ON CORNER LAYERS IN BALANCED EQUATIONS

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Abstract. We study equations of the form
\[ \varepsilon^2 u_{xx} = (u - a(x))(u - b(x))(u - c(x)), \quad x \in (0, 1) \]
with Neumann boundary conditions, where \( b \equiv \frac{a + c}{2} \) and \( 0 < \varepsilon \ll 1 \). The functions \( a, b, c \) are allowed to intersect and, thus, the degenerate relation obtained by setting \( \varepsilon = 0 \) can have a continuous but non-smooth solution \( u_0 \). We apply a variational argument to show that, for small \( \varepsilon \), there exists a solution \( u_\varepsilon \in C^2 \) such that \( u_\varepsilon \to u_0 \) as \( \varepsilon \to 0 \).

1. Introduction and main result

Consider the singularly perturbed boundary value problem
\[
\begin{align*}
\varepsilon^2 u_{xx} &= f(x, u), \quad x \in (0, 1), \\
u_x(0) &= 0 = u_x(1),
\end{align*}
\]
(1)
under the following hypothesis on \( f \in C([0, 1] \times \mathbb{R}) \) (\( f \) is independent of \( \varepsilon \)):

(I) There exists \( u_0 \in H^1(0, 1) \) such that
\[ f(x, u_0(x)) = 0, \quad x \in [0, 1]. \]

(II) There exists a \( \delta_0 > 0 \) such that
\[ F(x, u) := \int_{u_0(x)}^u f(x, t)dt \geq 0 \quad \text{if} \quad |u - u_0(x)| \leq \delta_0, \quad x \in [0, 1]. \]

(III) If \( v \in H^1(0, 1) \) satisfies
\[ \|v - u_0\|_{L^\infty(0, 1)} \leq \delta_0 \quad \text{and} \quad F(x, v(x)) = 0, \quad x \in [0, 1], \]
then \( u_0 \equiv v \).

Example. A typical example is
\[ f(x, u) = (u - a(x))(u - b(x))(u - c(x)) \quad \text{with} \quad b \equiv \frac{a + c}{2}, \]
and \( a, b, c \) intersecting at \( x = x_0 \). In the figures we sketch such a situation and illustrate the graph of \( F(x, \cdot) \) with \( x \) considered as a parameter. We can easily see that
\[ u_0 := \begin{cases} 
  c(x), & x \leq x_0, \\
  a(x), & x > x_0,
\end{cases} \]
satisfies (I)-(III) (other choices of \( u_0 \) are also possible).

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Remark 1. If \(a, b, c \in C^2\) and \(c(x_0) < b(x_0) < a(x_0)\) one can prove the existence of a solution \(u_\varepsilon\) such that

\[
\|u_\varepsilon - u_0\|_{L^\infty(0,1)} \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

and

\[
u_\varepsilon = \frac{\varepsilon^2}{2} U\left(\frac{x - x_0}{\varepsilon^2}\right) + a(x_0) + \text{H.O.T.}, \quad |x - x_0| \leq \varepsilon^2
\]

with \(U\) solving

\[
U_{xx} = (U - a_x(x_0)x)(U - b_x(x_0)x)(U - c_x(x_0)x), \quad x \in \mathbb{R},
\]

\[
U - c_x(x_0)x \to 0, \quad x \to -\infty, \quad U - a_x(x_0)x \to 0, \quad x \to \infty.
\]

This can be done without the assumption \(b \equiv \frac{a + c}{2}\) by borrowing techniques from [6] and proving the existence of \(U\) via upper and lower solutions (employing the eigenfunction method). Details will appear elsewhere. However even in this case our proof of Theorem 1 is still useful since, besides its simplicity, it can be applied (with only minor modifications) to the partial differential analog of (1) (it gives convergence only in \(L^2\) though). We should mention that the PDE problem of exactly two functions intersecting has been treated (in great generality) by constructing suitable upper and lower solutions in [1]. Note also that our proof below works if \(a = b = c\) over an interval.

