REMARKS ON THE INTERFACE LAYER OF A TWO-COMPONENT BOSE-EINSTEIN CONDENSATE

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Abstract. We consider the heteroclinic connection problem for a class of Hamiltonian systems, containing a large competition parameter, which arise in the study of the behaviour of the wave functions of a two-component Bose-Einstein condensate near the interface, in the case of segregation. This problem was studied in detail recently in [2] under a symmetry assumption. Here, we extend the approach of the latter paper to the nonsymmetric case, to recover essentially the same result. Moreover, we establish a uniqueness result for such solutions.

1. Introduction

We will consider the heteroclinic connection problem:

$$\begin{cases} -v_1'' + g_1 v_1^3 - \lambda_1 v_1 + \Lambda v_2^2 v_1 = 0, \\ \\ -v_2'' + g_2 v_2^3 - \lambda_2 v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{1.1}$$

$$(v_1, v_2) \to \left(0, \sqrt{\frac{\lambda_2}{g_2}}\right) \text{ as } z \to -\infty, \quad (v_1, v_2) \to \left(\sqrt{\frac{\lambda_1}{g_1}}, 0\right) \text{ as } z \to \infty, \tag{1.2}$$

with $v_1, v_2 > 0$, for values of the parameter

$$\Lambda > \sqrt{g_1 g_2}, \tag{1.3}$$

assuming that the positive constants $g_1, g_2, \lambda_1, \lambda_2$ satisfy

$$\frac{\lambda_1^2}{g_1} = \frac{\lambda_2^2}{g_2}. \tag{1.4}$$

By the conservation of the Hamiltonian

$$H = \sum_{i=1}^{2} \left[\frac{1}{2}(v_i')^2 - \frac{g_i}{4} v_i^4 + \frac{\lambda_i}{2} v_i^2 \right] - \frac{\Lambda}{2} v_1^2 v_2^2 - \frac{\lambda_1^2}{2g_1}, \tag{1.5}$$

we see that (1.4) is a necessary condition for the existence of solutions to (1.1)-(1.2). In particular, we observe that the Hamiltonian constant along solutions of (1.1)-(1.2) is equal to $-\frac{\lambda_1^2}{4g_1}$. Conversely, the aforementioned condition is also sufficient (see [3]).

The above problem arises in the study of two-component Bose-Einstein condensates in the case of segregation (see for example [3, 4, 9] and the references therein). In particular, such heteroclinic solutions are the blow-up profiles that govern the transition layer behaviour near the interface in the Thomas-Fermi limit (see [1, 10]).

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In [2], mainly motivated by [1, 4, 5], we recently studied in detail the strong separation limit $\Lambda \to \infty$ in the case where $g_1 = g_2$ and $\lambda_1 = \lambda_2$ (note that by (1.4) one of these equalities implies the other). The purpose of the current note is twofold. We will first show that the arguments of [2] can be extended to produce essentially the same result in the general case. Then, we will establish a uniqueness result for any Λ satisfying (1.3).

1.1. **Notation.** By c/C, we will denote small/large positive generic constants that are independent of large $\Lambda > 0$ and whose value will decrease/increase as the paper moves on. A number ρ will be of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$ if $\rho = \mathcal{O}(\Lambda^{-m})$, for any $m > 1$, as $\Lambda \to \infty$.

2. The strong separation limit $\Lambda \to \infty$

It is known that solutions of (1.1)-(1.2) are uniformly bounded independently of $\Lambda > \sqrt{g_1 g_2}$ (see [3]). Hence, by the general theory developed in [13] and the references therein, they are uniformly Lipschitz continuous and converge, uniformly as $\Lambda \to \infty$, to the Lipschitz continuous pair $\left(\chi_{(0,\infty)} U_1, \chi_{(-\infty,0)} U_2\right)$, where U_1 and U_2 denote the unique solutions respectively of the following problems:

$$u'' + \lambda_1 u - g_1 u^3 = 0,\ z > 0;\ u(0) = 0,\ u(z) \to \sqrt{\frac{\lambda_1}{g_1}} \text{ as } z \to \infty, \tag{2.1}$$

$$u'' + \lambda_2 u - g_2 u^3 = 0,\ z < 0;\ u(z) \to \sqrt{\frac{\lambda_2}{g_2}} \text{ as } z \to -\infty,\ u(0) = 0. \tag{2.2}$$

We note that these solutions are monotone. This implies that they are nondegenerate, in the sense that zero is the only bounded element in the kernel of the associated linearized operators (see the first assertion of Lemma 2.1 in [2] for more details). Another important observation is that (1.4) and the Hamiltonian structure of (2.1), (2.2) imply easily the reflection property

$$U_1'(0) + U_2'(0) = 0, \quad \text{with} \quad \psi_0 = U_1'(0) = \frac{\lambda_1}{\sqrt{2g_1}}. \tag{2.3}$$

The functions $\chi_{(0,\infty)} U_1, \chi_{(-\infty,0)} U_2$ do satisfy (1.1)-(1.2) for $z \neq 0$. However, their second derivatives blow-up at the origin as delta masses. To remedy this, as in [2], guided by formal matched asymptotics and (2.3), we will instead use the following approximate solution:

$$\Lambda^{-\frac{1}{4}} V_1\left(\Lambda^{\frac{1}{4}} z\right),\ \Lambda^{-\frac{1}{4}} V_2\left(\Lambda^{\frac{1}{4}} z\right) \quad \text{for} \quad |z| \le (\ln \Lambda)\Lambda^{-\frac{1}{4}},$$

where the pair (V_1, V_2) is provided by the following proposition.

Proposition 2.1. [5, 6] *There exists a unique solution (V_1, V_2) with positive components to the system*

$$\begin{cases} \ddot{V}_1 = V_2^2 V_1, \\ \ddot{V}_2 = V_1^2 V_2, \end{cases} \tag{2.4}$$

such that

$$\frac{V_1}{x} \to \psi_0 \quad \text{and} \quad V_2 \to 0 \text{ as } x \to \infty, \tag{2.5}$$

where $\psi_0 > 0$ is as in (2.3), and

$$V_1(-x) = V_2(x),\ x \in \mathbb{R}. \tag{2.6}$$

Furthermore, every other entire solution of (2.4) with positive components is given by

$$(\mu V_1(\mu(x-h)), \mu V_2(\mu(x-h))) \tag{2.7}$$

for some $\mu > 0$ and $h \in \mathbb{R}$.

