INTERFACE LAYER OF A TWO-COMPONENT BOSE-EINSTEIN CONDENSATE

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ABSTRACT. This paper deals with the study of the behaviour of the wave functions of a two-component Bose-Einstein condensate near the interface, in the case of strong segregation. This yields a system of two coupled ODE's for which we want to have estimates on the asymptotic behaviour, as the strength of the coupling tends to infinity. As in phase separation models, the leading order profile is a hyperbolic tangent. We construct an approximate solution and use the properties of the associated linearized operator to perturb it into a genuine solution for which we have an asymptotic expansion. We prove that the constructed heteroclinic solutions are linearly nondegenerate, in the natural sense, and that there is a spectral gap, independent of the large interaction parameter, between the zero eigenvalue (due to translations) at the bottom of the spectrum and the rest of the spectrum. Moreover, we prove a uniqueness result which implies that, in fact, the constructed heteroclinic is the unique minimizer (modulo translations) of the associated energy, for which we provide an expansion.

1. INTRODUCTION

1.1. **The problem.** A two-component condensate is described by two complex valued wave functions minimizing a Gross-Pitaevskii energy with a coupling term. According to the magnitude of the coupling parameter, the components can either coexist or segregate.

The segregation behaviour in two-component condensates has been widely studied in the mathematics literature: regularity of the wave function [13, 14, 30, 34, 35, 39, 40], regularity of the limiting interface [11, 37, 41], asymptotic behaviour near the interface [8, 9], Γ-convergence in the case of a trapped problem [2, 19, 20].

This paper deals with the case of segregation, and more precisely, the study of the behaviour of the wave functions near the interface. In the physics literature, there is a formal analysis of this small coexistence region which, at leading order, is given by a hyperbolic tangent [6, 7, 38]. Here, we want to derive a rigorous asymptotic expansion of this transition layer, which will be useful in the analysis of more complex patterns.

The aim of this paper is therefore to study the positive solutions of the system

$$\begin{cases} -v_1'' + v_1^3 - v_1 + \Lambda v_2^2 v_1 = 0, \\ \\ -v_2'' + v_2^3 - v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{1.1}$$

$$(v_1, v_2) \to (0, 1) \ \text{ as } \ z \to -\infty, \quad (v_1, v_2) \to (1, 0) \ \text{ as } \ z \to +\infty. \tag{1.2}$$

The segregation case corresponds to

$$\Lambda > 1 \tag{1.3}$$

and the limit $\Lambda \to \infty$.

The Hamiltonian

$$H = \sum_{i=1}^{2} \left[\frac{1}{2}(v_i')^2 - \frac{(1-v_i^2)^2}{4} \right] - \frac{\Lambda}{2} v_1^2 v_2^2, \tag{1.4}$$

is constant along solutions of (1.1)-(1.2) and is equal to

$$H = -\frac{1}{4}. \tag{1.5}$$

It is known that solutions of (1.1)-(1.2) are uniformly bounded independently of $\Lambda > 1$ (see (7.3) below). Hence, by the general theory developed in [35] and the references therein, they are uniformly Lipschitz continuous and converge, uniformly as $\Lambda \to \infty$, to the merely Lipschitz continuous pair $(\chi_{(0,\infty)} U_1, \chi_{(-\infty,0)} U_2)$, where U_1 and U_2 denote the unique solutions respectively of the following problems:

$$u'' + u - u^3 = 0, \ z > 0; \ u(0) = 0, \ u(z) \to 1 \text{ as } z \to +\infty, \tag{1.6}$$

$$u'' + u - u^3 = 0, \ z < 0; \ u(z) \to 1 \text{ as } z \to -\infty, \ u(0) = 0, \tag{1.7}$$

(χ_I stands for the characteristic function of a set I). In fact, we have the explicit formulas:

$$U_i(z) = \tanh\left(\frac{|z|}{\sqrt{2}} \right), \quad (-1)^i z \leq 0, \quad i = 1, 2.$$

A crucial observation is that the Hamiltonian structure of (1.6), (1.7) implies the reflection property

$$U_1'(0) + U_2'(0) = 0, \quad \text{with} \ \psi_0 = U_1'(0) = \frac{1}{\sqrt{2}}. \tag{1.8}$$

Of course, this follows at once from the explicit representations of U_1, U_2 but we would like to start convincing the reader that the specific form of the nonlinearity in (1.6), (1.7) is not of essential importance in the proofs. On the one hand, the functions $\chi_{(0,\infty)} U_1, \chi_{(-\infty,0)} U_2$ do satisfy (1.1)-(1.2) for $z \neq 0$. On the other hand, their second derivatives blow-up at the origin as delta masses. To remedy this, guided by formal matched asymptotics and (1.8) (see also [38]), we will instead use near the origin an approximate solution with leading term

$$\left(\Lambda^{-\frac{1}{4}} V_1 \left(\Lambda^{\frac{1}{4}} z \right), \ \Lambda^{-\frac{1}{4}} V_2 \left(\Lambda^{\frac{1}{4}} z \right) \right), \tag{1.9}$$

where the pair (V_1, V_2) is provided by the following proposition.

Proposition 1.1. [8, 9] *There exists a unique solution (V_1, V_2) with positive components to the system*

$$\begin{cases} V_1'' = V_2^2 V_1, \\ V_2'' = V_1^2 V_2, \end{cases} \tag{1.10}$$

such that

$$\frac{V_1}{x} \to \psi_0 \quad \text{and} \quad V_2 \to 0 \text{ as } x \to +\infty, \tag{1.11}$$

where $\psi_0 > 0$ is as in (1.8), and

$$V_1(-x) = V_2(x), \ x \in \mathbb{R}. \tag{1.12}$$

Moreover,

$$V_1(x) = \psi_0 x + \kappa + \mathcal{O}\left(e^{-cx^2} \right) \quad \text{and} \quad V_2(x) = \mathcal{O}\left(e^{-cx^2} \right) \quad \text{as } x \to +\infty, \tag{1.13}$$

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for some $\kappa \geq 0$, and these relations can be differentiated arbitrarily many times. Every other entire solution of (1.10) with positive components is given by

$$(\mu V_1 (\mu(x-h)), \mu V_2 (\mu(x-h))) \tag{1.14}$$

for some $\mu > 0$ and $h \in \mathbb{R}$.

