RESEARCH STATEMENT

CHRISTOS SOURDIS

My research interests are singular perturbation problems for differential equations, modelling phase separation. I have mainly considered slow-fast Hamiltonian systems and semilinear elliptic equations involving corner layers. The mathematical tools that I employ include the blow-up approach to geometric singular perturbation theory [42, 66], and infinite–dimensional Lyapunov Schmidt reductions [37, 71]. The physical motivation is diverse, stemming mainly from the study of crystalline grain boundaries, spatial ecology, and Bose-Einstein condensation.

In parallel, I am also interested in the qualitative properties of entire solutions to a class of semilinear elliptic partial differential equations that arise in phase transitions such as the well known Allen-Cahn equation and its vectorial counterpart. The motivation stems mainly from De Giorgi's conjecture [48] which uncovers a deep relation of these problems with the theory of minimal surfaces. In the scalar case, the main mathematical techniques that I employ for this purpose are based on applications of the maximum principle. In the vector case, modelling multiphase transitions, where this fundamental techique is no longer available, I mostly employ variational terchniques.

Recently, I have been interested in systems of hyperbolic conservation laws and their regularization through the addition of a self-similar viscosity (see [33]). In particular, I am interested in the formation of delta shocks in the corresponding singularly perturbed system of ordinary differential equations.

More recently, I have been interested in elliptic systems describing phase separation.

Description of obtained results

- N.D. Alikakos, P.C. Fife, G. Fusco, and C. Sourdis, *Analysis of the heteroclinic connection in a singularly perturbed system arising from the study of crystalline grain boundaries*, Interfaces Free Bound. **8** (2006), 159-183.

Mathematically, the problem considered here is that of heteroclinic connections for the system

$$\begin{aligned} \varepsilon^2 u'' &= g_u(u,v), \\ v'' &= g_v(u,v), \end{aligned} \tag{1}$$

where u, v are functions of x such that

$$(u,v) \to (u_\pm, v_\pm) \ \text{ as } \ x \to \pm\infty,$$

where the C^2 function $g : \mathbb{R}^2 \to [0,\infty)$ attains its global minima at the points $(u_\pm, v_\pm)$ (and perhaps elsewhere), which are assumed to be nondegenerate. The small parameter $\varepsilon > 0$ conveys a singular perturbation. The main assumption is that the level set $g_u = 0$ has a smooth connected component, containing $(u_\pm, v_\pm)$, whereon $g_{uu} > 0$. These solutions give rise to standing planar waves to a system of reaction-diffusion equations. The motivation comes from a multi-order-parameter phase field model developed by J. Cahn and his collaborators in [22] for the description of crystalline interphase boundaries (see also [8]). In this, the smallness of ε is related to large anisotropy. The existence of such a heteroclinic, and its dependence on the small parameter, is proved by a perturbation argument based on the contraction mapping principle. In addition, its stability is established within the context of the associated evolutionary phase field equations. Our approach uses a Lyapunov-Schmidt reduction procedure tailored to scalar elliptic singular perturbation problems (see [14], [80]). Since the problem at hand is a slow-fast system (see [64]), the analysis is not standard and there arise some extra difficulties (see also [56]).

- N.D. Alikakos, P.C. Fife, G. Fusco, and C. Sourdis, *Singular perturbation problems arising from the anisotropy of crystalline grain boundaries*, J. Dyn. Diff. Equations **19** (2007), 935-949.

In this paper, we reprove the same result of our previous paper by using geometric singular perturbation theory for ordinary differential equations (see [50], [64]). The heteroclinic orbit lies on a slow manifold which depends smoothly on the small parameter $\varepsilon > 0$ up to $\varepsilon = 0$. This reduces the singular perturbation problem to a regular perturbation problem on the slow manifold.

- C. Sourdis, and P.C. Fife, *Existence of heteroclinic orbits for a corner layer problem in anisotropic interfaces*, Adv. Differential Equations **12** (2007), 623-668.

In this paper, we consider the same problem as in the two above papers but now we seek standing planar waves oriented in a different direction of the crystal. The major difference is that now the sought after heteroclinic should lie close to a slow manifold, determined from the set $g_u = 0$, which has a pitchfork bifurcation along a submanifold. This prohibits the use of standard geometric singular perturbation theory. Nevertheless, we succeed in showing the existence of the desired “corner layered” heteroclinic orbit, and studying its behavior as $\varepsilon \to 0$, by introducing a non-standard perturbation scheme. Our result makes rigorous the formal asymptotic analysis of [8], which revealed that the fine behavior of the layer should be described by the Hastings-Mcleod solution of the Painlevé-II equation, i.e. the unique solution of the boundary value problem

$$u_{xx} - u(u^2 + x) = 0, \quad x \in \mathbb{R}; \;\; u(x) - \sqrt{-x} \to 0, \; x \to -\infty, \;\; u(x) \to 0, \; x \to \infty, \quad (2)$$

(see [58]). In particular, our main effort is in matching an inner solution, furnished by the Hastings-McLeod solution, with appropriate outer ones. This allows us to construct a good approximation to the problem. The fact that the Hastings-McLeod solution is non-degenerate, after some work, is translated in invertibility of the associated linearized operator to (1) on the approximate solution. In turn, this allows us to apply an unconventional implicit function type argument to capture a true solution near this approximation.