In fact, it follows from the analysis in [5] that

$$V_1(x) = \psi_0 x + \kappa + \mathcal{O}\left(e^{-cx^2}\right) \text{ and } V_2(x) = \mathcal{O}\left(e^{-cx^2}\right) \text{ as } x \to \infty, \tag{2.8}$$

for some

$$\kappa \geq 0,$$

and these relations can be differentiated arbitrarily many times. Moreover, it follows easily that

$$\dot{V}_1 > 0, \ \dot{V}_2 < 0, \ x \in \mathbb{R}. \tag{2.9}$$

Our goal is to refine these outer and inner approximate solutions, carefully glue them together and show that the resulting global approximate solution can be perturbed to a genuine one. We will accomplish this by exploiting the aforementioned non-degeneracy of the outer profiles U_1, U_2 and the following non-degeneracy property of the blow-up profile (V_1, V_2).

Proposition 2.2. [5] *If $\Phi_1, \Phi_2 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy*

$$L\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$L\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} -\ddot{\Phi}_1 + V_2^2\Phi_1 + 2V_1V_2\Phi_2 \\ -\ddot{\Phi}_2 + V_1^2\Phi_2 + 2V_1V_2\Phi_1 \end{pmatrix}, \tag{2.10}$$

then

$$(\Phi_1, \Phi_2) \equiv \lambda(\dot{V}_1, \dot{V}_2)$$

for some $\lambda \in \mathbb{R}$.

For future reference, we note that the invariance of system (2.4) under scalings, described in (2.7), implies that L has also the pair

$$(E_1, E_2) = (x\dot{V}_1 + V_1, x\dot{V}_2 + V_2) \tag{2.11}$$

amongst its four-dimensional kernel.

This programme was carried out successfully in [2] under the restriction that $\lambda_1 = \lambda_2$ and $g_1 = g_2$ (keep in mind (1.4)). The importance of the latter assumption was that it allows to work entirely within the mirror symmetric class of pairs (v_1, v_2) satisfying $v_1(-z) = v_2(z)$, $z \in \mathbb{R}$, which takes out of the picture the bounded element $(\dot{V}_1, \dot{V}_2)$ in the kernel of L.

Let us indicate how the arguments of [2] can be adapted for constructing the analogous solution to that of Theorems 1.1 and 1.2 for the general problem (1.1)-(1.2).

2.1. **Construction of the inner solution.** Based on the analysis of [2] for the symmetric case, we expect that the inner approximate solution will control the outer ones. So, let us begin by constructing the former.

As in [2], we seek an inner approximate solution $(v_{1,in}, v_{2,in})$ to (1.1) in the form

$$v_{i,in}(z) = \mu\Lambda^{-\frac{1}{4}}V_i(x) + \Phi_i(x), \quad |z| \leq (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \quad i = 1, 2, \tag{2.12}$$

where the functions Φ_1, Φ_2 are also to be determined, and x stands for the stretched coordinate

$$x = \Lambda^{\frac{1}{4}}z. \tag{2.13}$$

We point out that, similarly to [2], the scaling invariance of (2.4) (recall (2.7)) will be exploited by introducing a parameter B in (Φ_1, Φ_2). Actually, at first, we were tempted to exploit the translation invariance of (2.4) by introducing a shift parameter in the stretched variable x, similarly to (2.7), but then realized that this is not needed as the whole problem (1.1)-(1.2) is translation invariant; nevertheless, like in the symmetric case, we will shift slightly the outer profiles U_1 and U_2. The pair (Φ_1, Φ_2) should now satisfy

$$L\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \Lambda^{-\frac{3}{4}}\begin{pmatrix} \lambda_1 V_1 \\ \lambda_2 V_2 \end{pmatrix} \quad \text{for} \quad |x| < \ln\Lambda \quad \text{(at least)}. \tag{2.14}$$

Letting

$$(\Phi_1, \Phi_2) = \Lambda^{-\frac{3}{4}}(\lambda_1 Z_1, \lambda_2 Z_2) + (\tilde{\Phi}_1, \tilde{\Phi}_2),$$

where Z_1, Z_2 are some smooth, fixed functions that satisfy $Z_1(-x) \equiv Z_2(x)$ and

$$Z_1(x) = -\psi_0\frac{x^3}{6} - \kappa\frac{x^2}{2}, \; Z_2(x) = 0, \; x \geq 1, \tag{2.15}$$

we find that $(\tilde{\Phi}_1, \tilde{\Phi}_2)$ satisfies the same equation but with righthand side that is the product of $\Lambda^{-\frac{3}{4}}$ times a fixed function whose components converge super-exponentially fast to zero as $x \to \pm\infty$. However, we cannot use the linear analysis of the second section of [2] for the latter equation. Indeed, in the general class, the operator L with Neumann boundary conditions at $\pm\ln\Lambda$ is nearly non-invertible, as $(\dot{V}_1, \dot{V}_2)$ satisfies these conditions up to an $\mathcal{O}(\Lambda^{-\infty})$ small error. Instead, we will use the following variation with Dirichlet boundary conditions.

Proposition 2.3. *Given $\alpha > 0$, there exist $\Lambda_1, C > 0$ such that the boundary value problem*

$$L\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad |x| < \ln\Lambda; \quad \Phi_i(\pm\ln\Lambda) = 0, \; i = 1, 2,$$

where L is as in (2.10) and $G_1, G_2 \in C[-(\ln\Lambda), \ln\Lambda]$, has a unique solution such that

$$\sum_{i=1}^{2}\left(\|\dot{\Phi}_i\|_{L^\infty(-\ln\Lambda,\ln\Lambda)} + \|\Phi_i\|_{L^\infty(-\ln\Lambda,\ln\Lambda)}\right) \leq C(\ln\Lambda)\sum_{i=1}^{2}\|e^{\alpha|x|}G_i\|_{L^\infty(-\ln\Lambda,\ln\Lambda)}, \tag{2.16}$$

provided that $\Lambda \geq \Lambda_1$.

Proof. As usual, it suffices to verify the validity of the asserted uniform a-priori estimate. In contrast to Proposition 2.2 in [2], however, an important role here will be played by the 'blow-down' problem:

$$\begin{cases} -\frac{d^2\varphi_1}{dy^2} + (\ln\Lambda)^2 V_2^2\left((\ln\Lambda)y\right)\varphi_1 + 2(\ln\Lambda)^2 V_1V_2\left((\ln\Lambda)y\right)\varphi_2 = (\ln\Lambda)^2 g_1, \\ \\ -\frac{d^2\varphi_2}{dy^2} + (\ln\Lambda)^2 V_1^2\left((\ln\Lambda)y\right)\varphi_2 + 2(\ln\Lambda)^2 V_1V_2\left((\ln\Lambda)y\right)\varphi_1 = (\ln\Lambda)^2 g_2, \\ \\ \text{for } |y|<1; \quad \varphi_i(\pm 1) = 0, \quad i=1,2, \end{cases} \tag{2.17}$$

where $\varphi_i(y) = \Phi_i\left((\ln\Lambda)y\right)$ and $g_i(y) = G_i\left((\ln\Lambda)y\right)$, $i=1,2$.

