AN ASYMPTOTIC MONOTONICITY FORMULA FOR MINIMIZERS TO A CLASS OF ELLIPTIC SYSTEMS OF ALLEN-CAHN TYPE AND THE LIOUVILLE PROPERTY

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Abstract. We prove an asymptotic monotonicity formula for bounded, globally minimizing solutions to a class of semilinear elliptic systems of the form $\Delta u = W_u(u)$, $x \in \mathbb{R}^n$, $n \geq 2$, with $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, nonnegative and vanishing at exactly one point (at least in the closure of the image of the considered solution u). As an application, we can prove a Liouville type theorem under various assumptions.

1. Introduction and statement of the main results

Consider the semilinear elliptic system

$$\Delta u = W_u(u) \text{ in } \mathbb{R}^n, \quad n \geq 2, \tag{1.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and *nonnegative*. This system has variational structure, as solutions (in a smooth, bounded domain $\Omega \subset \mathbb{R}^n$) are critical points of the energy

$$E(v;\Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx \tag{1.2}$$

(subject to their own boundary conditions), where $|\nabla v|^2 = \sum_{i=1}^n |v_{x_i}|^2$. A solution $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$ is called *globally minimizing* if

$$E(u;\Omega) \leq E(u+\varphi;\Omega) \tag{1.3}$$

for every smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and for every $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ (see also [28] and the references therein).

If $m \geq 2$, two are the main categories of such potentials W:

- Those that vanish only on a discrete set of points (usually finite); in this case (1.1) is known as the vectorial Allen-Cahn equation and models multi-phase transitions (see [8], [14], [25] and the references that follow).
- Those that vanish on a continuum of points, as in the Ginzburg-Landau system (see [13]) or the elliptic system modeling phase-separation in [12] or the one in [18].

This article is motivated from the first class. In this setting, an effective way to construct entire, nontrivial solutions to (1.1) is to assume that W is symmetric with respect to a finite reflection group and to look for equivariant solutions. Under proper assumptions, this roughly amounts to studying bounded, globally minimizing solutions to (1.1) such that the

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closure of their image contains at most one global minimum of W. In the scalar case, that is $m = 1$, this approach has been utilized, among others, in [15] and [21]. On the other side, recent progress has been made in the vector case in [4], [10], [11] and [27]. In our opinion, the main obstruction in the vector case is the lack of the maximum principle. This short discussion motivates our main result:

Theorem 1.1. *Assume that $W \in C^1(\mathbb{R}^m;\mathbb{R})$, $m \geq 1$, and that there exists $a \in \mathbb{R}^m$ such that*

$$W > 0 \quad in \quad \mathbb{R}^m \setminus \{a\} \quad and \quad W(a) = 0. \tag{1.4}$$

If $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$, $n \geq 2$, is a bounded, globally minimizing solution to the elliptic system (1.1), we have that

$$\lim_{R\to\infty}\left(\frac{1}{R^{n-1}}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) = 0, \tag{1.5}$$

where B_R stands for the n-dimensional ball of radius R and center at the origin.

The above result may be interpreted as an *asymptotic monotonicity formula* (see (2.19) below). We emphasize that there is no assumption for the behavior of W near a. Our proof of Theorem 1.1 is based on an adaptation to this setting of the famous "bad discs" construction of [13] from the study of vortices in the Ginzburg-Landau model. Under very general assumptions on W, it is well known that every entire, bounded and globally minimizing solution to (1.1) satisfies

$$\limsup_{R\to\infty}\left(\frac{1}{R^{n-1}}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) < \infty,$$

(see for example [17]). The above relation can be proven by comparing the energy of u in B_R to that of a (not too bad) test function which agrees with u on ∂B_R and is equal to some zero of W in ∂B_{R-1}. This simple idea, which can actually be traced back to the theory of minimal surfaces (see [19]), will also play an important role in our analysis.

As an application of Theorem 1.1, we can prove the following Liouville type theorem.

