ON SOME SECOND ORDER SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH NON-DEGENERATE INNER SOLUTIONS

C. SOURDIS

Abstract. In this paper we study an inhomogeneous Allen-Cahn equation where the spatial reaction is positive but vanishes at a point. We also consider Fisher's equation and the nonlinear Schrödinger equation with critical frequency.

1. Introduction

Throughout this paper we will denote by $c/C, C'$ a small/large positive generic constant independent of ε whose value will change from line to line. Frequently we will not explicitly denote the obvious dependence of functions on ε.

2. Transition layer solutions in an Allen-Cahn equation with non-negative spatial reaction

In [9] the author studied the singularly perturbed boundary value problem

$$\begin{cases} \varepsilon^2 u'' + h(x)g(u) = 0, \quad 0 < x < 1, \\ u'(0) = 0 = u'(1), \end{cases} \tag{2.1}$$

under the hypotheses: $h \in C[0,1]$ is strictly positive, $g \in C^2(\mathbb{R})$ and

$$\begin{cases} g \text{ has precisely three zeros } a_- < 0 < a_+ \\ g_u(a_-) < 0,\ g_u(0) > 0,\ g_u(a_+) < 0 \\ \int_{a_-}^{a_+} g(u)du = 0 \\ \frac{g(u)}{u} > g_u(u), \quad u \neq 0. \end{cases} \tag{2.2}$$

Theorem 2.1. [8, 9] *Let $\{x_1, x_2, ..., x_m\}$ be an arbitrary subset of the set of local minimum points of h. Then there exists a stable solution of (2.1) which has one layer near each x_k $(k = 1, \cdots, m)$ and has no layer in the rest of the interval $(0, 1)$.*

The inner solution at each x_k is $U\left(\frac{x-x_k}{\varepsilon}\right)$ where

$$\begin{cases} U'' + h(x_k)g(u) = 0, \qquad x \in \mathbb{R}, \\ U(x) \to a_- \text{ as } x \to -\infty, \quad U(x) \to a_+ \text{ as } x \to \infty. \end{cases}$$

Partly supported by grant FONDECYT 3085026.

(the layer can also go in the opposite direction and is also stable). The above equation has a unique solution U (modulo translation) and Theorem 2.1 follows by constructing suitable upper and lower solutions.

Our interest is in the case when h is strictly positive and has a global minimum at $x_1 \in (0,1)$ with $h(x_1) = 0$. A similar investigation for the nonlinear Scrödinger equation was carried out in [2] (see also Section 4 of the present paper).

We assume that there exists $A > 0,\ \alpha > 0$ such that

$$h(x) = A|x - x_1|^\alpha + O(|x - x_1|^{\alpha+\delta}), \quad x \in [0,1], \tag{2.3}$$

for some $\delta > 0$ ($\forall R > 1,\ |t| \le R$ we have $|O(t)| \le C(R)|t|$). We consider only odd functions g and more precisely $g(u) = u - u^3$ (i.e., $a_- = 1,\ a_+ = -1$) but our approach works for odd functions g with $g_u(0) > 0,\ g(a_+) = 0,\ g_u(a_+) < 0$ (we don't need the last assumption in (2.2)).

Equation 2.1 models phase transitions between the two stable states $a_-,\ a_+$ in the spatial domain $(0,1)$. By allowing h to take the value 0 we can include more general situations.

We will consider the following equation and boundary conditions

$$\varepsilon^2 u'' + h(x)(u - u^3) = 0, \quad x \in (0,1), \tag{2.4}$$

$$u'(0) = 0 = u'(1). \tag{2.5}$$

2.1. **The inner approximate solution $u_{in,\varepsilon}$.** For every $u \in C^2[0,1]$, we have

$$\varepsilon^2 u'' + h(x)(u - u^3) \overset{(2.3)}{=} \varepsilon^2 u'' + A|x - x_1|^\alpha (u - u^3) + O(|x - x_1|^{\alpha+\delta})(u - u^3). \tag{2.6}$$

We will search the inner solution $u_{in,\varepsilon}$ as a suitable solution of

$$\varepsilon^2 u'' + A|x - x_1|^\alpha (u - u^3) = 0, \quad x \in \mathbb{R} \tag{2.7}$$

and then we will restrict its domain of definition to $[0,1]$. Using the transformation

$$u(x) = U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right), \quad x \in \mathbb{R} \tag{2.8}$$

we see that (2.7) becomes

$$U''(x) + A|x|^\alpha \left(U(x) - U^3(x)\right), \quad x \in \mathbb{R}. \tag{2.9}$$

Since u should be layered, the obvious boundary conditions for U are

$$\lim_{x \to -\infty} U(x) = -1 \ \text{ and } \ \lim_{x \to \infty} U(x) = 1.$$

Proposition 2.1. *There exists an odd solution U with $U'(x) > 0,\ x \in \mathbb{R}$ of*

$$u'' = A|x|^\alpha (u^3 - u) \quad \text{in } \mathbb{R} \tag{2.10}$$

satisfying

$$u(x) \to -1 \quad \text{as } x \to -\infty, \quad u(x) \to 1 \quad \text{as } x \to +\infty. \tag{2.11}$$

Proof. We follow the idea of [4]. Equation (2.10) is equivalent to the non-autonomous system

$$\begin{aligned} p' &= q, \\ q' &= A|x|^\alpha p(p+1)(p-1). \end{aligned} \tag{2.12}$$

Consider first the aspect of solution curves in (p, q, x)-space with $x \ge 0$. Let

$$W = \{(p, q, x) |\ 0 \le p \le 1,\ q \ge 0 \text{ and } x \ge 0\}.$$

First observe that any positive half orbit $\{(p(x), q(x), x)|\ x \geq 0\}$ with $p(0) = 0$, $q(0) > 0$ which lies totally in W gives us a solution that we seek. Indeed, by odd reflection we see that p satisfies (2.10). Moreover $p' = q > 0$ (if $q(x_0) = 0$ then $q \equiv 0$). It remains to show that p satisfies (2.11). p(x) must increase to some positive limit $p_\infty \leq 1$. Also since $q' = -A|x|^\alpha p(p+1)(1-p) < 0$, q must decrease to some nonnegative limit $q_\infty \geq 0$. Now if $q_\infty > 0$, then p_∞ would be $+\infty$, thus $q_\infty = 0$. If $p_\infty < 1$, then $\lim_{x\to+\infty} q'(x) = -\infty$ which implies that $\lim_{x\to+\infty} q(x) = -\infty$, thus $q_\infty = 0$.