Suppose that the a-priori estimate (2.16) were false. Then, there would exist $\Lambda_n \to \infty$ and pairs $(\varphi_{1,n}, \varphi_{2,n}) \in C^2[-1,1]\times C^2[-1,1]$, $(g_{1,n}, g_{2,n}) \in C[-1,1]\times C[-1,1]$, satisfying (2.17) with $\Lambda = \Lambda_n$, which violate it. In fact, there is no loss of generality in assuming that $\|\varphi_{1,n}\|_{L^\infty(-1,1)} \geq \|\varphi_{2,n}\|_{L^\infty(-1,1)}$. Dividing both equations by $\|\varphi_{1,n}\|_{L^\infty(-1,1)}$, we may further assume that

$$\begin{array}{c} \|\varphi_{1,n}\|_{L^\infty(-1,1)} = 1, \quad \|\varphi_{2,n}\|_{L^\infty(-1,1)} \leq 1 \\ \text{and} \quad (\ln\Lambda_n)\sum_{i=1}^{2} \|e^{\alpha|(\ln\Lambda_n)y|} g_{i,n}\|_{L^\infty(-1,1)} \to 0. \end{array} \tag{2.18}$$

Throughout the rest of the proof, $c \setminus C$ will stand for small\large positive generic constants that are independent of n. A standard barrier argument yields that

$$|\varphi_{1,n}(y)| \leq e^{c(\ln\Lambda_n)y}, \quad -1 \leq y \leq 0; \quad |\varphi_{2,n}(y)| \leq e^{-c(\ln\Lambda_n)y}, \quad 0 \leq y \leq 1.$$

In view of (2.17), (2.18) and the above relation, by the usual diagonal-compactness argument, passing to a subsequence if necessary, we find that

$$\varphi_{i,n} \to \varphi_{i,\infty} \quad \text{in} \quad C^2_{loc}\left([-1,1]\setminus\{0\}\right), \quad i=1,2, \tag{2.19}$$

where

$$\varphi_{1,\infty}(y) = 0, \ y \in [-1,0); \quad \frac{d^2\varphi_{1,\infty}}{dy^2} = 0, \ y \in (0,1],$$

and

$$\frac{d^2\varphi_{2,\infty}}{dy^2} = 0, \ y \in [-1,0); \quad \varphi_{2,\infty}(y) = 0, \ y \in (0,1].$$

Hence, we get that

$$\varphi_{1,\infty}(y) = a_1(y-1), \quad y \in (0,1], \quad \text{and} \quad \varphi_{2,\infty}(y) = a_2(y+1), \quad y \in [-1,0). \tag{2.20}$$

By virtue of Proposition 2.2, passing to a further subsequence if needed, we have that

$$\varphi_{i,n}\left((\ln\Lambda_n)^{-1}x\right) \to b\dot{V}_i(x) \quad \text{in} \quad C^1_{loc}(\mathbb{R}), \quad i=1,2,$$

for some $b \in \mathbb{R}$.

In the same manner as in Theorem 1.2 in [2], to reach a contradiction, it is enough to show that

$$a_1 = a_2 = b = 0. \tag{2.21}$$

Firstly, as in the end of the proof of the aforementioned theorem, we obtain that

$$\sqrt{2}b = -a_1 \quad \text{and} \quad -\sqrt{2}b = a_2. \tag{2.22}$$

By testing (2.17) with $\left(\dot{V}_1\left((\ln\Lambda_n)y\right), \dot{V}_2\left((\ln\Lambda_n)y\right)\right)$ and integrating by parts, we arrive at

$$\sum_{i=1}^{2}\left[\frac{d\varphi_{i,n}(1)}{dy}\dot{V}_i\left(\ln\Lambda_n\right)-\frac{d\varphi_{i,n}(-1)}{dy}\dot{V}_i\left(-\ln\Lambda_n\right)\right]=$$
$$-(\ln\Lambda_n)^2\sum_{i=1}^{2}\int_{-1}^{1}\dot{V}_i\left((\ln\Lambda_n)y\right)g_{i,n}(y)dy.$$

Letting $n\to\infty$ in the above relation, and using (2.18), (2.19), (2.20), we deduce that

$$a_1+a_2=0.$$

The desired relation (2.21) now follows at once from (2.22). □

Armed with the above proposition, in analogy to [2, Sec. 2], we define the inner approximate solution to (1.1) as $(v_{1,in}, v_{2,in})$, with

$$v_{i,in}(z)=\Lambda^{-\frac14}V_i(x)+\Lambda^{-\frac34}\left[H(x)Z_i(x)+\hat{\Phi}_i(x)\right]+BE_i(x),\quad |z|\leq(\ln\Lambda)\Lambda^{-\frac14},\ i=1,2,\quad (2.23)$$

where H is a smooth function such that

$$H(x)=\begin{cases}\lambda_2, & x\leq -1,\\ \lambda_1, & x\geq 1,\end{cases}$$

the pair $\Lambda^{-\frac34}\left(HZ_1+\hat{\Phi}_1, HZ_2+\hat{\Phi}_2\right)$ satisfies (2.14) with $\hat{\Phi}_i(\pm\ln\Lambda)=0$, $i=1,2$, and

$$\sum_{i=1}^{2}\left(\|\partial_x\hat{\Phi}_i\|_{L^\infty(-\ln\Lambda,\ln\Lambda)}+\|\hat{\Phi}_i\|_{L^\infty(-\ln\Lambda,\ln\Lambda)}\right)\leq C\ln\Lambda, \quad (2.24)$$

while the free parameter B will be chosen shortly so that the Hamiltonian (1.5) on $(v_{1,in}, v_{2,in})$ at $z=0$ is equal to $-\frac{\lambda_1^2}{4g_1}$. We point out that we cannot replace (E_1,E_2) by $(\dot{V}_1,\dot{V}_2)$ in the above definition for this purpose because, as it turns out, the corresponding terms in the expansion of the Hamiltonian at $z=0$ cancel at principal order. Similarly as in [2], we find that there exists

$$B=\mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac34}\right)\quad\text{as}\quad\Lambda\to\infty \quad (2.25)$$

such that the value of the Hamiltonian (1.5) on the inner approximate solution, as defined in (2.23), at $z=0$ is $-\frac{\lambda_1^2}{4g_1}$. We can now define, exactly as in [2], a one-parameter family of refined inner approximate solutions $\tilde{B}\mapsto(w_{1,in}, w_{2,in})$ as

$$w_{i,in}(z)=v_{i,in}(z)+\tilde{B}E_i(x),\quad |z|\leq(\ln\Lambda)\Lambda^{-\frac14},\ i=1,2, \quad (2.26)$$

for the parameter values:

$$\tilde{B}=\mathcal{O}(\Lambda^{-1})\ \ \text{as}\ \Lambda\to\infty. \quad (2.27)$$

Clearly the remainder left by this inner approximate solution in 1.1 is of the same order as the corresponding one in [2].

