INTERFACE LAYER OF A TWO-COMPONENT BOSE-EINSTEIN CONDENSATE

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Abstract. This paper deals with the study of the behaviour of the wave functions of a two-component Bose-Einstein condensate near the interface, in the case of strong segregation. This yields a system of two coupled ODE's for which we want to have estimates on the asymptotic behaviour, as the strength of the coupling tends to infinity. As in phase separation models, the leading order profile is a hyperbolic tangent. We construct an inner and outer solution that we match at a point using the fact that the Hamiltonian of the system is constant. Then, we use the properties of the associated linearized operator to perturb the constructed approximate solution into a genuine one for which we have an asymptotic expansion. Furthermore, we prove that the constructed heteroclinic solutions are linearly nondegenerate, in the natural sense, and that there is a spectral gap, independent of the large interaction parameter, between the zero eigenvalue (due to translations) at the bottom of the spectrum and the rest of the spectrum.

1. Introduction

1.1. **The problem.** A two-component condensate is described by two complex valued wave functions minimizing a Gross-Pitaevskii energy. The interaction is usually modelled by a term involving only the modulus of these wave functions. According to the magnitude of the interaction parameter, the components can either coexist or segregate. Even though the interaction is only through the modulus, it can produce effects on the phases of each component, and in particular on the singularities or vortices. A full phase diagram has been computed in [19].

This paper deals with the study of the behaviour of the wave functions near the interface, in the case of segregation. In the physics literature, there is a formal analysis of this small coexistence region which, at leading order, is given by a hyperbolic tangent [6]. Other works include [5, 26]. Here, we want to derive a rigorous asymptotic expansion of this transition layer, which will be useful in the analysis of more complex patterns.

The segregation behaviour in two-component condensates has been widely studied in the mathematics literature: regularity of the wave function [12, 13, 20, 22, 23, 27, 28], regularity of the interface [10, 25], asymptotic behaviour near the interface [7, 8], Γ-convergence in the case of a trapped problem [2, 14, 15].

The aim of this paper is to study the system

$$\begin{cases} -v_1'' + v_1^3 - v_1 + \Lambda v_2^2 v_1 = 0, \\ -v_2'' + v_2^3 - v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{1.1}$$

$$(v_1, v_2) \to (0,1) \text{ as } z \to -\infty, \quad (v_1, v_2) \to (1,0) \text{ as } z \to \infty. \tag{1.2}$$

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The segregation case corresponds to the limit $\Lambda \to \infty$. We will construct solutions to the above problem which satisfy the symmetry

$$v_1(-z) = v_2(z), \quad z \in \mathbb{R}. \tag{1.3}$$

We observe that (1.1) is such that the Hamiltonian

$$H = \sum_{i=1}^{2} \left[\frac{1}{2}(v_i')^2 - \frac{1}{4}(1 - v_i^2)^2 \right] - \frac{\Lambda}{2} v_1^2 v_2^2 \tag{1.4}$$

is constant along solutions. In particular, the heteroclinic orbits that we seek should have this constant equal to $-1/4$. Let us mention that the existence of such a heteroclinic solution has been shown to hold for any $\Lambda > 0$ by variational methods in [4].

The leading order behaviour of this system, for large Λ, is governed by the hyperbolic tangent, namely the unique solution U of

$$u'' + u - u^3 = 0, \ z > 0; \ u(0) = 0, \ u(z) \to 1 \text{ as } z \to \infty, \tag{1.5}$$

which, in fact, admits the explicit representation

$$U(z) = \tanh\left(\frac{z}{\sqrt{2}}\right), \text{ for } z \geq 0. \tag{1.6}$$

More precisely, solutions to (1.1)-(1.2)-(1.3) should converge uniformly to the merely Lipschitz continuous singular limit $|U(z)|$ as $\Lambda \to \infty$. An important role in the upcoming analysis will be played by the behavior of U near the origin, which, at leading order, is given by $\psi_0 z$ with

$$\psi_0 = U'(0) = \frac{1}{\sqrt{2}}. \tag{1.7}$$

Near the origin, a formal blow-up analysis and the aforementioned linear behaviour of the solution U predict that the interaction between the two components should be governed by special solutions of a limiting problem, described in the next proposition which is due to [7, 8]:

Proposition 1.1. *There exists a unique solution (V_1, V_2) with positive components to the system*

$$\begin{cases} \ddot{V}_1 = V_2^2 V_1, \\ \ddot{V}_2 = V_1^2 V_2, \end{cases} \tag{1.8}$$

such that

$$\frac{V_1}{x} \to \psi_0 \ \ \textit{and} \ \ V_2 \to 0 \ \textit{as} \ x \to \infty, \tag{1.9}$$

where $\psi_0 > 0$ is as in (1.7), and

$$V_1(-x) = V_2(x), \ x \in \mathbb{R}. \tag{1.10}$$

Furthermore, every other entire solution of (1.8) with positive components is given by

$$\left(\mu V_1\left(\mu(x-h)\right), \mu V_2\left(\mu(x-h)\right)\right) \tag{1.11}$$

for some $\mu > 0$ and $h \in \mathbb{R}$.

1.2. **Main results.** The main result of the paper is the following:

Theorem 1.1. *If $\Lambda > 0$ is sufficiently large, problem (1.1)-(1.2) has a solution $(v_{1,\Lambda}, v_{2,\Lambda})$ such that $v_{1,\Lambda}(-z) = v_{2,\Lambda}(z)$, $v'_{1,\Lambda}(z) > 0$ for $z \in \mathbb{R}$,*

$$\|v_{1,\Lambda} - U\|_{L^\infty(0,\infty)} \le C\Lambda^{-\frac{1}{4}}, \tag{1.12}$$

$$v_{1,\Lambda}(z) = \Lambda^{-\frac{1}{4}} V_1\left(\Lambda^{\frac{1}{4}} z\right) + \mathcal{O}\left(\Lambda^{-\frac{3}{4}} + |z|^3\right), \tag{1.13}$$

uniformly on $\left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right]$, as $\Lambda \to \infty$, and, for any m,

$$\|v_{1,\Lambda}\|_{L^\infty\left(-\infty, -(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right)} = \mathcal{O}(\Lambda^m) \quad as \quad \Lambda \to \infty, \tag{1.14}$$

where U is the unique solution of (1.5) and (V_1, V_2) is the solution of (1.8)-(1.9)-(1.10); see Subsection 1.5 for the notation. Moreover,

$$\|v'_{1,\Lambda} - U'\|_{L^\infty\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)} \le C\Lambda^{-\frac{1}{4}}, \tag{1.15}$$

$$v'_{1,\Lambda}(z) = \dot{V}_1\left(\Lambda^{\frac{1}{4}} z\right) + \mathcal{O}\left(\Lambda^{-\frac{1}{2}} + |z|^2\right), \tag{1.16}$$

uniformly on $\left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right]$, as $\Lambda \to \infty$.

We can also show that $(v_{1,\Lambda}, v_{2,\Lambda})$ is nondegenerate, in the natural sense, and that there is a spectral gap.

Theorem 1.2. *Let (v_1, v_2) be the heteroclinic solution to (1.1)-(1.2) which is constructed in Theorem 1.1. Then, if $\Lambda > 0$ is sufficiently large, the spectrum of the linearized operator*

$$\boldsymbol{M}\begin{pmatrix} \varphi_1 \\ \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3v_1^2 - 1)\varphi_1 + \Lambda v_2^2 \varphi_1 + 2\Lambda v_1 v_2 \varphi_2 \\ \\ -\varphi_2'' + (3v_2^2 - 1)\varphi_2 + \Lambda v_1^2 \varphi_2 + 2\Lambda v_1 v_2 \varphi_1 \end{pmatrix}, \tag{1.17}$$

in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ is structured as follows:

- *0 is the first eigenvalue and has (v_1', v_2') as the associated eigenfunction,*
- *the rest of the spectrum is discrete and continuous, and is contained in (c, ∞) for some $c > 0$.*

1.3. **Main steps of the proofs.** The idea of the proof of Theorem 1.1 is to construct an approximate solution and then to perturb it into a genuine solution, using the linearized operator. Firstly, we construct an outer solution of (1.1)-(1.2)-(1.3) for $|z| \ge (\ln \Lambda)\Lambda^{-1/4}$, which satisfies the expected asymptotic behaviour (1.12). We do not prescribe conditions at the end points. We point out that this construction is possible by the nondegeneracy of the solution U of (1.5) (see Lemma 2.1 below). Then, we construct an inner solution based on the blow-up profile (V_1, V_2) of Proposition 1.1 and on its nondegeneracy in the symmetric class (1.3) (see Proposition 2.1 below). In particular, we note that the linearized operator of the blow-up system (1.8) at (V_1, V_2) contains an element (E_1, E_2) in its kernel, which satisfies the symmetry (1.3). This is due to the invariance of (1.8) under scaling. We can use this element to define a one parameter family of inner solutions with the same Hamiltonian constant as the outer solution, namely $-1/4$. In order to efficiently glue together the inner and outer approximations, we need to adjust the constant parameters involved in their separate constructions. For technical reasons, which will be clear from the proofs, instead of matching these approximations in the C^1-sense over an intermediate zone, we match them continuously only at the points $\pm(\ln\Lambda)\Lambda^{-1/4}$. At first sight, this unconventional argument might look rather counterintuitive, as it would create jumps on the gradients at the gluing points. But, using the property that the Hamiltonian has the same value on each side, it turns out that these jumps on the gradient are actually transcendentally small. The resulting global approximation fails to be an exact solution to the problem by just a transcendentally small factor. To conclude, we perturb it to a genuine solution by obtaining a rough positive lower bound for the spectrum of the associated linearized operator about it.

The proof of Theorem 1.2 again relies on the nondegeneracy of the solution U, and on the blow-up profile (V_1, V_2) in the symmetric class. More attention is paid on the interaction between these two separate problems.

1.4. **Physical Motivation.** A rotating two component Bose-Einstein condensate is the ground state of the following energy

$$
\begin{aligned}
E(u_1,u_2) = \sum_{j=1}^{2}\int_{\mathbb{R}^2}\left\{\frac{|\nabla u_j|^2}{2} + \frac{V(|x|)}{2\varepsilon^2}|u_j|^2 + \frac{g_j}{4\varepsilon^2}|u_j|^4 - \Omega x^\perp\cdot(iu_j,\nabla u_j)\right\}dx \\
+\frac{g}{2\varepsilon^2}\int_{\mathbb{R}^2}|u_1|^2|u_2|^2\,dx
\end{aligned}
\tag{1.18}
$$

in the space

$$
\mathcal{H} = \left\{(u_1,u_2):\ u_j\in H^1(\mathbb{R}^2,\mathbb{C}),\ \int_{\mathbb{R}^2}|x|^2|u_j|^2\,dx<\infty,\ \|u_j\|_{L^2(\mathbb{R}^2)}=1,\ j=1,2\right\}. \tag{1.19}
$$

The trapping potential $V(|x|)$ is usually taken to be $|x|^2$, corresponding to the experiments. The parameters g_1, g_2, g, ε and Ω are positive: g_j is the self interaction of each component (intracomponent coupling) while g measures the effect of interaction between the two components (intercomponent coupling); Ω is the angular velocity corresponding to the rotation of the condensate, $x^\perp = (-x_2, x_1)$ and $\cdot$ is the scalar product for vectors, whereas $(\ ,\ )$ is the complex scalar product, so that we have

$$
x^\perp\cdot(iu,\nabla u) = x^\perp\cdot\frac{iu\nabla\bar{u}-i\bar{u}\nabla u}{2} = -x_2\frac{iu\partial_{x_1}\bar{u}-i\bar{u}\partial_{x_1}u}{2} + x_1\frac{iu\partial_{x_2}\bar{u}-i\bar{u}\partial_{x_2}u}{2}.
$$

The existence and behavior of the minimizers in the limit when ε is small, describing strong interactions, is also called the Thomas-Fermi limit. If the condition $g^2 < g_1g_2$ is satisfied, it means that the two components u_1 and u_2 of the minimizers can coexist, as opposed to the segregation case $g^2 > g_1g_2$. This is discussed and explained in [1, 19]. The ground state at $\Omega = 0$ in the coexistence case has been studied in the small ε limit in [1]. In this paper, we are interested in the segregation case. The Γ limit of (1.18) in the case where $g^2 > g_1g_2$ and $g = g_\varepsilon \to \infty$ (that is Λ large in our notations) has been studied in [2]. A change of functions is used, namely (v,φ), where

$$
v^2 = u_1^2 + u_2^2 \text{ and } \cos\varphi = \frac{u_1^2-u_2^2}{u_1^2+u_2^2}.
$$

A Γ limit is obtained on the functional for (v,φ) and links can be made with the results in [7, 8]. The limiting problem is given by two domains D_1 (for component 1) and D_2 (for component 2) for which the interface minimizes a perimeter type problem weighted by the trapping potential.