Theorem 1.2. *Assume that W and u are as in Theorem 1.1. Then, it holds that*

$$u \equiv a,$$

provided that one of the following additional conditions holds:

- **(a):** *$m = 1$ and $W \in C^{1,1}_{loc}(\mathbb{R};\mathbb{R})$; or $m \geq 1$ and u is radially symmetric; or $m \geq 1$ and Modica's gradient bound holds, that is*

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \quad in \quad \mathbb{R}^n. \tag{1.6}$$

- **(b):** *$n = 2$ and there exists small $r_0 > 0$ such that the functions*

$$r \mapsto W(a + r\nu), \quad where \quad \nu \in \mathbb{S}^{m-1}, \quad are\ strictly\ increasing\ for\ r \in (0, r_0]; \tag{1.7}$$

  or $n = 2$, $W \in C^{1,1}_{loc}(\mathbb{R}^m;\mathbb{R})$, and the above functions are nondecreasing for $r \in (0, r_0]$; or $n = 2$ and $m = 1$.

The above Liouville property was originally proven, for all $n \geq 2$, by different techniques in [28] (see also the earlier paper [27]), under the conditions that $W \in C^2(\mathbb{R}^m;\mathbb{R})$ and u satisfy the assumptions of Theorem 1.1, and that the functions in (1.7) have a *strictly positive second derivative* in $(0, r_0)$. In particular, the approach of the latter references is based on a quantitative refinement of the replacement lemmas in [4] and [26], combined with a rather involved iterative procedure. If W additionally satisfies the stronger assumption that a is a non-degenerate minimum, this theorem was recently re-proven in [7] by extending to this setting the density estimates of [17]. In the aforementioned references, the Liouville type theorem was proven by an application of a basic pointwise estimate. However, it is not difficult to convince oneself that the opposite direction is also possible, that is the pointwise estimate follows from the Liouville property (see also [39] for this viewpoint). We note that the pointwise estimate is the one that is directly applicable in relation to the discussion preceding Theorem 1.1. This pointwise estimate roughly says that if W (as in Theorem 1.1) is such that the Liouville type theorem holds, then a globally minimizing solution, defined in a sufficiently large ball (with the appropriate modifications in the definition), and bounded independently of the size of the ball, has to be close to a in the ball of half the radius.

In the scalar case, under the assumptions of the first part of Case (a) above, this theorem can also be proven by using radial barriers as in [39].

In our opinion, three are the main advantages of our approach: Firstly, we can treat in a unified and coordinate way various situations. Secondly, in our opinion, our approach is considerably simpler than those in the aforementioned references. Lastly, to the best of our knowledge, it provides the strongest available result when $n = 2$ for any $m \geq 1$, *even for the extensively studied scalar case*. This may seem too restrictive at first, but keep in mind that the dimensions $n = 2, 3$ are the ones with physical interest. In fact, the majority of papers on the subject deals exclusively with these dimensions (see [1], [2], [14], [37] for $n = 2$ and [31] for $n = 3$). If $n = 2$, our results imply that the convexity of W near its global minima, that was assumed in some of the aforementioned papers that deal with the existence of equivariant solutions to (1.1) (for instance in [11]), can be relaxed to the monotonicity condition that is described after (1.7). We emphasize that systems of the form (1.1) where the potentials have degenerate minima arise naturally in various physical models (see for example [9]).

The proof of Theorem 1.2 is based on combining Theorem 1.1 with a variety of diverse results that are available in the literature.

In the sequel, we will provide the proofs of our main results. We will close the paper with an appendix that is used in the proof of Theorem 1.2, but is also of independent interest as it contains a new result.