Let us mention that the behavior of the corner layered orbit, as $\varepsilon \to 0$, was studied by P. Fife in [51] by using a shooting argument. In contrast to our scheme, his approach used a monotonicity property of (1), due to the special form of g, and seems difficult to generalize. Moreover, no connection with the Hastings-Mcleod solution was made in [51].

- S. Schecter, and C. Sourdis, *Heteroclinic orbits in slow-fast Hamiltonian systems with slow manifold bifurcations*, J. Dyn. Diff. Equat. **22** (2010), 629-655.

Motivated by the previously discussed paper of mine with P. Fife, we consider a class of slow-fast Hamiltonian systems of the form (1) in which the slow manifold loses normal hyperbolicity due to a transcritical or pitchfork bifurcation as a slow variable changes. We show that under assumptions appropriate to the motivating problem, a singular heteroclinic solution gives rise to a true heteroclinic solution. In contrast to previous approaches to such problems, our approach uses blow-up of the bifurcation manifold, which allows geometric matching of inner and outer solutions. In particular, using the blow-up procedure for dealing with loss of normal hyperbolicity of the slow manifold [42], [66], we blow up the bifurcation line to a sphere crossed with a line. The inner solution, corresponds to a special solution of the blown-up system that connects two equilibria on one of these spheres. Loosely speaking, the latter special solution is the Hastings-McLeod solution (2) in the case of pitchfork bifurcation, and the unique positive solution of

$$u_{xx} = u^2 - x^2,\ x \in \mathbb{R};\quad u(x) - |x| \to 0 \text{ as } |x| \to \infty, \tag{3}$$

in the case of transcritical bifurcation. We emphasize that the connections on the sphere is transverse because the Hastings-McLeod solution as well as the positive solution of (3) are non-degenerate. Matching then becomes a geometric problem of identifying the behavior of certain manifolds of solutions, which is resolved using the so called "corner lemma", see [76]. Besides being more intuitive, the geometric approach produces optimal estimates which improve upon those from my paper with P. Fife (in the case of the pitchfork bifurcation) and those of [26] (in the case of transcritical bifurcation). In the latter reference, a special monotonicity property was assumed on (1) and no connection with (3) was made (recall the related discussion at the end of the above review of my paper with P. Fife).

- G. Karali, and C. Sourdis, *Radial and bifurcating non-radial solutions for a singular perturbation problem in the case of exchange of stabilities*, Ann. I. H. Poincaré - AN **29** (2012), 131-170.

We consider the singular perturbation problem $\varepsilon^2\Delta u = (u - a(r))\,(u - b(r))$ in the unit ball of $\mathbb{R}^N$, $N \geq 1$, under Neumann boundary conditions. We note that if $b = 0$, then the problem is equivalent to the study of steady state solutions of the well known logistic reaction-diffusion equation $u_t = \Delta u + u\,(a(x) - u)$ from spatial ecology (see [29]). The assumption that $a(r) - b(r)$ changes sign in $(0, 1)$, known as the case of exchange of stabilities, is the main source of difficulty. Note that this implies that the zero set of the corresponding limit $\varepsilon = 0$ algebraic equation admits a bifurcation. Under the assumption that $a - b$ has one simple zero in $0, 1)$ (i.e. the bifurcation is transcritical), we prove the existence of two radial solutions u_+ and u_- that converge uniformly to $\max\{a, b\}$, as $\varepsilon \to 0$. In the context of the logistic equation, the convergence of the positive solution u_+ to $\max a, 0$, as $\varepsilon \to 0$, corresponds to spatial segregation (see for example [32]). The solution u_+ is asymptotically stable, whereas u_- has Morse index one, in the radial class. The solution u_+ is constructed by using the positive solution U_+ of (2) in order to construct an inner solution near the "interface" whereon $a = b$ (we remark that U_+ is an asymptotically stable solution of (3)). On the other side, the inner solution that we use for u_- is an unstable solution U_- of (3) with Morse index one, found as $U_- = U_+ - \varphi$ where φ solves

$$\varphi_{xx} - 2U_+(x)\varphi + \varphi^2 = 0, \quad x \in \mathbb{R}; \quad \varphi(x) \to 0 \text{ as } |x| \to \infty. \tag{4}$$