If an orbit leaves W it must do so across one of two faces, namely, $\{p = 1,\ q > 0,\ x > 0\}$ or $\{q = 0,\ 0 < p < 1,\ x > 0\}$. This follows from $p' = q \geq 0$ and $q' = -A|x|^\alpha p(p+1)(1-p) \leq 0$. Furthermore each exit point is a point of strict exit meaning (here) that orbits cross the faces non-tangentially; namely on the first face $p' > 0$ and on the second $q' < 0$ (note that the faces are "open"). Wazewski's lemma implies that the map that assigns to each point of W its point of exit from W is continuous wherever it is defined. (This is easily proved directly in this simple case).

Consider the line $S_\delta^Q = \{(0, q, 0)|\ \delta \leq q \leq Q\}$ where $\delta,\ Q > 0$.

Claim 1: *If $Q > 0$ is sufficiently large, then the trajectory $\{(p(x), q(x), x)|\ 0 \leq x < x_{max}\}$ starting at the upper endpoint $(0, Q, 0)$ of S_δ^Q leaves W for the first time through the face $\{p = 1,\ q > 0,\ x > 0\}$.*

If not, then there exists a first $0 < x_0 < x_{\max}$ such that

$$q(x_0) = \frac{Q}{2} \quad \text{and} \quad (p(x), q(x), x) \in W,\ 0 \leq x \leq x_0.$$

Making use of (2.12) we obtain that $p(x) \geq \frac{Q}{2}x$, $x \in [0, x_0]$ and in particular $x_0 \leq \frac{2}{Q}$. Also $-q'(x) \leq 2Ax^\alpha$ in $(0, x_0)$, and by integrating over $(0, x_0)$,

$$\frac{Q}{2} \leq 2A\frac{{x_0}^{\alpha+1}}{\alpha+1} \leq \frac{2A}{\alpha+1}\frac{2^{\alpha+1}}{Q^{\alpha+1}}.$$

This is impossible if Q is chosen sufficiently large.

Claim 2: *If $\delta > 0$ is sufficiently small, then the trajectory $\{(p(x), q(x), x)|\ 0 \leq x < x_{max}\}$ starting at the lower endpoint $(0, \delta, 0)$ of S_δ^Q leaves W for the first time through the face $\{q = 0,\ 0 < p < 1,\ x > 0\}$.*

Indeed, p, q are $O(\delta^3)$ close to $\tilde{p}, \tilde{q}$ in compact intervals as $\delta \to 0$, where

$$\tilde{p}' = \tilde{q},$$

$$\tilde{q}' = -A|x|^\alpha \tilde{p},$$

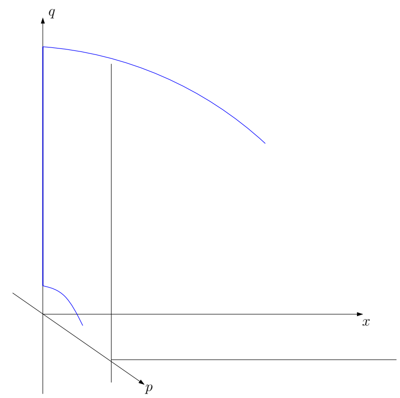
$\tilde{p}(0) = 0,\ \tilde{q}(0) = \delta$. Note that $\tilde{p}(x) = \delta v(x)$, $\tilde{q}(x) = \delta v'(x)$ where

$$v'' + A|x|^\alpha v = 0, \quad v(0) = 0,\ v'(0) = 1.$$

Since v is oscillatory, q becomes negative after an $O(1)$ x-interval whereas $0 < p = O(\delta)$ in this interval.

It follows that there exists a point of S_δ^Q whose positive half orbit is contained in W provided Q is chosen large and $\delta > 0$ small. If not, then the exit curve of S_δ^Q (which would be continuous) would contain a point on the line $\{(1, 0, x)|\ x > 0\}$ which is not possible.

The proof of the proposition is complete. □



Remark 2.1. *U is the only odd solution of (2.10), (2.11). Indeed, let us suppose that there exists another odd solution V of (2.10), (2.11). Then it is easy to see that θV, $0 \le \theta \le 1$ is a family of subsolutions of (2.10) in $(0,\infty)$ such that $\theta V(0) = U(0)$, $\theta V(\infty) < U(\infty)$, $0 \le \theta < 1$. Moreover, $\theta V \le U$ in $(0,\infty)$ when $\theta = 0$. By Serrin's sweeping technique (see* [10]*) we get $V \le U$ in $(0,\infty)$. Similarly, $U \le V$ in $(0,\infty)$.*

Proposition 2.2.

$$0 < 1 - U(x) \le C x^{-\frac{\alpha}{4}} e^{-\frac{2\sqrt{2A}}{\alpha+2} x^{\frac{\alpha}{2}+1}}, \quad x > 0. \tag{2.13}$$

Proof. The function $\varphi = 1 - U$ is bounded in $(0,\infty)$ and satisfies

$$-\varphi'' + q(x)\varphi = 0, \quad x > 0,$$

with $q(x) = A|x|^\alpha U(1+U)$. The bound now follows from the asymptotic theory of linear equations (see [1]). $\square$

We define the inner approximate solution of (2.4) by

$$u_{in,\varepsilon}(x) = U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right), \quad x \in [0,1].$$

Proposition 2.3. *For small $\varepsilon > 0$, we have*

$$\|\varepsilon^2 u_{in}'' + h(x)(u_{in} - u_{in}^3)\|_{L^\infty(0,1)} \le C\varepsilon^{\frac{2(\alpha+\delta)}{\alpha+2}}.$$