2. Proof of the main results

2.1. **Proof of Theorem 1.1.**

Proof. Throughout this proof, we will denote the energy density of u by

$$e(x) = \frac{1}{2}|\nabla u(x)|^2 + W\left(u(x)\right), \quad x \in \mathbb{R}^n. \tag{2.1}$$

Firstly, note that standard elliptic regularity theory and Sobolev imbeddings [23, 30], in combination with the fact that u is bounded, yield that

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^m)} \leq C_1, \tag{2.2}$$

for some $\alpha \in (0, 1)$ and $C_1 > 0$ (in fact, it holds for any $\alpha \in (0, 1)$ provided that $C_1 = C_1(\alpha) > 0$). We point out that this is the only place where we use that $W \in C^2$.

Since u is a globally minimizing solution, by comparing its energy to that of a suitable test function which agrees with u on ∂B_R and is identically a in B_{R-1}, we find that

$$\int_{B_R} e(x)dx \leq C_2 R^{n-1}, \quad R \geq 1, \tag{2.3}$$

for some $C_2 > 0$ (see also [17]).

Therefore, by (2.3), the co-area formula (see for instance [23, Ap. C]), the nonnegativity of W, and the mean value theorem, there exist

$$S_R \in (R, 2R) \tag{2.4}$$

such that

$$\int_{\partial B_{S_R}} e(x)dS(x) \leq C_3 R^{n-2}, \quad R \geq 1, \tag{2.5}$$

for some $C_3 > 0$ (actually, we can take $C_3 = \frac{C_2}{2}$).

Let $\epsilon > 0$ be any small number. By virtue of (2.2), we can infer that the subset of ∂B_{S_R} where $e(x)$ is above ϵ is contained in at most $\mathcal{O}(R^{n-2})$ number of geodesic balls of radius 1 as $R \to \infty$ (the so-called "bad discs", see [13]). More precisely, there exist $N_{\epsilon,R} \geq 0$ points $\{x_{R,1}, \cdots, x_{R,N_{\epsilon,R}}\}$ on ∂B_{S_R} such that

$$e(x) \geq \epsilon \ \text{ if } \ x \in U_R(x_{R,i}, 1), \quad i = 1, \cdots, N_{\epsilon,R},$$

and

$$e(x) \leq \epsilon \ \text{ if } \ x \in \partial B_{S_R} \setminus \bigcup_{i=1}^{N_{\epsilon,R}} U_R(x_{R,i}, 1), \tag{2.6}$$

for $R \gg 1$, where $U_R(p, r) \subset \partial B_{S_R}$ stands for the geodesic ball with center at p and radius r. Moreover, we have that

$$N_{\epsilon,R} \leq M_\epsilon R^{n-2}, \quad R \gg 1 \text{ (with } M_\epsilon > 0 \text{ independent of } R). \tag{2.7}$$

In the sequel, we will prove the above properties by adapting some arguments from [13]. Firstly, we prove a *clearing-out property*. Note that (2.2) implies that there exists $\mu_\epsilon < \epsilon$ such that

$$\int_{U_R(y,2)} e(x)dS(x) < \mu_\epsilon \ \text{ for some } y \in \partial B_{S_R}$$

implies that

$$e(x) \leq \epsilon, \quad x \in U_R(y, 1),$$

for $R \geq 1$. Indeed, suppose that

$$e(z) \geq \epsilon \text{ for some } y \in \partial B_{S_R} \text{ and } z \in U_R(y, 1). \tag{2.8}$$

From (2.2), there exists $C_4 > 0$ such that

$$\|e\|_{C^{0,\alpha}(\mathbb{R}^n;\mathbb{R})} \leq C_4.$$

It then follows that

$$e(x) \geq \epsilon - C_4 d^\alpha, \quad x \in B(z, d) = z + B_d,$$

for all $d < \min\left\{1, \left(\frac{\epsilon}{2C_4}\right)^{\frac{1}{\alpha}}\right\}$ (see also [42, Lem. 2.3]). Since $e \geq 0$, we find that

$$\int_{U_R(y,2)} e(x)dS(x) \geq \int_{U_R(z,d)} e(x)dS(x) \geq (\epsilon - C_4 d^\alpha) |U_R(z, d)| \geq \frac{\epsilon}{2} |U_R(z, d)| = \frac{\epsilon}{2} |\mathbb{S}^{n-1}| d^{n-1}.$$