We emphasize that we will not need the above symmetry and uniqueness properties of the *blow-up* profiles, which were shown in [8] and [9] respectively by a nontrivial sliding method.

Our goal is to refine the outer and inner approximate solutions in (1.6)-(1.7) and (1.9) respectively, carefully glue them together and show that the resulting global approximate solution can be perturbed to a genuine one.

1.2. **Main results.** The main result of the paper is the following:

Theorem 1.1. *If $\Lambda > 0$ is sufficiently large, problem (1.1)-(1.2) has a solution $(v_{1,\Lambda}, v_{2,\Lambda})$ such that*

$$v'_{1,\Lambda}(z) > 0, \quad v'_{2,\Lambda}(z) < 0 \quad \text{for} \ \ z \in \mathbb{R}, \tag{1.15}$$

$$v_{i,\Lambda}(z) = U_i \left(z - (-1)^i \psi_0^{-1} \kappa \Lambda^{-\frac{1}{4}} \right) + \mathcal{O} \left((\ln \Lambda) \Lambda^{-\frac{3}{4}} \right) e^{-c|z|}, \tag{1.16}$$

$$\text{uniformly for} \ \ (-1)^{i+1} z \geq (\ln \Lambda) \Lambda^{-\frac{1}{4}}, \quad \text{as} \ \ \Lambda \to \infty,$$

$$v_{i,\Lambda}(z) = \Lambda^{-\frac{1}{4}} V_i \left(\Lambda^{\frac{1}{4}} z \right) + \mathcal{O} \left(\Lambda^{-\frac{3}{4}} + |z|^3 \right), \tag{1.17}$$

uniformly on $\left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}} \right]$, as $\Lambda \to \infty$, $i = 1, 2$, where U_1, U_2 are the unique solutions of (1.6), (1.7) respectively, (V_1, V_2) is the solution of (1.10)-(1.11)-(1.12), and $\kappa > 0$ is as in (1.13). Furthermore, for any $m > 0$, we have

$$v_{i,\Lambda}(z) \leq C\Lambda^{-\frac{1}{4}} e^{-c\Lambda^{\frac{1}{2}} z^2} + \mathcal{O}(\Lambda^{-m}), \quad (-1)^i z \in \left[\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}} \right], \tag{1.18}$$

and

$$v_{i,\Lambda}(z) \leq \Lambda^{-m} e^{-c(\ln \Lambda)^{\frac{1}{2}} \Lambda^{\frac{1}{4}} |z|}, \quad (-1)^i z \geq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \quad i = 1, 2, \tag{1.19}$$

as $\Lambda \to \infty$. Moreover,

$$v'_{i,\Lambda}(z) = U'_i \left(z - (-1)^i \psi_0^{-1} \kappa \Lambda^{-\frac{1}{4}} \right) + \mathcal{O} \left((\ln \Lambda) \Lambda^{-\frac{3}{4}} \right) \left(|z| + \Lambda^{-\frac{1}{4}} \right) e^{-c|z|}, \tag{1.20}$$

$$\text{uniformly for} \ \ (-1)^{i+1} z \geq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \quad \text{as} \ \ \Lambda \to \infty,$$

$$v'_{i,\Lambda}(z) = V'_i \left(\Lambda^{\frac{1}{4}} z \right) + \mathcal{O} \left(\Lambda^{-\frac{1}{2}} + |z|^2 \right), \tag{1.21}$$

uniformly on $\left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}} \right]$, as $\Lambda \to \infty$, $i = 1, 2$.

We can also show that $(v_{1,\Lambda}, v_{2,\Lambda})$ is nondegenerate, in the natural sense, and that there is a spectral gap.

Theorem 1.2. *Let (v_1, v_2) be the heteroclinic solution to (1.1)-(1.2) which is constructed in Theorem 1.1. Then, if $\Lambda > 0$ is sufficiently large, the spectrum of the linearized operator*

$$\boldsymbol{M} \begin{pmatrix} \varphi_1 \\ \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3v_1^2 - 1)\varphi_1 + \Lambda v_2^2 \varphi_1 + 2\Lambda v_1 v_2 \varphi_2 \\ \\ -\varphi_2'' + (3v_2^2 - 1)\varphi_2 + \Lambda v_1^2 \varphi_2 + 2\Lambda v_1 v_2 \varphi_1 \end{pmatrix}, \tag{1.22}$$

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in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ is structured as follows:

- *0 is the first eigenvalue and has (v_1', v_2') as the associated eigenfunction,*
- *the rest of the spectrum is contained in (c, ∞) for some $c > 0$.*

Furthermore, we have the following uniqueness result.

Theorem 1.3. *If Λ is as in (1.3), there exists at most one solution (modulo translations) to problem (1.1)-(1.2) with positive components such that one of them is strictly monotone.*

An important consequence of the above theorem is that the minimization problem

$$\sigma_\Lambda = \inf_{(v_1, v_2) \in \mathcal{Y}} E_\Lambda(v_1, v_2), \tag{1.23}$$

where

$$E_\Lambda(v_1, v_2) = \int_{\mathbb{R}} \left\{ \sum_{i=1}^{2} \left[\frac{1}{2}(v_i')^2 + \frac{(1 - v_i^2)^2}{4} \right] + \frac{\Lambda}{2} v_1^2 v_2^2 - \frac{1}{4} \right\} dz, \tag{1.24}$$

and

$$\mathcal{Y} = \left\{ (v_1, v_2) \in H^1_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R}) \text{ which sastisfy } (1.2) \right\}. \tag{1.25}$$

has a unique solution (modulo translations), which is that of Theorem 1.1. In fact, this settles a conjecture from [4, Sec. 5], in relation to the stability properties of the family of minimizers of the complex valued version of (1.24) with respect to the associated nonlinear Schrödinger dynamics. Armed with the estimates provided by our main theorem, we can give an asymptotic expression for its minimal energy σ_Λ:

Corollary 1.1. *As $\Lambda \to \infty$,*

$$\sigma_\Lambda = \frac{2\sqrt{2}}{3} + 2\Lambda^{-\frac{1}{4}} \int_{-\infty}^{\infty} V_1' \left(V_1' - \psi_0 \right) dx + \mathcal{O}\left((\ln \Lambda)^3 \Lambda^{-\frac{3}{4}} \right).$$

The minimal energy σ_Λ in (1.23) represents the interface tension of the condensate. In [38, 29], a formal series expansion of σ_Λ in powers of $\Lambda^{-\frac{1}{4}}$ is given by using matched asymptotic analysis. The first term of this series was recovered rigorously in [19] via variational arguments. In comparison, our Corollary 1.1 recovers rigorously the first three terms of that prediction together with the correct order of the fourth term, modulo the 'artificial' logarithmic factor.