Further results include the Γ limit proved recently by [14]: in particular, in Theorem 1.8, they prove that the interface between the two components of the condensate is governed by the problem

$$-\varepsilon^2 u_1'' + g_1 u_1^2 u_1 + g u_2^2 u_1 = \lambda_1 u_1,$$
$$-\varepsilon^2 u_2'' + g_2 u_2^2 u_2 + g u_1^2 u_2 = \lambda_2 u_2,$$

which, in the case $g_1 = g_2$ and $g = \Lambda$ large, gives rise to our one dimensional system.

The basic interaction of a two component condensate is through a modulus term, but other interactions include a Rabi coupling or a spin orbit coupling. The precise knowledge of the interface behaviour will prove useful to analyze more complicated patterns:

- the segregation case in the spin orbit coupling where there are vortex sheets [17],
- in the case of Rabi coupling, a vortex and anti-vortex pair create a vortex molecule, where two vortices are eventually connected by a domain wall of relative phase [18, 24],
- half vortices, where a vortex in one component corresponds to a peak in the other component [11].

In order to analyze the singularity patterns in all these cases, one needs to make an energy expansion in order to determine the energy of the specific configuration. Because of the transition layer between the two species, one needs to have a precise estimate of the decrease of the modulus of the wave functions, which is precisely given by our main theorem. Therefore, the approximation that we prove in Theorem 1.1 is expected to be extremely useful for the construction of upper bounds for these patterns. The principal significance of Theorem 1.2 is that it allows for the heteroclinic connection to be used as a building block for possible constructions of solutions corresponding to critical points of the energy E.

1.5. **Notation.** By c/C, we will denote small/large positive generic constants that are independent of large $\Lambda > 0$ and whose value will decrease/increase as the paper moves on. A number ρ will be of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$ if $\rho = \mathcal{O}(\Lambda^{-m})$, for any $m > 1$, as $\Lambda \to \infty$.

2. The approximate solution

In this section, we will construct a sufficiently good approximate solution to problem (1.1)-(1.2), for large $\Lambda > 0$, satisfying the symmetry condition (1.3). In fact, with the exception of Propositions 1.1, 2.1 and Section 7 or unless specified otherwise, all the two-component functions in this article will satisfy this symmetry; therefore, we may consider them only for $z \geq 0$.

2.1. **The outer solution** $(v_{1,out}, v_{2,out})$**.** In this subsection, we will construct an approximate solution to problem (1.1)-(1.2) except at the origin, where it looses its smoothness.

2.1.1. *The outer profile* U. The building block of this construction will be the unique solution U of problem (1.5). For future reference, let us note that

$$U'(z) > 0, \ z \geq 0, \tag{2.1}$$

and

$$1 - U(z) + U'(z) - U''(z) \le Ce^{-cz}, \ z \ge 0. \tag{2.2}$$

We will also need the following properties for the associated linearized operator, which are well known and essentially follow from (2.1)-(2.2).

Lemma 2.1. *Let $\phi \in C^2[0,\infty)$ be bounded and satisfy*

$$-\phi'' + \left(3U^2(z) - 1\right)\phi = \mu\phi, \quad z > 0, \tag{2.3}$$

for some constant $\mu \le 0$. Then,

- *If $\phi(0) = 0$, then $\phi \equiv 0$.*
- *If $\phi'(0) = 0$, then $\phi \equiv cU'$ for some $c \in \mathbb{R}$.*

Proof. We only present the main ideas of the proof. Suppose that ϕ is nontrivial. We will first show that $\mu = 0$. If not, by virtue of (2.1) and the relation

$$\mu \int_{-\infty}^{\infty} \phi U' dz = 0,$$

which follows by testing (2.3) with U' and using (2.2), we deduce that ϕ has to change sign. Since $(3U^2(z) - 1) \to 2$ as $z \to \infty$, by Sturm's theorem and without loss of generality, we may assume that there exists a $z_0 \ge 0$ such that $\phi(z_0) = 0$ and $\phi(z) > 0$ for $z > z_0$. Then, we can multiply the equation by U', integrate by parts the resulting identity over (z_0, ∞), and make use of (2.1)-(2.2) to find that $\phi'(z_0) \le 0$. We have thus reached a contradiction to our assumption that $\mu < 0$. Now that we know that $\mu = 0$, the desired assertions of the lemma follow easily from the observation that, besides of U', the differential operator in the lefthand side of (2.3) also has an unbounded function in its two-dimensional kernel (otherwise the Wroskian would be zero, by (2.2)). □

2.1.2. *The construction of the outer approximate solution.* We can now define our outer approximate solution as

$$v_{1,out}(z) = U(z+\xi) + \tau U'(z+\xi), \ v_{2,out}(z) = 0 \text{ for } z \ge (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \tag{2.4}$$

with

$$\xi = \mathcal{O}(\Lambda^{-\frac{1}{4}}) \text{ and } \tau = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \text{ as } \Lambda \to \infty, \tag{2.5}$$

to be determined. We point out that the choice of the power $1/4$ in (2.4) is motivated by a formal blow-up analysis (see the next subsection), whereas the choice of powers in (2.5) is an a-posteriori result of matching considerations (see Subsection 3.1 below).

2.1.3. *The remainder of the outer approximate solution.* We note that $(v_{1,out}, v_{2,out})$ satisfies the desired asymptotic behavior (1.2) exactly, while it satisfies the system (1.1) approximately as is shown in the next lemma.

Lemma 2.2. *The remainder*

$$R(v_{1,out}, v_{2,out}) = \begin{pmatrix} -v''_{1,out} + v^3_{1,out} - v_{1,out} + \Lambda v^2_{2,out} v_{1,out} \\ \\ -v''_{2,out} + v^3_{2,out} - v_{2,out} + \Lambda v^2_{1,out} v_{2,out} \end{pmatrix}$$

that is left by $(v_{1,out}, v_{2,out})$ *in (1.1) satisfies*

$$R(v_{1,out}, v_{2,out}) = \left(\begin{array}{c} \mathcal{O}(\Lambda^{-\frac{3}{2}})e^{-cz} \\ \\ 0 \end{array} \right),$$

uniformly in $\left[(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)$, *as* $\Lambda \to \infty$.

Proof. Keeping in mind that $v_{2,out}$ is identically zero, we observe that, by virtue of (1.5), we have

$$-v''_{1,out} + v^3_{1,out} - v_{1,out} = \tau^3 \left(U'(z+\xi)\right)^3 + 3\tau^2 U(z+\xi) \left(U'(z+\xi)\right)^2,$$

and then use (2.2), (2.5). □

2.2. **The inner approximate solution** $(v_{1,in}, v_{2,in})$**.** In this subsection, we will construct an approximate solution to the system (1.1) which, however, is effective only in a small neighborhood of the origin. Nevertheless, it will have the appropriate behavior so as to be easily "continued" away from the origin by the outer solution of the previous subsection.

2.2.1. *The blow-up profile* (V_1, V_2)*.* Based on a formal blow-up analysis and the behavior of the outer approximate solution near the origin, the building blocks will be special solutions of a limiting problem, described in Proposition 1.1 which is due to [7, 8].

In fact, it follows from the analysis in [7] that

$$V_1(x) = \psi_0 x + \kappa + \mathcal{O}\left(e^{-cx^2}\right) \text{ and } V_2(x) = \mathcal{O}\left(e^{-cx^2}\right) \text{ as } x \to \infty, \tag{2.6}$$

for some

$$\kappa \geq 0,$$

and these relations can be differentiated arbitrarily many times. Moreover, it follows easily that

$$\dot{V}_1 > 0, \;\; \dot{V}_2 < 0, \;\; x \in \mathbb{R}. \tag{2.7}$$

The invariance of system (1.8) under translation and scaling, described in (1.11), implies that the associated linearized operator

$$L \left(\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right) = \left(\begin{array}{c} -\ddot{\Phi}_1 + V_2^2\Phi_1 + 2V_1V_2\Phi_2 \\ -\ddot{\Phi}_2 + V_1^2\Phi_2 + 2V_1V_2\Phi_1 \end{array} \right) \tag{2.8}$$

has

$$(\dot{V}_1, \dot{V}_2) \;\; \text{and} \;\; (x\dot{V}_1 + V_1, x\dot{V}_2 + V_2) \tag{2.9}$$

amongst its four-dimensional kernel. The next proposition, also proven in [7], will play a key role in what will follow.

Proposition 2.1. *If* $\Phi_1, \Phi_2 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfy*

$$L \left(\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right),$$

then

$$(\Phi_1, \Phi_2) \equiv \lambda(\dot{V}_1, \dot{V}_2)$$

for some $\lambda \in \mathbb{R}$.

2.2.2. *Construction of the inner approximate solution.* Motivated from the above and [7], we consider the stretched variable

$$x = \mu \Lambda^{\frac{1}{4}} z \tag{2.10}$$

with

$$\mu = 1 + \mathcal{O}(\Lambda^{-\frac{1}{2}}) \text{ as } \Lambda \to \infty \tag{2.11}$$

to be determined (the last relation is an a-posteriori consequence of matching considerations, see Subsection 3.1 below). Then, we seek an inner approximate solution $(v_{1,in}, v_{2,in})$ to (1.1) in the form

$$v_{i,in}(z) = \mu \Lambda^{-\frac{1}{4}} V_i(x) + \Phi_i(x), \quad |z| \leq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \quad i = 1, 2, \tag{2.12}$$

where the functions Φ_1, Φ_2 are also to be determined. Using (1.8), we find that the remainder which is left in the first equation of (1.1) by this approximation is

$$\begin{aligned} &-\mu^2 \Lambda^{\frac{1}{2}} \ddot{\Phi}_1 + \mu^3 \Lambda^{-\frac{3}{4}} V_1^3 + \Phi_1^3 + 3\mu^2 \Lambda^{-\frac{1}{2}} V_1^2 \Phi_1 + 3\mu \Lambda^{-\frac{1}{4}} V_1 \Phi_1^2 + \mu^2 \Lambda^{\frac{1}{2}} V_2^2 \Phi_1 \\ &+\mu \Lambda^{\frac{3}{4}} \Phi_2^2 V_1 + \Lambda \Phi_2^2 \Phi_1 + 2\mu^2 \Lambda^{\frac{1}{2}} V_1 V_2 \Phi_2 + 2\mu \Lambda^{\frac{3}{4}} V_2 \Phi_1 \Phi_2 - \mu \Lambda^{-\frac{1}{4}} V_1 - \Phi_1, \end{aligned} \tag{2.13}$$

and an analogous relation holds for the second equation. Hence, we would like for (Φ_1, Φ_2) to satisfy

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \mu^{-1} \Lambda^{-\frac{3}{4}} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

where the linear operator L is as in (2.8), for $x \in \mathbb{R}$ which is the natural domain of definition for Φ_1 and Φ_2.

In view of the righthand side of the above equation, (1.10) and (2.6), we are naturally led to seek (Φ_1, Φ_2) as

$$(\Phi_1, \Phi_2) = \mu^{-1} \Lambda^{-\frac{3}{4}} (Z_1, Z_2) + (\tilde{\Phi}_1, \tilde{\Phi}_2),$$

where Z_1, Z_2 are some smooth, fixed functions that satisfy the symmetry condition (1.3) and

$$Z_1(x) = -\psi_0 \frac{x^3}{6} - \kappa \frac{x^2}{2}, \; Z_2(x) = 0, \; x \geq 1. \tag{2.14}$$

Then, the fluctuation $(\tilde{\Phi}_1, \tilde{\Phi}_2)$ should satisfy

$$L \begin{pmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{pmatrix} = \mu^{-1} \Lambda^{-\frac{3}{4}} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{2.15}$$

for some fixed, smooth (F_1, F_2) which enjoys the same symmetry and

$$|F_1(x)| + |F_2(x)| \leq C e^{-cx^2}, \; x \in \mathbb{R}. \tag{2.16}$$

Since (1.10) implies that

$$\dot{V}_1(-x) = -\dot{V}_2(x), \quad x \in \mathbb{R}, \tag{2.17}$$

we know from Proposition 2.1 that the kernel of L has no nontrivial, bounded elements in the considered symmetry class (1.3). Unfortunately, this fact by itself does not seem to be enough for the development of a complete invertibility theory for L; we keep in mind that we only know two out of the four nontrivial elements of the kernel of L (in the general class which is, recall (2.9)). Nevertheless, we can use this fact in order to obtain a satisfactory estimate for the inhomogeneous problem (2.15). This is contained in the following proposition.

