Analysis of an irregular boundary layer behavior for the steady state flow of a Boussinesq fluid

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Turcotte, Spence and Bau *Int. J. Heat Mass Transfer* (1982) considered the vertical flow of an internally heated Boussinesq fluid in a vertical channel with viscous dissipation and pressure work.

Boundary value problem for the steady state flow:

\[
\begin{cases}
2u'' = u^2 - A(1 - x^2), & x \in (-1, 1), \\
u(-1) = u(1) = 0,
\end{cases}
\]

* A represents the (non-dimensional) heat addition  
* $u$ is the velocity  
* $x$ is the scaled position  
* $[-1, 1]$ is the horizontal cross section of the vertical channel
\[
\begin{cases}
2u'' = u^2 - A(1 - x^2), \quad x \in (-1, 1), \\
u(-1) = u(1) = 0,
\end{cases}
\]

Numerical simulations indicated that the number of (even) solutions diverges as \( A \to \infty \).

Setting
\[
\varepsilon^2 = \frac{2}{\sqrt{A}} \to 0 \quad \text{and} \quad v = \frac{1}{\sqrt{A}}u,
\]
the problem is equivalent to the singular perturbation problem:
\[
\begin{cases}
\varepsilon^2 v'' = v^2 - (1 - x^2), \quad x \in (-1, 1), \\
v(-1) = v(1) = 0.
\end{cases}
\]
The $\varepsilon = 0$ limit problem (outer problem)

\[
\begin{cases}
0 = v^2 - (1 - x^2), \quad x \in (-1, 1), \\
v(-1) = v(1) = 0.
\end{cases}
\]

Two continuous solutions : $v_0 = \pm \sqrt{1 - x^2}$. 
It is natural to expect that there exist solutions \( v_\varepsilon \to v_0 \) uniformly on \([-1,1]\) as \( \varepsilon \to 0 \).

However, a more complete formal analysis is required, with special attention at the boundary where such solutions should develop *irregular boundary layer behavior*. 
Blow-up problem

Near the boundary $x = -1$, say for $x \in [-1, -1 + \delta]$, a standard matching asymptotic analysis Turcotte, Spence and Bau *Int. J. Heat Mass Transfer* (1982) predicts that

$$v_\varepsilon(x) \sim \varepsilon^{2/5} 2^{5/3} Y(s), \quad s = 2^{-7/3}(x + 1)/\varepsilon^{4/5}$$

$$\begin{align*}
y'' &= y^2 - s, & s > 0, \\
y(0) &= 0, \quad y - \sqrt{s} \to 0 \text{ as } s \to \infty.
\end{align*}$$

The above differential equation is integrable, known as the Painlevé-I transcendent.
Holmes-Spence (1984) used dynamical systems techniques to find a unique increasing solution $Y_+$ and at least one solution $Y_-$ with exactly one local minimum (see also Dancer-Yan (2005) for variational methods).

Hastings-Troy (1989) showed that these two are the only solutions (their proof relied on some four decimal point numerical calculations!)
\[
\begin{cases}
y'' = y^2 - s, \ s > 0, \\
y(0) = 0, \ y - \sqrt{s} \to 0 \ \text{as} \ s \to \infty.
\end{cases}
\]

Nondegeneracy

\[-\phi'' + 2Y_{\pm}(s)\phi = 0, \ s > 0, \ \phi(0) = 0, \ \phi \in L^\infty(0, \infty) \Rightarrow \phi \equiv 0.\]
Proposition [Sourdis 2015]

\[-\phi'' + 2Y_-(s)\phi = 0, \ s > 0, \ \phi(0) = 0, \ \phi \in L^\infty(0, \infty) \Rightarrow \phi \equiv 0.\]

Idea of the proof

\[y'' = y^2 - s; \ y(0) = 0, \ y(s) \sim \sqrt{s}, \ s \rightarrow \infty.\]