2.2. **Solution of the inner problem.** Given $\tilde{B}$ satisfying (2.27), we seek an inner genuine solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ to (1.1) in the form

$$(\mathbf{v}_{1,in}, \mathbf{v}_{2,in}) = (w_{1,in}, w_{2,in}) + (\varphi_1, \varphi_2), \quad |z| \leq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \tag{2.28}$$

with (φ_1, φ_2) satisfying Dirichlet boundary conditions instead of Neumann which was the case in [2] (and of course without imposing any symmetry conditions). We then write this inner problem in the corresponding abstract form to that in [2, Sec. 4] with Dirichlet boundary conditions and with the new operators $\mathcal{L}, R, Q, N$ having just more general constant coefficients (the linear operator $\mathcal{L}$ actually remains the same). Taking into account Proposition 2.3, we find that the properties established in [2] for these operators are still valid, provided that we change the Neumann boundary conditions to Dirichlet and drop all the symmetry conditions. Hence, analogously to [2], we have the following proposition.

Proposition 2.4. *Given $\alpha \in (0,1)$ and $\tilde{B}$ satisfying (2.27), there exists $C > 0$ such that problem (1.1) admits a unique solution $(\boldsymbol{v}_{1,in}, \boldsymbol{v}_{2,in})$ in the form (2.28) with*

$$\sum_{i=1}^{2} \left(\Lambda^{-\frac{1}{4}} \|\varphi_i'\|_{L^\infty(I_\Lambda)} + \|\varphi_i\|_{L^\infty(I_\Lambda)} \right) \leq C\Lambda^{-\frac{5}{4}+\alpha},$$

where

$$I_\Lambda = \left(-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}} \right),$$

and

$$\varphi_i \left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}} \right) = 0, \quad i = 1, 2,$$

provided that Λ is sufficiently large. Moreover, this solution depends continuously, with respect to the $C^1(\overline{I_\Lambda})$-norm, on $\tilde{B}$ as in (2.27) (for fixed Λ).

The corresponding analysis in [2] applies unchanged and guarantees the existence of a

$$\tilde{B} = \mathcal{O}\left(\Lambda^{-\frac{5}{4}+\alpha}\right) \quad \text{as} \quad \Lambda \to \infty, \tag{2.29}$$

such that the Hamiltonian constant of the solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$, provided by Proposition 2.4, is equal to $-\frac{\lambda_1^2}{4g_1}$.

Clearly, as was also the case in [2], it holds

$$\mathbf{v}_{2,in} \geq c_L \Lambda^{-\frac{1}{4}} \quad \text{and} \quad -\mathbf{v}_{2,in}' \geq c_L \quad \text{on} \quad \left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, L\Lambda^{-\frac{1}{4}} \right], \tag{2.30}$$

and $\mathbf{v}_{2,in}$ satisfies a linear equation of the form

$$-v'' + P(z)v = 0 \quad \text{with} \quad P(z) \geq c\Lambda z^2, \quad z \in \left(L\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}} \right), \tag{2.31}$$

with $c > 0$ independent of both Λ, L, provided that $\Lambda > 0$ is sufficiently large (the analogous relations hold for $\mathbf{v}_{1,in}$). The difference compared to [2] is that now we have

$$\mathbf{v}_{2,in} \left((\ln \Lambda)\Lambda^{-\frac{1}{4}} \right) = \Lambda^{-\frac{1}{4}} V_2 (\ln \Lambda) + (B + \tilde{B}) E_2(\ln \Lambda) = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty.$$

So, by a barrier argument and standard elliptic estimates, we deduce that

$$\mathbf{v}_{2,in}' \left((\ln \Lambda)\Lambda^{-\frac{1}{4}} \right) = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty. \tag{2.32}$$

Analogously, we get

$$\mathbf{v}_{1,in}'\left(-(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)=\mathcal{O}(\Lambda^{-\infty})\ \text{ as }\ \Lambda\to\infty. \tag{2.33}$$

2.3. **Construction of the outer solution.** In view of the corresponding analysis in [2], and taking into account that we have used Dirichlet boundary conditions for the above constructions of the inner solutions, we define our outer approximate solution for $z\geq(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ as

$$w_{1,out}(z)=U_1(z+\xi_1)+(\tau_1+\tilde{\tau}_1)U_1'(z+\xi_1),$$

$$w_{2,out}(z)=\left[\Lambda^{-\frac{1}{4}}V_2(\ln\Lambda)+(B+\tilde{B})E_2(\ln\Lambda)\right]\zeta(z),$$

with

$$\xi_1=\psi_0^{-1}\kappa\Lambda^{-\frac{1}{4}}+2B(\ln\Lambda)+\tfrac{1}{2}(\ln\Lambda)\Lambda^{-\frac{1}{4}}\xi_1^2,$$

$$\tau_1=\tfrac{\xi_1^3}{6}+\psi_0^{-1}B\kappa\overset{(2.25)}{=}\mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac{3}{4}}\right),$$

$$\tilde{\tau}_1=\frac{\left[\Lambda^{-\frac{1}{4}}V_1(\ln\Lambda)+(B+\tilde{B})E_1(\ln\Lambda)-U_1\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}+\xi_1\right)\right]}{U_1'\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}+\xi_1\right)}-\tau_1=\mathcal{O}\left((\ln\Lambda)^5\Lambda^{-\frac{5}{4}}\right),$$

and $\zeta\in C_0^\infty(\mathbb{R})$ is a fixed cutoff function which is equal to one on $[-1,1]$. Analogously we can define the corresponding outer approximate solution $(\tilde{w}_{1,out},\tilde{w}_{2,out})$ for $z\leq-(\ln\Lambda)\Lambda^{-\frac{1}{4}}$. The remainder left by these outer approximations in (1.1) is of the same order as in [2, Sec. 2].