Proposition 2.2. *Given $\alpha > 0$, there exist $M_0, C > 0$ such that the boundary value problem*

$$L\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \ |x| < M; \quad \dot{\Phi}_i(\pm M) = 0, \quad i = 1, 2, \tag{2.18}$$

where $G_1, G_2 \in C[-M, M]$ satisfy

$$G_1(-x) = G_2(x), \quad -M \le x \le M, \tag{2.19}$$

has a unique solution such that

$$\Phi_1(-x) = \Phi_2(x), \quad -M \le x \le M, \tag{2.20}$$

and

$$\sum_{i=1}^{2} \|\Phi_i\|_{L^\infty(-M,M)} \le C \sum_{i=1}^{2} \|e^{\alpha|x|} G_i\|_{L^\infty(-M,M)},$$

provided that $M \ge M_0$.

Proof. Since the linear operator L is self-adjoint (in the natural Sobolev spaces associated to the boundary value problem), it suffices to verify the validity of the asserted a-priori estimate. To this end, we will argue by contradiction. So, let us suppose that there exist $M_n \to \infty$ as $n \to \infty$ and $\Phi_{i,n} \in C^2[-M_n, M_n]$, $G_{i,n} \in C[-M_n, M_n]$, $i = 1, 2$, satisfying (2.18)-(2.19)-(2.20) while

$$\sum_{i=1}^{2} \|\Phi_{i,n}\|_{L^\infty(-M_n,M_n)} = 1, \quad n \ge 1, \tag{2.21}$$

$$\sum_{i=1}^{2} \|e^{\alpha|x|} G_{i,n}\|_{L^\infty(-M_n,M_n)} \to 0 \ \text{ as } n \to \infty, \tag{2.22}$$

(for the above two relations, we also used (2.19)-(2.20)).

By standard elliptic estimates and the usual diagonal Cantor-type argument, passing to a subsequence if necessary, we find that

$$\Phi_{i,n} \to \Phi_{i,\infty} \ \text{ in } \ C^2_{loc}(\mathbb{R}) \ \text{ as } n \to \infty, \quad i = 1, 2,$$

where $\Phi_{1,\infty}, \Phi_{2,\infty}$ satisfy

$$L\begin{pmatrix} \Phi_{1,\infty} \\ \Phi_{2,\infty} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Phi_{1,\infty}(-x) = \Phi_{2,\infty}(x), \quad x \in \mathbb{R}, \quad \text{and} \quad \sum_{i=1}^{2} \|\Phi_{i,\infty}\|_{L^\infty(\mathbb{R})} \le 1.$$

So, by Proposition 2.1 and (2.17), we infer that $\Phi_{i,\infty} \equiv 0$, $i = 1, 2$, that is

$$\Phi_{i,n} \to 0 \ \text{ in } \ C^2_{loc}(\mathbb{R}) \ \text{ as } n \to \infty, \quad i = 1, 2. \tag{2.23}$$

To conclude, we will show that

$$|\Phi_{1,n}(x)| \le Ce^{-c|x|} + o(1), \quad \text{uniformly on } [-M_n, M_n], \tag{2.24}$$

for some fixed constants $c, C > 0$, as $n \to \infty$, which clearly comes into contradiction with (2.20), (2.21) and (2.23). However, this task will require some work and will take up the rest of the proof.

Firstly, we note that (1.10), (2.6), (2.21) and (2.22) imply that

$$\left| -\ddot{\Phi}_{1,n} + V_2^2(x)\Phi_{1,n} \right| \le Ce^{-\alpha|x|}, \quad x \in [-M_n, M_n], \tag{2.25}$$

where $C>0$ is independent of $n\geq 1$. Since

$$V_2^2(x)\geq V_2^2(0)>0 \ \text{ for } \ x\leq 0 \ \text{ (recall (2.7))},$$

by means of a standard barrier argument (similarly to [9, Lem. 2]), we deduce that

$$|\Phi_{1,n}(x)|\leq \|\Phi_{1,n}\|_{L^\infty(-M_n,M_n)}e^{cx} \overset{(2.21)}{\leq} e^{cx}, \quad x\in[-M_n,0], \tag{2.26}$$

for some $c>0$ which is independent of $n\geq 1$. In particular, this implies the validity of the desired estimate (2.24) on $[-M_n,0]$. By integrating the first equation of $(2.18)_n$, and using the Neumann boundary condition at $-M_n$, we obtain that

$$\dot{\Phi}_{1,n}(x)=\int_{-M_n}^{x}\left[V_2^2\Phi_{1,n}+2V_1V_2\Phi_{2,n}-G_{1,n}\right]dy, \quad x\in[-M_n,M_n]. \tag{2.27}$$

For future reference, we note that the Neumann boundary condition at M_n implies that

$$\int_{-M_n}^{M_n}\left[V_2^2\Phi_{1,n}+2V_1V_2\Phi_{2,n}-G_{1,n}\right]dy=0. \tag{2.28}$$

Integrating (2.27) by parts, it follows readily that

$$\begin{aligned}\Phi_{1,n}(x) \ &= \ \Phi_{1,n}(0)-\textstyle\int_0^x y\left[V_2^2\Phi_{1,n}+2V_1V_2\Phi_{2,n}-G_{1,n}\right]dy\\ &\quad +x\textstyle\int_{-M_n}^{x}\left[V_2^2\Phi_{1,n}+2V_1V_2\Phi_{2,n}-G_{1,n}\right]dy,\end{aligned}$$

for $x\in[-M_n,M_n]$. In particular, thanks to (2.28), we have

$$\Phi_{1,n}(M_n)=\Phi_{1,n}(0)-\int_0^{M_n}y\left[V_2^2\Phi_{1,n}+2V_1V_2\Phi_{2,n}-G_{1,n}\right]dy.$$

Then, by (1.10), (2.6), (2.20), (2.21), (2.22), (2.23), (2.26), and Lebesgue's dominated convergence theorem, we deduce that

$$\Phi_{1,n}(M_n)\to 0 \ \text{ as } \ n\to\infty. \tag{2.29}$$

On the other hand, from (2.6), (2.21), (2.22) and (2.25), we get that

$$\left|\ddot{\Phi}_{1,n}(x)\right|\leq Ce^{-\alpha x}, \quad x\in[0,M_n],$$

for some $C>0$ independent of $n\geq 1$. In turn, integrating twice the above relation, making use of the Neumann boundary condition at M_n and of (2.29), we infer that

$$|\Phi_{1,n}(x)|\leq Ce^{-cx}+o(1),$$

uniformly on $[0,M_n]$, for some fixed $c,C>0$ (depending only on α), as $n\to\infty$, thus completing the proof of the desired estimate (2.24) as well as that of the proposition. □

Corollary 2.1. *Under the assumptions of Proposition 2.2, we have*

$$\sum_{i=1}^{2}\|\dot{\Phi}_i\|_{L^\infty(-M,M)}\leq C\sum_{i=1}^{2}\|e^{\alpha|x|}G_i\|_{L^\infty(-M,M)},$$

for $M\geq M_0$.

Proof. In view of the assertion of Proposition 2.2, the asymptotic behavior of V_1, V_2, and the analogous exponential decay estimate to (2.26), we find that

$$\sum_{i=1}^{2}\|\ddot{\Phi}_i\|_{L^\infty(-M,M)} \leq C\sum_{i=1}^{2}\|e^{\alpha|x|}G_i\|_{L^\infty(-M,M)}.$$

The desired estimate now follows by plainly interpolating between the assertion of Proposition 2.2 and the above estimate, for example using the elementary inequality

$$\|\dot{\Phi}\|_{L^\infty(-M,M)} \leq 2\|\Phi\|_{L^\infty(-M,M)} + \|\ddot{\Phi}\|_{L^\infty(-M,M)}$$

and the symmetry (2.20). □

Applying Proposition 2.2 to the case where the righthand side of (2.18) is (F_1, F_2), as defined through (2.15) (which is independent of Λ), and letting $M \to \infty$, we obtain the existence of a solution $(\hat{\Phi}_1, \hat{\Phi}_2)$, with bounded components (recall (2.16)), to the system

$$L\begin{pmatrix}\hat{\Phi}_1\\ \hat{\Phi}_2\end{pmatrix} = \begin{pmatrix}F_1\\ F_2\end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.30}$$

such that

$$\hat{\Phi}_1(-x) = \hat{\Phi}_2(x), \quad x \in \mathbb{R}. \tag{2.31}$$

This is indeed possible by standard elliptic estimates and the usual diagonal argument. In fact, by virtue of Proposition 2.1 and (2.17), such a solution is unique. We observe that $\hat{\Phi}_1$ satisfies

$$-\frac{d^2}{dx^2}\hat{\Phi}_1 + V_2^2\hat{\Phi}_1 = \mathcal{O}\left(e^{-cx^2}\right) \quad \text{as} \quad |x| \to \infty.$$

Then, the asymptotic behavior of V_2 from (1.10), (2.6), yields that there exists an $a \in \mathbb{R}$ such that, for any $D > 0$, we have

$$\hat{\Phi}_1(x) = \mathcal{O}(e^{Dx}) \text{ as } x \to -\infty, \quad \hat{\Phi}_1(x) = a + \mathcal{O}(e^{-Dx}) \text{ as } x \to \infty, \tag{2.32}$$

and these relations can be differentiated arbitrary many times (for the last relation, see also [3, Lem. 3.2]). Of course, thanks to the symmetry (2.31), the asymptotic behavior of $\hat{\Phi}_2$ follows immediately.

This allows us to improve our inner approximate solution (2.12):

Definition 1. *We define the inner approximate solution to (1.1) as $(v_{1,in}, v_{2,in})$, with*

$$v_{i,in}(z) = \mu\Lambda^{-\frac{1}{4}}V_i(x) + \mu^{-1}\Lambda^{-\frac{3}{4}}\left[Z_i(x) + \hat{\Phi}_i(x)\right] + BE_i(x), \quad |z| \leq (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \ i = 1, 2, \tag{2.33}$$

where x is the stretched variable (2.10), $V_i, Z_i, \hat{\Phi}_i$ are defined in Proposition 1.1, (2.14), (2.30) respectively and

$$(E_1, E_2) = (x\dot{V}_1 + V_1, x\dot{V}_2 + V_2) \tag{2.34}$$

denotes the aforementioned element of the kernel of L with the desired symmetry. The constants μ and B will be determined later, subject to the constraints (2.11) and

$$B = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \quad \text{as} \quad \Lambda \to \infty, \tag{2.35}$$

respectively.

2.2.3. *The remainder of the inner approximate solution.* In view of (2.10), (2.11), (2.13), (2.35), and the construction of $(\hat{\Phi}_1, \hat{\Phi}_2)$, we have the validity of the following lemma.

Lemma 2.3. *In equation (1.1), the remainder*

$$R(v_{1,in}, v_{2,in}) = \begin{pmatrix} -v''_{1,in} + v_{1,in}^3 - v_{1,in} + \Lambda v_{2,in}^2 v_{1,in} \\ \\ -v''_{2,in} + v_{2,in}^3 - v_{2,in} + \Lambda v_{1,in}^2 v_{2,in} \end{pmatrix}$$

which is left by the solution $(v_{1,in}, v_{2,in})$ *of Definition 1 satisfies*

$$R(v_{1,in}, v_{2,in}) = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \begin{pmatrix} \Lambda^{\frac{3}{4}} z^3 + 1 \\ \\ e^{-D\Lambda^{\frac{1}{4}} z} \end{pmatrix},$$

for any $D \geq 1$, *uniformly on* $\left[0, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right]$, *as* $\Lambda \to \infty$.

3. Matching the outer and inner approximate solutions

In this section, we will adjust the parameters ξ, τ, μ, B in the definitions of the outer and inner approximate solutions in (2.4) and (2.33), subject to the constraints (2.5), (2.11) and (2.35), so that the aforementioned approximations are equal at leading order at the points $z_\pm = \pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}$. By the symmetry property, it is enough to do this at the point z_+. Note that the Hamiltonian on the inner approximate solution at the point $z = 0$ is equal to $-\psi_0^2/2$, which is exactly the Hamiltonian constant of the heteroclinic connection we expect from (1.7) and the comments related to (1.4). In some sense, this is equivalent to matching the solutions in a C^1 fashion at the points $\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}$.