1.3. **Main steps of the proofs.** The idea of the proof of Theorem 1.1 is to construct an approximate solution and then to perturb it into a genuine solution, using the linearized operator. This approach has been extensively pursued and thoroughly developed in the past years for constructing localized solutions to elliptic problems, mainly involving spike-transition layer or bubbling phenomena. However, as will be apparent, some important differences occur with respect to the standard technique. This should already be expected from the irregular form of the singular limit solution of the problem in hand. In particular, its corner layered structure at the origin forces the corresponding blow-up profiles to be unbounded, in sharp contrast to the situation in the aforementioned widely studied concentration problems.

Firstly, it is not difficult to construct an outer (exact) solution of (1.1)-(1.2) for $|z| \geq (\ln \Lambda) \Lambda^{-1/4}$, which satisfies the expected asymptotic behaviour (1.16). We do not prescribe conditions at the end points, as these will be controlled by the inner solution that we construct

next. We point out that this construction is possible by the nondegeneracy of the solutions U_1, U_2 of (1.6), (1.7) respectively (see Lemma 2.1 below). Then, we construct an inner solution for $|z| \leq (\ln \Lambda) \Lambda^{-\frac{1}{4}}$ based on the blow-up profile (V_1, V_2) of Proposition 1.1 and on its nondegeneracy (in the natural sense, see Proposition 2.1 below). Actually, by exploiting the scaling invariance of (1.10), illustrated by the parameter μ in (1.14), we can define a one-parameter family of such inner solutions. Let us note that there is no gain in exploiting the translation invariance of (1.10) for this purpose, as the whole problem (1.1)-(1.2) is itself translation invariant. We emphasize that the construction of the inner solution relies heavily on the study of the linearization of the blow-up problem (1.10) (see Subsection 2.2), which could also prove useful in other settings. More precisely, the linearized operator of the blow-up system (1.10) at (V_1, V_2) contains an element (E_1, E_2) with linear growth in its (formal) kernel, due to the aforementioned invariance of (1.10) under scaling. We can use a constant multiple of this element as the parameter in the inner solutions. In order to efficiently glue together the inner and outer approximations, we need to adjust the constant parameters involved in their separate constructions. For technical reasons, which will be clear from the proofs, instead of matching these approximations in the C^1-sense over an intermediate zone, we match them continuously only at the points $\pm (\ln \Lambda) \Lambda^{-1/4}$. At first sight, this unconventional argument might look rather counterintuitive, as it would create jumps on the gradients at the gluing points. But, by choosing the free parameter in the inner solution so that the Hamiltonian has the same value on each side, it turns out that these jumps on the gradient are actually transcendentally small. The resulting global approximation fails to be an exact solution to the problem by essentially just a transcendentally small factor. Naturally, the first thing that comes to mind is to try to perturb it to a genuine solution by some type of local inversion argument, through the study of the associated linearized operator about it. However, the translation invariance of (1.1)-(1.2) implies that the latter operator is nearly non-invertible, as the derivative of the approximate solution fails to be in its kernel by at most a transcendentally small factor. Nevertheless, by combining the linear analysis that we developed separately for the inner and outer problems, we can show that the global linearized operator does not have other elements in its spectrum tending to zero, as $\Lambda \to \infty$, besides the transcendentally small eigenvalue at the bottom. In fact, the latter eigenvalue turns out to be simple (see Theorem 1.2 and Proposition 6.1). Consequently, we are led to use a Lyapunov-Schmidt variational reduction method for the perturbation argument.

The proof of our uniqueness result rests upon a homotopy argument, taking advantage of the nondegeneracy property of this type of monotone solutions to (1.1)-(1.2), which holds for Λ in the range (1.3).

1.4. Physical motivation and known results.

A rotating two-component Bose-Einstein condensate is described by the ground state of the following energy

$$
\mathcal{E}(u_1, u_2) = \sum_{j=1}^{2} \int_{\mathbb{R}^2} \left\{ \frac{|\nabla u_j|^2}{2} + \frac{V(|x|)}{2\varepsilon^2} |u_j|^2 + \frac{g_j}{4\varepsilon^2} |u_j|^4 - \Omega x^\perp \cdot (iu_j, \nabla u_j) \right\} dx
$$
$$
+ \frac{g}{2\varepsilon^2} \int_{\mathbb{R}^2} |u_1|^2 |u_2|^2 \, dx
$$

(1.26)

in the set

$$\mathcal{H} = \left\{ (u_1, u_2) : \ u_j \in H^1(\mathbb{R}^2; \mathbb{C}), \ \int_{\mathbb{R}^2} V(|x|)|u_j|^2 \, dx < \infty, \ \|u_j\|_{L^2(\mathbb{R}^2)} = 1, \ j = 1, 2 \right\}. \tag{1.27}$$

The trapping potential $V(|x|)$ is usually taken to be $|x|^2$, corresponding to the experiments. The parameters g_1, g_2, g, ε and Ω are positive: g_j is the self-interaction of each component (intracomponent coupling) while g measures the effect of interaction between the two components (intercomponent coupling); Ω is the angular velocity corresponding to the rotation of the condensate, $x^\perp = (-x_2, x_1)$ and $\cdot$ is the scalar product for vectors, whereas $(\, , \,)$ is the complex scalar product, so that we have

$$x^\perp \cdot (iu, \nabla u) = x^\perp \cdot \frac{iu\nabla\bar{u} - i\bar{u}\nabla u}{2} = -x_2 \frac{iu\partial_{x_1}\bar{u} - i\bar{u}\partial_{x_1}u}{2} + x_1 \frac{iu\partial_{x_2}\bar{u} - i\bar{u}\partial_{x_2}u}{2}.$$

The existence and behaviour of the minimizers in the limit when ε is small, describing strong interactions, is also called the Thomas-Fermi limit. Even though the interaction is only through the modulus, it can produce effects on the phases of each component, and in particular on the singularities or vortices. A full phase diagram has been computed in [28].

