A WEIGHTED HARDY-SOBOLEV-MAZ'YA INEQUALITY

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Abstract. We provide a weighted extension of a Hardy-Sobolev-Maz'ya inequality that is due to Filippas, Maz'ya and Tertikas.

If Ω is a smooth, bounded and convex domain in $\mathbb{R}^n$, $n \geq 3$, it has been proven in [2] that there exists a positive constant C such that

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C\left(\int_\Omega |u|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} \quad \forall\ u \in C_0^\infty(\Omega), \tag{0.1}$$

where $d(x) = \text{dist}(x, \partial\Omega)$ (see also [3] for more general results). In fact, the constant depends only on n, as has been proven in [4].

In this note, based on the above inequality, we will prove the following weighted extension.

Theorem 0.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a smooth bounded and convex domain. It holds that*

$$\int_\Omega d^\beta |\nabla u|^2 dx - \frac{(1-\beta)^2}{4}\int_\Omega d^\beta \frac{u^2}{d^2}dx \geq C\left(\int_\Omega d^{\frac{\beta n}{n-2}}|u|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} \quad \forall\ u \in C_0^\infty(\Omega), \tag{0.2}$$

for every $u \in C_0^\infty(\Omega)$ and $\beta \leq 0$, where C is as in (0.1).

Inequality (0.2) has been shown to hold for $C = 0$ in [1] for $\beta < 1$ (see also [6] for the case of mean convex domains). It has been shown there that $(1-\beta)^2/4$ is the best constant. We believe that the inequality (0.2) also holds for $\beta < 1$.

As will be apparent from the proof, our approach can easily provide weighted versions of other similar inequalities in convex or mean convex domains.

Proof of Theorem 0.1. It is well known that $-d(\cdot)$ is a convex Lipschitz function. Therefore, $-\Delta d$ is a nonnegative Radon measure, that is

$$\int_\Omega \nabla d \nabla \varphi dx = -\int_\Omega \varphi(\Delta d)dx \ \ \forall \varphi \in C_0^\infty(\Omega), \tag{0.3}$$

and

$$-\Delta d \geq 0 \quad \text{in the sense of distributions.} \tag{0.4}$$

For more details, we refer to [5, 6] (in fact, these references consider mean convex domains).

For every $u \in C_0^\infty(\Omega)$ and $\alpha \leq 0$, we set

$$v = d^\alpha u. \tag{0.5}$$

We have

$$\nabla v = \alpha d^{\alpha-1} u \nabla d + d^\alpha \nabla u.$$

So, using the well known property that $|\nabla d| = 1$ almost everywhere in Ω, we find that

$$|\nabla v|^2 = \alpha^2 d^{2\alpha-2}u^2 + d^{2\alpha}|\nabla u|^2 + \alpha d^{2\alpha-1}\nabla d \nabla(u^2),$$

a.e in Ω. Next, we write

$$|\nabla v|^2 = \alpha^2 d^{2\alpha-2}u^2 + d^{2\alpha}|\nabla u|^2 + \alpha\nabla\left(d^{2\alpha-1}u^2\nabla d\right) - \alpha\nabla\left(d^{2\alpha-1}\nabla d\right)u^2.$$

Then, since

$$\nabla\left(d^{2\alpha-1}\nabla d\right) = (2\alpha-1)d^{2\alpha-2} + d^{2\alpha-1}\Delta d \overset{(0.4)}{\leq} (2\alpha-1)d^{2\alpha-2}, \tag{0.6}$$

and $\alpha \leq 0$, we deduce that

$$|\nabla v|^2 \leq d^{2\alpha}|\nabla u|^2 + (\alpha-\alpha^2)d^{2\alpha-2}u^2 + \alpha\nabla\left(d^{2\alpha-1}u^2\nabla d\right). \tag{0.7}$$

Finally, substituting (0.5) and (0.7) into (0.1), using (0.3), and rearranging terms, we arrive at

$$\int_\Omega d^{2\alpha}|\nabla u|^2 dx - \left(\frac{1}{4}+\alpha^2-\alpha\right)\int_\Omega d^{2\alpha-2}u^2 dx \geq C\left(\int_\Omega |d^\alpha u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

The assertion of the theorem now follows by letting $\alpha = \frac{\beta}{2}$. □

Remark 0.1. Let

$$\underline{\mathcal{H}} = \min_{x\in\partial\Omega}\mathcal{H}(x) \geq 0,$$

where $\mathcal{H}(x) \geq 0$ stands for the mean curvature of $\partial\Omega$ at a point $x \in \partial\Omega$. The following properties hold in the sense of distributions:

$$\Delta d \leq -(n-1)\underline{\mathcal{H}} \quad \text{and} \quad \Delta d \leq -\frac{4}{n}\underline{\mathcal{H}}^2 d,$$

see [5, 6]. If we use these refinements in (0.6), instead of the rough estimate $\Delta d \leq 0$, the righthand side of (0.2) may be strengthened by the addition of the nonnegative terms

$$-\frac{\beta}{2}(n-1)\underline{\mathcal{H}}\int_\Omega d^{\beta-1}u^2 dx \quad \text{and} \quad -\frac{2\beta}{n}\underline{\mathcal{H}}^2\int_\Omega d^\beta u^2 dx$$

respectively.

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