3.1. **Matching** $(v_{1,out}, v_{2,out})$ **and** $(v_{1,in}, v_{2,in})$ **continuously at** $\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}$**.** In view of (1.7), (2.4), (2.5) and the facts that

$$U''(0) = 0, \quad U'''(0) = -\psi_0, \quad U^{(4)}(0) = 0, \tag{3.1}$$

we find that

$$\begin{aligned} v_{1,out}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) &= \psi_0\left((\ln \Lambda)\Lambda^{-\frac{1}{4}} + \xi\right) - \tfrac{\psi_0}{6}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}} + \xi\right)^3 \\ &\quad + \mathcal{O}\left((\ln \Lambda)^5 \Lambda^{-\frac{5}{4}}\right) + \tau\psi_0 + \tau\mathcal{O}\left((\ln \Lambda)^2 \Lambda^{-\frac{1}{2}}\right) \end{aligned}$$

as $\Lambda \to \infty$, where the quantities in the Landau symbols are independent of τ. In turn, by expanding, we get that

$$\begin{aligned} v_{1,out}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) &= \psi_0\xi + \psi_0(\ln \Lambda)\Lambda^{-\frac{1}{4}} + \tau\psi_0 - \tfrac{\psi_0}{6}\xi^3 - \tfrac{\psi_0}{2}(\ln \Lambda)\Lambda^{-\frac{1}{4}}\xi^2 \\ &\quad - \tfrac{\psi_0}{6}(\ln \Lambda)^3\Lambda^{-\frac{3}{4}} - \tfrac{\psi_0}{2}(\ln \Lambda)^2\Lambda^{-\frac{1}{2}}\xi \\ &\quad + \mathcal{O}\left((\ln \Lambda)^5 \Lambda^{-\frac{5}{4}}\right) + \tau\mathcal{O}\left((\ln \Lambda)^2 \Lambda^{-\frac{1}{2}}\right) \end{aligned}$$

as $\Lambda \to \infty$.

On the other side, from (2.6), (2.14), (2.32) and (2.33), we obtain that

$$v_{1,in}\left((\ln\Lambda)\Lambda^{-\frac14}\right) = \mu^2\psi_0(\ln\Lambda)\Lambda^{-\frac14}+\mu\kappa\Lambda^{-\frac14}+B\kappa+2B\psi_0\mu(\ln\Lambda)+\mu^{-1}\Lambda^{-\frac34}a$$

$$-\tfrac{\psi_0}{6}\mu^2(\ln\Lambda)^3\Lambda^{-\frac34}-\tfrac12\kappa\mu(\ln\Lambda)^2\Lambda^{-\frac34}+\mathcal{O}(\Lambda^{-\infty})$$

as $\Lambda\to\infty$.

The first thing that comes to mind is to take $\mu=1$, $\xi=\psi_0^{-1}\kappa\Lambda^{-\frac14}$, $B=-\frac14\Lambda^{-\frac14}\xi^2$ and then choose $\tau=\frac16\xi^3+\psi_0^{-1}B\kappa+\psi_0^{-1}\Lambda^{-\frac34}a$ to take care of the rest. Actually, this amounts to matching $v_{1,in}$ with $v_{1,out}$ in the C^1 sense over an intermediate zone. However, it is more convenient to just match them continuously at the point $(\ln\Lambda)\Lambda^{-\frac14}$ and to exploit the Hamiltonian structure of the system. The main reasons for this are the following. The first one has to do with the extension of the current approach to the general nonsymmetric case. There, it is not clear how to establish the existence of a solution to the linear inhomogeneous problem corresponding to (2.30) with 'controlled' asymptotic behaviour as $x\to\pm\infty$. Secondly, this type of matching leads us naturally to building a solution of (1.1)-(1.2) in the same spirit, that is by constructing separately inner and outer genuine solutions which match continuously at $\pm(\ln\Lambda)\Lambda^{-\frac14}$. In particular, the latter strategy allows us to use directly the first assertion of Lemma 2.1 and that of Proposition 2.1 for this purpose, as we did for the construction of the approximate solutions.

Given B satisfying (2.35), we take

$$\mu=1,\quad \xi=\psi_0^{-1}\kappa\Lambda^{-\frac14}+2B(\ln\Lambda)+\frac12(\ln\Lambda)\Lambda^{-\frac14}\xi^2, \tag{3.2}$$

which is indeed possible by the implicit function theorem for Λ large (we can even find an explicit formula for ξ by solving the above trinomial). In turn, we choose

$$\tau=\frac{\xi^3}{6}+\psi_0^{-1}B\kappa+\psi_0^{-1}a\Lambda^{-\frac34}.$$

Then, using that

$$\xi=\psi_0^{-1}\kappa\Lambda^{-\frac14}+\mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac34}\right)\quad\text{as}\quad\Lambda\to\infty,$$

it follows readily that

$$(v_{1,out}-v_{1,in})\left((\ln\Lambda)\Lambda^{-\frac14}\right)=\mathcal{O}\left((\ln\Lambda)^5\Lambda^{-\frac54}\right)\quad\text{as}\quad\Lambda\to\infty. \tag{3.3}$$

We have

$$(v_{2,out}-v_{2,in})\left((\ln\Lambda)\Lambda^{-\frac14}\right)=-v_{2,in}\left((\ln\Lambda)\Lambda^{-\frac14}\right)=\mathcal{O}\left(\Lambda^{-\infty}\right)\quad\text{as}\quad\Lambda\to\infty. \tag{3.4}$$

3.2. **Adjusting the value of the Hamiltonian on the inner approximate solution at $z=0$.** In this subsection, we will choose B, under the constraint (2.35), so that the value of the Hamiltonian on $(v_{1,in},v_{2,in})$ at $z=0$ is equal to the Hamiltonian constant of the expected heteroclinic connection, namely $-\psi_0^2/2$.

Firstly, from (2.10), (2.33) and (2.35), we note that

$$\left[v_{1,in}'(0)\right]^2=\left[\dot V_1(0)\right]^2+2\dot V_1(0)\left[O(\Lambda^{-\frac12})+2\dot V_1(0)B\Lambda^{\frac14}\right]+\mathcal{O}(\Lambda^{-1})$$

as $\Lambda \to \infty$, with $O(\Lambda^{-\frac{1}{2}})$ being independent of B. Furthermore, it is clear that

$$v_{1,in}^4(0) = \Lambda^{-1}V_1^4(0) + \mathcal{O}(\Lambda^{-\frac{1}{2}})B^2 + 4\Lambda^{-\frac{3}{4}}V_1^4(0)B + O(\Lambda^{-\frac{3}{2}}) \ \text{ as } \Lambda \to \infty,$$

where $O(\Lambda^{-\frac{3}{2}})$ is independent of B, and that

$$\frac{\left(1 - v_{1,in}^2(0)\right)^2}{4} = \frac{1}{4} + O(\Lambda^{-\frac{1}{2}}) + \mathcal{O}(\Lambda^{-\frac{1}{4}})B \overset{(1.7)}{=} \frac{\psi_0^2}{2} + O(\Lambda^{-\frac{1}{2}}) + \mathcal{O}(\Lambda^{-\frac{1}{4}})B$$

as $\Lambda \to \infty$, where $O(\Lambda^{-\frac{1}{2}})$ is independent of B. Now, using that $V_1(0) = V_2(0)$, $\dot{V}_1(0) = -\dot{V}_2(0)$ and

$$\dot{V}_1^2 + \dot{V}_2^2 - V_1^2V_2^2 = \psi_0^2, \quad x \in \mathbb{R},$$

it follows readily that the equation

$$H(v_{1,in}(0), v_{2,in}(0)) = -\frac{\psi_0^2}{2} \ \text{ (with the obvious notation, keep in mind (1.4))}$$

takes the form

$$2\psi_0^2 B\Lambda^{\frac{1}{4}} = O(\Lambda^{-\frac{1}{2}}) + \mathcal{O}(\Lambda^{-\frac{1}{4}})B + \mathcal{O}(\Lambda^{-1}) \ \text{ as } \Lambda \to \infty,$$

which has a unique solution

$$B = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \ \text{ as } \Lambda \to \infty, \tag{3.5}$$

as desired (keep in mind that, according to our notation, the term $O(\Lambda^{-\frac{1}{2}})$ above does not contain B).

3.3. **A refined inner approximate solution** $(w_{1,in}, w_{2,in})$**.** For $\tilde{B} \in \mathbb{R}$ to be chosen later, subject to the constraint

$$\tilde{B} = \mathcal{O}(\Lambda^{-1}) \ \text{ as } \Lambda \to \infty, \tag{3.6}$$

we consider the more refined inner approximate solution $(w_{1,in}, w_{2,in})$

$$w_{i,in}(z) = v_{i,in}(z) + \tilde{B}E_i(x), \quad |z| \leq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \ i = 1, 2, \tag{3.7}$$

where $(v_{1,in}, v_{2,in})$ is defined in Definition 1 and E_i comes from (2.34). Actually, $\tilde{B}$ will turn out to be chosen much smaller than in (3.6).

It is easy to see that the assertion of Lemma 2.3, concerning the remainder of this refined inner solution, continues to hold for $(w_{1,in}, w_{2,in})$.

4. Solution of the inner problem

In this section, we will show that the one-parameter family of refined inner approximate solutions $(w_{1,in}, w_{2,in})$, described in the previous section (parameterized by $\tilde{B}$), can be perturbed smoothly to a one-parameter family of inner genuine solutions to the system (1.1), for large $\Lambda > 0$. Then, we will show that there exists at least one value of $\tilde{B}$, in the range (3.6), for which the corresponding inner genuine solution to (1.1) has a Hamiltonian constant equal to $-\psi_0^2/2$.

4.1. **The perturbation argument.** Given $\tilde{B}$ satisfying (3.6), we seek a solution of system (1.1) as

$$(\mathbf{v}_{1,in}, \mathbf{v}_{2,in}) = (w_{1,in}, w_{2,in}) + (\varphi_1, \varphi_2), \quad |z| \le (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \tag{4.1}$$

with

$$\varphi_i'\left(\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad i = 1, 2,$$

and

$$\varphi_1(-z) = \varphi_2(z), \quad |z| \le (\ln\Lambda)\Lambda^{-\frac{1}{4}}.$$

After rearranging terms, we find that (φ_1, φ_2) has to satisfy

$$\begin{cases} \mathcal{L}(\varphi_1, \varphi_2) = -R(w_{1,in}, w_{2,in}) - Q(\varphi_1, \varphi_2) - N(\varphi_1, \varphi_2), \\ \\ \varphi_i'\left(\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad i = 1, 2, \end{cases} \tag{4.2}$$

where

$$\mathcal{L}(\varphi_1, \varphi_2) = \begin{pmatrix} -\varphi_1'' + \Lambda^{\frac{1}{2}} V_2^2(x)\varphi_1 + 2\Lambda^{\frac{1}{2}} V_1(x)V_2(x)\varphi_2 \\ \\ -\varphi_2'' + \Lambda^{\frac{1}{2}} V_1^2(x)\varphi_2 + 2\Lambda^{\frac{1}{2}} V_1(x)V_2(x)\varphi_1 \end{pmatrix},$$

the term $R(w_{1,in}, w_{2,in})$ denotes the remainder which is left by $(w_{1,in}, w_{2,in})$ in (1.1) (analogously to Lemma 2.3),

$$Q(\varphi_1, \varphi_2) =$$

$$\begin{pmatrix} (3w_{1,in}^2 - 1)\varphi_1 + \Lambda\left(w_{2,in}^2 - \Lambda^{-\frac{1}{2}}V_2^2\right)\varphi_1 + 2\Lambda\left(w_{1,in}w_{2,in} - \Lambda^{-\frac{1}{2}}V_1V_2\right)\varphi_2 \\ \\ (3w_{2,in}^2 - 1)\varphi_2 + \Lambda\left(w_{1,in}^2 - \Lambda^{-\frac{1}{2}}V_1^2\right)\varphi_2 + 2\Lambda\left(w_{1,in}w_{2,in} - \Lambda^{-\frac{1}{2}}V_1V_2\right)\varphi_1 \end{pmatrix}$$

and

$$N(\varphi_1, \varphi_2) = \begin{pmatrix} \varphi_1^3 + 3w_{1,in}\varphi_1^2 + \Lambda w_{1,in}\varphi_2^2 + \Lambda\varphi_2^2\varphi_1 + 2\Lambda w_{2,in}\varphi_1\varphi_2 \\ \\ \varphi_2^3 + 3w_{2,in}\varphi_2^2 + \Lambda w_{2,in}\varphi_1^2 + \Lambda\varphi_1^2\varphi_2 + 2\Lambda w_{1,in}\varphi_1\varphi_2 \end{pmatrix}.$$

Concerning the linear operator $\mathcal{L}$, we observe that Proposition 2.2 and Corollary 2.1, after a simple re-scaling (recall that $x = \Lambda^{\frac{1}{4}}z$), yield the following.