If the condition $g^2 < g_1 g_2$ is satisfied, it means that the two components u_1 and u_2 of the minimizers can coexist, as opposed to the segregation case $g^2 > g_1 g_2$. This is discussed and explained in [1, 28]. The ground state at $\Omega = 0$ in the coexistence case has been studied in the small ε limit in [1, 18].

In the present paper, we are interested in the segregation case. The Γ limit of (1.26)-(1.27) with $\Omega = 0$ in the case where $g^2 > g_1 g_2$, $g_1 = g_2$ and $g = g_\varepsilon \to \infty$ has been studied in [2]. A change of functions is used, namely (v, φ), where

$$v^2 = u_1^2 + u_2^2 \text{ and } \cos\varphi = \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2}. \tag{1.28}$$

A Γ limit is obtained on the functional for (v, φ). The limiting problem is given by two domains D_1 (for component 1) and D_2 (for component 2) for which the interface minimizes a perimeter type problem weighted by the trapping potential.

As conjectured in [8], the minimum of v satisfies

$$c\Lambda^{-\frac{1}{4}} \le \inf_{\mathbb{R}} v \le C\Lambda^{-\frac{1}{4}},$$

for some constants $c, C > 0$ independent of large Λ. Actually, an analogous estimate was established very recently in [34, 39] for uniformly bounded solutions to a broad class of elliptic systems. Clearly, the above estimate is a particular consequence of our Theorem 1.1 and the comments preceding Corollary 1.1.

Further Γ convergence results for $g = g_\varepsilon$ fixed have been proved by [20], and extended for $g_1 \ne g_2$ by [19]. In these two papers, near the interface, the interaction between the two components of the condensate is governed by the system

$$-u_1'' + g_1 u_1^2 u_1 + g u_2^2 u_1 = \lambda_1 u_1,$$
$$-u_2'' + g_2 u_2^2 u_2 + g u_1^2 u_2 = \lambda_2 u_2,$$

for some constants $\lambda_1, \lambda_2 > 0$ corresponding to the Lagrange multipliers of (1.26)-(1.27). In the case where $g = \Lambda$ and g_i, λ_i are equal to 1, this gives rise to our one dimensional problem

and they prove that the Γ limit of the full 2D problem in a bounded domain is given by the minimization of σ_Λ.

Remark 1.1. *In (1.1), we have taken all the constants in front of the non-coupled terms to be equal to one. However, all of our arguments carry over easily to the positive solutions of*

$$\begin{cases} -v_1'' + g_1 v_1^3 - \lambda_1 v_1 + \Lambda v_2^2 v_1 = 0, \\ \\ -\nu v_2'' + g_2 v_2^3 - \lambda_2 v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{1.29}$$

$$(v_1, v_2) \to \left(0, \sqrt{\frac{\lambda_2}{g_2}}\right) \; as \; z \to -\infty, \quad (v_1, v_2) \to \left(\sqrt{\frac{\lambda_1}{g_1}}, 0\right) \; as \; z \to +\infty, \tag{1.30}$$

for values of the parameter

$$\Lambda > \sqrt{g_1 g_2}, \tag{1.31}$$

assuming that $\nu > 0$, and the positive constants $g_1, g_2, \lambda_1, \lambda_2$ satisfy

$$\frac{\lambda_1^2}{g_1} = \frac{\lambda_2^2}{g_2}. \tag{1.32}$$

Thanks to (1.32), all the constants g_i, λ_i, $i = 1, 2$, and ν can be scaled out since a solution is given by

$$\sqrt{\frac{g_1}{\lambda_1}} v_1 \left(\sqrt{\frac{1}{\lambda_1}} z\right), \quad \sqrt{\frac{g_2}{\lambda_2}} v_2 \left(\sqrt{\frac{\nu}{\lambda_2}} z\right),$$

where v_1, v_2 satisfy (1.1)-(1.2), while the new coupling constant is $\tilde{\Lambda} = \frac{\lambda_2 \Lambda}{\lambda_1 g_2}$. We point out that (1.32) is a necessary condition for the existence of solutions to (1.29)-(1.30) since the corresponding Hamiltonian is conserved and therefore the limit at $+/-\infty$ has to be the same.

Conversely, by means of variational arguments, it was shown recently in [4] and [19] that condition (1.32) is also sufficient (see also [5, 42]) for the existence of solutions.

The basic interaction of a two component condensate is through a modulus term, but other interactions include a Rabi coupling or a spin orbit coupling. The precise knowledge of the interface behaviour will prove useful to analyze more complicated patterns:

- the segregation case in the spin orbit coupling where there are vortex sheets [24],
- in the case of Rabi coupling, a vortex and anti-vortex pair create a vortex molecule, where two vortices are eventually connected by a domain wall of relative phase [25, 36],
- half vortices, where a vortex in one component corresponds to a peak in the other component [12].

In order to analyze the singularity patterns in all these cases, one needs to make an energy expansion in order to determine the energy of the specific configuration. Because of the transition layer between the two species, one needs to have a precise estimate of the decrease of the modulus of the wave functions, which is precisely given by our main theorem. Therefore, what we prove in Theorems 1.1, 1.2 is expected to be extremely useful for the construction of upper bounds for these patterns.

1.5. **Outline of the paper.** In Section 2, we construct our approximate solution to the problem, in the outer and inner regions separately. Then, in Section 3 we adjust them further so that they match conveniently for our purposes. In Sections 4 and 5, we perturb the resulting inner and outer approximations respectively to genuine ones . In Section 6, we prove Theorems 1.1 and 1.2. In Section 7 we prove Theorem 1.3. Finally, in Section 8 we show Corollary 1.1.