The above equation arises, when seeking standing waves of the well known nonlinear Schrödinger equation; the existence and non-degeneracy of φ, which at once imply the same properties for U_-, have been studied in [74] and [65] (we point out that $U_+ > 0$ is even and $U'(x) > 0$ for $x > 0$). Our approach for the singular perturbation problem is a considerable refinement of that in my previously discussed paper with P. Fife. In particular, we introduce a new delicate matching procedure which yields optimal estimates, as those that can be obtained via geometric singular perturbation theory, similarly to the paper of mine with S. Schecter, for the one dimensional problem. In [13], the authors constructed radial "spike layered" solutions to the semiclassical nonlinear Schrödinger equation, concentrating along a submanifold of B_1. These solutions have inner profile which is given by (4) with U_+ constant (and arbitrary power nonlinearity), but there the phenomenon is exponentially localized and gluing together inner and outer solutions is by now a standard procedure.

If $N \geq 2$, we prove that the Morse index of u_-, in the general class, is asymptotically given by $[c + o(1)]\varepsilon^{-\frac{2}{3}(N-1)}$ as $\varepsilon \to 0$, with $c > 0$ a certain positive constant. Having the precise asymptotic behavior of $u_\pm$, as $\varepsilon \to 0$, at our disposal, this task is accomplished by adapting to our setting the arguments of [41], dealing with the linearization of the spatially inhomogeneous Allen-Cahn equation on its unstable radially symmetric transition layered solution. The main difference with the latter reference is that here the radial linearized operator on $u_\pm$ has many small (but stable) eigenvalues as $\varepsilon \to 0$. Furthermore, we prove the existence of a decreasing sequence of $\varepsilon_k > 0$, with $k \to 0$ as $k \to \infty$, such that non-radial solutions bifurcate from the unstable branch $(u_-(\varepsilon)), \;\; \varepsilon > 0)$ at $\varepsilon = \varepsilon_k$, $k = 1, 2, \cdots$. We accomplish this by equivariant bifurcation theory, as in [75].

- G. Karali, and C. Sourdis, *Resonance phenomena in a singular perturbation problem in the case of exchange of stabilities*, Comm. Partial Diff. Eqns. **37** (2012), 1620–1667.

Here, restricting ourselves to two dimensions, we remove the radial symmetry assumption from our previous paper. More precisely, we consider the following singularly perturbed elliptic problem:

$$\varepsilon^2 \Delta u = (u - a(\mathbf{y}))\,(u - b(\mathbf{y})) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \;\; \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\varepsilon > 0$ is a small parameter, $\mathbf{n}$ denotes the outward normal of $\partial\Omega$, and a, b are smooth functions. We assume that the zero set of $a - b$ is a simple closed curve Γ, contained in Ω, and $\nabla(a - b) \neq 0$ on Γ. We construct solutions u_ε that converge in the Hölder sense to $\max\{a, b\}$ in Ω, and their Morse index tends to infinity, as $\varepsilon \to 0$, provided that ε stays away from certain *critical numbers*. Even in the case of stable solutions, whose existence is well established for *all* small $\varepsilon > 0$, our estimates improve previous results in [25], [61] and [69]. We adapt to our setting the infinite–dimensional Lyapunov-Schmidt reduction procedure that was introduced in [37]. As expected, the presence of the "corner layer" along the curve Γ requires some new ideas in order to implement the approach of [37], originally devised for the purpose of removing the radial symmetry assumption from the paper [13] that we mentioned earlier (see also the related approach of [70]).

- G. Karali, and C. Sourdis, *The ground state of a Gross–Pitaevskii energy with general potential in the Thomas–Fermi limit*, Archive for Rational Mechanics and Analysis, 217, no 2, 439-523, (2015).