Proof. Note that, by its construction, $u_{in,\varepsilon}$ satisfies

$$\begin{aligned} |\varepsilon^2 u_{in}'' + h(x)(u_{in} - u_{in}^3)| &\overset{(2.6)}{\le} C|x - x_1|^{\alpha+\delta}\left[1 + U\left(\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right)\right]\left[1 - U\left(\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right)\right] \\ &\overset{(2.13)}{\le} C\varepsilon^{\frac{2(\alpha+\delta)}{\alpha+2}}\left|\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right|^{\alpha+\delta} e^{-c\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}} \\ &\le C\|z^{\alpha+\delta}e^{-cz}\|_{L^\infty(0,\infty)}\varepsilon^{\frac{2(\alpha+\delta)}{\alpha+2}}, \end{aligned}$$

$x \in [0,1]$. The proof is complete. $\square$

2.2. **The approximate solution $u_{ap,\varepsilon}$.** The function $U\left(\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right)$ is certainly a good candidate for an approximate transition layer solution to the equation (2.4). However, it does not satisfy the boundary conditions. We, therefore, adjust the approximate solution to

$$u_{ap,\varepsilon}(x) = \zeta_0(x-x_1)U\left(\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right) + \zeta_+(x-x_1), \tag{2.14}$$

where $\zeta_0,\ \zeta_+$ are cutoff functions of class $C^\infty(\mathbb{R})$ with

$$\zeta_0(x) = \begin{cases} 1 & |x| \le d \\ 0 & |x| \ge 2d \\ 0 \le \zeta_0(x) \le 1 & x \in \mathbb{R}, \end{cases} \qquad \zeta_+(x) = \begin{cases} \zeta_0(x) - 1 & x \le 0 \\ 1 - \zeta_0(x) & x \ge 0, \end{cases} \tag{2.15}$$

for some small fixed $d > 0$ (see also [6]).

Proposition 2.4. *For small $\varepsilon > 0$, we have*

$$\|\varepsilon^2 u''_{ap} + h(x)(u_{ap} - u^3_{ap})\|_{L^\infty(0,1)} \le C\varepsilon^{\frac{2(\alpha+\delta)}{\alpha+2}}.$$

Proof. In the interval $[x_1-d, x_1+d]$, $u_{ap} = u_{in}$ and the bound follows from Proposition 2.3.

In $[0,1]\backslash(x_1-d, x_1+d)$ the bound holds as well. In the interval $[x_1-2d, x_1-d]$ we have that $1+u_{ap},\ |u''_{ap}|$ are transcedentally small in ε (note that Proposition 2.2 implies that $U'(z),\ U''(z)$ also converge to zero super-exponentially as $|z| \to \infty$). In $[x_1+d, x_1+2d]$, $1-u_{ap},\ |u''_{ap}|$ are transcedentally small in ε. In $[0, x_1-2d]$ we have $u_{ap} = -1$ and in $[x_1+2d, 1]$, $u_{ap} = 1$.

The proof of the proposition is complete. □

2.3. **Linear analysis: Asymptotic stability of U.** In Proposition 2.1 we showed that the limiting problem (2.10), (2.11) has a solution U. In this subsection we show that U is non-degenerate and actually asymptotically stable.

The spectrum, in $L^2(\mathbb{R})$, of the linear operator

$$M(\varphi) = -\varphi'' + A|x|^\alpha(3U^2-1)\varphi, \tag{2.16}$$

is discrete and consists of simple eigenvalues $\mu_1 < \mu_2 < \cdots$ (recall that $|x|^\alpha(3U^2-1) \to \infty$ as $|x| \to \infty$). We denote the corresponding L^2-normalized eigenfunctions by $\psi_1, \psi_2, \cdots$. Each ψ_i, $i \ge 1$ has exactly $i-1$ zeroes and we may assume that $\psi_1 > 0$. Moreover each ψ_i satisfies the bound (**??**) (with constants depending on i). Without loss of generality we can assume that $\psi_1 > 0$ is an even function of x.

Proposition 2.5. *We have that $\mu_1 > 0$.*

Proof. We have

$$-\psi''_1 + A|x|^\alpha(3U^2-1)\psi_1 = \mu_1\psi_1, \quad x \in \mathbb{R}. \tag{2.17}$$

Note that $w = U' > 0$ solves

$$-w'' + A|x|^\alpha(3U^2-1)w = -A\alpha|x|^{\alpha-2}x(U^3-U), \quad x \in \mathbb{R}. \tag{2.18}$$

By multiplying (2.17) with w, (2.18) with ψ_1 and subtracting, we find that

$$\mu_1\int_{-\infty}^{\infty}\psi_1 w dx = -A\alpha\int_{-\infty}^{\infty}|x|^{\alpha-2}x(U^3-U)\psi_1 dx = -2A\alpha\int_0^\infty x^{\alpha-1}(U^3-U)\psi_1 dx > 0.$$

The proof is complete. □

2.4. **Linear analysis: The linearized operator in a small neighborhood of the approximate solution.** In this subsection we will study the linear operator

$$L_\varepsilon(\varphi) = -\varepsilon^2\varphi'' + h(x)(3u_\varepsilon^2 - 1)\varphi,$$

with $\varphi \in H^2(0,1)$, $\varphi'(0) = \varphi'(1) = 0$ and $u_\varepsilon \in C[0,1]$ satisfying

$$\|u_\varepsilon - u_{ap,\varepsilon}\|_{L^\infty(0,1)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.19}$$

This operator arises when we linearize (2.4), (2.5) on the approximate solution $u_{ap,\varepsilon}$.

Note that there exists a large number $D > 0$ such that

$$3u_\varepsilon^2(x) - 1 \geq 1, \quad |x - x_1| \geq D\varepsilon^{\frac{2}{\alpha+2}}, \tag{2.20}$$

if $\varepsilon > 0$ is sufficiently small. Hence,

$$h(x)\left(3u_\varepsilon^2(x) - 1\right) \geq c|x - x_1|^\alpha, \quad |x - x_1| \geq D\varepsilon^{\frac{2}{\alpha+2}}, \tag{2.21}$$

for small ε. Moreover, it is easy to see that

$$h(x)\left(3u_\varepsilon^2(x) - 1\right) \geq -C\varepsilon^{\frac{2\alpha}{\alpha+2}}, \quad x \in [0,1]. \tag{2.22}$$

By (2.3), the definition of u_{ap}^ε and our assumption on u_ε, we have

$$h(x_1 + \varepsilon^{\frac{2}{\alpha+2}}x)\left(3u_\varepsilon^2(x_1 + \varepsilon^{\frac{2}{\alpha+2}}x) - 1\right) \to A|x|^\alpha(3U^2(x) - 1) \quad \text{in} \quad C_{loc}(\mathbb{R}) \tag{2.23}$$

as $\varepsilon \to 0$.

