Analysis of an irregular boundary layer behavior for the steady state flow of a Boussinesq fluid

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Turcotte, Spence and Bau *Int. J. Heat Mass Transfer* (1982) considered the vertical flow of an internally heated Boussinesq fluid in a vertical channel with viscous dissipation and pressure work.

Boundary value problem for the steady state flow:

$$\begin{cases} 2u'' = u^2 - A(1-x^2), \quad x \in (-1,1), \\ \\ u(-1) = u(1) = 0, \end{cases}$$

A represents the (non-dimensional) heat addition
u is the velocity
x is the scaled position
$[-1,1]$ is the horizontal cross section of the vertical channel

$$\begin{cases} 2u'' = u^2 - A(1-x^2), \quad x \in (-1,1), \\ \\ u(-1) = u(1) = 0, \end{cases}$$

Numerical simulations indicated that the number of (even) solutions diverges as $A \to \infty$.

Setting

$$\varepsilon^2 = \frac{2}{\sqrt{A}} \to 0 \text{ and } v = \frac{1}{\sqrt{A}} u,$$

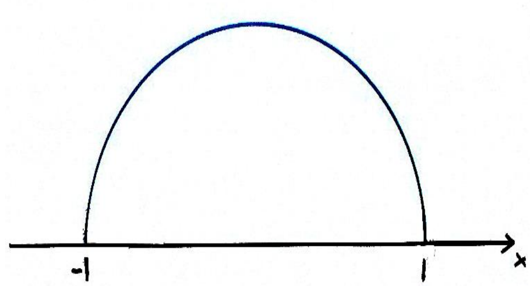
the problem is equivalent to the singular perturbation problem:

$$\begin{cases} \varepsilon^2 v'' = v^2 - (1-x^2), \quad x \in (-1,1), \\ \\ v(-1) = v(1) = 0. \end{cases}$$

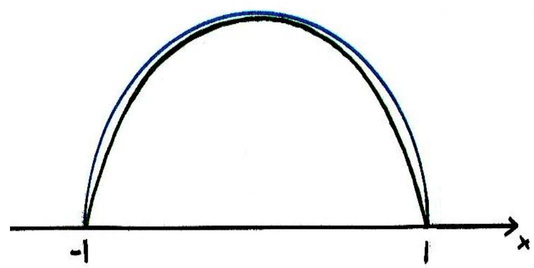
The $\varepsilon = 0$ limit problem (outer problem)

$$\begin{cases} 0 = v^2 - (1 - x^2), \quad x \in (-1, 1), \\ \\ v(-1) = v(1) = 0. \end{cases}$$

Two continuous solutions : $v_0 = \pm\sqrt{1 - x^2}$.



It is natural to expect that there exist solutions $v_\varepsilon \to v_0$ uniformly on [-1,1] as $\varepsilon \to 0$.



However, a more complete formal analysis is required, with special attention at the boundary where such solutions should develop irregular boundary layer behavior.

Blow-up problem

Near the boundary $x = -1$, say for $x \in [-1, -1 + \delta]$, a standard matching asymptotic analysis **Turcotte, Spence and Bau** *Int. J. Heat Mass Transfer* (1982) predicts that

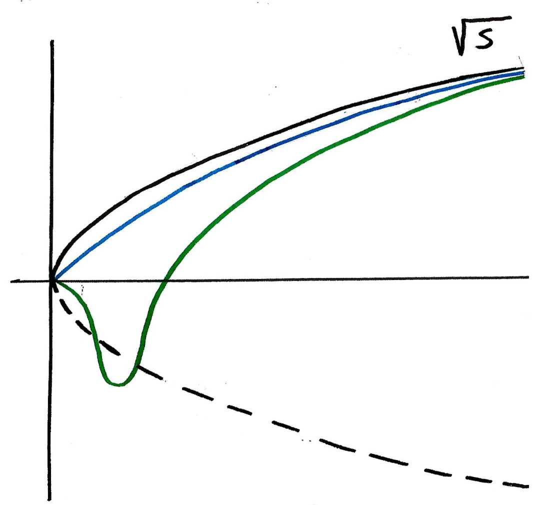
$$v_\varepsilon(x) \sim \varepsilon^{2/5} 2^{5/3} Y(s), \quad s = 2^{-7/3}(x+1)/\varepsilon^{4/5}$$

$$\begin{cases} y'' = y^2 - s, \quad s > 0, \\ y(0) = 0, \qquad y - \sqrt{s} \to 0 \quad \text{as} \quad s \to \infty. \end{cases}$$

The above differential equation is integrable, known as the Painlevé-I transcendent.