We study the ground state which minimizes the Gross–Pitaevskii energy

$$G_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}|u|^4 + \frac{1}{2\varepsilon^2}W(\mathbf{y})|u|^2 \right\} d\mathbf{y} \tag{5}$$

in

$$\mathcal{H} \equiv \left\{ u \in W^{1,2}(\mathbb{R}^2;\mathbb{C}) : \int_{\mathbb{R}^2} W(\mathbf{y})|u|^2 d\mathbf{y} < \infty, \ \int_{\mathbb{R}^2} |u|^2 d\mathbf{y} = 1 \right\}, \tag{6}$$

with *general non-radial* trapping potential W such that $W(\mathbf{y}) \to \infty$ as $|\mathbf{y}| \to \infty$, in the Thomas–Fermi limit where the small parameter ε tends to 0. This ground state plays an important role in the mathematical treatment of recent experiments on the phenomenon of Bose–Einstein condensation, and in the study of various types of solutions of nonhomogeneous defocusing nonlinear Schrödinger equations. Many of these applications require delicate estimates for the $\varepsilon \to 0$ limiting behavior of the ground state near the boundary of the condensate, namely $W^{-1}(\lambda_0)$ where $\int_{\mathbb{R}^2}(\lambda_0 - W)^+ = 1$, in the vicinity of which the ground state has irregular behavior in the form of a steep *corner layer*. In particular, the role of this layer is important in order to detect the presence of vortices in the small density region of the condensate, understand the superfluid flow around an obstacle, and also has a leading order contribution in the energy. In contrast to previous approaches (see [1, 6, 62, 63]), we utilize a perturbation argument, adapting techniques from our previously discussed paper, to go beyond the classical Thomas–Fermi approximation and *accurately* approximate the layer by the Hastings–McLeod solution of the Painlevé–II equation (2). An additional ingredient of our proof, not present in our previous works, is that we invert the linearized operator, about the approximate solution, in suitably chosen weighted spaces, motivated from [72]. This settles an open problem (cf. [2, pg. 13 or Open Problem 8.1]), answered very recently *only* for the special case of the model harmonic potential $W(\mathbf{y}) = |\mathbf{y}|^2$ in [55]. In fact, we even improve upon previous results that relied heavily on the radial symmetry of the potential trap. Moreover, we show that the ground state has the *maximal regularity* available, namely it remains uniformly bounded in the $\frac{1}{2}$-Hölder norm, which is the *exact* Hölder regularity of the singular limit profile, as $\varepsilon \to 0$. Our study is highly motivated by an interesting open problem posed recently by Aftalion, Jerrard, and Royo-Letelier [3], and an open question of Gallo and Pelinovsky [54], concerning the removal of the radial symmetry assumption from the potential trap. We emphasize that, by a remarkable identity due to [67] (a division trick), the ground state under investigation plays an important role in the study of vortices of minimizers of the energy

$$E_\varepsilon(v) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}|v|^4 + \frac{1}{2\varepsilon^2}W(\mathbf{y})|v|^2 - \Omega \mathbf{y}^\perp \cdot (iv, \nabla v) \right\} d\mathbf{y}, \quad v \in \mathcal{H},$$

where Ω is the angular velocity, $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$, and $\mathbf{y}^\perp = (-\mathbf{y}_2, \mathbf{y}_1)$.

- *Thomas-Fermi approximation for two component Bose-Einstein condensates and nonexistence of vortices for small rotation*, Comm. Math. Physics **336** (2015), 509–579.

In this paper, we study the behavior, as $\varepsilon \to 0$, of the minimizers of the following energy functional which descibe the a pair of rotated coupled Bose-Einstein condensates.

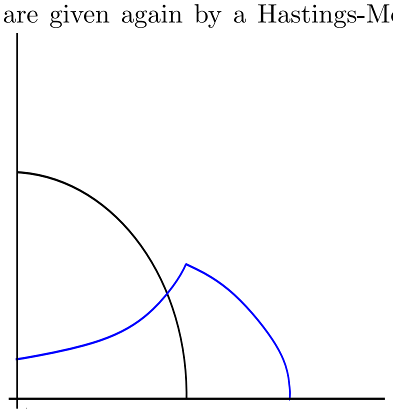
$$E_\varepsilon^\Omega(u_1,u_2)=\sum_{i=1}^{2}\int_{\mathbb{R}^2}\left\{\frac{|\nabla u_i|^2}{2}+\frac{|x|^2}{2\varepsilon^2}|u_i|^2+\frac{g_i}{4\varepsilon^2}|u_i|^4-\Omega x^\perp\cdot(iu_i,\nabla u_i)+\frac{g}{2\varepsilon^2}|u_1|^2|u_2|^2\right\}dx \tag{7}$$

in the set

$$\mathcal{H}=\left\{(u_1,u_2):\ u_i\in H^1(\mathbb{R}^2,\mathbb{C}),\ \int_{\mathbb{R}^2}|x|^2|u_i|^2\,dx<\infty,\ \|u_i\|_{L^2(\mathbb{R}^2)}=1,\ i=1,2\right\}. \tag{8}$$

The parameters g_1, g_2 and g are positive and independent of the other positive parameter ε. Here $\Omega > 0$ is the angular velocity, $x^\perp = (-x_2, x_1)$ and $\cdot$ is the real scalar product, whereas $(\ ,\ )$ is the complex scalar product. Our standing assumption is that $g^2 < g_1 g_2$. The latter assumptions allows for the two components to coexist in certain regions, as opposed to segregate.