Proposition 2.6. *The exists $\varepsilon_0 > 0$, $C > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, $f \in C[0,1]$ and $\varphi \in C^2[0,1]$ satisfy*

$$\begin{cases} L_\varepsilon(\varphi) = f & x \in (0,1) \\ \varphi'(0) = \varphi'(1) = 0, \end{cases}$$

then

$$\|\varphi\|_{L^\infty(0,1)} \leq C\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_{L^\infty(0,1)}.$$

Proof. We will argue by contradiction. Assuming the opposite means that there are sequences $\varepsilon_n \to 0$, $\varphi_n \in C^2[0,1]$, $f_n \in C[0,1]$ such that

$$\|\varphi_n\|_{L^\infty(0,1)} = 1, \ n \geq 1, \quad \varepsilon_n^{-\frac{2\alpha}{\alpha+2}}\|f_n\|_{L^\infty(0,1)} \to 0 \ \text{ as } n \to \infty, \tag{2.24}$$

$$\begin{cases} -\varepsilon_n^2\varphi_n'' + h(x)(3u_{\varepsilon_n}^2 - 1)\varphi_n = f_n \quad \text{in} \quad (0,1), \\ \varphi_n'(0) = \varphi_n'(1) = 0. \end{cases} \tag{2.25}$$

In view of the boundary conditions, without loss of generality, we assume that there exist $\xi_n \in [0,1]$ such that

$$\varphi_n(\xi_n) = 1, \ \varphi_n'(\xi_n) = 0, \ \varphi_n''(\xi_n) \leq 0, \quad n \geq 1.$$

We claim that

$$\varepsilon_n^{-\frac{2}{\alpha+2}}|\xi_n - x_1| \leq C, \quad n \geq 1. \tag{2.26}$$

Indeed, by (2.21), (2.25), we deduce that, for each $n \geq 1$,

$$h(\xi_n)\left(3u_{\varepsilon_n}^2(\xi_n) - 1\right) \leq \|f_n\|_{L^\infty(0,1)}.$$

Relation (2.26) now follows immediately from (2.21), (2.24).

Let

$$\tilde{\varphi}_n(x) = \varphi_n(x_1 + \varepsilon_n^{\frac{2}{\alpha+2}}x), \quad \tilde{f}_n(x) = f_n(x_1 + \varepsilon_n^{\frac{2}{\alpha+2}}x), \quad x \in I_n = \left[-\frac{x_1}{\varepsilon_n^{\frac{2}{\alpha+2}}}, \frac{1 - x_1}{\varepsilon_n^{\frac{2}{\alpha+2}}}\right].$$

We see that

$$\|\tilde{\varphi}_n\|_{L^\infty(I_n)} = 1,\ n \geq 1,\quad \varepsilon_n^{-\frac{2\alpha}{\alpha+2}} \|\tilde{f}_n\|_{L^\infty(I_n)} \to 0 \ \text{ as } n \to \infty, \tag{2.27}$$

$$\begin{cases} -\tilde{\varphi}_n'' + \varepsilon_n^{-\frac{2\alpha}{\alpha+2}} h(x_1 + \varepsilon_n^{\frac{2}{\alpha+2}} x) \left(3u_{\varepsilon_n}^2 (x_1 + \varepsilon_n^{\frac{2}{\alpha+2}} x) - 1 \right) \tilde{\varphi}_n = \varepsilon_n^{-\frac{2\alpha}{\alpha+2}} \tilde{f}_n \quad \text{in } \ I_n, \\ \tilde{\varphi}_n' \left(-\frac{x_1}{\varepsilon_n^{\frac{2}{\alpha+2}}} \right) = \tilde{\varphi}_n' \left(\frac{1-x_1}{\varepsilon_n^{\frac{2}{\alpha+2}}} \right) = 0. \end{cases} \tag{2.28}$$

The above two relations and (2.23) imply that for any $L > 0$ we have

$$\|\tilde{\varphi}_n\|_{C^2[-L,L]} \leq C(L) \quad \text{if } \ n \geq N(L).$$

Using the Arczela-Ascoli theorem, (2.23), (2.28) and the standard diagonal argument we obtain that, for a subsequence,

$$\tilde{\varphi}_n \to \bar{\varphi} \ \text{ in } \ C^2_{loc}(\mathbb{R}) \ \text{ as } \ n \to \infty,$$

where $\bar{\varphi}$ is bounded in $\mathbb{R}$ and satisfies

$$-\bar{\varphi}'' + A|x|^\alpha \left(3U^2(x) - 1 \right) \bar{\varphi} = 0 \quad \text{in } \ \mathbb{R}.$$

Since $\bar{\varphi} \in L^\infty(\mathbb{R})$ and $A|x|^\alpha(3U^2 - 1) \to \infty$ as $|x| \to \infty$, it is easy to see that $\bar{\varphi} \to 0$ as $|x| \to \infty$ super-exponentially. Hence, by Proposition 2.5,

$$\bar{\varphi} \equiv 0.$$

On the other hand, since

$$\tilde{\varphi}_n \left(\frac{\xi_n - x_1}{\varepsilon_n^{\frac{2}{\alpha+2}}} \right) = 1, \quad n \geq 1,$$

we obtain, via (2.26) and the C^2_{loc} convergence of $\tilde{\varphi}_n$, that $\bar{\varphi}(\bar{x}) = 1$ for some $\bar{x}$; a contradiction and the proof is complete. □

Proposition 2.7. *The exists $\varepsilon_0 > 0$, $C > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, $f \in C[0,1]$ and $\varphi \in C^2[0,1]$ satisfy*

$$\begin{cases} L_\varepsilon(\varphi) = hf & x \in (0,1) \\ \varphi'(0) = \varphi'(1) = 0, \end{cases}$$

then

$$\|\varphi\|_{L^\infty(0,1)} \leq C\|f\|_{L^\infty(0,1)}.$$

Proof. We will argue by contradiction. Assuming the opposite means that there are sequences $\varepsilon_n \to 0$, $\varphi_n \in C^2[0,1]$, $f_n \in C[0,1]$ such that

$$\|\varphi_n\|_{L^\infty(0,1)} = 1,\ n \geq 1,\quad \|f_n\|_{L^\infty(0,1)} \to 0 \ \text{ as } n \to \infty, \tag{2.29}$$

$$\begin{cases} -\varepsilon_n^2 \varphi_n'' + h(x)(3u_{\varepsilon_n}^2 - 1)\varphi_n = h(x) f_n \quad \text{in } \ (0,1), \\ \varphi_n'(0) = \varphi_n'(1) = 0. \end{cases} \tag{2.30}$$

