ON CORNER LAYERS IN BALANCED EQUATIONS

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Abstract. We study equations of the form

$$\varepsilon^2 u_{xx} = (u - a(x))(u - b(x))(u - c(x)), \quad x \in (0,1)$$

with Neumann boundary conditions, where $b \equiv \frac{a+c}{2}$ and $0 < \varepsilon \ll 1$. The functions a, b, c are allowed to intersect and, thus, the degenerate relation obtained by setting $\varepsilon = 0$ can have a continuous but non-smooth solution u_0. We apply a variational argument to show that, for small ε, there exists a solution $u_\varepsilon \in C^2$ such that $u_\varepsilon \to u_0$ as $\varepsilon \to 0$.

1. Introduction and main result

Consider the singularly perturbed boundary value problem

$$\begin{cases} \varepsilon^2 u_{xx} = f(x,u), & x \in (0,1), \\ u_x(0) = 0 = u_x(1), \end{cases} \tag{1}$$

under the following hypothesis on $f \in C([0,1] \times \mathbb{R})$ (f is independent of ε):

(I) There exists $u_0 \in H^1(0,1)$ such that

$$f(x, u_0(x)) = 0, \quad x \in [0,1].$$

(II) There exists a $\delta_0 > 0$ such that

$$F(x,u) := \int_{u_0(x)}^{u} f(x,t)dt \geq 0 \quad \text{if} \quad |u - u_0(x)| \leq \delta_0, \quad x \in [0,1].$$

(III) If $v \in H^1(0,1)$ satisfies

$$\|v - u_0\|_{L^\infty(0,1)} \leq \delta_0 \ \text{ and } \ F(x, v(x)) = 0, \quad x \in [0,1],$$

then $u_0 \equiv v$.

Example. A typical example is

$$f(x,u) = (u - a(x))(u - b(x))(u - c(x)) \quad \text{with} \ b \equiv \frac{a+c}{2}, \tag{2}$$

and a, b, c intersecting at $x = x_0$. In the figures we sketch such a situation and illustrate the graph of $F(x, \cdot)$ with x considered as a parameter. We can easily see that

$$u_0 := \begin{cases} c(x), & x \leq x_0, \\ a(x), & x > x_0, \end{cases}$$

satisfies (I)-(III) (other choices of u_0 are also possible).

1991 *Mathematics Subject Classification.* Primary: 34D15 ; Secondary: 35A15.
Key words and phrases. Singular perturbation, Corner layer, Variational methods.
Partly supported by grant FONDECYT 3085026.

Remark 1. If $a, b, c \in C^2$ and $c_x(x_0) < b_x(x_0) < a_x(x_0)$ one can prove the existence of a solution u_ε such that

$$\|u_\varepsilon - u_0\|_{L^\infty(0,1)} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and

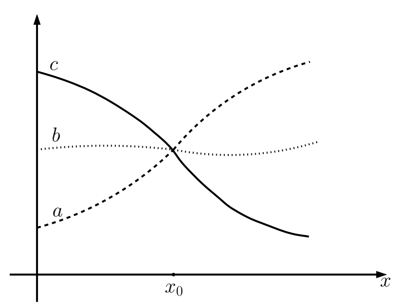
$$u_\varepsilon = \varepsilon^{\frac{2}{5}} U\left(\frac{x - x_0}{\varepsilon^{\frac{2}{5}}}\right) + a(x_0) + \text{H.O.T.}, \quad |x - x_0| \le \varepsilon^{\frac{2}{5}}$$

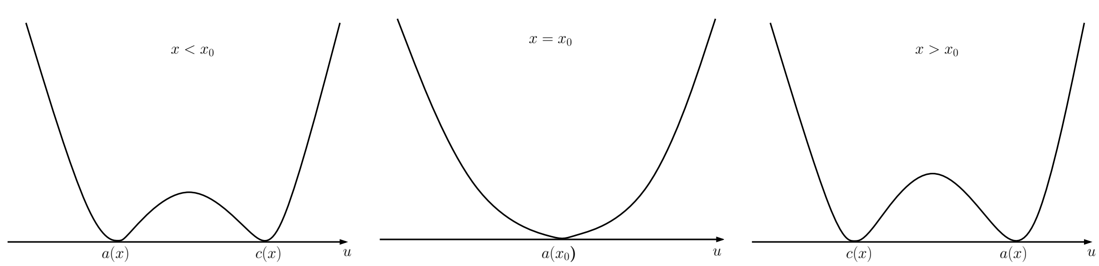
with U solving

$$U_{xx} = (U - a_x(x_0)x)(U - b_x(x_0)x)(U - c_x(x_0)x), \quad x \in \mathbb{R},$$

$$U - c_x(x_0)x \to 0, \quad x \to -\infty, \quad U - a_x(x_0)x \to 0, \quad x \to \infty.$$

This can be done without the assumption $b \equiv \frac{a+c}{2}$ by borrowing techniques from [6] and proving the existence of U via upper and lower solutions (employing the eigenfunction method). Details will appear elsewhere. However even in this case our proof of Theorem 1 is still useful since, besides its simplicity, it can be applied (with only minor modifications) to the partial differential analog of (1) (it gives convergence only in L^2 though). We should mention that the PDE problem of exactly two functions intersecting has been treated (in great generality) by constructing suitable upper and lower solutions in [1]. Note also that our proof below works if $a = b = c$ over an interval.