Firstly we show that existence and uniqueness of minimizers holds (for any $\varepsilon > 0$ in fact). Then, we provide a very detailed description of the ground state when $\Omega = 0$. This is proved by linearizing around an approximate solution to construct solutions of the Euler-Lagrange equations, which (by the earlier uniqueness results) are then shown to coincide with the minimizer. The construction of the approximate solution, which relies on my previously described paper with G. Karali, is nontrivial. Moreover, the estimates needed to carry out the linearization argument, as well as other, more refined estimates, are quite delicate. A typical singular limiting profile, where both components are supported on two co-centric discs, is shown in the figure. Clerly, at the boundaries of the supports, the blow-up profile of the $\varepsilon > 0$ minimizer are given again by a Hastings-McLeod type solution (2).



Using the estimates for the unrotated ground state, and ideas from the paper [3], we can then show that the exact same ground state arises when one considers the same problem with rotational forcing below a certain critical threshold.

- C. Sourdis, *Analysis of an irregular boundary layer behavior for the steady state flow of a Boussinesq fluid*, to appear in Discrete and Continuous Dynamical Systems-A (2016).

We consider the following singular perturbation problem

$$\begin{cases}\varepsilon^2 v''=v^2-(1-x^2), & x\in(-1,1),\\[2ex] v(-1)=v(1)=0.\end{cases} \tag{9}$$

This problem arises in the study of the vertical flow of an internally heated Boussinesq fluid in a vertical channel with viscous dissipation and pressure work, see [79]. The candidate for the singular limit is $\sqrt{1-x^2}$ but its gradient blows up at the boundary. So, this problem has the same character as the ones in my previously described joint papers. However, as predicted by a formal blow-up analysis in the latter reference, the profile of the boundary layers should be governed by a solution of the following problem:

$$\begin{cases} y'' = y^2 - s, \quad s > 0, \\ \\ y(0) = 0, \qquad y - s^{\frac{1}{2}} \to 0 \text{ as } s \to \infty. \end{cases} \tag{10}$$

We note that the above ODE is the first Painlevé transcedent. Existence of two solutions, one increasing and one which changes monotonicity only once, has been shown in [60]. Subsequently, it was shown in [57] with a very involved proof (which remarkably relied on some four decimal point calculations!) that these two are in fact the only solutions. As may be anticipated from the description of my above joint papers, essentially the only thing that is missing for us to be able to treat the singular perturbation problem (9) is the nondegeneracy of these two solutions. The nondegeneracy of the increasing solution is evident. On the other hand, that of the sign changing solution is not at all clear. Nevertheless, we are able to establish the desired nondegeneracy property by relating (10) to recent studies on the radially symmetric ground states of the nonlinear Schrödinger equation in an annulus [49]. Moreover, we make a connection of (9) with the special case of the famous Lazer-McKenna conjecture that was treated in [35].

- A. Aftalion and C. Sourdis, *Interface layer of a two-component Bose-Einstein condensate*, preprint (2015).

The model problem that we consider is

$$\begin{cases} -v_1'' + v_1^3 - v_1 + \Lambda v_2^2 v_1 = 0, \\ \\ -v_2'' + v_2^3 - v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{11}$$

$$(v_1, v_2) \to (0, 1) \text{ as } z \to -\infty, \quad (v_1, v_2) \to (1, 0) \text{ as } z \to +\infty, \tag{12}$$

with coupling parameter $\Lambda > 1$. Solutions of this problem should govern the behaviour of a two-component Bose-Einstein condensate, near the interface, in the case of segregation. Mainly motivated from [4], we consider the case $\Lambda \to \infty$ (see also the physics paper [15]). The leading order behaviour of this system, for large Λ, is governed by the hyperbolic tangent, namely the unique solution U of

$$u'' + u - u^3 = 0, \; z > 0; \; u(0) = 0, \; u(z) \to 1 \text{ as } z \to \infty. \tag{13}$$

More precisely, in the singular limit, the two components should have disjoint supports (each suported on a half-line); for $z > 0$, the first component should be $U(z)$, while the second $U(-z)$. This phenomenon is called *phase separation.* Using a perturbation argument, we can construct a solution to (11)-(12) based on the blow-up problem

$$\begin{cases} \ddot{V}_1 = V_2^2 V_1, \\ \\ \ddot{V}_2 = V_1^2 V_2, \end{cases} \tag{14}$$

which we use to build an inner approximate solution near $z = 0$ for large Λ. A crucial ingredient is the nondegeneracy of the special solution with linear growth of the above problem that we use as the building block, which was proven in [20]. Moreover, we can prove the uniqueness of monotone solutions to (11)-(12), for all $\Lambda < 1$, which was left open in [7].