1.6. **Notation.** By c/C we will denote small/large positive generic constants that are independent of large $\Lambda > 0$ and whose value will decrease/increase as the paper moves on. The value of $\Lambda > 0$ will also increase from line to line so that all the previous relations hold. According to the Landau notation, a number ρ will be of order $\mathcal{O}(\Lambda^{-m})$ as $\Lambda \to \infty$, for some $m \in \mathbb{R}$, if $|\rho| \leq C\Lambda^{-m}$ for Λ sufficiently large; a number ρ will be of order $o(\Lambda^{-m})$ as $\Lambda \to \infty$, for some $m \in \mathbb{R}$, if $\Lambda^m \rho \to 0$ as $\Lambda \to \infty$; a number ρ will be of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$ if $\rho = \mathcal{O}(\Lambda^{-m})$, for any $m > 1$, as $\Lambda \to \infty$. We will remove the obvious dependence on Λ of various functions.

2. THE APPROXIMATE SOLUTION

In this section, we will construct a sufficiently good approximate solution to problem (1.1)-(1.2) for large $\Lambda > 0$.

2.1. **The outer solution** $(v_{1,out}, v_{2,out})$. In this subsection, we will construct an approximate solution to problem (1.1)-(1.2) except at the origin, where it looses its smoothness.

2.1.1. *The outer profiles U_1, U_2.* The building blocks of this construction will be the unique solutions U_1, U_2 of problems (1.6), (1.7) respectively. Actually, we will restrict our attention to U_1, as the corresponding analysis for U_2 is completely analogous. For future reference, let us note that

$$U_1'(z) > 0, \ z \geq 0, \tag{2.1}$$

and

$$1 - U_1(z) + U_1'(z) - U_1''(z) \leq Ce^{-cz}, \ z \geq 0. \tag{2.2}$$

We will also need the following properties for the associated linearized operator, which are well known and essentially follow from (2.1)-(2.2).

Lemma 2.1. *Let $\phi \in C^2[0, \infty)$ be bounded and satisfy*

$$-\phi'' + \left(3U_1^2(z) - 1\right)\phi = 0, \quad z > 0. \tag{2.3}$$

Then, we have that

$$\phi \equiv cU_1' \ \text{for some} \ c \in \mathbb{R}.$$

In particular, if $\phi(0) = 0$, then $\phi \equiv 0$.

Proof. The desired assertions of the lemma follow immediately from the observation that, besides of U_1', the differential operator in the lefthand side of (2.3) has also an unbounded function in its two-dimensional kernel. Indeed, otherwise the Wronskian would be zero, by (2.2) and the fact that bounded elements in the kernel also have bounded derivatives (by a standard interpolation argument, see for instance (2.38) below). $\square$

2.1.2. *The construction of the outer approximate solution.* We can now define our outer approximate solution as

$$v_{1,out}(z) = U_1(z + \xi_1) + \tau_1 U_1'(z + \xi_1), \ v_{2,out}(z) = 0 \text{ for } z \geq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \qquad (2.4)$$

with

$$\xi_1 = \mathcal{O}(\Lambda^{-\frac{1}{4}}) \text{ and } \tau_1 = \mathcal{O}\left((\ln \Lambda)\Lambda^{-\frac{3}{4}}\right) \text{ as } \Lambda \to \infty, \qquad (2.5)$$

to be determined. Analogously we define it for $z \leq -(\ln \Lambda)\Lambda^{-\frac{1}{4}}$. We point out that the choice of the power $1/4$ in (2.4) is motivated by a formal blow-up analysis (see the next subsection), whereas the choice of powers in (2.5) is an a-posteriori result of matching considerations (see Subsection 3.1 below).

2.1.3. *The remainder of the outer approximate solution.* We note that $(v_{1,out}, v_{2,out})$ satisfies the desired asymptotic behaviour (1.2) exactly, while it satisfies the system (1.1) approximately as is shown in the next lemma.

Lemma 2.2. *The remainder*

$$R(v_{1,out}, v_{2,out}) = \begin{pmatrix} -v_{1,out}'' + v_{1,out}^3 - v_{1,out} + \Lambda v_{2,out}^2 v_{1,out} \\ \\ -v_{2,out}'' + v_{2,out}^3 - v_{2,out} + \Lambda v_{1,out}^2 v_{2,out} \end{pmatrix}$$

that is left by $(v_{1,out}, v_{2,out})$ in (1.1) satisfies

$$R(v_{1,out}, v_{2,out}) = \begin{pmatrix} \mathcal{O}\left((\ln \Lambda)^2 \Lambda^{-\frac{3}{2}}\right) e^{-cz} \\ \\ 0 \end{pmatrix},$$

uniformly for $z \geq (\ln \Lambda)\Lambda^{-\frac{1}{4}}$, as $\Lambda \to \infty$, and an analogous estimate holds for $z \leq -(\ln \Lambda)\Lambda^{-\frac{1}{4}}$.

Proof. Keeping in mind that $v_{2,out}$ is identically zero, we observe that, by virtue of (1.6), we have

$$-v_{1,out}'' + v_{1,out}^3 - v_{1,out} = \tau_1^3 \left(U_1'(z + \xi_1)\right)^3 + 3\tau_1^2 U_1(z + \xi_1) \left(U_1'(z + \xi_1)\right)^2,$$

and then use (2.2), (2.5). The proof for $z \leq -(\ln \Lambda)\Lambda^{-\frac{1}{4}}$ is analogous. $\qquad \square$

2.2. **The inner approximate solution** $(v_{1,in}, v_{2,in})$. In this subsection, we will construct an approximate solution to the system (1.1) which, however, is effective only in a small neighborhood of the origin. Nevertheless, it will have the appropriate behaviour so as to be easily "continued" away from the origin by the outer solution of the previous subsection.

2.2.1. *The blow-up profile (V_1, V_2).* Based on a formal blow-up analysis and the behaviour of the outer approximate solution near the origin, the building blocks will be special solutions of a limiting problem, described in Proposition 1.1 which is due to [8, 9].

Moreover, by the convexity of V_1, V_2, it follows easily that

$$V_1' > 0, \ V_2' < 0, \ x \in \mathbb{R}. \qquad (2.6)$$

Actually, it is not hard to show that $\kappa > 0$. Indeed, we observe that the auxiliary function

$$R(x) = V_1(x) - \psi_0 x - V_2(x), \quad x \geq 0,$$

satisfies

$$R'' = V_1 V_2 (V_2 - V_1) < 0, \quad x > 0,$$

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while

$$R(0) = 0, \quad \lim_{x \to +\infty} R(x) = \kappa.$$

The invariance of system (1.10) under translation and scaling, described in (1.14), implies that the associated linearized operator

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} -\Phi_1'' + V_2^2 \Phi_1 + 2V_1 V_2 \Phi_2 \\ -\Phi_2'' + V_1^2 \Phi_2 + 2V_1 V_2 \Phi_1 \end{pmatrix} \tag{2.7}$$

has

$$(V_1', V_2') \quad \text{and} \quad (xV_1' + V_1, xV_2' + V_2) \tag{2.8}$$

amongst its four-dimensional kernel. The next proposition, also proven in [8], will play a key role in what will follow.