Hence, we can exclude the scenario (2.8) by choosing

$$\mu_\epsilon = \frac{\epsilon}{2^n}|\mathbb{S}^{n-1}|\left(\min\left\{1, \left(\frac{\epsilon}{2C_4}\right)^{\frac{1}{\alpha}}\right\}\right)^{n-1}. \tag{2.9}$$

Next, consider a finite family of geodesic balls $U_R(x_i,1)_{i\in I_R}$ such that

$$U_R\left(x_i,\frac{1}{4}\right)\cap U_R\left(x_k,\frac{1}{4}\right)=\emptyset \ \text{ if } \ i\neq k, \tag{2.10}$$

$$\bigcup_{i\in I_R} U_R(x_i,1)\supset \partial B_{S_R}, \tag{2.11}$$

for all $R\geq 1$ (having suppressed the obvious dependence of x_i on R). This is indeed possible by the Vitali's covering theorem (see [22, Sec. 1.5] and keep in mind that ∂B_{S_R} becomes a metric space when equipped with the geodesic distance). We say that the ball $U_R(x_i,1)$ is a **good ball** if

$$\int_{U_R(x_i,2)} e(x)dS(x)<\mu_\epsilon,$$

and that $U_R(x_i,1)$ is a **bad ball** if

$$\int_{U_R(x_i,2)} e(x)dS(x)\geq\mu_\epsilon.$$

The collection of bad balls is labeled by

$$J_R=\{i \ : \ U_R(x_i,1) \text{ is a bad ball}\}.$$

The main observation is that, by virtue of (2.10), there is a universal constant $C_5>0$ (independent of both ϵ and R) such that

$$\sum_{i\in I_R}\int_{U_R(x_i,2)} e(x)dS(x)\leq C_5\int_{\partial B_{S_R}} e(x)dS(x),$$

since each point on ∂B_{S_R} is covered by at most C_5 geodesic balls $U_R(x_i,2)$. The latter property plainly follows by observing that all such balls that contain the same point are certainly contained in a geodisc ball of radius 10, and from the basic fact that any $(n-1)$-dimensional ball of radius 10 can contain only a finite number of disjoint balls of radius $\frac{1}{4}$. Making use of (2.5), we infer that

$$\text{card}J_R\leq\frac{C_5C_3}{\mu_\epsilon}R^{n-2}, \quad R\geq 1. \tag{2.12}$$

Now, let $x\in\partial B_{S_R}\setminus\bigcup_{i\in J_R}U_R(x_i,1)$. By (2.11), there exists some $k\in I_R\setminus J_R$ such that $x\in U_R(x_k,1)$ which is a good ball. It follows from the definition of μ_ϵ that

$$e(x)\leq\epsilon,$$

as desired.

In view of (1.4) and (2.6), we have that

$$|\nabla u(x)|^2\leq 2\epsilon \ \text{ and } \ |u(x)-a|\leq m_\epsilon \ \text{ if } \ x\in\partial B_{S_R}\setminus\bigcup_{i=1}^{N_{\epsilon,R}}U_R(x_{R,i},1), \quad R\gg 1, \tag{2.13}$$

where

$$m_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{2.14}$$

(we point out that m_ϵ depends only on ϵ).

We consider the function $v_R \in W^{1,2}(B_{S_R};\mathbb{R}^m) \cap L^\infty(B_{S_R};\mathbb{R}^m)$ which is defined, in terms of polar coordinates, as follows:

$$v_R(r,\theta) = \begin{cases} u(S_R,\theta) + (a - u(S_R,\theta))(S_R - r), & r \in [S_R - 1, S_R],\ \theta \in \mathbb{S}^{n-1}, \\ a, & r \in [0, S_R - 1],\ \theta \in \mathbb{S}^{n-1}, \end{cases}$$