Holmes-Spence (1984) used dynamical systems techniques to find a unique increasing solution Y_+ and at least one solution Y_- with exactly one local minimun (see also **Dancer-Yan (2005)** for variational methods).

Hastings-Troy (1989) showed that these two are the only solutions (their proof relied on some four decimal point numerical calculations!)



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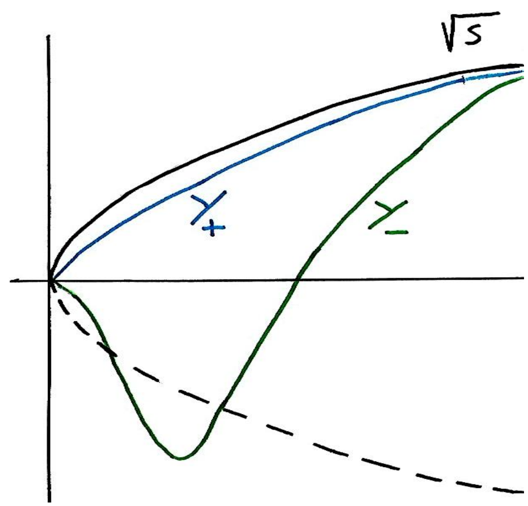


Nondegeneracy

$$-\phi'' + 2Y_\pm(s)\phi = 0, \;\; s > 0, \; \phi(0) = 0, \;\; \phi \in L^\infty(0,\infty) \Rightarrow \phi \equiv 0.$$

Proposition [Sourdis 2015]

$-\phi''+2Y_-(s)\phi = 0,\ \ s > 0,\ \phi(0) = 0,\ \ \phi \in L^\infty(0,\infty) \Rightarrow \phi \equiv 0.$

Idea of the proof

$$y'' = y^2 - s;\ \ y(0) = 0,\ \ y(s) \sim \sqrt{s},\ \ s \to \infty.$$

We know that there is a unique positive solution Ψ_+, with $\Psi_+' > 0$. Following **Dancer-Yan (2005)**, we search a sign changing solution $Y_- < Y_+$ as $Y_- = Y_+ - w$, where w is a positive solution of the 'Nonlinear Schrödinger Equation'

$$w'' - V(s)w + w^2 = 0,\ \ s > 0, w(0) = 0,\ \ w(\infty) = 0,$$

with potential $V(s) = 2Y_+(s)$

$$V(0) = 0,\ \ V' > 0,\ \ V \sim 2\sqrt{s} \text{ as } s \to \infty$$

w is a mountain pass solution to the above problem

$$w'' - V(s)w + w^2 = 0,\ \ s > 0, w(0) = 0,\ \ w(\infty) = 0,$$

$$V(0) = 0,\ \ V' > 0,\ \ V \sim 2\sqrt{s} \text{ as } s \to \infty$$

Felmer-Martinez-Tanaka (2008) Uniqueness and non-degeneracy result includes:

$$w'' + (\nu/s)w' - V(s)w + w^2 = 0,\ s > 0, w(a) = 0\ (a > 0),\ w(\infty) = 0,$$

$$\nu > 0, \quad 0 < \inf V < \sup V < \infty, \quad V' \geq 0$$

Modelled after $\Delta u - u + u^p = 0$, $x \in \mathbb{R}^n$, $|x| > a$ $(\nu = n - 1)$.

Long history: Coffman, Kwong, McLeod-Serrin, Peletier-Serrin, Ni-Takagi, Tang ...

Outer solution: $\sqrt{1-x^2}$

Near $x = -1$, two possible inner solutions:

$$\varepsilon^{2/5} 2^{5/3} Y_{\pm}\left(2^{-7/3}(x+1)/\varepsilon^{4/5}\right)$$

Glue them together and then show that the resulting global approximate solution can be perturbed to a genuine solution

PDE approach: **Sourdis-Fife** (2007), **Karali-Sourdis** (2012)
Blow-up approach to geometric singular perturbation theory: **Krupa-Szmolyan** (2001), **Schecter** (2004) **Schecter-Sourdis** (2009)

Theorem [Sourdis (2015)]

$$\begin{cases} \varepsilon^2 v'' = v^2 - (1 - x^2), \quad x \in I = (-1, 1), \\ \\ v(-1) = v(1) = 0. \end{cases}$$