The analysis in [2, Sec. 5], which was based on the aforementioned non-degeneracy of U_1, carries over almost word for word to give us a solution $(\mathbf{v}_{1,out},\mathbf{v}_{2,out})$ of the outer problem for $z\geq(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ which agrees with $(\mathbf{v}_{1,in},\mathbf{v}_{2,in})$ at $(\ln\Lambda)\Lambda^{-\frac{1}{4}}$, satisfies the desired behavior as $z\to\infty$, and is uniformly close to $(w_{1,out},w_{2,out})$ according to the obvious analog of the corresponding proposition in [2]. Analogously we can define the corresponding outer genuine solution $(\tilde{\mathbf{v}}_{1,out},\tilde{\mathbf{v}}_{2,out})$ for $z\leq-(\ln\Lambda)\Lambda^{-\frac{1}{4}}$. Obviously the Hamiltonian constant of both these outer genuine solutions is equal to $-\frac{\lambda_1^2}{4g_1}$.

2.4. **A gluing argument: The global approximate solution** $(\mathbf{w}_{1,ap},\mathbf{w}_{2,ap})$**.** Analogously to [2], we define a continuous and piecewise smooth approximate solution $(\mathbf{v}_{1,ap},\mathbf{v}_{2,ap})$ with

$$\mathbf{v}_{i,ap}(z)=\begin{cases}\tilde{\mathbf{v}}_{i,out}(z), & z\leq-(\ln\Lambda)\Lambda^{-\frac{1}{4}}\\ \mathbf{v}_{i,in}(z), & |z|\leq(\ln\Lambda)\Lambda^{-\frac{1}{4}}\\ \mathbf{v}_{i,out}(z), & z\geq(\ln\Lambda)\Lambda^{-\frac{1}{4}}\end{cases},\quad i=1,2. \tag{2.34}$$

As in [2], since all three solution branches share the same Hamiltonian constant, and (2.32)-(2.33) hold, we infer that the jumps in the gradient of $(\mathbf{v}_{1,ap},\mathbf{v}_{2,ap})$ at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ are at most of $\mathcal{O}\left(\Lambda^{-\infty}\right)$ order.

We are now ready to define our global C^1-smooth approximate solution to problem (3.1)-(3.2) as

$$\mathbf{w}_{i,ap}(z)=\mathbf{v}_{i,ap}(z)+(s_i)_-e^{-\left|z+(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right|}+(s_i)_+e^{-\left|z-(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right|}, \tag{2.35}$$

where the numbers

$$(s_i)_\pm = \mathcal{O}\left(\Lambda^{-\infty}\right) \quad \text{as} \quad \Lambda \to \infty, \quad i = 1, 2, \tag{2.36}$$

are chosen so that $\mathbf{w}_{i,ap}$, $i = 1, 2$, are C^1 at $\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}$. This approximate solution leaves a remainder in (1.1) which is uniformly of order $\mathcal{O}(\Lambda^{-\infty})$ (keep in mind, however, that it may have finite jump discontinuities at the two gluing points), while the asymptotic behaviour (1.2) as $z \to \pm\infty$ is fulfilled exactly.

2.5. **Perturbing the approximate solution to a genuine one.** Even though $(\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap})$ is an extremely good approximate solution, perturbing it to a genuine one by some type of local inversion argument is subtle. Indeed, the associated linearized operator

$$\mathcal{M}\begin{pmatrix} \varphi_1 \\ \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3\mathbf{w}_{1,ap}^2 - 1)\varphi_1 + \Lambda \mathbf{w}_{2,ap}^2 \varphi_1 + 2\Lambda \mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\varphi_2 \\ \\ -\varphi_2'' + (3\mathbf{w}_{2,ap}^2 - 1)\varphi_2 + \Lambda \mathbf{w}_{1,ap}^2 \varphi_2 + 2\Lambda \mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\varphi_1 \end{pmatrix} \tag{2.37}$$

is nearly non-invertible because $(\mathbf{w}_{1,ap}', \mathbf{w}_{2,ap}')$ is extremely close to being in the kernel. We point out that this was not an issue in the symmetric case treated in [2], as $(\mathbf{w}_{1,ap}', \mathbf{w}_{2,ap}')$ lies outside of the symmetry class. Nevertheless, guided by [2, Thm. 1.2], we will surpass this difficulty by adapting to our setting a well known *variational Lyapunov-Schmidt method* (see [8] and the references therein).

Naturally, we seek a solution of (1.1)-(1.2) in the form

$$(v_1, v_2) = (\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap}) + (\varphi_1, \varphi_2), \tag{2.38}$$

with fluctuations satisfying the orthogonality condition

$$\int_{-\infty}^{\infty} \left(\mathbf{w}_{1,ap}'\varphi_1 + \mathbf{w}_{2,ap}'\varphi_2\right) dz = 0. \tag{2.39}$$

The following proposition, concerning the so-called *linear projected problem*, makes it legitimate to apply the aforementioned Lyapunov-Schmidt method.

Proposition 2.5. *There exist constants $\Lambda_2, C > 0$ such that if $\Lambda \geq \Lambda_2$ and $(h_1, h_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with $\|(h_1, h_2)\|_* < \infty$, the problem*

$$\mathcal{M}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + c_\Lambda \begin{pmatrix} \boldsymbol{w}_{1,ap}' \\ \boldsymbol{w}_{2,ap}' \end{pmatrix} \tag{2.40}$$

has a unique solution $(\phi_1, \phi_2) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $c_\Lambda \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \left(\boldsymbol{w}_{1,ap}'\phi_1 + \boldsymbol{w}_{2,ap}'\phi_2\right) dz = 0 \tag{2.41}$$

and

$$\|(\phi_1, \phi_2)\|_* \leq C\|(h_1, h_2)\|_*, \tag{2.42}$$

where $\|\cdot\|_*$ stands for the weighted norm

$$\|(\phi_1, \phi_2)\|_* = \sum_{i=1}^{2} \|\rho\left((-1)^i \Lambda^{\frac{1}{4}} z\right) \phi_i\|_{L^\infty(\mathbb{R})}$$

with

$$\rho(x)=\begin{cases}1+x^2, & x\geq 0,\\ \\ 1, & x\leq 0,\end{cases}$$

and the operator $\mathcal{M}$ is as in (2.37).

Proof. The proof will be divided into three steps.