In view of the boundary conditions, without loss of generality, we assume that there exist $\xi_n \in [0,1]$ such that

$$\varphi_n(\xi_n) = 1,\ \varphi_n'(\xi_n) = 0,\ \varphi_n''(\xi_n) \leq 0, \quad n \geq 1.$$

We claim that

$$\varepsilon_n^{-\frac{2}{\alpha+2}} |\xi_n - x_1| \leq C, \qquad n \geq 1. \tag{2.31}$$

Indeed, by (2.30) we deduce that, for each $n \geq 1$,

$$h(\xi_n)\left(3u_{\varepsilon_n}^2(\xi_n) - 1\right) \leq h(\xi_n)\|f_n\|_{L^\infty(0,1)}.$$

Relation (2.31) now follows immediately from (2.20).

The rest of the argument is similar to the one in the proof of Proposition 2.6. □

Similarly we can show (see [13])

Proposition 2.8. *For any fixed integer $m \geq 1$ the first m eigenvalues $\lambda_{1,\varepsilon} < \cdots < \lambda_{m,\varepsilon}$ and the corresponding L^2-normalized eigenfunctions $\varphi_{i,\varepsilon}$ of L_ε satisfy*

$$\varepsilon^{-\frac{2\alpha}{\alpha+2}}\lambda_{i,\varepsilon} \to \mu_i$$

$$\|\varphi_{i,\varepsilon}(x) - \varepsilon^{-\frac{1}{\alpha+2}}\psi_i\left(\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right)\|_{L^2(0,1)} \to 0$$

as $\varepsilon \to 0$, $i = 1, \cdots, k$, where μ_i, ψ_i are as in Section 2.3.

2.5. **Existence and stability for the nonlinear problem.**

Theorem 2.2. *If $\varepsilon > 0$ is sufficiently small, there exists a solution $\boldsymbol{u}_\varepsilon$ of (2.4), (2.5) such that*

$$\|\boldsymbol{u}_\varepsilon - u_{ap,\varepsilon}\|_{L^\infty(0,1)} \leq C\varepsilon^{\frac{2\delta}{\alpha+2}}. \tag{2.32}$$

Moreover, the smallest eigenvalue $\lambda_{1,\varepsilon}$ of the eigenvalue problem

$$\begin{cases} -\varepsilon^2\phi'' + h(x)(3\boldsymbol{u}_\varepsilon^2 - 1)\phi = \lambda\phi \quad in \ (0,1), \\ \phi'(0) = \phi'(1) = 0, \end{cases} \tag{2.33}$$

is positive and satisfies the assertion of Proposition 2.8.

Proof. We look for a solution to (2.4), (2.5) in the form

$$u = u_{ap,\varepsilon} + \varphi,$$

where $\varphi \in C^2[0,1]$ with $\varphi'(0) = \varphi'(1) = 0$ is a small perturbation. Thus the equation for u is equivalent to

$$\begin{cases} L_\varepsilon(\varphi) = h(x)N_\varepsilon(\varphi) + E_\varepsilon \quad \text{in} \ (0,1), \\ \varphi'(0) = \varphi'(1) = 0, \end{cases} \tag{2.34}$$

where

$$L_\varepsilon(\varphi) = -\varepsilon^2\varphi'' + h(x)(3u_{ap}^2 - 1)\varphi,$$

$$N_\varepsilon(\varphi) = -\varphi^3 - 3u_{ap}\varphi^2,$$

$$E_\varepsilon = \varepsilon^2 u_{ap}'' - h(x)(u_{ap}^3 - u_{ap}).$$

Note that L_ε satisfies the hypotheses of Section 2.4 (see (2.19)).

In view of Proposition 2.6, for small ε, we can define a mapping $T_\varepsilon : C[0,1] \to C[0,1]$ by the relation

$$\begin{cases} L_\varepsilon\left(T_\varepsilon(\varphi)\right) = h(x)N_\varepsilon(\varphi) + E_\varepsilon \quad \text{in} \ (0,1), \\ \left(T_\varepsilon(\varphi)\right)'(0) = \left(T_\varepsilon(\varphi)\right)'(1) = 0. \end{cases} \tag{2.35}$$

We will show that we can chose a large M (indep. of ε) such that, for small ε, the nonlinear operator T_ε maps

$$B_{\varepsilon,M} := \{\varphi \in C[0,1] : \|\varphi\|_{L^\infty(0,1)} \leq M\varepsilon^{\frac{2\delta}{\alpha+2}}\}$$

into itself and is a contraction with respect to the L^∞ norm.
If $\varphi \in B_{\varepsilon,M}$, by Propositions 2.4, 2.6, 2.7 we get

$$\begin{aligned} \|T_\varepsilon(\varphi)\|_{L^\infty(0,1)} &\leq C\|N_\varepsilon(\varphi)\|_{L^\infty(0,1)} + C\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|E_\varepsilon\|_{L^\infty(0,1)} \\ &\leq C\|\varphi\|^2_{L^\infty(0,1)} + C\varepsilon^{\frac{2\delta}{\alpha+2}} \\ &\leq C\left(M^2\varepsilon^{\frac{2\delta}{\alpha+2}} + 1\right)\varepsilon^{\frac{2\delta}{\alpha+2}} \end{aligned}$$

provided ε is small (independently of $\varphi,\ M$). (The constant C in the above relation is independent of $\varphi,\ M$). By choosing $M = 2C$ and ε_0 such that $4C^2\varepsilon_0^{\frac{2\delta}{\alpha+2}} < 1$ we deduce that $T_\varepsilon(\varphi) \in B_{\varepsilon,M}$ if $0 < \varepsilon < \varepsilon_0$.
If $\varphi_1,\ \varphi_2 \in B_\varepsilon$ (we dropped the subscript M since we fixed it), we have

$$\begin{aligned} \|T_\varepsilon(\varphi_1) - T_\varepsilon(\varphi_2)\|_{L^\infty(0,1)} &\leq C\|N_\varepsilon(\varphi_1) - N_\varepsilon(\varphi_2)\|_{L^\infty(0,1)} \\ &\leq C\varepsilon^{\frac{2\delta}{\alpha+2}}\|\varphi_1 - \varphi_2\|_{L^\infty(0,1)}. \end{aligned}$$

It follows that, if ε is sufficiently small, $T_\varepsilon : B_\varepsilon \to B_\varepsilon$ is a contraction.