Proposition 2.1. *If* $\Phi_1, \Phi_2 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfy*

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then

$$(\Phi_1, \Phi_2) \equiv \lambda(V_1', V_2')$$

for some $\lambda \in \mathbb{R}$.

We emphasize that the proof of the above proposition is based on the monotonicity property (2.6). In particular, no use is made of the symmetry (1.14) or the uniqueness property of (V_1, V_2). In fact, the latter properties are considerably harder to establish.

2.2.2. *Construction of the inner approximate solution.* Motivated from the above and [8], we consider the stretched variable

$$x = \mu \Lambda^{\frac{1}{4}} z \tag{2.9}$$

with

$$\mu = 1 + \mathcal{O}(\Lambda^{-\frac{1}{2}}) \text{ as } \Lambda \to \infty \tag{2.10}$$

to be determined (the last relation is an a-posteriori consequence of matching considerations, see Subsection 3.1 below). Then, we seek an inner approximate solution $(v_{1,in}, v_{2,in})$ to (1.1) in the form

$$v_{i,in}(z) = \mu \Lambda^{-\frac{1}{4}} V_i(x) + \Phi_i(x), \quad |z| \le (\ln \Lambda) \Lambda^{-\frac{1}{4}}, \quad i = 1, 2, \tag{2.11}$$

where the functions Φ_1, Φ_2 are also to be determined. Actually, at first, we were tempted to also exploit the translation invariance of (1.10) by introducing a shift parameter in the stretched variable x, similarly to (1.14), but then realized that this is not needed as the problem (1.1)-(1.2) itself is translation invariant. Using (1.10), we find that the remainder which is left in the first equation of (1.1) by this approximation is

$$-\mu^2 \Lambda^{\frac{1}{2}} \Phi_1'' + \mu^3 \Lambda^{-\frac{3}{4}} V_1^3 + \Phi_1^3 + 3\mu^2 \Lambda^{-\frac{1}{2}} V_1^2 \Phi_1 + 3\mu \Lambda^{-\frac{1}{4}} V_1 \Phi_1^2 + \mu^2 \Lambda^{\frac{1}{2}} V_2^2 \Phi_1$$

$$\tag{2.12}$$

$$+\mu \Lambda^{\frac{3}{4}} \Phi_2^2 V_1 + \Lambda \Phi_2^2 \Phi_1 + 2\mu^2 \Lambda^{\frac{1}{2}} V_1 V_2 \Phi_2 + 2\mu \Lambda^{\frac{3}{4}} V_2 \Phi_1 \Phi_2 - \mu \Lambda^{-\frac{1}{4}} V_1 - \Phi_1,$$

and an analogous relation holds for the second equation. Hence, we would like for (Φ_1, Φ_2) to satisfy

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \mu^{-1} \Lambda^{-\frac{3}{4}} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

where the linear operator L is as in (2.7), for $x \in \mathbb{R}$ which is the natural domain of definition for Φ_1 and Φ_2.

In view of the righthand side of the above equation and the asymptotic behaviour (1.13) (keep in mind (1.12)), we are naturally led to seek (Φ_1, Φ_2) as

$$(\Phi_1, \Phi_2) = \mu^{-1} \Lambda^{-\frac{3}{4}} (Z_1, Z_2) + (\tilde{\Phi}_1, \tilde{\Phi}_2),$$

where Z_1, Z_2 are some smooth, fixed functions that satisfy

$$Z_1(x) = -\psi_0 \frac{x^3}{6} - \kappa \frac{x^2}{2}, \ Z_2(x) = 0, \ x \geq 1, \ \text{and (say)} \ Z_1(-x) \equiv Z_2(x). \tag{2.13}$$

Then, the fluctuation $(\tilde{\Phi}_1, \tilde{\Phi}_2)$ should satisfy

$$L \begin{pmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{pmatrix} = \mu^{-1} \Lambda^{-\frac{3}{4}} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{2.14}$$

for some fixed, smooth pair (F_1, F_2) such that

$$|F_1(x)| + |F_2(x)| \leq C e^{-cx^2}, \ x \in \mathbb{R}. \tag{2.15}$$

In the sequel, we will show how the injectivity result in Proposition 2.1 can be used to establish the existence of solutions with *linear growth* to the inhomogeneous linear problem:

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.16}$$

with H_1, H_2 being smooth and decaying exponentially fast. We remark that it is not clear to us how to use tools from functional analysis to achieve this because, as we expect from the general theory in [23], the continuous spectrum of L (when defined in the natural Hilbert space) should be the interval $[0, \infty)$ (it was also shown in [8] that the whole spectrum of L is nonnegative). Another obstruction is that, even though we are aware of two elements in the (formal) kernel of L (recall (2.8)), the remaining two elements or their asymptotic behaviour are not known to us (in fact, we suspect that the latter should involve a super-exponential growth which is not useful for matching purposes). Therefore, in contrast to related scalar second order problems (see for example [21, Lem. 4.1]), it is not clear how to derive conclusions from the corresponding variations of constants formula. Lastly, in relation to the cooperative character of L (see [9]), let us mention that we have not been able to construct appropriate upper and lower solution pairs to (2.16).

Motivated by the existence proof of [8] for the nonlinear problem (1.10), we will first solve (2.16) in large bounded intervals and then obtain the sought solution via a limiting procedure. In this direction, we have the following result. Let us point out that the following estimates for the problems in the bounded intervals do not only serve as stepping stones to reach this goal but will also play a crucial role in the upcoming singular perturbation analysis of (1.1)-(1.2).