Step 1. We will first establish the validity of the a-priori estimate (2.42) when the constant c_Λ in (2.40) is equal to zero. To this end, as usual, we will argue by contradiction. So, let us suppose that there are $\Lambda_n\to\infty$, $(\phi_{1,n},\phi_{2,n})\in H^2(\mathbb{R})\times H^2(\mathbb{R})$, $(h_{1,n},h_{2,n})\in L^2(\mathbb{R})\times L^2(\mathbb{R})$ satisfying (2.40) with $c_{\Lambda_n}=0$ and (2.41), while

$$\|(\phi_{1,n},\phi_{2,n})\|_*=1,\quad \|(h_{1,n},h_{2,n})\|_*\to 0. \tag{2.43}$$

By a standard barrier argument, we readily find that

$$|\phi_{i,n}(z)|\leq 2\frac{\|\phi_{i,n}\|_{L^\infty(\mathbb{R})}}{1+\Lambda_n^{\frac{1}{2}}z^2},\quad (-1)^i z\geq \Lambda_n^{-\frac{1}{4}},\quad i=1,2. \tag{2.44}$$

We can now proceed analogously to the proof of [2, Thm. 1.2] to get that

$$\phi_{1,n}\to 0\ \text{ in }\ C^1_{loc}(-\infty,0),\qquad \phi_{1,n}\to a_1U_1'\ \text{ in }\ C^1_{loc}(0,\infty);$$

$$\phi_{2,n}\to a_2U_2'\ \text{ in }\ C^1_{loc}(-\infty,0),\quad \phi_{2,n}\to 0\ \text{ in }\ C^1_{loc}(0,\infty),$$

having possibly passed to a subsequence. Moreover, passing to the limit in the orthogonality relation (2.41), with the help of Lebesgue's dominated convergence theorem, we obtain that

$$a_1\int_0^\infty (U_1')^2dz+a_2\int_{-\infty}^0 (U_2')^2dz=0.$$

In fact, thanks to (2.43), similar arguments as in [2, Thm. 1.2] apply to give that

$$|\phi_{i,n}-a_iU_i'|\leq Ce^{-c\Lambda_n^{\frac{1}{4}}|z|}+o(1),\quad i=1,2,$$

uniformly for $0\leq(-1)^{i+1}z\leq 1$, as $n\to\infty$ (as usual, the generic constants do not depend on n). On the other side, we have that

$$\varphi_i(\Lambda_n^{-\frac{1}{4}}x)\to b\dot{V}_i(x)\ \text{ in }\ C^1_{loc}(\mathbb{R}),\quad i=1,2.$$

Then, by the same method as in the proof of [2, Thm. 1.2], we deduce that $a_1=a_2=b=0$, and that

$$\|\phi_{i,n}\|_{L^\infty(\mathbb{R})}\to 0,\quad i=1,2.$$

In turn, relation (2.44) yields that

$$\|(\phi_{1,n},\phi_{2,n})\|_*\to 0,$$

which contradicts (2.43) and completes the proof of Step 1.

Step 2. We will show that the a-priori estimate (2.42) holds for the full problem (2.40)-(2.41). Testing (2.40) by $(\mathbf{w}'_{1,ap},\mathbf{w}'_{2,ap})$ gives that

$$|c_\Lambda|\leq C\|(h_1,h_2)\|_*+C\left|\left\langle\mathcal{M}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\begin{pmatrix}\mathbf{w}'_{1,ap}\\ \mathbf{w}'_{2,ap}\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\times L^2(\mathbb{R})}\right|. \tag{2.45}$$

Unfortunately, since $\mathbf{w}'_{1,ap}, \mathbf{w}'_{2,ap}$ are merely in $H^1(\mathbb{R})$, we cannot use directly the self-adjointness of $\mathcal{M}$ in the last term of the above relation. Nevertheless, from the nonsymmetric analog of (2.35)-(2.36) and the fact that

$$\mathbf{v}''_{i,ap}\left(\left[\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]^-\right) = \mathbf{v}''_{i,ap}\left(\left[\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]^+\right)$$

(keep in mind (2.34)), we find that the jumps of $\mathbf{w}''_{i,ap}$ at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ are of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$, $i = 1, 2$. Hence, splitting the integral under consideration into three parts, integrating each one by parts using the self-adjointness of $\mathcal{M}$ and the previous observation to estimate the boundary terms, we arrive readily at the bound

$$\left\langle \mathcal{M}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \begin{pmatrix}\mathbf{w}'_{1,ap}\\ \mathbf{w}'_{2,ap}\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\times L^2(\mathbb{R})} = \mathcal{O}(\Lambda^{-\infty})\|(\phi_1,\phi_2)\|_* \ \text{ as } \Lambda \to \infty.$$

In turn, via (2.45), we obtain that

$$|c_\Lambda| \le C\|(h_1,h_2)\|_* + \mathcal{O}(\Lambda^{-\infty})\|(\phi_1,\phi_2)\|_*.$$

On the other hand, applying the conclusion of Step 1 to (2.40)-(2.41), it holds

$$\|(\phi_1,\phi_2)\|_* \le C\|(h_1,h_2)\|_* + C|c_\Lambda|.$$

The desired a-priori estimate now follows at once by combining the above two relations.

Step 3. We will establish the existence of a unique solution $(\phi_1,\phi_2) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $c_\Lambda \in \mathbb{R}$ to the problem (2.40)-(2.41), given (h_1,h_2) as in the assertion of the proposition. Let $\mathcal{X}$ denote the subspace of $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ that consists of pairs $\Phi = (\phi_1,\phi_2)$ satisfying the orthogonality condition (2.41). The problem (2.40)-(2.41) admits the following weak formulation: find $\Phi \in \mathcal{X}$ such that

$$\langle \mathcal{M}(\Phi), \Psi\rangle_{L^2(\mathbb{R})\times L^2(\mathbb{R})} = \langle H, \Psi\rangle_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \quad \forall\ \Psi \in \mathcal{X},$$

(where $H = (h_1,h_2)$). This weak formulation can then be readily put in the operator form

$$\mathbb{M}(\Phi) = \hat{H},$$

where $\mathbb{M} : \mathcal{X} \to \mathcal{X}$ is self-adjoint, and $\hat{H} \in \mathcal{X}$ depends linearly on H. The a-priori estimate of Step 2 implies that, for $\hat{H} = 0$, there is only the trivial solution. Consequently, by the self-adjoint property of $\mathbb{M}$, we infer that the above problem has a solution $\Phi \in \mathcal{X}$, which is clearly unique. This completes the proof of Step 3 and also of the proposition. $\square$