(having slightly abused notation, keep in mind that $x = r\theta$). We note that v_R belongs in $W^{1,2}$ because it is the composition of a smooth function with a Lipschitz continuous one (see [33, pg. 54] and keep in mind that we only use the polar coordinates away from the origin). Clearly, we have that

$$v_R = u \quad \text{on} \quad \partial B_{S_R}. \tag{2.15}$$

Let

$$\mathcal{A}_R = B_{S_R} \setminus B_{(S_R-1)} \quad \text{and} \quad \mathcal{C}_R = \bigcup_{i=1}^{N_{\epsilon,R}} \left(\bar{B}_{10}(x_{R,i}) \cap \bar{A}_R \right).$$

If $x = r\theta \in \mathcal{A}_R \setminus \mathcal{C}_R$, via (2.13), it holds that

$$|v_R(x) - a| \le 2\,|u(S_R,\theta) - a| \le 2m_\epsilon. \tag{2.16}$$

Moreover, for such x, we find that

$$\begin{aligned} |\nabla_{\mathbb{R}^n} v_R|^2 &= |u(S_R,\theta) - a|^2 + \tfrac{1}{r^2}\,|(1 + r - S_R)\nabla_{\mathbb{S}^{n-1}} u(S_R,\theta)|^2 \\ \text{using (2.4), (2.13)}: \quad &\le m_\epsilon^2 + \tfrac{9}{S_R^2}\,|\nabla_{\mathbb{S}^{n-1}} u(S_R,\theta)|^2 \\ &\le m_\epsilon^2 + 9\,|\nabla_{\mathbb{R}^n} u(S_R\theta)|^2 \\ \text{using again (2.13)}: \quad &\le m_\epsilon^2 + 18\epsilon, \end{aligned} \tag{2.17}$$

where we made repeated use of the identity

$$|\nabla_{\mathbb{R}^n} v|^2 = |v_r|^2 + \frac{1}{R^2}|\nabla_{\mathbb{S}^{n-1}} v|^2 \quad \text{on } \partial B_R, \quad R > 0,$$

(see [43, Ch. 8]). It follows that

$$\begin{aligned} \int_{B_{S_R}} \left\{ \tfrac{1}{2}|\nabla v_R|^2 + W(v_R) \right\} dx &= \int_{\mathcal{A}_R} \left\{ \tfrac{1}{2}|\nabla v_R|^2 + W(v_R) \right\} dx \\ \text{using (2.2)}: \quad &\le C_6 N_{\epsilon,R} + \int_{\mathcal{A}_R \setminus \mathcal{C}_R} \left\{ \tfrac{1}{2}|\nabla v_R|^2 + W(v_R) \right\} dx \\ \text{using (2.16), (2.17)}: \quad &\le C_6 N_{\epsilon,R} + \left(\tfrac{m_\epsilon^2}{2} + 9\epsilon + C_7 m_\epsilon \right) |\mathcal{A}_R \setminus \mathcal{C}_R| \\ &\le C_6 N_{\epsilon,R} + C_8 (m_\epsilon + \epsilon) S_R^{n-1}, \end{aligned}$$

where $C_6, C_7, C_8 > 0$ are independent of both ϵ and R. Since u is a globally minimizing solution, thanks to (2.15), we obtain that

$$\begin{aligned} \textstyle\int_{B_{S_R}} e(x)dx &\leq C_6 N_{\epsilon,R} + C_8(m_\epsilon+\epsilon)S_R^{n-1} \\ \text{using (2.4), (2.7)}: &\leq 2^{n-2}C_6 M_\epsilon R^{n-2} + 2^{n-1}C_8(m_\epsilon+\epsilon)R^{n-1}, \quad R \gg 1. \end{aligned} \tag{2.18}$$

Since $\epsilon > 0$ is arbitrary, in light of (2.14), we infer that (1.5) holds, as desired. □

2.2. Proof of Theorem 1.2.

Proof. Case (a) If u satisfies (1.6), since $W \geq 0$, it is known that the following strong monotonicity formula holds:

$$\frac{d}{dR}\left(\frac{1}{R^{n-1}}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) \geq 0, \quad R > 0, \tag{2.19}$$

(see [16], [36] for $m = 1$ and [3], [18] for arbitrary $m \geq 1$). We point out that the fact that u is minimal is not used for this. Hence, for any positive $r < R$, we have that

$$\frac{1}{r^{n-1}}\int_{B_r}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx \leq \frac{1}{R^{n-1}}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx.$$

By virtue of Theorem 1.1, letting $R \to \infty$ in the above relation yields that $u \equiv a$.