- C. Sourdis, *Uniform estimates for positive solutions of a class of semilinear elliptic equations*, preprint (2012).

We consider the semilinear elliptic equation $\Delta u = W'(u)$ with Dirichlet boundary conditions in a smooth, possibly unbounded, domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions on the potential W, including the double well potential $\frac{1}{4}(u^2-1)^2$ that gives rise to the Allen-Cahn equation, we deduce a condition on the size of the domain that implies the existence of a positive solution satisfying a uniform pointwise estimate. Here, uniform means that the estimate is independent of Ω. The main advantage of our approach is that it allows us to remove a restrictive monotonicity assumption on W that was imposed in the recent paper by G. Fusco, F. Leonetti and C. Pignotti [52]. In addition, we can remove a non-degeneracy condition on the global minimum of W that was assumed in the latter reference. Furthermore, we can generalize an old result of P. Hess [59] and D. G. De Figueiredo [36], concerning semilinear elliptic nonlinear eigenvalue problems. Moreover, we study the boundary layer of global minimizers of the corresponding singular perturbation problem. For the above applications, our approach is based on a refinement of a useful result that dates back to P. Clément and G. Sweers [31], concerning the behavior of global minimizers of the associated energy over large balls, subject to Dirichlet conditions. Combining this refinement with global bifurcation theory and the celebrated sliding method, we can prove uniform estimates for solutions away from their nodal set, refining a lemma from a well known paper of H. Berestycki, L. A. Caffarelli and L. Nirenberg [17]. In particular, combining our approach with a-priori estimates that we obtain by blow-up, the doubling lemma of P. Polacik, P. Quittner, and P. Souplet [73] and known Liouville type theorems, we can give a new proof of a Liouville type theorem of Y. Du and L. Ma [43], without using boundary blow-up solutions. We can also provide an alternative proof of a Liouville theorem of H. Berestycki, F. Hamel, and H. Matano [19], involving the presence of an obstacle. Making use of the latter extension, we consider the singular perturbation problem with mixed boundary conditions. Furthermore, we prove some new one-dimensional symmetry properties of certain entire solutions to Allen-Cahn type equations, by exploiting for the first time an old result of Caffarelli, Garofalo, and Segála [27], and we suggest a connection with the theory of minimal surfaces. More precisely, our main result, in the case of the Allen-Cahn equation, in this direction says that if $u \in C^2(\mathbb{R}^n)$, $-1 < u < 1$, satisfies

$$\Delta u + u - u^3 = 0, \quad x \in \mathbb{R}^n, \quad u < 0 \text{ in } \{x_1 < 0\} \cap \mathbb{R}^n, \quad u(0) = 0,$$

then u is one dimensional. Our strategy is to use barriers in order to show that the inequality

$$\frac{|\nabla u|^2}{2} \le \frac{(1-u^2)^2}{4}, \quad x \in \mathbb{R}^n, \tag{15}$$

holds as equality at $x = 0$. We stress that, in contrast to [44], we do not use that the nonlinearity is odd. Using this approach, we also provide new proofs of well known symmetry results in half-spaces with Dirichlet boundary conditions. Moreover, we can generalize a rigidity result due to A. Farina [46] which says that if $\Delta u + u - u^3 = 0$, in $\mathbb{R}^n$, $|u| \le 1$, and $u(x_1, x') \to 1$ as $|x_1| \to \infty$, uniformly in $x' \in \mathbb{R}^{n-1}$, then $u \equiv 1$. In contrast to the latter reference, we do not use the odd symmetry of the nonlinearity. A new proof of Gibbons' conjecture in this spirit is also given, which works at least in two dimensions. The latter conjecture has been proven in all dimensions (see [16, 18, 45]) and asserts that any solution of the previous problem is one dimensional if the assumption $u(x_1, x') \to \pm 1$ as $x_1 \to \pm\infty$ uniformly in $x' \in \mathbb{R}^{n-1}$ is assumed instead. Lastly, we study the one-dimensional symmetry of solutions in convex cylindrical domains with Neumann boundary conditions that was originally studied by rearrangement arguments in [30]. In contrast to ([30]), we do not assume that the solution is a minimizer.

- C. Sourdis, *On the profile of globally and locally minimizing solutions of the spatially inhomogeneous Allen–Cahn and Fisher–KPP equation*, to appear in Advanced Nonlinear Studies (2016).