Corollary 4.1. *Given $\alpha > 0$, there exist $\Lambda_0, C > 0$ such that the boundary value problem*

$$\mathcal{L}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad |z| < (\ln\Lambda)\Lambda^{-\frac{1}{4}}; \quad \varphi_i'\left(\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \ i = 1, 2,$$

where $g_1, g_2 \in C\left[-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]$ with

$$g_1(-z) = g_2(z) \quad \textit{for} \quad |z| \le (\ln\Lambda)\Lambda^{-\frac{1}{4}},$$

has a unique solution such that

$$\varphi_1(-z) = \varphi_2(z) \quad \textit{for} \quad |z| \le (\ln\Lambda)\Lambda^{-\frac{1}{4}}$$

and

$$\sum_{i=1}^{2}\left(\Lambda^{-\frac{1}{4}}\|\varphi_i'\|_{L^\infty(I_\Lambda)} + \|\varphi_i\|_{L^\infty(I_\Lambda)}\right) \le C\Lambda^{-\frac{1}{2}+\alpha}\sum_{i=1}^{2}\|g_i\|_{L^\infty(I_\Lambda)},$$

where

$$I_\Lambda = \left(-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right),$$

provided that $\Lambda \geq \Lambda_0$.

The required estimates for the remainder $R(w_{1,in}, w_{2,in})$ hold as they follow from Lemma 2.3. Furthermore, using that

$$|v_{i,in}(z)| \leq C(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \quad |E_i(x)| \leq C(\ln\Lambda),$$

together with the easy to prove estimates

$$\left|v_{i,in}^2(z) - \Lambda^{-\frac{1}{2}}V_i^2(x)\right| + \left|v_{1,in}v_{2,in}(z) - \Lambda^{-\frac{1}{2}}V_1V_2(x)\right| \leq C(\ln\Lambda)^4\Lambda^{-\frac{5}{4}},$$

for $|z| \leq (\ln\Lambda)\Lambda^{-\frac{1}{4}}$, $i = 1, 2$, and (3.6), it follows readily that there exists $C > 0$ such that

$$\sum_{i=1}^{2} \|Q_i(\varphi_1, \varphi_2)\|_{L^\infty(I_\Lambda)} \leq C \sum_{i=1}^{2} \|\varphi_i\|_{L^\infty(I_\Lambda)}, \tag{4.3}$$

for any $\varphi_1, \varphi_2 \in C(\overline{I_\Lambda})$. Moreover, there exists a $C > 0$ such that

$$\|N_i(\varphi_1, \varphi_2)\|_{L^\infty(I_\Lambda)} \leq C \sum_{i=1}^{2} \left\{\|\varphi_i\|_{L^\infty}^3 + \Lambda^{\frac{3}{4}}(\ln\Lambda)\|\varphi_i\|_{L^\infty}^2 + \Lambda\|\varphi_i\|_{L^\infty}\|\varphi_{i+1}\|_{L^\infty}^2\right\}, \tag{4.4}$$

$i = 1, 2$, for any $\varphi_1, \varphi_2 \in C(\overline{I_\Lambda})$ (with the obvious notation), and

$$\begin{aligned} &\textstyle\sum_{i=1}^2 \|N_i(\varphi_1, \varphi_2) - N_i(\psi_1, \psi_2)\|_{L^\infty(I_\Lambda)} \leq \\ &C \textstyle\sum_{i=1}^2 \left\{\Lambda\left(\|\varphi_i\|_{L^\infty}^2 + \|\psi_i\|_{L^\infty}^2\right) + \Lambda^{\frac{3}{4}}(\ln\Lambda)\left(\|\varphi_i\|_{L^\infty} + \|\psi_i\|_{L^\infty}\right)\right\} \\ &\qquad \times \left(\textstyle\sum_{i=1}^2 \|\varphi_i - \psi_i\|_{L^\infty}\right), \end{aligned} \tag{4.5}$$

for any $\varphi_1, \varphi_2, \psi_1, \psi_2 \in C(\overline{I_\Lambda})$.

In view of the above, and paying attention to the dependence on $\tilde{B}$, a standard application of the contraction mapping principle yields the following.

Proposition 4.1. *Given* $\alpha \in (0, 1)$, *there exists* $C > 0$ *such that problem (4.2) has a unique solution satisfying*

$$\sum_{i=1}^{2} \left(\Lambda^{-\frac{1}{4}}\|\varphi_i'\|_{L^\infty(I_\Lambda)} + \|\varphi_i\|_{L^\infty(I_\Lambda)}\right) \leq C\Lambda^{-\frac{5}{4}+\alpha},$$

and $\varphi_1(-z) = \varphi_2(z)$, $z \in I_\Lambda$, *provided that* Λ *is sufficiently large. Moreover, this solution depends continuously, with respect to the* $C^1(\overline{I_\Lambda})$*-norm, on* $\tilde{B}$ *as in (3.6) (for fixed* Λ*).*

We point out that the aforementioned continuous dependence on $\tilde{B}$ can be proven easily as follows. Let $\tilde{B}_n$ satisfy (3.6), for fixed Λ as in the above proposition, and $\tilde{B}_n \to \tilde{B}_\infty$ as $n \to \infty$. We denote by $(\varphi_{1,n}, \varphi_{2,n})$ and $(\varphi_{1,\infty}, \varphi_{2,\infty})$ the solutions of (4.2) corresponding to $\tilde{B}_n$ and $\tilde{B}_\infty$ respectively, as provided by the first part of the above proposition. Then, thanks to Arzela-Ascoli's theorem, passing to a subsequence if necessary, and utilizing the uniqueness assertion of the aforementioned proposition, we find that $\varphi_{i,n} \to \varphi_{i,\infty}$ in $C^1(\overline{I_\Lambda})$ as $n \to \infty$, $i = 1, 2$. Finally, by employing once more the uniqueness property of $(\varphi_{1,\infty}, \varphi_{2,\infty})$, we deduce that the previous convergence holds for the original sequence.

4.1.1. *Some preliminary positivity and monotonicity properties of the inner genuine solution* $(\boldsymbol{v}_{1,in}, \boldsymbol{v}_{2,in})$. It is clear from the construction of the refined inner approximate solution $(w_{1,in}, w_{2,in})$ and Proposition 4.1 that, given any $L > 1$, there exists $c_L > 0$ such that

$$\mathbf{v}_{2,in} \geq c_L \Lambda^{-\frac{1}{4}} \quad \text{and} \quad -\mathbf{v}'_{2,in} \geq c_L \quad \text{on} \quad \left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, L\Lambda^{-\frac{1}{4}}\right], \tag{4.6}$$

provided that $\Lambda > 0$ is sufficiently large. On the other side, we observe that $\mathbf{v}_{2,in}$ satisfies a linear equation of the form

$$-v'' + P(z)v = 0 \quad \text{with} \quad P(z) \geq c\Lambda z^2, \quad z \in \left(L\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right), \tag{4.7}$$

with $c > 0$ independent of both Λ, L. Unfortunately, it is not clear to us how to use the maximum principle to deduce the positivity and monotonicity of $\mathbf{v}_{2,in}$ in this remaining interval without too much effort. A possible way would be to show that $\mathbf{v}'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = w'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) < 0$. This last task, however, would require us to keep track of the sharp super-exponential decay of the various functions involved in the construction of $w_{2,in}$. Nevertheless, since

$$\mathbf{v}'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = w'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty, \tag{4.8}$$

by a standard barrier argument, we can easily infer that

$$\mathbf{v}_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty. \tag{4.9}$$

The above two estimates will be used as stepping stones for 'extending' $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ as a heteroclinic solution to (1.1)-(1.2), which can then easily be shown to have positive components with the right monotonicity properties. Needless to say that the corresponding properties for $\mathbf{v}_{1,in}$ follow at once from the imposed symmetry.

4.2. **Adjusting the Hamiltonian constant of the inner genuine solution.** From the calculations of Subsection 3.2 and Proposition 4.1, it follows that, if α therein is inside $(0, 1/4)$ and $\tilde{B}$ satisfies (3.6), the equation for $\tilde{B}$ such that the Hamiltonian constant of the exact solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ of the inner problem is equal to $-\psi_0^2/2$ has the form

$$2\psi_0^2 \tilde{B}\Lambda^{\frac{1}{4}} + \Lambda^{-1+\alpha}h(\Lambda, \tilde{B}) = 0,$$

where the function h is uniformly bounded in Λ and continuous. Consequently, by the Bolzano-Weistrass theorem, there exists at least one

$$\tilde{B} = \mathcal{O}\left(\Lambda^{-\frac{5}{4}+\alpha}\right) \quad \text{as} \quad \Lambda \to \infty, \tag{4.10}$$

which satisfies the above equation ($\alpha \in (0, 1/4)$ is still as in Proposition 4.1).

Clearly, our working assumption (3.6) is satisfied for $\alpha > 0$ sufficiently small.

5. Solution of the outer problem

In this section, we will construct a symmetric solution $(\mathbf{v}_{1,out}, \mathbf{v}_{2,out})$ to system (1.1) outside of the interval I_Λ which, however, agrees on ∂I_Λ with the already constructed solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ of the inner problem (in the C^0 sense) and satisfies the desired asymptotic behavior in (1.2). Naturally, thanks to the symmetry assumption, we can restrict ourselves to z positive.

5.1. **A refined outer approximate solution** $(w_{1,out}, w_{2,out})$**.** We first consider a refinement of the outer approximate solution $(v_{1,out}, v_{2,out})$ that was constructed in Subsection 2.1, defined as

$$w_{1,out}(z) = v_{1,out}(z) + \tilde{\tau} U'(z+\xi), \quad w_{2,out}(z) = \mathbf{v}_{2,in}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)\zeta(z),$$

for $z \geq (\ln\Lambda)\Lambda^{-\frac{1}{4}}$; where

$$\tilde{\tau} = -\frac{(v_{1,out}-\mathbf{v}_{1,in})\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)}{U'\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}+\xi\right)} \overset{(2.1),(3.3),\mathrm{Prop.}4.1,(4.10)}{=} \mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac{5}{4}+\alpha}\right),$$

as $\Lambda \to \infty$ ($\alpha \in (0, \frac{1}{4})$ still as in Proposition 4.1), and $\zeta \in C_0^\infty(\mathbb{R})$ is a fixed cutoff function which is equal to one on $[-1,1]$ (in this regard, keep in mind (3.4), (4.9)). By construction, we have

$$w_{i,out}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathbf{v}_{i,in}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right), \quad i=1,2,$$

and that the asymptotic behavior (1.2) is still satisfied.

5.2. **The remainder of the refined outer approximate solution.** By recalling the proof of Lemma 2.2 and relations (3.4), (4.9), we find that the assertion of the aforementioned lemma continues to hold for $(w_{1,out}, w_{2,out})$, except that now the uniform norm of the remainder of the second equation is bounded by a $\Lambda^{-\infty}$-small number times a fixed, compactly supported function.

5.3. **The perturbation argument.** We seek a solution of system (1.1) as

$$\begin{aligned} &(\mathbf{v}_{1,out}, \mathbf{v}_{2,out}) = (w_{1,out}, w_{2,out}) + (\varphi_1, \varphi_2), \quad z \geq (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \\ &\varphi_i\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad \lim_{z\to\infty}\varphi_i(z) = 0, \quad i=1,2. \end{aligned} \tag{5.1}$$

Proceeding as in Subsection 4.1, we find that now the corresponding linear operator is

$$(\mathcal{L}+Q)\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3w_{1,out}^2-1)\varphi_1 + \Lambda w_{2,out}^2\varphi_1 + 2\Lambda w_{1,out}w_{2,out}\varphi_2 \\ \\ -\varphi_2'' + (3w_{2,out}^2-1)\varphi_2 + \Lambda w_{1,out}^2\varphi_2 + 2\Lambda w_{1,out}w_{2,out}\varphi_1 \end{pmatrix}.$$

The invertibility properties of the above operator that we will need are contained in the following proposition.