To complete the proof in this case, we note that the gradient estimate (1.6) was shown in [24] to hold for *any* bounded, entire solution when $m = 1$ and $W \in C^{1,1}_{loc}(\mathbb{R};\mathbb{R})$ is nonnegative (see [16], [35] for earlier proofs which required higher regularity on W). Lastly, it is easy to show that any radially symmetric solution satisfies this gradient bound for any $m \geq 1$ and $W \in C^1$ nonnegative (see [40]).

Case (b) Here we partly follow [40]. Since $n = 2$, by working as in (2.5), and using the assertion of Theorem 1.1, we arrive at

$$\int_{\partial B_{S_R}} W(u(x))\,dS(x) \to 0, \quad \text{for some } \ S_R \in (R, 2R), \quad \text{as } \ R \to \infty.$$

By making use of just the C^1-bound in (2.2), and working as we did in order to exclude (2.8), we deduce that

$$\max_{|x|=S_R}|u(x) - a| \to 0 \ \text{ as } \ R \to \infty. \tag{2.20}$$

Under the assumptions of the first part of Case (b), a recent variational maximum principle from [6] implies that

$$\max_{|x|\leq S_R}|u(x) - a| \leq \max_{|x|=S_R}|u(x) - a|,$$

(see also Appendix A herein). Moreover, as we will prove in Appendix A, this variational maximum principle also holds under the assumptions of the second part of Case (b). In light of (2.20), by letting $R \to \infty$ in the above relation, we can conclude that the first two assertions in Case (b) hold.

We will establish the validity of the last assertion of Case (b) by borrowing some ideas from [44], while adopting a more explanatory viewpoint. To this end, we will argue by contradiction. Without loss of generality, we may assume that there exists a sequence $R_j \to \infty$ and a $\delta > 0$ such that

$$u(x_j) = \max_{|x| \leq S_{R_j}} u(x) \geq a + \delta, \quad j \geq 1,$$

for some $x_j \in B_{S_{R_j}}$. In particular, there exists $d \in (0, \delta)$ such that

$$W(a+d) < W\left(u(x_j)\right), \quad j \geq 1.$$

By virtue of (2.20), we may further assume that

$$\max_{|x| = S_{R_j}} u(x) \leq a + \frac{d}{2}, \quad j \geq 1. \tag{2.21}$$

Let $u_j \in [a+d, u(x_j))$ be such that

$$W(u_j) = \min_{u \in [a+d, u(x_j)]} W(u). \tag{2.22}$$

Consider the competitor function

$$V_j(x) = \min\left\{u(x), u_j\right\}, \quad x \in B_{S_{R_j}},$$

which belongs in $W^{1,2}\left(B_{S_{R_j}}; \mathbb{R}^m\right) \cap L^\infty\left(B_{S_{R_j}}; \mathbb{R}^m\right)$ (see for instance [20]) and, thanks to (2.21), agrees with u on $\partial B_{S_{R_j}}$. To conclude, we will show that

$$E\left(V_j; B_{S_{R_j}}\right) < E\left(u; B_{S_{R_j}}\right),$$

which contradicts the minimality character of u. To this aim, let

$$\mathcal{D}_j = \left\{x \in B_{S_{R_j}} \ : \ u(x) > u_j\right\}.$$

Observe that $\mathcal{D}_j$ is nonempty (since it contains x_j) and strictly contained in $B_{S_{R_j}}$ (from (2.21)). Then, note that

$$E\left(V_j; B_{S_{R_j}} \setminus \mathcal{D}_j\right) = E\left(u; B_{S_{R_j}} \setminus \mathcal{D}_j\right) \ \text{ and } \ E\left(V_j; \mathcal{D}_j\right) = E\left(u_j; \mathcal{D}_j\right) < E\left(u; \mathcal{D}_j\right),$$

since (2.22) holds and there exists a connected component $\mathcal{E}_j$ of $\mathcal{D}_j$, say the one containing x_j, where u is nonconstant (note that $u = u_j$ on $\partial \mathcal{D}_j$). □