By the Banach fixed point theorem, there exists a unique

$$\boldsymbol{\varphi}_\varepsilon \in B_\varepsilon$$

such that

$$T_\varepsilon(\boldsymbol{\varphi}_\varepsilon) = \boldsymbol{\varphi}_\varepsilon.$$

The function

$$\mathbf{u}_\varepsilon := u_{ap,\varepsilon} + \boldsymbol{\varphi}_\varepsilon$$

is a solution of (2.4), (2.5) satisfying estimate (2.32). The principal eigenvalue of the linearized problem (3.3) satisfies the assertion of Proposition 2.8 because $\mathbf{u}_\varepsilon$ satisfies (2.19). The proof of the theorem is complete. □

3. Layered solutions in a problem arising in population genetics

We consider the singularly perturbed boundary value problem

$$\begin{cases} \varepsilon^2\Delta u + h(x)g(u) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$, $h \in C(\bar{\Omega})$ and $g \in C^2(\mathbb{R})$ satisfies

$$\begin{cases} g(0) = 0, & g(1) = 0, \\ g_u(0) > 0, & g_u(1) < 0, \\ g(u) > 0, & \forall u \in (0,1). \end{cases}$$

Note that (3.1) includes the important Fisher's equation (with $g(u) = u(1-u)$).

This problem was treated in [7] in the context of population genetics. The existence of a family of global minimizers u_ε of the corresponding energy functional which develop inner transition layers with a single interface given by

$$S = \{x \in \Omega : h(x) = 0\}$$

was proved if h changes sign at S. More precisely, the author assumed that $h \in L^\infty(\Omega)$, $h \neq 0$ a.e., and that the set

$$P = \{x \in \Omega : \; h(x) > 0 \; \text{ a.e.}\}$$

has finite capacity. He showed that any global minimizer u_ε satisfies

$$u_\varepsilon \to \chi_P \;\text{ as }\; \varepsilon \to 0 \;\text{ in }\; L^q(\Omega) \;\; \forall\, q < \infty,$$

(see [7] Chapter 10, Exercise 3).

In the case where the interface S is smooth, one can apply the arguments of the previous sections and give a perturbation proof which yields fine estimates (the one dimensional profile turns out to be asymptotic stable). For simplicity we focus on the one dimensional case and we assume that h changes sign only once. We will assume that $h \in C[0,1]$ satisfies

$$h(x) < 0, \quad x \in [0, x_1), \quad h(x) > 0, \quad x \in (x_1, 1],$$

$$h(x) = A\text{sign}(x - x_1)|x - x_1|^\alpha + O(|x - x_1|^{\alpha+\delta}) \quad \text{as} \quad x \to x_1,$$

for some $x_1 \in (0,1)$ and $A, \; \alpha, \; \delta > 0$.

Theorem 3.1. *If ε is sufficiently small, there exists a layered solution $\boldsymbol{u}_\varepsilon$ of*

$$\begin{cases} \varepsilon^2 u'' + h(x) g(u) = 0, \quad x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.2}$$

such that

$$\left| \boldsymbol{u}_\varepsilon(x) - U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right) \right| \le C\varepsilon^{\frac{2\delta}{\alpha+2}}, \quad x \in [0,1],$$

where U is the unique monotone solution of

$$\begin{cases} u'' + A sign(x)|x|^\alpha g(u) = 0, \quad x \in \mathbb{R}, \\ u(x) \to 0 \;\; as\; x \to -\infty, \qquad u(x) \to 1 \;\; as\; x \to \infty, \end{cases}$$

(see [5]). Moreover, the smallest eigenvalue $\lambda_{1,\varepsilon}$ of the eigenvalue problem

$$\begin{cases} -\varepsilon^2 \phi'' + h(x) g_u(\boldsymbol{u}_\varepsilon)\phi = \lambda\phi \quad in \;\; (0,1), \\ \phi'(0) = \phi'(1) = 0, \end{cases} \tag{3.3}$$

is positive and satisfies

$$\varepsilon^{-\frac{2\alpha}{\alpha+2}} \lambda_{1,\varepsilon} \to \mu_1 \quad as \;\; \varepsilon \to 0,$$

where $\mu_1 > 0$ is the principal eigenvalue of

$$-\psi'' - A sign(x)|x|^\alpha g_u(U)\psi = \mu\psi, \quad \psi \in L^2(\mathbb{R}).$$

We refer to [5] for the asymptotic stability of U (we note that $U' > 0$). The proof follows almost word by word the arguments of Sections 2.2, 2.4, 2.5 and, thus, is omitted.

4. Standing waves with critical frequency for the nonlinear Schrödinger equation

In [2] the authors studied the problem

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + u^p = 0, \quad x \in \mathbb{R}^N, \\ \\ u(x) \to 0, \quad |x| \to \infty, \end{cases} \tag{4.1}$$

under the hypotheses:

$$\begin{cases} V \in C(\mathbb{R}^N) \\ \\ \liminf_{|x|\to\infty} V(x) > 0 \\ \\ \inf_{x\in\mathbb{R}^N} V(x) = 0 \\ \\ 1 < p < \frac{N+2}{N-2} \ \text{ if } N \geq 3, \quad p > 1 \ \text{ if } N = 1, 2. \end{cases} \tag{4.2}$$

In the case when V has an isolated minimum at $x_1 \in \mathbb{R}^N$ such that

$$V(x) = A|x - x_1|^\alpha + o(|x - x_1|^\alpha) \quad \text{as} \quad x \to x_1,$$

for some constants $A, \ \alpha > 0$, their general result says that (4.1) has a positive solution u_ε such that

$$\varepsilon^{-\frac{2\alpha}{(p-1)(\alpha+2)}} u_\varepsilon(x_1 + \varepsilon^{\frac{2}{\alpha+2}} x) \to U(x) \quad \text{uniformly in } \mathbb{R}^N \text{ as } \varepsilon \to 0, \tag{4.3}$$

where U is the unique positive solution of

$$\begin{cases} \Delta u - A|x|^\alpha u + u^p = 0, \quad x \in \mathbb{R}^N, \\ \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

(see [3]). Their approach is variational (equation (4.1) under (4.2) has mountain-pass structure).