For small $\varepsilon > 0$ has four solutions such that

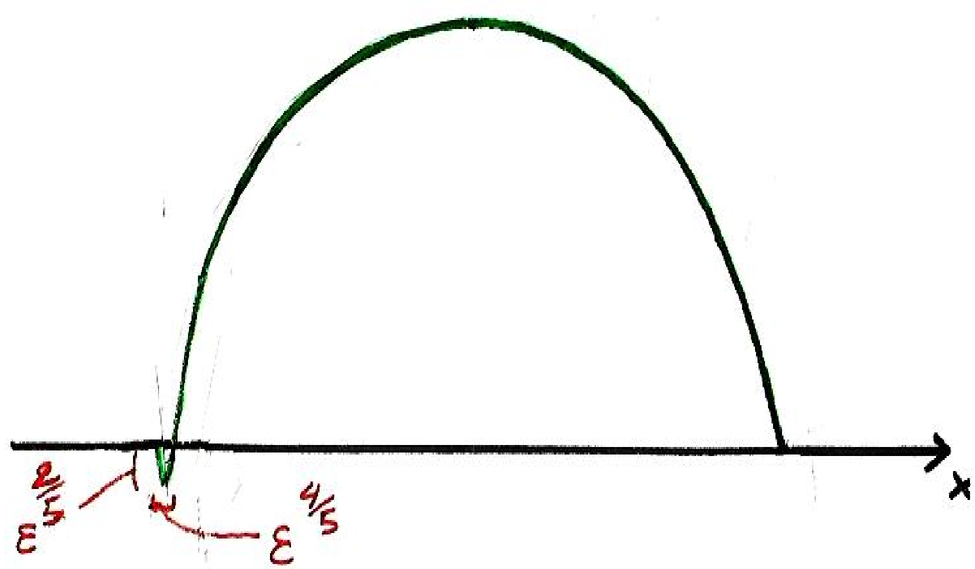
$$\|v_\varepsilon - \sqrt{1 - x^2}\|_{L^\infty(-1,1)} \leq C\varepsilon^{2/5}$$

$$v_\varepsilon(x) = 2^{5/3}\varepsilon^{2/5} Y_i \left(2^{-7/3}\mathrm{dist}(x, \partial I)/\varepsilon^{4/5}\right) + O\left(\varepsilon^{6/5}\right),$$

for $\mathrm{dist}(x, \partial I) \leq C\varepsilon^{4/5}$, $i = \pm$.

Moreover, they are nondegenerate:

$$-\varepsilon^2\phi'' + 2v_\varepsilon\phi = f, \;\; \phi(-1) = \phi(1) = 0 \;\Rightarrow\; \|\phi\|_{L^\infty} \leq C\varepsilon^{-2/5}\|f\|_{L^\infty}$$



Morse index 1 solution (see also **Dancer-Yan** (2005))

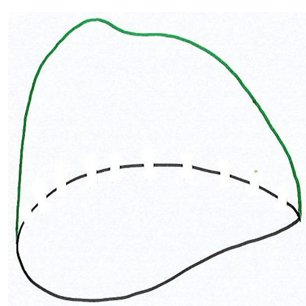
Dancer-Yan (2005) gave the first proof of the Lazer-McKenna conjecture from 1983 in the PDE case by establishing that the number of solutions diverges as $\varepsilon \to 0$ for the singular perturbation problem:

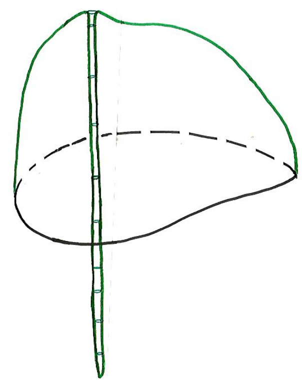
$$\varepsilon^2 \Delta u = |u|^p - \phi_1(x), \; x \in \Omega \subset \mathbb{R}^n; \; u = 0, \; x \in \partial\Omega$$

$\phi_1 > 0$ eigenfunction associated to $\lambda_1(\Omega)$ ($\partial_\nu < 0$ on $\partial\Omega$

$$1 < p < (n+2)/(n-2) \text{ if } n \geq 3, \quad p > 1 \text{ if } n = 1, 2.$$

Outer solution: $\phi_1^{1/p}$





$$\varepsilon^2 \Delta w - 2u_\varepsilon(x)w + w^2 = 0,\ x \in \Omega,\ w > 0,\ w = 0\ x \in \partial\Omega$$