Let us now return to our original problem. Our existence result is

Main Theorem. Let \(f, u_0\) satisfy assumptions (I)-(III). Then there exists an \(\varepsilon_0 > 0\) such that for \(0 < \varepsilon < \varepsilon_0\) problem (1) has a solution \(u_\varepsilon \in C^2[0,1]\) such that

\[
\|u_\varepsilon - u_0\|_{L^\infty(0,1)} = o_\varepsilon(1),
\]

where \(o_\varepsilon(1) \to 0\) as \(\varepsilon \to 0\).
Proof. Let

\[ J_\varepsilon(u) = \int_0^1 \varepsilon^2 x^2 + F(x, u) \, dx, \]

for 

\[ u \in U_\delta = \{ u \in H^1(0, 1) : \|u - u_0\|_{L^\infty(0, 1)} \leq \delta \}, \quad 0 < \delta \leq \delta_0 \]

(our motivation for this comes from [3]). Obviously \( J_\varepsilon \) is bounded from below (it is nonnegative) and coercive in \( U_\delta \) (with respect to the \( H^1 \) norm), and \( U_\delta \) is weakly closed in \( H^1(0, 1) \). Hence, by standard arguments, there exists \( u_{\varepsilon, \delta} \in U_\delta \) such that

\[ J_\varepsilon(u_{\varepsilon, \delta}) = \inf_{u \in U_\delta} J_\varepsilon(u). \]

The assertion of the theorem will follow immediately if we show that for any \( 0 < \delta \leq \delta_0 \) there exists \( \varepsilon(\delta) > 0 \) such that

\[ \|u_{\varepsilon, \delta} - u_0\|_{L^\infty(0, 1)} \leq \delta \]

if \( 0 < \varepsilon < \varepsilon(\delta) \).

We argue by contradiction. Suppose that for some \( \delta \in (0, \delta_0] \) there exist \( \varepsilon_n \to 0 \) as \( n \to \infty \) such that

\[ \|u_{\varepsilon_n, \delta} - u_0\|_{L^\infty(0, 1)} = \delta, \quad n \geq 1. \]

Since \( u_0 \in U_\delta \), we have (dropping the subscript \( \delta \))

\[ J_{\varepsilon_n}(u_{\varepsilon_n}) \leq J_{\varepsilon_n}(u_0), \quad n \geq 1, \]

i.e.,

\[ \frac{\varepsilon_n^2}{2} \int_0^1 \left( \frac{du_{\varepsilon_n}}{dx} \right)^2 \, dx + \int_0^1 F(x, u_{\varepsilon_n}) \, dx \leq \frac{\varepsilon_n^2}{2} \int_0^1 \left( \frac{du_0}{dx} \right)^2 \, dx, \quad n \geq 1, \]

(recall that \( F(x, u_0) \equiv 0 \)) and by using (II) we obtain that

\[ \|u_{\varepsilon_n}\|_{H^1(0, 1)} \leq C, \quad n \geq 1 \quad (C \text{ indep. of } n). \]

By the reflexivity of \( H^1(0, 1) \) and its compact embedding in \( L^\infty(0, 1) \) we may assume, by passing to a subsequence, that

\[ \|u_{\varepsilon_n} - u_*\|_{L^\infty(0, 1)} \to 0 \quad \text{as } n \to \infty, \]

for some \( u_* \in H^1(0, 1) \) satisfying

\[ 0 < \|u_* - u_0\|_{L^\infty(0, 1)} = \delta \leq \delta_0. \]

By letting \( n \to \infty \) in (3) we infer that

\[ \int_0^1 F(x, u_*(x)) \, dx \leq 0. \]

The above relation combined with (II), (III), (4) implies that

\[ u_* \equiv u_0 \]

which contradicts (4). The proof is complete. \( \square \)

Remark 2. The variational approach gives a general existence result but does not provide estimates for \( u_\varepsilon \) or for the spectrum of the linearized operator on \( u_\varepsilon \) (this is important in the case of systems). Also it does not work in the case

\[ f(x, u) = u(u^2 - p(x)) \quad \text{with } p \geq 0, \quad p(x_0) = 0, \quad p'(x_0) > 0, \]

for some \( x_0 \in (0, 1) \); this is because

\[ u_0 = \begin{cases} \sqrt{p(x)}, & x \geq x_0, \\ 0, & x < x_0. \end{cases} \]
is not in $H^1$ (cf. [2] for a shooting argument and [6] for a proof via the contraction mapping principle). (It works if $p(x) \sim c(x-x_0)^{\alpha+1}$ as $x \to x_0^+$ for some $\alpha > 0$).

**Remark 3.** Solutions with transition layers for (1), (2) have been studied in [4], [7] and in the recent work [5] (unstable solutions).

**References**


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