Proposition 2.2. *Given $\alpha > 0$, there exist $M_0, C > 0$ such that the boundary value problem*

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad |x| < M; \quad \Phi_i(\pm M) = 0, \ i = 1, 2, \tag{2.17}$$

where L is as in (2.7) and $H_1, H_2 \in C[-M, M]$, has a unique solution such that

$$\sum_{i=1}^{2} \left(\|\Phi_i'\|_{L^\infty(-M,M)} + \|\Phi_i\|_{L^\infty(-M,M)} \right) \leq CM \sum_{i=1}^{2} \|e^{\alpha|x|} H_i\|_{L^\infty(-M,M)}, \qquad (2.18)$$

provided that $M \geq M_0$.

If we further assume one of the following orthogonality conditions:

$$\int_{-M}^{M} (V_1' H_1 + V_2' H_2) \, dx = 0 \quad or \quad \int_{-M}^{M} ((xV_1' + V_1)H_1 + (xV_2' + V_2)H_2) \, dx = 0, \qquad (2.19)$$

we get that

$$\sum_{i=1}^{2} \left(\|\Phi_i'\|_{L^\infty(-M,M)} + \|\Phi_i\|_{L^\infty(-M,M)} \right) \leq C \sum_{i=1}^{2} \|e^{\alpha|x|} H_i\|_{L^\infty(-M,M)}. \qquad (2.20)$$

Proof. Since the linear operator L is self-adjoint (in the natural Sobolev spaces associated to the boundary value problem), we only need to verify the validity of the asserted a-priori estimates. An important role will be played by the 'blow-down' problem:

$$\begin{cases} -\frac{d^2\varphi_1}{dy^2} + M^2 V_2^2 (My) \varphi_1 + 2M^2 V_1 V_2 (My) \varphi_2 = M^2 h_1, \\[2mm] -\frac{d^2\varphi_2}{dy^2} + M^2 V_1^2 (My) \varphi_2 + 2M^2 V_1 V_2 (My) \varphi_1 = M^2 h_2, \\[2mm] \text{for } |y| < 1; \quad \varphi_i(\pm 1) = 0, \quad i = 1, 2, \end{cases} \qquad (2.21)$$

where

$$\varphi_i(y) = \Phi_i (My) \quad \text{and} \quad h_i(y) = H_i (My), \quad i = 1, 2.$$

Let us start by establishing the a-priori estimate

$$\sum_{i=1}^{2} \|\Phi_i\|_{L^\infty(-M,M)} \leq CM \sum_{i=1}^{2} \|e^{\alpha|x|} H_i\|_{L^\infty(-M,M)}. \qquad (2.22)$$

Suppose, to the contrary, that the above a-priori estimate were false. Then, there would exist $M_n \to \infty$ and pairs $(\varphi_{1,n}, \varphi_{2,n}) \in C^2[-1, 1] \times C^2[-1, 1]$, $(h_{1,n}, h_{2,n}) \in C[-1, 1] \times C[-1, 1]$, satisfying (2.21) with $M = M_n$, which violate it. In fact, there is no loss of generality in assuming that $\|\varphi_{1,n}\|_{L^\infty(-1,1)} \geq \|\varphi_{2,n}\|_{L^\infty(-1,1)}$. Dividing both equations by $\|\varphi_{1,n}\|_{L^\infty(-1,1)}$, we may further assume that

$$\|\varphi_{1,n}\|_{L^\infty(-1,1)} = 1, \quad \|\varphi_{2,n}\|_{L^\infty(-1,1)} \leq 1$$

$$\text{and} \quad M_n \sum_{i=1}^{2} \|e^{\alpha|M_n y|} h_{i,n}\|_{L^\infty(-1,1)} \to 0. \qquad (2.23)$$

Throughout the rest of the proof, $c \setminus C$ will stand for small\large positive generic constants that are independent of n. A standard barrier argument yields that

$$|\varphi_{1,n}(y)| \leq e^{cM_n y}, \quad -1 \leq y \leq 0; \quad |\varphi_{2,n}(y)| \leq e^{-cM_n y}, \quad 0 \leq y \leq 1. \qquad (2.24)$$

In view of (2.21), (2.23) and the above relation, by a standard diagonal-compactness argument, passing to a subsequence if necessary, we find that

$$\varphi_{i,n} \to \varphi_{i,\infty} \quad \text{in } C_{loc}^2 ([-1, 1] \setminus \{0\}), \quad i = 1, 2, \qquad (2.25)$$

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where the limiting functions satisfy

$$\varphi_{1,\infty}(y) = 0, \ y \in [-1,0); \quad \frac{d^2\varphi_{1,\infty}}{dy^2} = 0, \ y \in (0,1],$$

and

$$\frac{d^2\varphi_{2,\infty}}{dy^2} = 0, \ y \in [-1,0); \quad \varphi_{2,\infty}(y) = 0, \ y \in (0,1].$$

Hence, we get that

$$\varphi_{1,\infty}(y) = a_1(y-1), \quad y \in (0,1], \quad \text{and} \quad \varphi_{2,\infty}(y) = a_2(y+1), \quad y \in [-1,0). \tag{2.26}$$

We will next show that the convergence in (2.25) can be strengthened to

$$|\varphi_{i,n}(y) - \varphi_{i,\infty}(y)| \le Ce^{-cM_n|y|} + o(1), \tag{2.27}$$

uniformly for $(-1)^{i+1}y \in (0,1]$, as $n \to \infty$, $i = 1,2$ (keep in mind (2.24) for the remaining intervals). To this end, let us consider the difference

$$\psi_{i,n} = \varphi_{i,n} - \varphi_{i,\infty} \quad \text{for} \ \ y \in (0,1].$$

Then, in view of (1.13), (2.21) and (2.23), we find that

$$\left|\frac{d^2\psi_{i,n}}{dy^2}\right| = \left|\frac{d^2\varphi_{i,n}}{dy^2}\right| \le CM_n^2 e^{-cM_n y}, \quad y \in (0,1]. \tag{2.28}$$

Note that we only made mild use of the last assumption in (2.23) at this point (in particular, there was no need here for the term M_n in front of the sum). In turn, integrating twice the above relation, and making use of (2.25) only at $y = 1$, yields estimate (2.27) for $i = 1$. The case $i = 2$ is completely analogous. Observe that (2.27) provides useful information only for $|y| \gg M_n^{-1}$.