Let us now return to our original problem. Our existence result is

Main Theorem. *Let f, u_0 satisfy assumptions (I)-(III). Then there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ problem (1) has a solution $u_\varepsilon \in C^2[0,1]$ such that*

$$\|u_\varepsilon - u_0\|_{L^\infty(0,1)} = o_\varepsilon(1),$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$.

Proof. Let

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2} u_x^2 + F(x,u)dx,$$

for

$$u \in U_\delta = \{u \in H^1(0,1) : \ \|u - u_0\|_{L^\infty(0,1)} \le \delta\}, \quad 0 < \delta \le \delta_0$$

(our motivation for this comes from [3]). Obviously J_ε is bounded from below (it is nonnegative) and coercive in U_δ (with respect to the H^1 norm), and U_δ is weakly closed in $H^1(0,1)$. Hence, by standard arguments, there exists $u_{\varepsilon,\delta} \in U_\delta$ such that

$$J_\varepsilon(u_{\varepsilon,\delta}) = \inf_{u \in U_\delta} J_\varepsilon(u).$$

The assertion of the theorem will follow immediately if we show that for any $0 < \delta \le \delta_0$ there exists $\varepsilon(\delta) > 0$ such that

$$\|u_{\varepsilon,\delta} - u_0\|_{L^\infty(0,1)} < \delta \quad \text{if} \ \ 0 < \varepsilon < \varepsilon(\delta).$$

We argue by contradiction. Suppose that for some $\delta \in (0, \delta_0]$ there exist $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\|u_{\varepsilon_n,\delta} - u_0\|_{L^\infty(0,1)} = \delta, \quad n \ge 1.$$

Since $u_0 \in U_\delta$, we have (dropping the subscript δ)

$$J_{\varepsilon_n}(u_{\varepsilon_n}) \le J_{\varepsilon_n}(u_0), \quad n \ge 1,$$

i.e.,

$$\frac{\varepsilon_n^2}{2} \int_0^1 \left(\frac{du_{\varepsilon_n}}{dx}\right)^2 dx + \int_0^1 F(x, u_{\varepsilon_n}) dx \le \frac{\varepsilon_n^2}{2} \int_0^1 \left(\frac{du_0}{dx}\right)^2 dx, \quad n \ge 1, \tag{3}$$

(recall that $F(x,u_0) \equiv 0$) and by using (II) we obtain that

$$\|u_{\varepsilon_n}\|_{H^1(0,1)} \le C, \quad n \ge 1 \ \ (C \text{ indep. of } n).$$

By the reflexivity of $H^1(0,1)$ and its compact embedding in $L^\infty(0,1)$ we may assume, by passing to a subsequence, that

$$\|u_{\varepsilon_n} - u_*\|_{L^\infty(0,1)} \to 0 \quad \text{as} \ \ n \to \infty,$$

for some $u_* \in H^1(0,1)$ satisfying

$$0 < \|u_* - u_0\|_{L^\infty(0,1)} = \delta \le \delta_0. \tag{4}$$

By letting $n \to \infty$ in (3) we infer that

$$\int_0^1 F(x, u_*(x)) dx \le 0.$$

The above relation combined with (II), (III), (4) implies that

$$u_* \equiv u_0$$

which contradicts (4). The proof is complete. □

Remark 2. The variational approach gives a general existence result but does not provide estimates for $u_\varepsilon - u_0$ or for the spectrum of the linearized operator on u_ε (this is important in the case of systems). Also it does not work in the case

$$f(x,u) = u(u^2 - p(x)) \ \text{ with } p \ge 0, \ p(x_0) = 0, \ p'(x_0) > 0,$$

for some $x_0 \in (0,1)$; this is because

$$u_0 = \begin{cases} \sqrt{p(x)}, & x \ge x_0, \\ 0, & x \le x_0, \end{cases}$$

is not in H^1 (cf. [2] for a shooting argument and [6] for a proof via the contraction mapping principle). (It works if $p(x) \sim c(x-x_0)^{\alpha+1}$ as $x \to x_0^+$ for some $\alpha > 0$).

Remark 3. Solutions with transition layers for (1), (2) have been studied in [4], [7] and in the recent work [5] (unstable solutions).

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