Armed with the above proposition, we can apply the contraction mapping theorem in these weighted spaces to show that the *nonlinear projected problem*

$$\begin{cases} -v_1'' + g_1 v_1^3 - \lambda_1 v_1 + \Lambda v_2^2 v_1 = c_\Lambda \mathbf{w}'_{1,ap}, \\ \\ -v_2'' + g_2 v_2^3 - \lambda_2 v_2 + \Lambda v_1^2 v_2 = c_\Lambda \mathbf{w}'_{2,ap}, \end{cases}$$

has a solution (v_1,v_2) and c_Λ such that

$$v_i = \mathbf{w}_{i,ap} + \varphi_i \ \text{ with } \ \varphi_i \in H^2(\mathbb{R}) \ \text{ and } \ \|\varphi_i\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\Lambda^{-\infty}) \ \text{ as } \ \Lambda \to \infty, \ \ i = 1,2. \quad (2.46)$$

Moreover, the fluctuation (φ_1,φ_2) satisfies the orthogonality condition (2.39), while the constant c_Λ is of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$. To finish, we will show that the latter constant is in fact zero. Then, elliptic regularity theory would imply that the solution is smooth (up to this moment, we know that $v_1', v_2' \in H^2(\mathbb{R})$). To this end, testing the above nonlinear projected problem with (v_1', v_2'), thanks to (1.4), we arrive at

$$\begin{aligned} 0 &= c_\Lambda \textstyle\sum_{i=1}^{2} \int_{-\infty}^{\infty} \left[\left(\mathbf{w}_{i,ap}'\right)^2 + \mathbf{w}_{i,ap}'\varphi_i'\right] dz \\ &= c_\Lambda \textstyle\sum_{i=1}^{2} \int_{-\infty}^{\infty} \left[\left(\mathbf{w}_{i,ap}'\right)^2 - \mathbf{w}_{i,ap}''\varphi_i\right] dz. \end{aligned}$$

In turn, using the rough estimates

$$\int_{-\infty}^{\infty} \left(\mathbf{w}_{i,ap}'\right)^2 dz \geq c, \quad \left|\mathbf{w}_{i,ap}''(z)\right| \leq C\Lambda^{\frac{1}{4}} e^{-c|z|}, \ z \in \mathbb{R}, \ i = 1, 2,$$

and (2.46), we can conclude that $c_\Lambda = 0$ for Λ sufficiently large, as desired.

In summary, we have that the obvious non-symmetric analogs of Theorems 1.1 and 1.2 in [2] hold. However, taking into account that the estimates (2.24) and (2.25) are not as sharp as the corresponding ones in [2] due to the presence of the logarithm, we just have to change the corresponding estimates in [2, Thm. 1.1] to

$$v_{i,\Lambda}(z) = \Lambda^{-\frac{1}{4}} V_i\left(\Lambda^{\frac{1}{4}} z\right) + \mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac{3}{4}} + |z|^3\right), \tag{2.47}$$

$$v_{i,\Lambda}'(z) = \dot{V}_i\left(\Lambda^{\frac{1}{4}} z\right) + \mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac{1}{2}} + |z|^2\right), \tag{2.48}$$

uniformly on $\left[-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]$, as $\Lambda \to \infty$. Furthermore, the monotonicity property

$$v_1'(z) > 0, \quad v_2'(z) < 0, \quad z \in \mathbb{R}, \tag{2.49}$$

and the analogous exponential decay estimates hold. As is apparent, the appearance of the logarithm in the estimate (2.47) is only due to technical reasons and we believe that it can be removed.

3. Uniqueness of solutions

In this section, we will establish the uniqueness (modulo translations) of solutions to (1.1)-(1.2), satisfying the natural monotonicity property (2.49), for any Λ in the range (1.3). In particular, the latter monotonicity property is satisfied by stable solutions (see [3, Thm. 3.1]). To the best of our knowledge, this type of uniqueness was not known, even in the case of minimizing solutions (see also [3, Rem. 1.2]).

3.1. **The symmetric case.** Let us first consider the symmetric case where $g_1 = g_2$ and $\lambda_1 = \lambda_2$ (keep in mind (1.4)). In fact, without loss of generality, we may restrict our attention to the model problem:

$$\begin{cases} -v_1'' + v_1^3 - v_1 + \Lambda v_2^2 v_1 = 0, \\ -v_2'' + v_2^3 - v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{3.1}$$

$$(v_1, v_2) \to (0, 1) \text{ as } z \to -\infty, \quad (v_1, v_2) \to (1, 0) \text{ as } z \to \infty, \tag{3.2}$$

$$v_1'(z) > 0, \quad v_2'(z) < 0, \quad z \in \mathbb{R}, \tag{3.3}$$

as was the case in [2]. We point out, however, that we do not a-priori assume any symmetry between the components v_1 and v_2.

The main result of this section is the following.

Proposition 3.1. *If $\Lambda > 1$, there exists a unique solution (modulo translations) to (3.1)-(3.2)-(3.3).*

Proof. The proof is based on the observation that uniqueness holds for $\Lambda = 3$ (see also [3] and the references therein) and a continuation argument.

Throughout, we shall assume the 'pinning' condition:

$$v_1(0) = v_2(0). \tag{3.4}$$

Firstly, we note that any solution of (3.1)-(3.2) with $\Lambda > 1$ satisfies

$$v_1^2 + v_2^2 \leq 1, \quad z \in \mathbb{R}, \tag{3.5}$$

(see [3, Thm. 2.4]).

We next claim that the following localization property holds: Let $\underline{\lambda} > 1$ and $\varepsilon > 0$, then there exists $M > 0$ such that any solution of (3.1)-(3.2)-(3.4) with $\Lambda \geq \underline{\lambda}$ such that

$$v_1'(z) \geq 0, \quad v_2'(z) \leq 0, \quad z \in \mathbb{R}, \tag{3.6}$$

satisfies

$$1 - v_1(z) + v_2(z) < \varepsilon \ \text{ for } \ z \geq M, \tag{3.7}$$

and the analogous relation for $z \leq -M$. Indeed, in view of the conservation of the Hamiltonian, it is enough to verify that, given $\epsilon > 0$, there exists $L > 0$ so that any such solution satisfies

$$v_1'(z_0) - v_2'(z_0) < \epsilon \ \text{ for some } \ z_0 \in [0, L].$$

If not, for any $L > 0$, there would exist at least one such solution satisfying

$$v_1'(z) - v_2'(z) \geq \epsilon \ \text{ for } z \in [0, L],$$

i.e.,

$$v_1(L) - v_2(L) \geq \epsilon L,$$

which is clearly not possible for large L by virtue of (3.5). The proof of the claim is complete. By an easy maximum principle argument, we may further assume that any solution of (3.1)-(3.2)-(3.4)-(3.6) satisfies

$$v_1'(z) > 0 \ \text{ and } \ v_2'(z) < 0 \ \text{ for } \ |z| \geq M.$$

Moreover, similarly to [7, Thm. 2.8], for any $1 < \underline{\lambda} < \bar{\lambda}$, there exist constants $c, C > 0$ such that any solution of (3.1)-(3.2)-(3.4)-(3.6) with $\Lambda \in \left[\underline{\lambda}, \bar{\lambda}\right]$ satisfies

$$\sum_{i=1}^{2} \left\{ |v_i''| + |v_i'| + |v_i - 2 + i| \right\} \leq C e^{-cz}, \quad z \geq M, \tag{3.8}$$

and the analogous estimate for $z \leq -M$.