Proposition 5.1. *Given $m>1$, there exist $\Lambda_2, C > 0$ such that the boundary value problem*

$$(\mathcal{L}+Q)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}g_1\\ g_2\end{pmatrix}, \quad \varphi_i\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad \lim_{z\to\infty}\varphi_i(z) = 0, \quad i=1,2, \tag{5.2}$$

where $g_1, g_2 \in C\left[(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)$ decay exponentially fast to zero, has a unique solution such that

$$\|\varphi_1'\|_{L^\infty(J_\Lambda)} + \|\varphi_1\|_{L^\infty(J_\Lambda)} \leq C\|g_1\|_{L^\infty(J_\Lambda)} + \Lambda^{-m}\|g_2\|_{L^\infty(J_\Lambda)},$$

and

$$(\ln\Lambda)^{-1}\Lambda^{-\frac{1}{4}}\|\varphi_2'\|_{L^\infty(J_\Lambda)} + \|\varphi_2\|_{L^\infty(J_\Lambda)} \leq \Lambda^{-m}\|g_1\|_{L^\infty(J_\Lambda)} + C(\ln\Lambda)^{-2}\Lambda^{-\frac{1}{2}}\|g_2\|_{L^\infty(J_\Lambda)},$$

where

$$J_\Lambda = \left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty \right),$$

provided that $\Lambda \geq \Lambda_2$.

Proof. As in Proposition 2.2, it is enough to establish the validity of the asserted a-priori estimates.

We will first show that there exist constants $\Lambda_1, C > 0$ such that the following a-priori estimate holds: If $\phi \in C^2(\bar{J}_\Lambda)$ and $g \in C(\bar{J}_\Lambda)$ satisfy

$$-\phi'' + (3w_{1,out}^2 - 1)\phi + \Lambda w_{2,out}^2 \phi = g, \quad z \in J_\Lambda,$$

$$\phi\left((\ln \Lambda)\Lambda^{-\frac{1}{4}} \right) = 0, \quad \lim_{z \to \infty} \phi(z) = 0,$$

for $\Lambda \geq \Lambda_1$, then

$$\|\phi'\|_{L^\infty(J_\Lambda)} + \|\phi\|_{L^\infty(J_\Lambda)} \leq C\|g\|_{L^\infty(J_\Lambda)}.$$

Since

$$\|w_{2,out}\|_{L^\infty(J_\Lambda)} = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty, \tag{5.3}$$

it is clearly sufficient to show the a-priori estimate

$$\|\phi\|_{L^\infty(J_\Lambda)} \leq C\|g\|_{L^\infty(J_\Lambda)}.$$

To this end, we will argue by contradiction. So, let us suppose that there exist $\Lambda_n \to \infty$, $\phi_n \in C^2(\overline{J_{\Lambda_n}})$ and $g_n \in C(\overline{J_{\Lambda_n}})$ such that

$$-\phi_n'' + (3w_{1,out}^2 - 1)\phi_n + \Lambda_n w_{2,out}^2 \phi_n = g_n, \quad z \in J_{\Lambda_n},$$

$$\phi_n\left((\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}} \right) = 0, \quad \lim_{z \to \infty} \phi_n(z) = 0,$$

while

$$\|\phi_n\|_{L^\infty(J_{\Lambda_n})} = 1 \quad \text{and} \quad \|g_n\|_{L^\infty(J_{\Lambda_n})} \to 0.$$

Let

$$\tilde{\phi}_n(z) = \phi_n\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}} \right), \quad \tilde{g}_n(z) = g_n\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}} \right), \quad z \geq 0.$$

Then, we have that

$$-\tilde{\phi}_n'' + \left[3w_{1,out}^2\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}} \right) - 1 \right] \tilde{\phi}_n + \Lambda_n w_{2,out}^2 \left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}} \right) \tilde{\phi}_n = \tilde{g}_n,$$

$z \in [0, \infty)$, $\tilde{\phi}_n(0) = 0$, $\lim_{z \to \infty} \tilde{\phi}_n(z) = 0$, while

$$\|\tilde{\phi}_n\|_{L^\infty(0,\infty)} = 1 \quad \text{and} \quad \|\tilde{g}_n\|_{L^\infty(0,\infty)} \to 0.$$

Keeping in mind (5.3), thanks again to standard elliptic estimates and the usual diagonal argument, passing to a subsequence if necessary, and recalling the construction of $w_{1,out}$, we find that

$$\tilde{\phi}_n \to \tilde{\phi}_\infty \quad \text{in} \quad C^1_{loc}[0, \infty),$$

for some $\tilde{\phi}_\infty$ satisfying

$$-\tilde{\phi}_\infty'' + \left(3U^2(z) - 1 \right) \tilde{\phi}_\infty = 0, \quad z > 0; \quad \tilde{\phi}_\infty(0) = 0 \quad \text{and} \quad \|\tilde{\phi}_\infty\|_{L^\infty(0,\infty)} \leq 1.$$

Moreover, it is easy to see that $\tilde{\phi}_\infty$ is nontrivial since

$$\left[3w_{1,out}^2 - 1 + \Lambda_n w_{2,out}^2 \right] \left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}} \right) \to 2, \text{ as } z \to \infty, \text{ uniformly in } n,$$

which implies that the points where $|\tilde{\phi}_n|$ attains its maximum cannot escape at infinity. On the other hand, the first case in Lemma 2.1 implies that $\tilde{\phi}_\infty$ is identically equal to zero which is a contradiction.

Applying the previously proven a-priori estimate to the first equation of (5.2), and recalling (5.3), we obtain that

$$\|\varphi_1'\|_{L^\infty(J_\Lambda)} + \|\varphi_1\|_{L^\infty(J_\Lambda)} \leq C\|g_1\|_{L^\infty(J_\Lambda)} + \mathcal{O}(\Lambda^{-\infty})\|\varphi_2\|_{L^\infty(J_\Lambda)}. \tag{5.4}$$

The situation in the second equation is considerably simpler. Indeed, observing that

$$w_{1,out}(z) \geq c(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \quad z \in J_\Lambda, \tag{5.5}$$

it follows easily that

$$(\ln\Lambda)^{-1}\Lambda^{-\frac{1}{4}}\|\varphi_2'\|_{L^\infty(J_\Lambda)} + \|\varphi_2\|_{L^\infty(J_\Lambda)} \leq$$
$$\mathcal{O}(\Lambda^{-\infty})\|\varphi_1\|_{L^\infty(J_\Lambda)} + C(\ln\Lambda)^{-2}\Lambda^{-\frac{1}{2}}\|g_2\|_{L^\infty(J_\Lambda)}.$$

The assertion of the proposition now follows directly by combining (5.4) and the above relation. □

Armed with the above proposition and the observation made in Subsection 5.2, concerning the remainder left by $(w_{1,out}, w_{2,out})$, we can use the contraction mapping principle to capture the desired (φ_1, φ_2) in (5.1) and arrive at the main result of this section.

Proposition 5.2. *If $\Lambda > 0$ is sufficiently large, system (1.1) has a solution $(\mathbf{v}_{1,out}, \mathbf{v}_{2,out})$ of the form (5.1) with*

$$\|\varphi_1'\|_{L^\infty\left((\ln\Lambda)\Lambda^{-\frac{1}{4}},\infty\right)} + \|\varphi_1\|_{L^\infty\left((\ln\Lambda)\Lambda^{-\frac{1}{4}},\infty\right)} \leq C\Lambda^{-\frac{3}{2}},$$

$$\|\varphi_2'\|_{L^\infty\left((\ln\Lambda)\Lambda^{-\frac{1}{4}},\infty\right)} + \|\varphi_2\|_{L^\infty\left((\ln\Lambda)\Lambda^{-\frac{1}{4}},\infty\right)} = \mathcal{O}(\Lambda^{-\infty}).$$

6. Existence of the heteroclinic orbit: proof of Theorem 1.1

In this section, we will prove our main result. So far, we have solved exactly the problem in the inner interval $\left(-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)$ by $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$, in the outer interval $\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)$ by its continuous extension $(\mathbf{v}_{1,out}, \mathbf{v}_{2,out})$, and in the other outer interval $\left(-\infty, -(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)$ by the 'reflection' $(\mathbf{v}_{2,out}(-z), \mathbf{v}_{1,out}(-z))$. Furthermore, the asymptotic behavior in (1.2) is satisfied. In other words, the continuous and piecewise smooth pair

$$\mathbf{v}_{1,ap}(z) = \begin{cases} \mathbf{v}_{2,out}(-z), & z \leq -(\ln\Lambda)\Lambda^{-\frac{1}{4}} \\ \mathbf{v}_{1,in}(z), & |z| \leq (\ln\Lambda)\Lambda^{-\frac{1}{4}} \\ \mathbf{v}_{1,out}(z), & z \geq (\ln\Lambda)\Lambda^{-\frac{1}{4}} \end{cases}, \quad \mathbf{v}_{2,ap}(z) = \mathbf{v}_{1,ap}(-z), \quad z \in \mathbb{R}, \tag{6.1}$$

is an exact solution to problem (1.1)-(1.2) with the exception of the two points $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$. In some vague sense, the above pair can be considered as an approximate solution to (1.1)-(1.2) for large Λ. As we will see next, this can indeed be made rigorous.

6.1. **A gluing argument: The global approximate solution** $(\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap})$**.** We recall that we have constructed the solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ of the inner problem such that its Hamiltonian constant is equal to $-\psi_0^2/2$, which clearly is that of the aforementioned solutions of the outer problems. The main observation is that this implies that the jumps in the derivative of $\mathbf{v}_{1,ap}$ are transcendentally small (and in turn so are those of $\mathbf{v}_{2,ap}$). Indeed, by the equality of the Hamiltonian constants at $(\ln\Lambda)\Lambda^{-\frac{1}{4}}$, and the fact that $\mathbf{v}_{i,in} = \mathbf{v}_{i,out}$, $i = 1, 2$, at that point, we have that

$$\left[\mathbf{v}'_{1,in}\right]^2 + \left[\mathbf{v}'_{2,in}\right]^2 = \left[\mathbf{v}'_{1,out}\right]^2 + \left[\mathbf{v}'_{2,out}\right]^2 \quad \text{at} \quad (\ln\Lambda)\Lambda^{-\frac{1}{4}}.$$

In turn, (4.8), (4.9) and Proposition 5.2 yield that

$$\left[\mathbf{v}'_{1,in}\right]^2 - \left[\mathbf{v}'_{1,out}\right]^2 = \mathcal{O}\left(\Lambda^{-\infty}\right) \quad \text{at} \quad (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \quad \text{as} \quad \Lambda \to \infty.$$

Consequently, since $\mathbf{v}'_{1,in}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) \geq c$ and $\mathbf{v}'_{1,out}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) \geq c$ (keep in mind Propositions 4.1 and 5.2), we infer that

$$\left(\mathbf{v}'_{1,in} - \mathbf{v}'_{1,out}\right)\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}\left(\Lambda^{-\infty}\right), \quad \text{as} \quad \Lambda \to \infty,$$

as desired.

We are now ready to define our global C^1-smooth approximate solution to problem (1.1)-(1.2) as

$$\mathbf{w}_{1,ap}(z) = \mathbf{v}_{1,ap}(z) + s_- e^{-\left|z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right|} + s_+ e^{-\left|z-(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right|}, \tag{6.2}$$

and $\mathbf{w}_{2,ap}(z) = \mathbf{w}_{1,ap}(-z)$, $z \in \mathbb{R}$, where the numbers

$$s_\pm = \mathcal{O}\left(\Lambda^{-\infty}\right) \quad \text{as} \quad \Lambda \to \infty \tag{6.3}$$

are chosen so that $\mathbf{w}_{i,ap}$, $i = 1, 2$, are C^1 at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$. We note that the asymptotic behavior as $z \to \pm\infty$ has not been affected.

6.1.1. *The remainder of the global approximate solution* $(\boldsymbol{w}_{1,ap}, \boldsymbol{w}_{2,ap})$. Clearly, the remainder which is left by $(\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap})$ in (1.1) is at least of order $\mathcal{O}(\Lambda^{-\infty})$, uniformly in $\mathbb{R}$, as $\Lambda \to \infty$ and has finite jump discontinuities at the aforementioned two points.

6.2. **The perturbation argument.** We seek the desired heteroclinic orbit as

$$(v_1, v_2) = (\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap}) + (\varphi_1, \varphi_2), \tag{6.4}$$

with φ_1, φ_2 in the Sobolev space $H^2(\mathbb{R})$ and $\varphi_1(-z) = \varphi_2(z)$, $z \in \mathbb{R}$. The main reason for choosing this space is that it allows finite jump discontinuities in the second derivatives and, at the same time, does not affect the good asymptotic behavior of the approximate solution.