We know that there is a unique positive solution \(\Psi_+\), with \(\Psi'_+ > 0\). Following Dancer-Yan (2005), we search a sign changing solution \(Y_- < Y_+\) as \(Y_- = Y_+ - w\), where \(w\) is a positive solution of the 'Nonlinear Schrödinger Equation'

\[w'' - V(s)w + w^2 = 0, \ s > 0, w(0) = 0, \ w(\infty) = 0,\]

with potential \(V(s) = 2Y_+(s)\)

\[V(0) = 0, \ V' > 0, \ V \sim 2\sqrt{s} \text{ as } s \rightarrow \infty\]

\(w\) is a mountain pass solution to the above problem
\[ w'' - V(s)w + w^2 = 0, \ s > 0, w(0) = 0, \ w(\infty) = 0, \]
\[ V(0) = 0, \ V' > 0, \ V \sim 2\sqrt{s} \text{ as } s \to \infty \]

Felmer-Martinez-Tanaka (2008) Uniqueness and non-degeneracy result includes:
\[ w'' + (\nu/s)w' - V(s)w + w^2 = 0, \ s > 0, w(a) = 0 (a > 0), \ w(\infty) = 0, \]
\[ \nu > 0, \ 0 < \inf V < \sup V < \infty, \ V' \geq 0 \]

Modelled after \( \Delta u - u + u^p = 0, \ x \in \mathbb{R}^n, \ |x| > a (\nu = n - 1) \).

Long history: Coffman, Kwong, McLeod-Serrin, Peletier-Serrin, Ni-Takagi, Tang ...
Outer solution: \( \sqrt{1 - x^2} \)

Near \( x = -1 \), two possible inner solutions:

\[ \varepsilon^{2/5} 2^{5/3} Y_{\pm} \left( 2^{-7/3} (x + 1) / \varepsilon^{4/5} \right) \]

Glue them together and then show that the resulting global approximate solution can be perturbed to a genuine solution


Theorem [Sourdis (2015)]

\[
\begin{aligned}
\varepsilon^2 v'' &= v^2 - (1 - x^2), \quad x \in I = (-1, 1), \\
v(-1) &= v(1) = 0.
\end{aligned}
\]

For small $\varepsilon > 0$ has four solutions such that

\[
\|v_\varepsilon - \sqrt{1 - x^2}\|_{L^\infty(-1,1)} \leq C\varepsilon^{2/5}
\]

\[
v_\varepsilon(x) = 2^{5/3}\varepsilon^{2/5} Y_i \left(2^{-7/3}\text{dist}(x, \partial I)/\varepsilon^{4/5}\right) + O\left(\varepsilon^{6/5}\right),
\]

for $\text{dist}(x, \partial I) \leq C\varepsilon^{4/5}, \ i = \pm$.

Moreover, they are nondegenerate:

\[
-\varepsilon^2 \phi'' + 2v_\varepsilon \phi = f, \quad \phi(-1) = \phi(1) = 0 \Rightarrow \|\phi\|_{L^\infty} \leq C\varepsilon^{-2/5}\|f\|_{L^\infty}
\]
Morse index 1 solution (see also Dancer-Yan (2005))
Dancer-Yan (2005) gave the first proof of the Lazer-McKenna conjecture from 1983 in the PDE case by establishing that the number of solutions diverges as $\varepsilon \to 0$ for the singular perturbation problem:

$$\varepsilon^2 \Delta u = |u|^p - \phi_1(x), \; x \in \Omega \subset \mathbb{R}^n; \; u = 0, \; x \in \partial \Omega$$

$\phi_1 > 0$ eigenfunction associated to $\lambda_1(\Omega)$ ($\partial_\nu < 0$ on $\partial \Omega$)

$$1 < p < \frac{(n + 2)}{(n - 2)} \text{ if } n \geq 3, \; p > 1 \text{ if } n = 1, 2.$$
\( \varepsilon^2 \Delta w - 2u_\varepsilon(x)w + w^2 = 0, \ x \in \Omega, \ w > 0, \ w = 0 \ x \in \partial \Omega \)