The previous observations have the following interesting implication: Let $(v_{1,n}, v_{2,n})$ be a sequence of solutions of (3.1)-(3.2)-(3.3)-(3.4) with $\Lambda = \Lambda_n$, such that

$$v_{i,n} - v_{i,\infty} \to 0 \text{ in } H^2(\mathbb{R}), \ i = 1, 2, \ \text{and } \Lambda_n \to \Lambda_\infty \in (1, \infty).$$

Then, the limit $(v_{1,\infty}, v_{2,\infty})$ satisfies (3.1)-(3.2)-(3.3)-(3.4) with $\Lambda = \Lambda_\infty$. In view of our aforementioned observation, without loss of generality, it is enough to exclude the scenario where

$$v'_{1,\infty}(z_*) = 0 \quad \text{for some } z_* \in (-M, M). \tag{3.9}$$

To this end, we note that $\varphi \equiv v'_{1,\infty} \geq 0$ satisfies

$$-\varphi'' + P(z)\varphi = -2\Lambda_\infty v_{1,\infty} v_{2,\infty} v'_{2,\infty} \geq 0, \quad z \in \mathbb{R},$$

for some smooth function P. Thus, the above scenario (3.9) cannot happen, as it would violate a version of Hopf's boundary point lemma (see for example [11, Thm. 2.8.4]).

It follows from [3, Thm. 3.1] that the linearized operator of (3.1) about a solution of (3.1)-(3.2)-(3.3)-(3.4), defined on $H^2(\mathbb{R}) \times H^2(\mathbb{R})$, has a one-dimensional kernel spanned by (v'_1, v'_2). We also note that this linear operator is self-adjoint in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and its continuous spectrum is contained in $[\Lambda^2 - 1, \infty)$ (see again [3]). Therefore, by the variational Lyapunov-Schmidt procedure of Proposition 2.5 (in a regular perturbation setting) or the dynamical systems approach of [12], and the observation in the previous paragraph, we deduce the following: Each solution $(v_{1,\Lambda_0}, v_{2,\Lambda_0})$ of (3.1)-(3.2)-(3.3)-(3.4), for some $\Lambda_0 > 1$, is contained in a locally unique and smooth for $|\Lambda - \Lambda_0|$ sufficiently small (with respect to variations from $(v_{1,\Lambda_0}, v_{2,\Lambda_0})$ in the $H^2(\mathbb{R}) \times H^2(\mathbb{R})$-norm) branch of solutions of (3.1)-(3.2)-(3.3)-(3.4). In fact, if the aforementioned local uniqueness property failed, there would be such a solution with the associated linearized operator having a nontrivial element (Z_1, Z_2) in its kernel such that $Z_1(0) = Z_2(0)$, which is impossible by the opposite sign of v'_1 and v'_2. We observe next that, thanks to (3.5), (3.7) and (3.8), any solution to (3.1)-(3.2)-(3.3)-(3.4) with $\Lambda \in [\underline{\lambda}, \bar{\lambda}]$ satisfies

$$\|v_1 - v_{1,\Lambda_0}\|_{H^2(\mathbb{R})} + \|v_2 - v_{2,\Lambda_0}\|_{H^2(\mathbb{R})} \leq C,$$

where C depends only on $\underline{\lambda}, \bar{\lambda} > 1$. Therefore, the aforementioned solution branch of (3.1)-(3.2)-(3.3)-(3.4) can be extended smoothly and uniquely for all $\Lambda > 1$.

The key observation is that for $\Lambda = 3$ there exists a unique solution of (3.1)-(3.2)-(3.3)-(3.4). Indeed, letting $u \equiv v_1 + v_2$ yields that

$$u'' + u - u^3 = 0, \quad z \in \mathbb{R}; \quad u \to 1, \quad z \to \pm\infty,$$

that is $u \equiv 1$ and the aforementioned uniqueness follows at once. Hence, in the case where there was non-uniqueness of solutions to (3.1)-(3.2)-(3.3)-(3.4) for some $\Lambda > 1$, we would have two of the previously described solution branches meeting at some $\Lambda_* > 1$. However, this is not possible from the local uniqueness of the solution branches. □

3.2. **The general case.** It remains to establish the following.

Theorem 3.1. *If Λ is as in (1.3) and (1.4) holds, there exists a unique solution (modulo translations) to (1.1)-(1.2)-(2.49).*

Proof. Suppose, to the contrary, that there are two different solutions of (1.1)-(1.2)-(2.49)-(3.4) for some $\bar{\Lambda} > \sqrt{g_1 g_2}$. We may assume without loss of generality that $g_1 > g_2$. We then consider a new one-parameter family of equations (1.1), which we denote by $(1.1)_\Lambda$, where g_1 is a linear function of Λ such that $g_1(\bar{\Lambda}) = g_1$, $g_1(\bar{\Lambda} + 1) = g_2$, while λ_1 is such that (1.4) is satisfied (g_2, λ_2 remain fixed). In turn, since $g_1 > g_2$, it is clear that (1.3) holds for $\Lambda \in [\bar{\Lambda}, \bar{\Lambda}+1]$. It is easy to see that solutions of $(1.1)_\Lambda$-$(1.2)_\Lambda$-(2.49)-(3.4) for $\Lambda \in [\bar{\Lambda}, \bar{\Lambda}+1]$ satisfy the obvious analog of the localization property in the proof of Proposition 3.1, as well as its monotonicity and exponential decay implications. Moreover, we can analogously deduce the persistence of solutions to $(1.1)_\Lambda$-$(1.2)_\Lambda$-(2.49)-(3.4) with respect to small variations in Λ. We are now in position to conclude. Indeed, by virtue of Proposition 3.1, which guarantees uniqueness for the aforementioned problem for $\Lambda = \bar{\Lambda}+1$, there will be two solution branches meeting at some $\Lambda_* \in (\bar{\Lambda}, \bar{\Lambda}+1]$ which contradicts the local uniqueness of the branches. □

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