From [3] we know that U is non-degenerate, i.e.,

$$\begin{cases} \Delta\varphi - A|x|^\alpha \varphi + pU^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N, \\ \\ \lim_{|x|\to\infty} \varphi(x) = 0, \end{cases}$$

has only the trivial solution $\varphi \equiv 0$.

Here we assume that $V, \ p \geq 2$ satisfy (4.2),

$$V(x) = A|x - x_1|^\alpha + O(|x - x_1|^{\alpha+\delta}) \quad \text{as} \quad x \to x_1,$$

for some $A, \ \alpha, \ \delta > 0$ and $V(x) > 0$, $x \neq x_1$.

We set

$$u_{ap,\varepsilon}(x) = \varepsilon^{\frac{2\alpha}{(p-1)(\alpha+2)}} U\left(\frac{x - x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right) \eta(x - x_1), \quad x \in \mathbb{R}^N, \tag{4.4}$$

where $\zeta \in C_0(\mathbb{R}^N)$ is a smooth cut-off (independent of ε) such that

$$\zeta(x) = \begin{cases} 1 & |x| \leq 1 \\ \\ 0 & |x| \geq 2. \end{cases}$$

We introduce the weighted norm

$$\|u\|_w = \| \exp\left\{\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}\right\} u\|_{L^\infty(\mathbb{R}^N)},$$

and the Banach space

$$W = \{u \in C(\mathbb{R}^N): \ \|u\|_w < \infty\}$$

equipped with this norm.

Proposition 4.1. *If $\varepsilon > 0$ is sufficiently small,*

$$\|\varepsilon^2 \Delta u_{ap} - V(x)u_{ap} + u_{ap}^p\|_w \le C\varepsilon^{\frac{2\alpha p+2\delta(p-1)}{(p-1)(\alpha+2)}}.$$

Proof. By the definition of U and its super-exponential decay, we have

$$\left|\varepsilon^2 \Delta u_{ap} - A|x-x_1|^\alpha u_{ap} + u_{ap}^p\right| \le Ce^{-\frac{c}{\varepsilon}} e^{-\frac{|x-x_1|}{\varepsilon}} \quad \text{in } \mathbb{R}^N.$$

So,

$$\begin{aligned}
\left|\varepsilon^2 \Delta u_{ap} - V(x)u_{ap}u_{ap} + u_{ap}^p\right| &\le |(A|x-x_1|^\alpha - V(x))\, u_{ap}| + Ce^{-\frac{c}{\varepsilon}} e^{-\frac{|x-x_1|}{\varepsilon}} \\
&\le C\varepsilon^{\frac{2\alpha p+2\delta(p-1)}{(p-1)(\alpha+2)}} \left|\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right| U\left(\frac{x-x_1}{\varepsilon^{\frac{2}{\alpha+2}}}\right) + Ce^{-\frac{c}{\varepsilon}} e^{-\frac{|x-x_1|}{\varepsilon}} \qquad (4.5) \\
&\le C\varepsilon^{\frac{2\alpha p+2\delta(p-1)}{(p-1)(\alpha+2)}} e^{-\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}}, \qquad (4.6)
\end{aligned}$$

in $\mathbb{R}^N$, and the desired bound follows immediately. □

Remark 4.1. *We used the cut-off in case V increases exponentially fast at as $|x| \to \infty$.*

Remark 4.2. *The estimate $0 < U(x) \le Ce^{-2|x|}$ was all that we used.*

Let

$$L_\varepsilon(\varphi) = -\varepsilon^2 \Delta\varphi + \left(V(x) - pu_{ap}^{p-1}\right)\varphi.$$

Note that for some fixed $D > 1$ independent of ε,

$$V(x) - pu_{ap}^{p-1} \ge \begin{cases} c|x-x_1|^\alpha - \frac{c}{2}\varepsilon^{\frac{2\alpha}{\alpha+2}} & D\varepsilon^{\frac{2}{\alpha+2}} \le |x-x_1| \le 1 \\ c & |x-x_1| \ge 1, \end{cases} \qquad (4.7)$$

$$\varepsilon^{-\frac{2\alpha}{\alpha+2}} V(x_1 + \varepsilon^{\frac{2}{\alpha+2}} x) \to A|x|^\alpha \quad \text{in } \ C_{loc}(\mathbb{R}^N)$$

as $\varepsilon \to 0$ (these are the analogs of (2.21), (2.23)).

As in Proposition 2.6, we have

Proposition 4.2. *There exist constants ε_0, $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, $\varphi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfy*

$$L_\varepsilon(\varphi) = f \quad \text{in } \mathbb{R}^N,$$

then

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)} \le C\varepsilon^{-\frac{2\alpha}{\alpha+2}} \|f\|_{L^\infty(\mathbb{R}^N)}.$$

Proposition 4.3. *There exist constants ε_0, $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, and $f \in W$, then there exists a unique $\varphi \in W \cap C^2(\mathbb{R}^N)$ such that*

$$L_\varepsilon(\varphi) = f \quad in\ \mathbb{R}^N.$$

Moreover,

$$\|\varphi\|_w \le C\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_w.$$

Proof. The existence and uniqueness of φ can be deduced from Proposition 4.3 and a standard barrier argument. We already know that

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)} \le C\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_w. \tag{4.8}$$

So,

$$\exp\left\{\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}\right\}|\varphi(x)| \le C\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_w \quad \text{if } |x-x_1| \le D\varepsilon^{\frac{2}{\alpha+2}},$$

(D is as in (4.7)). Let

$$\psi_M(x) = M\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_w \exp\left\{-\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}\right\}, \quad |x-x_1| \ge D\varepsilon^{\frac{2}{\alpha+2}}.$$