In view of the previous observation on the remainder of the global approximate solution $(\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap})$, it is enough to show that the linearized operator about this approximation, which is

$$\mathcal{M}\begin{pmatrix}\varphi_1 \\ \\ \varphi_2\end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3\mathbf{w}_{1,ap}^2 - 1)\varphi_1 + \Lambda\mathbf{w}_{2,ap}^2\varphi_1 + 2\Lambda\mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\varphi_2 \\ \\ -\varphi_2'' + (3\mathbf{w}_{2,ap}^2 - 1)\varphi_2 + \Lambda\mathbf{w}_{1,ap}^2\varphi_2 + 2\Lambda\mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\varphi_1 \end{pmatrix}, \tag{6.5}$$

restricted to the symmetry class (1.3), is invertible and find an algebraically large (in terms of Λ) bound for the norm of its inverse (in the natural spaces). Indeed, in that case, the desired pair (φ_1, φ_2) in (6.4) can easily be captured by the contraction mapping principle (applied in a ball of $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ of radius, say, of order $\mathcal{O}(\Lambda^{-100})$ as $\Lambda \to \infty$).

6.2.1. *The lower bound for the spectrum of the linearized operator $\mathcal{M}$.* Corollary 4.1 and the fact that the solution (V_1, V_2) of (1.8) is stable (see [7]) predict the following.

Lemma 6.1. *For any fixed $\alpha \in (0, \frac{1}{4})$, the principal eigenvalue of $\mathcal{M}$, restricted to the symmetry class (1.3), in $I_\Lambda = \left(-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)$ with Neumann boundary conditions satisfies*

$$\lambda_{in} \geq c\Lambda^{\frac{1}{2}-\alpha},$$

provided that Λ is sufficiently large.

Proof. The desired result will follow from the construction of the inner approximate solution, after a simple re-scaling, once we show that the principal eigenvalue μ_1 of L (defined in (2.8)), restricted to the symmetry class (2.20), in $(-\ln\Lambda, \ln\Lambda)$ with Neumann boundary conditions satisfies

$$\mu_1 \geq c\Lambda^{-\alpha},$$

for large Λ (also keep in mind (4.3)). In light of Proposition 2.2, we find that

$$|\mu_1| \geq c\Lambda^{-\alpha}. \tag{6.6}$$

So, if $\mu_1 \geq 0$ we are done.

Let us suppose, to the contrary, that

$$\mu_1 < 0. \tag{6.7}$$

The variational characterization of μ_1 implies the rough lower bound

$$\mu_1 \geq -\|V_1V_2\|_{L^\infty(-\ln\Lambda,\ln\Lambda)} \geq -C.$$

In turn, assuming that the associated eigenfunctions (Φ_1, Φ_2) (actually μ_1 is simple, see [8] for the case of Dirichlet boundary conditions) have been normalized so that

$$\int_{|x|\leq\ln\Lambda} (\Phi_1^2 + \Phi_2^2)dx = 1, \tag{6.8}$$

we find that

$$\int_{|x|\leq\ln\Lambda} (\ddot{\Phi}_i)^2 dx \leq C(\ln\Lambda)^4, \quad i = 1, 2.$$

So, using the Neumann boundary conditions, we find that

$$\left|\dot{\Phi}_i(x)\right| \leq C(\ln\Lambda)^{\frac{5}{2}}, \quad |x| \leq \ln\Lambda, \quad i = 1, 2.$$

Therefore, since (6.8) implies that there exists some point where $|\Phi_i| \leq C(\ln\Lambda)^{-1/2}$, we obtain that

$$|\Phi_i(x)| \leq C(\ln\Lambda)^{\frac{7}{2}}, \quad |x| \leq \ln\Lambda, \quad i = 1, 2.$$

In turn, a simple barrier argument (recall also the related remarks in the proof of Proposition 2.2) yields that, for any $m \geq 1$,

$$\left|\dot{\Phi}_2(\ln\Lambda)\right| + |\Phi_2(\ln\Lambda)| \leq C_m\Lambda^{-m}.$$

Of course, by symmetry, the analogous relation holds for Φ_1 at $-\ln\Lambda$. We can now test (in the $L^2\times L^2$ sense) the eigenvalue equation of (Φ_1,Φ_2) by (Φ_1,Φ_2) and, as in [7, Thm. 1.3] (see also [4, Thm. 3.1]), write

$$\Phi_1=\dot V_1\bar\Phi_1,\quad \Phi_2=\dot V_2\bar\Phi_2,\quad \text{(recall (2.7))},$$

to find that

$$\mu_1\geq 2\left(\frac{\ddot V_1}{\dot V_1}\Phi_1^2\right)(\ln\Lambda)-2\left(\frac{\ddot V_1}{\dot V_1}\Phi_1^2\right)(-\ln\Lambda)\geq -C_m\Lambda^{-m},$$

for any $m\geq 1$ (where we also used the symmetry and boundary conditions). On the other hand, this is clearly contradictory to (6.6) and (6.7) for large $\Lambda>0$. □

The variational characterization of λ_{in} implies at once the following.

Corollary 6.1. *Let $\alpha>0$ be fixed sufficiently small. Then, there exists $c>0$ such that, if $\Lambda>0$ is sufficiently large, then*

$$\int_{I_\Lambda}\left\{\sum_{i=1}^{2}\left[(\varphi_i')^2+\left(3\boldsymbol{w}_{i,ap}^2-1+\Lambda\boldsymbol{w}_{i+1,ap}^2\right)\varphi_i^2\right]+4\Lambda\boldsymbol{w}_{1,ap}\boldsymbol{w}_{2,ap}\varphi_1\varphi_2\right\}dz$$
$$\geq c\Lambda^{\frac12-\alpha}\sum_{i=1}^{2}\int_{I_\Lambda}\varphi_i^2dz,$$

for any $(\varphi_1,\varphi_2)\in H^1(I_\Lambda)\times H^1(I_\Lambda)$ (with the obvious notation).

On the other side, in the outer region, we can show the following rough lower bound.

Lemma 6.2. *There exists $c>0$ such that, if $\Lambda>0$ is sufficiently large, then*

$$\int_{J_\Lambda}\left\{\sum_{i=1}^{2}\left[(\varphi_i')^2+\left(3\boldsymbol{w}_{i,ap}^2-1+\Lambda\boldsymbol{w}_{i+1,ap}^2\right)\varphi_i^2\right]+4\Lambda\boldsymbol{w}_{1,ap}\boldsymbol{w}_{2,ap}\varphi_1\varphi_2\right\}dz$$
$$\geq c(\ln\Lambda)\Lambda^{-\frac14}\sum_{i=1}^{2}\int_{J_\Lambda}\varphi_i^2dz,$$

for any $(\varphi_1,\varphi_2)\in H^1(J_\Lambda)\times H^1(J_\Lambda)$, where $J_\Lambda=\left((\ln\Lambda)\Lambda^{-\frac14},\infty\right)$.

Proof. In view of the construction of the outer solution, (5.3) and (5.5), it suffices to show that

$$\int_{J_\Lambda}\left[(\phi')^2+\left(3U^2(z+\xi)-1\right)\phi^2\right]dz\geq c(\ln\Lambda)\Lambda^{-\frac14}\int_{J_\Lambda}\phi^2dz,$$

for any $\phi\in H^1(J_\Lambda)$. For this purpose, we consider the self-adjoint operator in $L^2(J_\Lambda)$ defined as

$$\mathtt{L}(\phi)=-\phi''+\left(3U^2(z+\xi)-1\right)\phi$$

with domain $\phi\in H^2(J_\Lambda)$ with $\phi'\left((\ln\Lambda)\Lambda^{-\frac14}\right)=0$. The desired lower bound will follow if we show that the spectrum of $\mathtt{L}$ is contained in $\left(c(\ln\Lambda)\Lambda^{-\frac14},\infty\right)$ for some $c>0$.

Firstly, we observe that, since

$$\left(3U^2(z+\xi)-1\right)\to 2\quad\text{as}\quad z\to\infty,$$

the continuous spectrum of $\mathtt{L}$ is contained in $[2,\infty)$ (see [16]). So, let $\mu_\Lambda < 2$ denote the principal eigenvalue of $\mathtt{L}$ and ϕ_Λ the associated eigenfunction, which is simple and sign-definite (by the variational characterization of μ_Λ). Without loss of generality, we may assume that ϕ_Λ is positive and has L^∞-norm equal to one.

Using the Rayleigh quotient to bound μ_Λ from below and above (with $U'(z+\xi)$ as a competitor), it follows readily that

$$-1 \leq \mu_\Lambda \leq C(\ln\Lambda)\Lambda^{-\frac{1}{4}}.$$

We point out that in the above calculation we also used that $U''(0)=0$.

Then, similarly to the proof of Proposition 5.1, using (2.5) and the second case in Lemma 2.1, we get that

$$\mu_\Lambda \to 0 \quad\text{and}\quad \phi_\Lambda\left(z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) \to \frac{U'}{\|U'\|_{L^\infty(0,\infty)}} \quad\text{in } C^1_{loc}[0,\infty) \quad\text{as}\quad \Lambda\to\infty.$$

We stress that the above convergence holds for $\Lambda\to\infty$ by the uniqueness of the limit.

Multiplying the eigenvalue equation for $\phi_\Lambda\left(z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)$ by $U'\left(z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}+\xi\right)$, and integrating by parts the resulting identity over $[0,\infty)$, we arrive at

$$-\phi_\Lambda\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)U''\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}+\xi\right) = \mu_\Lambda\int_0^\infty \phi_\Lambda\left(z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)U'\left(z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}+\xi\right)dz.$$

Hence, letting $\Lambda\to\infty$ in the above relation, with the help of (2.2) and Lebesgue's dominated convergence theorem, as well as (3.1), we infer that

$$\mu_\Lambda = \frac{\psi_0^2}{\int_0^\infty (U')^2dz}(\ln\Lambda)\Lambda^{-\frac{1}{4}} + o\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) \quad\text{as}\quad \Lambda\to\infty.$$

The above relation clearly implies the desired lower bound for the spectrum of $\mathtt{L}$. ☐

By combining Corollary 6.1 and Lemma 6.2 above, we can deduce the following rough lower bound for the spectrum of the linear operator $\mathcal{M}$ in (6.5) which, however, as we have already remarked, is enough to give the main result of the paper.

Corollary 6.2. *If $\Lambda>0$ is sufficiently large, we have*

$$\sigma(\mathcal{M}) \subset \left[c(\ln\Lambda)\Lambda^{-\frac{1}{4}},\infty\right),$$

for some $c>0$.

6.2.2. *Positivity, monotonicity and decay properties of the heteroclinic orbit.* Armed with the previously proven C^1-uniform estimates for the constructed heteroclinic solution (v_1,v_2), we can complete the qualitative analysis of Subsubsection 4.1.1 and the proof of Theorem 1.1.

The main observation is that v_2 satisfies a linear equation of the form

$$-v''+\tilde{P}(z)v=0, \quad z\geq L\Lambda^{-\frac{1}{4}},$$

with

$$\tilde{P}(z) \geq \begin{cases} c\Lambda z^2, & z \in (L\Lambda^{-\frac{1}{4}}, \delta), \\ \\ c\Lambda, & z \in [\delta, \infty), \end{cases}$$

for some fixed small $\delta > 0$ (keep in mind (4.7)). Hence, since $v_2(L\Lambda^{-\frac{1}{4}}) \geq c\Lambda^{-\frac{1}{4}}$, $v_2'(L\Lambda^{-\frac{1}{4}}) \leq -c$ for large Λ (recall (4.6)) and $\lim_{z\to\infty} v_2(z) = 0$, we deduce by the maximum principle that

$$v_2 > 0 \text{ and } v_2' < 0 \text{ on } [L\Lambda^{-\frac{1}{4}}, \infty).$$

In summary, so far we have shown that

$$v_2 > 0 \text{ and } v_2' < 0 \text{ on } \left[-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \infty\right).$$

In fact, by the use of barriers and standard elliptic estimates, it follows readily that

$$v_2(z) - \Lambda^{-\frac{1}{4}} v_2'(z) \leq C\Lambda^{-\frac{1}{4}} e^{-c\Lambda^{\frac{1}{4}} z}, \quad z \geq 0. \tag{6.9}$$

Moreover, the above estimate can be improved for large z: Given any fixed $d > 0$, it holds

$$v_2(z) - \Lambda^{-\frac{1}{2}} v_2'(z) \leq C v_2(d) e^{-c\Lambda^{\frac{1}{2}} z}, \quad z \geq d.$$

The previously proven C^1-uniform estimates for the convergence of v_2 to $-U$ on $\left(-\infty, -(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)$ guarantee that the same holds on any fixed interval of the form $[-M, \infty)$, provided that Λ is sufficiently large. In particular, $v_2(-M) \to U(M) < 1$ and $v_2'(-M) \to -U'(M) < 0$ as $\Lambda \to \infty$. To conclude that v_2 is still decreasing in $(-\infty, -M)$, it is enough to apply the maximum principle to the linear equation that is satisfied by v_2'. Indeed, using the symmetry (1.3) and our previous observation, we find that the function $\psi \equiv v_2'$ satisfies

$$-\psi'' + \left(3v_2^2 - 1 + \Lambda v_1^2\right)\psi = -2\Lambda v_2 v_1 v_1' \leq 0, \quad z \leq -M; \quad \psi(-\infty) = 0, \ \psi(-M) < 0,$$

with $3v_2^2 - 1 + \Lambda v_1^2 > 0$ (having increased the value of M if necessary).