Remark 2.1. *We refer to* [29] *for a class of systems (1.1) of Allen-Cahn type whose solutions satisfy Modica's gradient bound (1.6).*

Remark 2.2. *To the best of our knowledge, there are no counterexamples to Modica's gradient bound for systems of Allen-Cahn type in the case of minimizing solutions. In this regard, we refer the interest reader to* [38].

Remark 2.3. *The assertion of Theorem 1.2 holds* for any entire, bounded solution *of (1.1), provided that W is assumed to be globally convex (see for example* [41]*).*

Appendix A. On a maximum principle for vector minimizers to the Allen-Cahn energy

In the recent paper [6], the authors proved the following variational maximum principle:

Theorem A.1. *Let $W : \mathbb{R}^m \to \mathbb{R}$ be C^1 and* nonnegative*. Assume that $W(a) = 0$ for some $a \in \mathbb{R}^m$ and that there is $r_0 > 0$ such that (1.7) holds. Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded set, with $\partial\Omega$ minimally smooth (Lipschitz continuous is enough), and suppose that $u \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ is minimal, in the sense that (1.3) is satisfied.*

If there holds

$$|u(x) - a| \le r \quad \text{for } x \in \partial\Omega, \tag{A.1}$$

for some $r \in \left(0, \frac{r_0}{2}\right)$, then it also holds that

$$|u(x) - a| \le r \quad \text{for } x \in \Omega. \tag{A.2}$$

The main idea of the proof is that if the assertion is violated at some point, then one can construct a suitable competitor function which agrees with u on $\partial\Omega$ and has *strictly less* energy, which is impossible.

In this appendix, under the slight additional regularity assumption that $W \in C^{1,1}_{loc}$ (which is consistent with most applications), we will show that one can conclude just by showing that the aforementioned competitor function has *less or equal* energy. Our main observation is to apply the unique continuation principle for linear elliptic systems (see [34] for other applications). As a result, under the slight additional assumption that $W \in C^{1,1}_{loc}$, we can simplify the corresponding proof in [6]. Moreover, we can allow for the functions in (1.7) to be merely nondecreasing which is crucial for establishing the second assertion of Case (b) in Theorem 1.2. More precisely, we have the following theorem.

Theorem A.2. *Assume that $W : \mathbb{R}^m \to \mathbb{R}$ is $C^{1,1}_{loc}$, nonnegative, such that $W(a) = 0$ for some $a \in \mathbb{R}^m$ and that the functions in (1.7) are nondecreasing. Moreover, assume that*

$$W(u) > 0 \quad \text{if} \quad |u - a| < 2r_0 \quad \text{and} \quad u \ne a.$$

Then, the assertion of Theorem A.1 remains true.

Proof. Firstly, by standard elliptic regularity theory, we have that u is a smooth solution to the elliptic system in (1.1) in Ω and continuous up to the boundary (under reasonable assumptions on $\partial\Omega$). Without loss of generality, we take $a = 0$. As in [6], we set

$$\rho(x) = |u(x)| \text{ in } \Omega \text{ and } \nu(x) = \frac{u(x)}{\rho(x)} \text{ in } \Omega_+ = \{x \in \Omega \ : \ \rho > 0\}.$$