In view of (4.7), (4.8) it is easy to check that

$$\begin{aligned} &L_\varepsilon(\psi_M) \ge f && |x-x_1| \ge D\varepsilon^{\frac{2}{\alpha+2}} \\ &\psi_M = \varphi && |x-x_1| = D\varepsilon^{\frac{2}{\alpha+2}}, \end{aligned}$$

provide ε is small and M is large (indep. of ε). By the maximum principle (recall also that φ, ψ_M vanish at infinity) we deduce that

$$\varphi(x) \le M\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_w \exp\left\{-\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}\right\}.$$

Doing the same for $-\varphi$ yields

$$\exp\left\{\frac{|x-x_1|}{\varepsilon^{\frac{2}{\alpha+2}}}\right\}|\varphi(x)| \le C\varepsilon^{-\frac{2\alpha}{\alpha+2}}\|f\|_w,$$

and the proof is complete. □

Remark 4.3. *As in Section 2.4, the above two propositions also hold true when we linearize on functions u_ε satisfying $\varepsilon^{-\frac{2\alpha}{(p-1)(\alpha+2)}}\|u_\varepsilon - u_{ap}\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$.*

Theorem 4.1. *If $\varepsilon > 0$ is sufficiently small, there exists a positive solution $\boldsymbol{u}_\varepsilon > 0$ of (4.1) such that*

$$\|\boldsymbol{u}_\varepsilon - u_{ap,\varepsilon}\|_w \le C\varepsilon^{\frac{2\alpha+2\delta(p-1)}{(p-1)(\alpha+2)}}, \tag{4.9}$$

or equivalently

$$\left|\varepsilon^{-\frac{2\alpha}{(p-1)(\alpha+2)}}u_\varepsilon(x_1+\varepsilon^{\frac{2}{\alpha+2}}x) - U(x)\right| \le C\varepsilon^{\frac{2\delta}{\alpha+2}}e^{-|x|}, \quad x \in \mathbb{R}^N. \tag{4.10}$$

Proof. We consider the equation

$$\varepsilon^2\Delta u - V(x)u + |u|^p = 0 \quad \text{in } \mathbb{R}^N. \tag{4.11}$$

We search for a solution u of (4.11) in the form $u = u_{ap,\varepsilon} + \varphi$. In terms of φ equation (4.11) becomes

$$L_\varepsilon(\varphi) = N_\varepsilon(\varphi) + E_\varepsilon,$$

where

$$N_\varepsilon(\varphi) = |u_{ap} + \varphi|^p - u_{ap}^p - pu_{ap}^{p-1}\varphi,$$

$$E_\varepsilon = \varepsilon^2\Delta u_{ap} - V(x)u_{ap} + u_{ap}^p.$$

Since $p \geq 2$, there exists a $C > 0$ such that

$$\big| |y+y_0|^p - |y_0|^p - p\text{sign}(y_0)|y_0|^{p-1}y \big| \leq C(|y|+|y_0|)^{p-2}y^2, \quad \forall y, y_0 \in [-1,1]. \tag{4.12}$$

In view of Propositions 4.1, 4.3 and (4.12), we can define a mapping $T_\varepsilon : W \to W$ by the relation

$$L_\varepsilon \left(T_\varepsilon(\varphi)\right) = N_\varepsilon(\varphi) + E_\varepsilon.$$

Let,

$$B_{\varepsilon,M} := \{\varphi \in W : \|\varphi\|_w \leq M\varepsilon^{\frac{2\alpha+2\delta(p-1)}{(p-1)(\alpha+2)}}\}$$

where $M > 0$ is to be chosen independently of $\varepsilon > 0$ small. Note that, via (4.12), for every $\varphi \in B_{\varepsilon,M}$ we have

$$\|N_\varepsilon(\varphi)\|_w \leq C\varepsilon^{\frac{2\alpha(p-2)}{(p-1)}}\|\varphi\|_w^2$$

with C independent of ε, M, φ provided ε is small. We leave it to the interested reader to check that we can fix a large $M > 0$ such that, if $\varepsilon > 0$ is sufficiently small, T_ε maps B_ε into itself and is a contraction with respect to the norm of W(see also the proof of Theorem 2.2). Hence, T_ε has a unique fixed point $\boldsymbol{\varphi}_\varepsilon$ in B_ε. Then $\boldsymbol{u}_\varepsilon := u_{ap} + \boldsymbol{\varphi}_\varepsilon$ solves equation (4.11) and satisfies estimates (4.9), (4.10). To conclude it remains to show that $\boldsymbol{u}_\varepsilon$ is positive in $\mathbb{R}^N$ (so that it solves (4.1)). By (4.9) we see that for every $L > 0$ independent of ε,

$$\boldsymbol{u}_\varepsilon(x) > 0, \quad |x| \leq L\varepsilon^{\frac{2}{\alpha+2}},$$

provided $0 < \varepsilon < \varepsilon(L)$. Note that $\boldsymbol{u}_\varepsilon$ satisfies

$$-\varepsilon^2\Delta\boldsymbol{u}_\varepsilon + \left(V(x) - \text{sign}(\boldsymbol{u}_\varepsilon)|\boldsymbol{u}_\varepsilon|^{p-1}\right)\boldsymbol{u}_\varepsilon = 0, \quad x \in \mathbb{R}^N,$$

and

$$V(x) - \text{sign}(\boldsymbol{u}_\varepsilon)|\boldsymbol{u}_\varepsilon|^{p-1} \geq cL^\alpha\varepsilon^{\frac{2\alpha}{\alpha+2}} - C\varepsilon^{\frac{2\alpha}{\alpha+2}}, \quad |x| \geq L\varepsilon^{\frac{2}{\alpha+2}},$$

provided $0 < \varepsilon < \varepsilon_1(L)$ (c, C independent of ε, L). By fixing a large $L > c^{-1}C^{\frac{1}{\alpha}}$, we conclude by the maximum principle that

$$\boldsymbol{u}_\varepsilon(x) > 0, \quad |x| \geq L\varepsilon^{\frac{2}{\alpha+2}},$$

and the proof is complete. □

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C. Sourdis, Departamento de Ingenieria Matematica, Universidad de Chile, Santiago, Chile

E-mail address: schristos@dim.uchile.cl