We show that the spatially inhomogeneous Allen-Cahn equation

$$-\varepsilon^2 \Delta u = u\,(u - a(x))\,(1-u)$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $u = 0$ on $\partial\Omega$, with $0 < a(\cdot) < 1$ continuous and $\varepsilon > 0$ a small parameter, cannot have globally minimizing solutions with transition layers in a smooth subdomain of Ω whereon $a - \frac{1}{2}$ does not change sign and $a - \frac{1}{2} \neq 0$ on that subdomain's boundary. Under the assumption of radial symmetry, this property was shown by E. N. Dancer and S. Yan in [34]. Our approach, which is in the spirit of local replacement lemmas of the calculus of variations (see [10]), may also be used to simplify some parts of the latter and related references. In particular, for this model, we can give a streamlined new proof of the existence of locally minimizing transition layered solutions with non-smooth interfaces, considered originally by M. del Pino in [40] using different techniques. Besides of its simplicity, the main advantage of our proof is that it allows one to deal with more degenerate situations. We also establish analogous results for a class of problems that includes the spatially inhomogeneous Fisher–KPP equation $-\varepsilon^2\Delta u = \rho(x)u(1-u)$ with ρ sign-changing.

- *The heteroclinic connection problem for general double-well potentials*

We study the existence of solutions $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the following problem:

$$u_{xx} = \nabla_u W(u), \quad x \in \mathbb{R}, \quad \lim_{x\to\pm\infty} u(x) = a_\pm, \tag{16}$$

where

$$W \in C^1(\mathbb{R}^n),\ n \geq 1,\ \text{satisfy}\ W(a_-) = W(a_+) = 0,\ W(u) > 0 \text{ if } u \neq a_\pm, \tag{17}$$

for some $a_- \neq a_+$, and

$$\liminf_{|u|\to\infty} W(u) > 0. \tag{18}$$

Even though there is an effective variational approach to this problem which deals with the lack of compactness due to translations (see [21] and the references therein), we base our approach on the recently devised constrained minimization approach of [**?**] that we refine in the process. Moroever, by adapting some ideas from the recent note [68], we are able to relax condition (18). Besides of its simplicity, this approach may also be extended to the study of heteroclinic solutions for the corresponding elliptic system (see also).

We also study the inhomogeneous problem

$$u_{xx} = h(x)\nabla W(u), \quad \lim_{x\to\pm\infty} u(x) = a_\pm, \tag{19}$$

under various assumptions on h.

The interest in this problem stems mainly from the study of the vectorial Allen-Cahn equation that models multi-phase transitions (see [5], [12], [9], [24], and the references therein). The heteroclinic connections are expected to describe the way in which the solutions to the multi-dimensional parabolic system

$$u_t = \varepsilon^2 \Delta u - \nabla W(u),$$

for small $\varepsilon > 0$, transition from one state to the other (see [23]).

A related problem from two component Bose-Einstein condensates has been recently studied in [7].

- C. Sourdis, *On the existence of dark solitons of the defocusing cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity*, Applied Mathematics Letters **46** (2015), 123–126.

The main subject of this paper is the nonlinear Schrödinger equation (NLSE):

$$i\psi_t = -\frac{1}{2}\psi_{xx} + g(x)|\psi|^2\psi,$$

with the spatial inhomogeneity $g > 0$ being reasonably smooth and periodic. This problem appears in a variety of physical situations.

Solitary waves are special solutions of the NLSE of form $\psi(x,t) = e^{i\lambda t}\phi(x)$, where

$$-\frac{1}{2}\phi_{xx} + \lambda\phi + g(x)\phi^3 = 0.$$

Such a wave ψ is called a *dark soliton* if ϕ "connects" two different periodic solutions of the above equation as x goes from $-\infty$ to ∞.

The existence of a dark soliton follows essentially from a result of Torres and Konotop [78] which relies on advanced tools such as topological degree theory and free homeomorphisms. Actually, several further technical restrictions on g have to be imposed.

In this paper we prove the existence of a dark soliton by a simple and elementary variational argument which *does not require any technical restriction on g*. Moreover, we can prove a monotonicity property of the obtained dark solitons. The main idea is to apply a remarkable identity, coming from the study of vortices in inhomogeneous Ginzburg-Landau equations, to reduce the problem to one where rather standard variational techniques are applicable.

The stability of the dark soliton in the case where g is a constant has been the subject of intensive studies in the last years. The approach of the current paper could be essential in adapting these studies to the inhomogeneous problem considered here and may even be the point of departure for future studies.

- C. Sourdis, *On the growth of the energy of entire solutions to the vector Allen-Cahn equation*, Comm. Pure Appl. Anal. **14** (2015), 577-584.