We also set $\Omega_0 = \{x \in \Omega \ : \ \rho = 0\}$. It has been shown in [6] that the energy of u equals

$$E(u; \Omega) = \frac{1}{2}\int_\Omega |\nabla\rho|^2 dx + \frac{1}{2}\int_{\Omega_+} \rho^2 |\nabla\nu|^2 dx + \int_\Omega W(\rho\nu) dx.$$

Let

$$\tilde{u}(x) = \begin{cases} \min\{\rho(x), r\}\, \alpha\left(\rho(x)\right) \nu(x), & x \in \Omega_+ \cap \{\rho < 2r\}, \\ 0, & x \in \Omega_0 \cup \{\rho \ge 2r\}, \end{cases}$$

where $\alpha(\cdot)$ is the auxiliary function

$$\alpha(\tau) = \begin{cases} 1, & \tau \leq r, \\ \frac{2r-\tau}{r}, & r \leq \tau \leq 2r, \\ 0, & \tau \geq 2r. \end{cases}$$

It was shown in [6] that $\tilde{u} \in W^{1,2}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and that its energy equals

$$E(\tilde{u};\Omega) = \frac{1}{2}\int_\Omega |\nabla\tilde{\rho}|^2 dx + \frac{1}{2}\int_{\tilde{\Omega}_+} \tilde{\rho}^2|\nabla\nu|^2 dx + \int_\Omega W(\tilde{\rho}\nu)dx,$$

where $\tilde{\rho}(x) = |\tilde{u}(x)|$ and $\tilde{\Omega}_+ = \{x \in \Omega \ : \ \tilde{\rho} > 0\}$. Note that, thanks to (A.1), we have

$$u = \tilde{u} \ \text{ on } \ \partial\Omega \ \text{ and } \ |\tilde{u}| \leq r \ \text{ a.e. in } \ \Omega. \tag{A.3}$$

It follows readily that

$$E(\tilde{u};\Omega) \leq E(u;\Omega),$$

see also the proof in [6]. Consequently, $\tilde{u}$ is also a minimizer in Ω subject to the same boundary conditions as u. Hence, the function $\tilde{u}$ is smooth and satisfies

$$\Delta\tilde{u} = W_u(\tilde{u}) \ \text{ in } \ \Omega.$$

We are now set to show that assertion (A.2) holds. Suppose, to the contrary, that

$$|u(x_0)| > r \ \text{ for some } \ x_0 \in \Omega. \tag{A.4}$$

We will first exclude the case

$$r \leq \rho(x) \leq 2r \ \text{ for all } \ x \in \Omega.$$

If not, the function

$$\hat{u} = r\nu(x) \in W^{1,2}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$$

would have strictly less energy then u (because $\int_\Omega |\nabla\rho|^2 dx > 0$) while $\hat{u} = u$ on $\partial\Omega$, which is impossible. Next, we exclude entirely the case

$$r \leq \rho(x), \quad x \in \Omega.$$

If not, there would exist $x_1 \in \Omega$ such that $\rho(x_1) > 2r$. This implies that $\tilde{u} = 0$ on a set of positive measure containing x_1. Since $W_u(u)$ is locally Lipschitz continuous, we see that $\tilde{u}$ satisfies the linear system

$$\Delta\tilde{u} = Q(x)\tilde{u} \ \text{ in } \ \Omega, \ \text{ for some } \ Q(x) \in L^\infty(\Omega;\mathbb{R}^{m\times m}).$$

On the other hand, because $\tilde{u} = 0$ on a set of positive measure, by the unique continuation principle for linear elliptic systems (see [32]), we infer that $\tilde{u} \equiv 0$ which is clearly impossible (otherwise $|u| \geq 2r$ in Ω). Therefore, we may assume that there exists a set $D \subset \Omega$ with positive measure such that

$$u = \tilde{u} \ \text{ in } \ D.$$

As before, by considering the linear system for the difference $u - \tilde{u}$, we conclude that $\tilde{u} \equiv u$. We have thus arrived at a contradiction, because of (A.3) and (A.4). □

Remark A.3. *The above approach may also be applied to the more general variational maximum principle in* [5].

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