A LIOUVILLE THEOREM FOR MINIMIZERS WITH FINITE POTENTIAL ENERGY FOR THE VECTORIAL ALLEN-CAHN EQUATION

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Abstract. We prove that if a globally minimizing solution to the vectorial Allen-Cahn equation has finite potential energy, then it is a constant.

Consider the semilinear elliptic system

$$\Delta u = \nabla W(u) \text{ in } \mathbb{R}^n, \quad n \geq 1, \tag{0.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and nonnegative. It has been recently shown in [1] that each nonconstant solution to the system (0.1) satisfies:

$$\int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx \geq \begin{cases} cR^{n-2} & \text{if } n \geq 3, \\ c \ln R & \text{if } n = 2, \end{cases} \tag{0.2}$$

for all $R > 1$, and some $c > 0$, where B_R stands for the n-dimensional ball of radius R, centered at the origin.

On the other side, if additionally W vanishes at least at one point, it is easy to see that

$$\int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx \leq CR^{n-1}, \quad R > 0, \tag{0.3}$$

for some $C > 0$ (see [4]).

The system (0.1) with $W \geq 0$ vanishing at a finite number of global minima (typically nondegenerate), and coercive at infinity, is used to model multi-phase transitions (see [4] and the references therein). In this case, the system (0.1) is frequently referred to as the vectorial Allen-Cahn equation. In [7], we showed the following theorem for globally minimizing solutions (see [5, 7] for the precise definition).

Theorem 0.1. Assume that $W \in C^1(\mathbb{R}^m; \mathbb{R})$, $m \geq 1$, and that there exist finitely many $N \geq 1$ points $a_i \in \mathbb{R}^m$ such that

$$W(u) > 0 \text{ in } \mathbb{R}^m \setminus \{a_1, \cdots, a_N\}, \tag{0.4}$$

and there exists small $r_0 > 0$ such that the functions

$$r \mapsto W(a_i + r\nu), \text{ where } \nu \in \mathbb{S}^1, \text{ are strictly increasing for } r \in (0, r_0), \ i = 1, \cdots, N. \tag{0.5}$$

Moreover, we assume that

$$\liminf_{|u| \to \infty} W(u) > 0. \tag{0.6}$$

If $u \in C^2(\mathbb{R}^2; \mathbb{R}^m)$ is a bounded, nonconstant, and globally minimizing solution to the elliptic system (0.1) with $n = 2$, there exist constants $c_0, R_0 > 0$ such that

$$\int_{B_R} W(u(x))\, dx \geq c_0 R \text{ for } R \geq R_0.$$

In view of (0.3), the above result captures the optimal growth rate in the case $n = 2$. The purpose of this note is to establish the following Liouville type theorem which holds in any dimension. Similarly to [7], we combine ideas from the study of vortices in the Ginzburg-Landau model [3] with variational maximum principles from the study of the vector Allen-Cahn equation [2].

Theorem 0.2. Let W be as in Theorem 0.1. Suppose that $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$, $n \geq 2$, is a bounded and globally minimizing solution to the elliptic system (0.1) such that

$$\int_{\mathbb{R}^n} W\left(u(x)\right) dx < \infty.$$

Then, we have that

$$u \equiv a_i \ \text{ for some } i \in \{1, \cdots, N\}.$$

Proof. It follows that there exists a constant $C_0 > 0$ such that

$$\int_{B_R} W\left(u(x)\right) dx \leq C_0, \quad R > 0. \tag{0.7}$$

Let

$$\varepsilon = \frac{1}{R} \ \text{ and } \ u_\varepsilon(y) = u\left(\frac{y}{\varepsilon}\right), \quad y \in B_1.$$

Then, relation (0.7) becomes

$$\int_{B_1} W\left(u_\varepsilon(y)\right) dy \leq C_1 \varepsilon^n, \quad \varepsilon > 0, \tag{0.8}$$

for some $C_1 > 0$. Note that, by standard elliptic regularity estimates [6], we have that

$$|u_\varepsilon| + \varepsilon |\nabla u_\varepsilon| \leq C_2 \ \text{ in } \mathbb{R}^n, \ \varepsilon > 0, \tag{0.9}$$

for some $C_2 > 0$.

Let $d > 0$ be any small number. As in [3], by combining (0.8) and (0.9), we deduce that the set where $W(u_\varepsilon)$ is above $d > 0$ is included in a uniformly bounded number of balls of radius ε, as $\varepsilon \to 0$. Certainly, there exists $r_\varepsilon \in (\frac{1}{4}, \frac{3}{4})$ such that

$$W\left(u_\varepsilon(y)\right) \leq d \ \text{ if } |y| = r_\varepsilon.$$

Since $d > 0$ is arbitrary, we are led to $\tilde{r}_\varepsilon \in (\frac{1}{4}, \frac{3}{4})$ such that

$$\max_{|y| = \tilde{r}_\varepsilon} W\left(u_\varepsilon(y)\right) \to 0 \ \text{ as } \varepsilon \to 0.$$

In terms of u and R, we have

$$\max_{|x| = s_R} W\left(u(x)\right) \to 0 \ \text{ as } R \to \infty, \ \text{ for some } s_R \in \left(\frac{1}{4}R, \frac{3}{4}R\right).$$

In view of (0.6), the above relation implies that there exist $i_j \in \{1, \cdots, N\}$ such that

$$\max_{|x| = s_R} \left|u(x) - a_{i_j}\right| \to 0 \ \text{ as } R \to \infty.$$

By virtue of (0.5), as in [7], exploiting the fact that u is a globally minimizing solution, we can apply a recent variational maximum principle from [2] to deduce that

$$\max_{|x| \leq s_R} \left|u(x) - a_{i_j}\right| \leq \max_{|x| = s_R} \left|u(x) - a_{i_j}\right| \ \text{ for } \ R \gg 1.$$

The above two relations imply the existence of an $i_0 \in \{1, \cdots, N\}$ such that

$$\max_{|x| \le s_R} |u(x) - a_{i_0}| \to 0 \quad \text{as} \quad R \to \infty.$$

Since $s_R \to \infty$ as $R \to \infty$, we conclude that $u \equiv a_{i_0}$. □

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