{(p - 1)(p + \alpha)}} \varepsilon^{\frac{1}{2}},$$ 

(4.9)

or equivalently

$$\left|\varepsilon^{-\frac{2\alpha}{(p - 1)(p + \alpha)}} u_\varepsilon(x_1 + \varepsilon^\frac{1}{p + \alpha} x) - U(x)\right| \leq C\varepsilon^{\frac{1}{2(p + \alpha)}} \varepsilon^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}, \quad x \in \mathbb{R}^N. \quad (4.10)$$

Proof. We consider the equation

$$\varepsilon^2 \Delta u - V(x)u + |u|^p = 0 \quad \text{in } \mathbb{R}^N. \quad (4.11)$$

We search for a solution $u$ of (4.11) in the form $u = u_{ap, \varepsilon} + \varphi$. In terms of $\varphi$ equation (4.11) becomes

$$L_\varepsilon(\varphi) = N_\varepsilon(\varphi) + E_\varepsilon,$$

where

$$N_\varepsilon(\varphi) = |u_{ap} + \varphi|^p - u_{ap}^p - pu_{ap}^{p-1} \varphi,$$

$$E_\varepsilon = \varepsilon^2 \Delta u_{ap} - V(x)u_{ap} + u_{ap}^p.$$
Since $p \geq 2$, there exists a $C > 0$ such that

$$\|y + y_0\|^p - |y_0|^p - p \text{sign}(y_0)|y_0|^{p-1}y| \leq C(|y| + |y_0|)^{p-2}y^2, \quad \forall y, y_0 \in [-1, 1].$$

(4.12)

In view of Propositions 4.1, 4.3 and (4.12), we can define a mapping $T_\varepsilon : W \to W$ by the relation

$$L_\varepsilon (T_\varepsilon (\varphi)) = N_\varepsilon (\varphi) + E_\varepsilon.$$

Let,

$$B_{\varepsilon,M} := \{ \varphi \in W : \| \varphi \|_W \leq M \varepsilon^{\frac{2\alpha + 2d(p-1)}{(p-1)(\alpha + 1)}} \}$$

where $M > 0$ is to be chosen independently of $\varepsilon > 0$ small. Note that, via (4.12), for every $\varphi \in B_{\varepsilon,M}$ we have

$$\|N_\varepsilon (\varphi)\|_W \leq C \varepsilon^{\frac{2\alpha(p-2)}{(p-1)(\alpha + 1)}} \|\varphi\|_W^2$$

with $C$ independent of $\varepsilon, M, \varphi$ provided $\varepsilon$ is small. We leave it to the interested reader to check that we can fix a large $M > 0$ such that, if $\varepsilon > 0$ is sufficiently small, $T_\varepsilon$ maps $B_\varepsilon$ into itself and is a contraction with respect to the norm of $W$ (see also the proof of Theorem 2.2). Hence, $T_\varepsilon$ has a unique fixed point $\varphi_\varepsilon$ in $B_\varepsilon$. Then $u_\varepsilon := u_{ap} + \varphi_\varepsilon$ solves equation (4.11) and satisfies estimates (4.9), (4.10). To conclude it remains to show that $u_\varepsilon$ is positive in $\mathbb{R}^N$ (so that it solves (4.1)). By (4.9) we see that for every $L > 0$ independent of $\varepsilon$,

$$u_\varepsilon (x) > 0, \quad |x| \leq L \varepsilon^{\frac{2}{\alpha + 2}},$$

provided $0 < \varepsilon < \varepsilon(L)$. Note that $u_\varepsilon$ satisfies

$$-\varepsilon^2 \Delta u_\varepsilon + (V(x) - \text{sign}(u_\varepsilon)|u_\varepsilon|^{p-1}) u_\varepsilon = 0, \quad x \in \mathbb{R}^N,$$

and

$$V(x) - \text{sign}(u_\varepsilon)|u_\varepsilon|^{p-1} \geq c L^0 \varepsilon^{\frac{2\alpha}{\alpha + 2}} - C \varepsilon^{\frac{2\alpha}{\alpha + 2}}, \quad |x| \geq L \varepsilon^{\frac{2}{\alpha + 2}},$$

provided $0 < \varepsilon < \varepsilon_1(L)$ ($c, C$ independent of $\varepsilon, L$). By fixing a large $L > c^{-1}C^{\frac{1}{2}}$, we conclude by the maximum principle that

$$u_\varepsilon (x) > 0, \quad |x| \geq L \varepsilon^{\frac{2}{\alpha + 2}},$$

and the proof is complete.

References


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