We consider entire, nonconstant, bounded solutions $u : \mathbb{R}^n \to \mathbb{R}^m$ to the semilinear elliptic system $\Delta u = W_u(u)$ with the potential W being sufficiently smooth and nonnegative.

In the scalar case $m = 1$, Modica proved that such solutions satisfy the following pointwise gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u), \tag{20}$$

from which one can derive a strong monotonicity formula. In particular, this implies a lower bound of order R^{n-1} for the growth of the energy of such solutions over balls of radius R as $R \to \infty$.

In the vector case $m \geq 2$, the analog of the aforementioned gradient bound does not hold in general. Nevertheless, one can prove a weak monotonicity formula which implies a lower bound of order R^{n-2} for the corresponding growth of the energy as $R \to \infty$.

In this article, particularizing to the case of phase transition potentials which have a finite number of nondegenerate global minima, we show that this growth rate is actually greater than $[\ln(\ln R)]^k (\ln R) R^{n-2}$, for any $k > 0$, as $R \to \infty$. The main advance is that we consider general solutions. In fact, to the best of our knowledge, besides trivial situations (radial or n-periodic solutions), the only exception up to now where there was an improvement of the R^{n-2} growth rate is in the case of minimal solutions, and this was accomplished very recently. A particular feature of our proofs is that we also employ techniques from the well studied Ginzburg-Landau system, where W attains its global minimum on the sphere $\mathbb{S}^{m-1}$. In particular, we employ a useful lemma from [77].

- *An asymptotic monotonicity formula for minimizers to a class of elliptic systems of Allen-Cahn type and the Liouville property*, preprint (2014).

This paper considers elliptic systems of gradient form

$$\Delta u = W_u(u),$$

having hence a variational formulation. In the case where the potential W has appropriate symmetries and only a finite number of global minima, one can construct equivariant entire solutions by employing the following Liouville property which has been proven recently by G. Fusco [53]: *If the potential has a unique minimizer and is strictly convex near it, then the entire, bounded minimizing solutions are constants equal to the aforementioned minimizer of the potential.* The proof of this result is quite involved and, in fact, is close in spirit to the well known work of Caffarelli and Córdoba [28] (see also the recent paper [11] for more elaborations on this point). We believe that our main result, which is based on a new asymptotic monotonicity formula, provides a considerably simpler proof in the case of two spatial dimensions. Moreover, our proof requires less regularity on W and allows the convexity assumption on W near the minimum to be relaxed to plain monotonicity. In addition, we present several variants of this result which follow from our asymptotic monotonicity formula (which holds in any dimension).

- *Optimal growth lower bounds for the potential energy of minimal solutions to the vectorial Allen-Cahn equation in two spatial dimensions*, preprint (2015).

This paper considers elliptic systems of gradient form

$$\Delta u = W_u(u), \quad u : \mathbb{R}^n \to \mathbb{R}^m, \tag{21}$$

with $n \geq 2$, $m \geq 1$, where the potential W is nonnegative and vanishes at a finite number of global, non-degenerate minimizers. Such systems typically arise as blow-up limits in the study of multi-phase transitions.

Among the many interesting results in the recent paper of Alikakos and Fusco [11], the authors prove that bounded, entire, nonconstant and minimal solutions (in the sense of Morse) satisfy

$$\liminf_{R\to\infty} R^{1-n} \int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx > 0,$$

where B_R is the n-dimensional ball centered at the origin. To illustrate the delicacy of this result, let us mention that its proof in the scalar case ($m = 1$) relies on Modica's gradient bound (20), which in turn relies on the maximum principle and is not known to hold in the vector case. Their approach is based on their extension to this setting of the classical density estimates of Caffarelli and Córdoba [28].

In the current note, using a considerably simpler argument, we prove that in two spatial dimensions, namely $n = 2$, it actually holds that

$$\liminf_{R\to\infty} R^{1-n} \int_{B_R} W(u)dx > 0.$$

In fact, we can allow W to have degenerate minima. Our proof makes use of a clearing-out argument from the study of vortices in the Ginzburg-Landau system and ideas coming from a recent variational maximum principle of [10]. It is also natural to *conjecture that our result continues to hold for* $n \geq 3$.

- C. Sourdis, *A Liouville type result for bounded, entire solutions to a class of variational semilinear elliptic systems*, preprint (2015).

We provide a sufficient condition on $W \geq 0$ such that all bounded solutions to (21) with finite potential energy $\int W(u) < \infty$ (or not too small, in some sense) are constants if $n \geq 4$. Our result is based on the weak monotonicity formula and, in particular, gives a new proof of a result of Farina [47].

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