The fact that

$$0 < v_1 < 1 \text{ and } v_1' > 0 \text{ in } \mathbb{R}$$

follows at once from the imposed symmetry.

Remark 6.1. *An effective approach for constructing heteroclinic orbits in singularly perturbed systems of ordinary differential equations is to make use of geometric singular perturbation theory. Here problem (1.8) can be viewed as a blow-up problem as in the case of loss of normal hyperbolicity (see* [21] *and the references therein). However, we have not been able to put system (1.1) in the slow-fast form that is required for the aforementioned machinery to apply.*

7. Non-degeneracy of the heteroclinic solution in the general class: proof of Theorem 1.2

Proof of Theorem 1.2. It has been shown in [4] that the lowest point in the spectrum of $\mathbf{M}$ is 0 which is a simple eigenvalue with (v_1', v_2') as the associated eigenfunction. Moreover, the continuous spectrum is contained in an interval of the form $[c, \infty)$ (we stress that $c > 0$ is independent of large Λ by the analysis in [4]). So, it is enough to show that the second eigenvalue $\mu > 0$ of $\mathbf{M}$ (should it exist) is bounded away from 0 independently of large Λ. To this end, we will argue by contradiction.

Suppose, to the contrary, that there are $\Lambda_n \to \infty$ such that the second eigenvalue $\mu_n > 0$ of $\mathbf{M}$ exists and satisfies

$$\mu_n \to 0. \tag{7.1}$$

Then, there would exist an associated eigenfunction $(\varphi_{1,n}, \varphi_{2,n}) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ such that

$$\begin{cases} -\varphi_{1,n}'' + (3v_1^2 - 1)\varphi_{1,n} + \Lambda_n v_2^2 \varphi_{1,n} + 2\Lambda_n v_1 v_2 \varphi_{2,n} = \mu_n \varphi_{1,n}, \\ -\varphi_{2,n}'' + (3v_2^2 - 1)\varphi_{2,n} + \Lambda_n v_1^2 \varphi_{2,n} + 2\Lambda_n v_1 v_2 \varphi_{1,n} = \mu_n \varphi_{2,n}, \end{cases} \quad z \in \mathbb{R}, \tag{7.2}$$

$$\|\varphi_{1,n}\|_{L^\infty(\mathbb{R})} + \|\varphi_{2,n}\|_{L^\infty(\mathbb{R})} = 1, \tag{7.3}$$

and

$$\int_{-\infty}^{\infty} (v_1' \varphi_{1,n} + v_2' \varphi_{2,n})\, dz = 0. \tag{7.4}$$

In the sequel, for notational convenience, we will often omit the index n, and all the generic constants $c, C > 0$ will be independent of n.

By writing

$$\varphi_1(z) = \frac{\varphi_1(z) + \varphi_2(-z)}{2} + \frac{\varphi_1(z) - \varphi_2(-z)}{2},$$
$$\varphi_2(z) = \frac{\varphi_2(z) + \varphi_1(-z)}{2} + \frac{\varphi_2(z) - \varphi_1(-z)}{2},$$

it is enough to exclude separately the cases where the components satisfy the mirror symmetry $\varphi_1(-z) \equiv \varphi_2(z)$ and the antisymmetry $\varphi_1(-z) \equiv -\varphi_2(z)$.

We start by assuming that

$$\varphi_1(-z) = -\varphi_2(z), \quad z \in \mathbb{R}. \tag{7.5}$$

In particular, by virtue of (1.3), the orthogonality condition (7.4) reduces to

$$\int_{-\infty}^{\infty} v_1' \varphi_{1,n} dz = 0. \tag{7.6}$$

Using that

$$c\Lambda^{-\frac{1}{4}} \le v_2(z) \le C(\Lambda^{-\frac{1}{4}} + |z|), \quad z \le 0, \tag{7.7}$$

together with the corresponding decay to (6.9) for v_1, (7.1) and (7.3), via the first equation in (7.2), we find that φ_1 satisfies a linear inhomogeneous equation of the form

$$-\varphi_1'' + p(z)\varphi_1 = f(z), \quad z \le 0,$$

with

$$p(z) \ge c\Lambda_n^{\frac{1}{2}} \text{ and } |f(z)| \le C\Lambda_n^{\frac{1}{2}} e^{c\Lambda_n^{\frac{1}{4}} z}, \quad z \le 0.$$

Hence, in view of the normalization (7.3), by a barrier argument, we deduce that

$$|\varphi_1(z)| \le e^{c\Lambda_n^{\frac{1}{4}} z}, \quad z \le 0. \tag{7.8}$$

Using (7.1), (7.2), (7.3), (7.5), (7.8), together with standard elliptic estimates and the familiar diagonal argument, passing to a subsequence if necessary, we find that

$$\varphi_{1,n} \to \varphi_{1,\infty} \text{ in } C^1_{loc}(\mathbb{R} \setminus \{0\}). \tag{7.9}$$

We have

$$\varphi_{1,\infty}(z) = 0 \text{ for } z < 0, \tag{7.10}$$

while

$$-\varphi_{1,\infty}'' + \big(3U^2(z) - 1\big)\,\varphi_{1,\infty} = 0, \quad z > 0.$$

Since $\varphi_{1,\infty}$ is bounded for $z \neq 0$, we must have that

$$\varphi_{1,\infty}(z) = aU'(z), \quad z > 0, \tag{7.11}$$

(recall the comments in the proof of Lemma 2.1, this a should not be confused with that in (2.32)). On the other hand, by (7.6) and Lebesgue's dominated convergence theorem (keep in mind Theorem 1.1 and the decay estimate (6.9)), we obtain that

$$a\int_0^\infty \left[U'(z)\right]^2 dz = 0.$$

Thus, by combining the above, we find

$$\varphi_{1,n} \to 0 \ \text{ in } \ C^1_{loc}\left(\mathbb{R}\setminus\{0\}\right). \tag{7.12}$$

The main effort will now be placed in showing that $\varphi_{1,n} \to 0$ uniformly on $[-1,1]$ which, in view of (7.5) and the above relation, will certainly come in contradiction with (7.3) (note also that, as in Proposition 5.1, the points where $|\varphi_{1,n}|$ achieves its maximum cannot escape at infinity). For this purpose, it will be convenient to write

$$\varphi_1(z) = \tilde{\varphi}_1(z) + \hat{\varphi}_1(z) \ \text{ for } \ z \in [-1,1], \tag{7.13}$$

where $\tilde{\varphi}_1$ is the unique solution of the following boundary value problem:

$$\begin{cases} \tilde{\varphi}_1'' = \left(3v_1^2(z) - 1\right)\varphi_1 - \mu_n\varphi_1, \quad z \in (-1,1), \\ \\ \tilde{\varphi}_1(\pm 1) = 0. \end{cases}$$

We observe that standard elliptic estimates imply that

$$\|\tilde{\varphi}_1\|_{H^2(-1,1)} \leq C\|(3v_1^2 - 1)\varphi_1 - \mu_n\varphi_1\|_{L^2(-1,1)} \to 0,$$

where the last relation holds by virtue of (7.1), (7.3), (7.12) and Lebesgue's dominated convergence theorem. In particular, thanks to the well known Sobolev embedding theorem in one dimension, we obtain that

$$\|\tilde{\varphi}_1\|_{C^1(-1,1)} \to 0. \tag{7.14}$$

In view of (7.12) and (7.14), the function $\hat{\varphi}_1$ in the decomposition (7.13) satisfies

$$\hat{\varphi}_1 \to 0 \ \text{ in } \ C^1_{loc}\left([-1,1]\setminus\{0\}\right). \tag{7.15}$$

Next, we will exploit that $\hat{\varphi}_1$ satisfies

$$\hat{\varphi}_1'' = \Lambda_n v_2^2(z)\varphi_1(z) + 2\Lambda_n v_1(z)v_2(z)\varphi_2(z), \quad z \in (-1,1). \tag{7.16}$$

Using (6.9), (7.3) and the analog of (7.7) for v_1 on $[0,\infty)$, we see that

$$|\hat{\varphi}_1''| \leq C\Lambda_n^{\frac{1}{2}}e^{-c\Lambda_n^{\frac{1}{4}}z}, \quad z \in [-1,1].$$

In turn, by integrating the above relation twice and using (7.15) only at $z = 1$, we deduce that

$$|\hat{\varphi}_1(z)| \leq Ce^{-c\Lambda_n^{\frac{1}{4}}z} + o(1), \quad \text{uniformly on } \ [0,1], \ \text{ as } \ n \to \infty. \tag{7.17}$$

Since

$$\Lambda^{\frac{1}{4}}v_i(\Lambda^{-\frac{1}{4}}x) \to V_i(x) \ \text{ in } \ C_{loc}(\mathbb{R}) \ \text{ as } \ \Lambda \to \infty, \quad i = 1,2, \tag{7.18}$$

using Proposition 2.1, (7.1), (7.2) and (7.3), passing to a subsequence if needed, we find that

$$\varphi_i(\Lambda_n^{-\frac{1}{4}}x) \to b\dot{V}_i(x) \ \text{ in } \ C^1_{loc}(\mathbb{R}), \quad i=1,2, \tag{7.19}$$

for some $b \in \mathbb{R}$. We claim that

$$b = 0. \tag{7.20}$$

Indeed, let $K>0$ be a number to be chosen, independently of n, such that

$$\dot{V}_1(K) \geq \frac{\psi_0}{2}$$

(recall 2.6). On the one hand, thanks to (7.19), the above relation implies that

$$\left|\varphi_1(\Lambda_n^{-\frac{1}{4}}K)\right| \geq |b|\frac{\psi_0}{2} + o(1) \ \text{ as } \ n\to\infty.$$

On the other hand, by (7.13), (7.14) and (7.17), we have that

$$\left|\varphi_1(\Lambda_n^{-\frac{1}{4}}K)\right| \leq Ce^{-cK} + o(1) \ \text{ as } \ n\to\infty,$$

($c, C > 0$ independent of both K, n). Thus, we can fix a sufficiently large $K > 0$ to get (7.20).

Consequently, by (7.8), (7.13), (7.14), (7.17), (7.19) and (7.20), we deduce that $\varphi_1 \to 0$ uniformly in $\mathbb{R}$ which contradicts (7.3) and (7.5).

It remains to exclude the case of mirror symmetry, namely (1.3). As before, we have the validity of (7.9), (7.10), and (7.11) for some $a \in \mathbb{R}$. Next, letting

$$\psi_1 = \varphi_1 - aU' \to 0 \ \text{ in } \ C^1_{loc}(0,\infty),$$

we observe that

$$\begin{aligned} -\psi_1'' + (3v_1^2-1)\psi_1 + \Lambda_n v_2^2\psi_1 &= -2\Lambda_n v_1 v_2 \varphi_2 + \mu_n\varphi_1 - a\Lambda_n v_2^2 U' \\ &\quad -3a(v_1^2-U^2)U', \end{aligned}$$

for $z \geq 0$. Then, on the interval $[0,1]$, we can decompose ψ_1 analogously to (7.13) and as before arrive at

$$|\varphi_1 - aU'| \leq Ce^{-c\Lambda_n^{\frac{1}{4}}z} + o(1), \ \text{ uniformly on } [0,1], \ \text{ as } n\to\infty. \tag{7.21}$$

On the other side, by the mirror symmetry (1.3), we must have that (7.19) holds with $b = 0$. Hence, since $U'(0) > 0$, we infer that $a = 0$. Finally, by combining the above, we are led to the desired contradiction and to the completion of the proof of the theorem. □

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