Actually, there are certain 'reflection laws' that have to be satisfied by $\varphi_{1,\infty}$ and $\varphi_{2,\infty}$ at $y = 0$. Indeed, by testing (2.21) with $(V_1'(M_n y), V_2'(M_n y))$ and integrating by parts, we arrive at

$$\sum_{i=1}^{2}\left[\frac{d\varphi_{i,n}(1)}{dy}V_i'(M_n) - \frac{d\varphi_{i,n}(-1)}{dy}V_i'(-M_n)\right] = \\ -M_n^2 \sum_{i=1}^{2}\int_{-1}^{1} V_i'(M_n y)\, h_{i,n}(y)dy. \tag{2.29}$$

Letting $n \to \infty$ in the above relation, and using (2.23), (2.25), (2.26), we deduce that

$$a_1 + a_2 = 0. \tag{2.30}$$

Likewise, testing by $(M_n y V_1'(M_n y) + V_1(M_n y), M_n y V_2'(M_n y) + V_2(M_n y))$ yields

$$a_1 = a_2. \tag{2.31}$$

Hence, we get that

$$a_1 = a_2 = 0. \tag{2.32}$$

A-posteriori, it turns out that only one of the relations (2.30), (2.31) will be needed to reach our eventual goal (see (2.35) below). So, with the next assertion of the proposition in our mind, let us ignore (2.32) and, say, (2.31).

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On the other side, again by the standard diagonal-compactness argument, passing to a further subsequence if necessary, we find that

$$\Phi_{i,n} \to \Phi_{i,\infty} \quad \text{in} \quad C_{loc}^2(\mathbb{R}) \quad \text{as } n \to \infty, \quad i = 1, 2,$$

where $\Phi_{1,\infty}, \Phi_{2,\infty}$ satisfy

$$L \begin{pmatrix} \Phi_{1,\infty} \\ \Phi_{2,\infty} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}, \quad \text{and} \quad \sum_{i=1}^2 \|\Phi_{i,\infty}\|_{L^\infty(\mathbb{R})} \le 2. \tag{2.33}$$

Note that again we did not use the full strength of the last assumption in (2.23) (this argument goes through without the factor M_n in front of the sum). Thus, by virtue of Proposition 2.1, we infer that

$$\varphi_{i,n}\left((\ln \Lambda_n)^{-1} x\right) \to b V_i'(x) \quad \text{in} \quad C_{loc}^1(\mathbb{R}), \quad i = 1, 2, \tag{2.34}$$

for some $b \in \mathbb{R}$.

In light of (2.24), (2.27) and (2.34), to reach a contradiction, it is enough to show that

$$a_1 = a_2 = b = 0. \tag{2.35}$$

For this purpose, we will exploit that (2.27) and (2.34) should match where their domains of effectiveness overlap. More precisely, we will focus our attention to the points $\pm K M_n^{-1}$ in the latter intermediate zone, where $K > 0$ is any sufficiently large positive number (independent of n). On the one hand, relations (2.26) and (2.27) give us that

$$\varphi_{1,n}(K M_n^{-1}) = -a_1 + \mathcal{O}\left(e^{-cK}\right) + o(1) \quad \text{as} \quad n \to \infty.$$

On the other hand, we obtain from (2.34) that

$$\varphi_{1,n}(K M_n^{-1}) = b V_1'(K) + o(1) \quad \text{as} \quad n \to \infty.$$

Equating the righthand sides of the above two relations, then letting $n \to \infty$ and subsequently $K \to \infty$ in the resulting identity, yields that

$$- a_1 = \psi_0 b. \tag{2.36}$$

In the same manner we can see that

$$a_2 = -\psi_0 b. \tag{2.37}$$

The desired relation (2.35) now follows at once from (2.30), (2.36) and (2.37). The proof of the a-priori estimate (2.22) is therefore complete.

We are now in position to establish the validity of the full a-priori estimate (2.17). In view of (2.17), (2.22), (2.24), and the asymptotic behaviour of V_1, V_2, we find that

$$\sum_{i=1}^2 \|\Phi_i''\|_{L^\infty(-M,M)} \le CM \sum_{i=1}^2 \|e^{\alpha|x|} H_i\|_{L^\infty(-M,M)}.$$

The desired estimate follows at once by plainly interpolating between (2.22) and the above estimate, for example using the elementary inequality

$$\|\Phi'\|_{L^\infty(-M,M)} \le 2\|\Phi\|_{L^\infty(-M,M)} + \|\Phi''\|_{L^\infty(-M,M)}, \tag{2.38}$$

(which holds for $M \ge 1$).

The proof of the second assertion is completely analogous (recall the comments below (2.28) and (2.33)). The only essential difference is in (2.29) or the corresponding relation

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that gives (2.31), where now the respective orthogonality condition in (2.19) implies that the righthand side is zero. $\qquad\square$

Remark 2.1. *An examination of the above proof reveals that the righthand side of (2.18) may also be replaced by*

$$C \sum_{i=1}^{2} \left\{ \|e^{\alpha|x|} H_i\|_{L^\infty(-M,M)} + M\|H_i\|_{L^1(-M,M)} \right\}.$$

Remark 2.2. *It is worth noting that in the proof of the above proposition we could have also tested (2.21) plainly by (y,y) and, using (2.24), (2.25), (2.26), (2.34) together with Lebesgue's dominated convergence theorem, arrive at the relation*

$$a_1 + a_2 + \psi_0 b = 0.$$

Remark 2.3. *In contrast, the operator L with Neumann boundary conditions at $\pm M$ becomes nearly non-invertible as $M \to \infty$ because (V_1', V_2') satisfies these conditions up to an $\mathcal{O}(e^{-M})$ small error. Nevertheless, the a-priori estimate (2.20) still holds, provided that we restrict ourselves within the mirror symmetric class (1.12).*

The following simple result will prove extremely useful in the sequel.

Lemma 2.3. *Suppose that u, q are smooth and satisfy*

$$-u'' + q(x)u = \mathcal{O}(e^{-c_0 x}) \quad as \quad x \to +\infty,$$

for some constant $c_0 > 0$. Then, the following properties hold.

- $\liminf_{x \to +\infty} q(x) = +\infty$ *and u has at most algebraic growth*
  $\implies u = \mathcal{O}(e^{-c_0 x})$ *as $x \to +\infty$,*
- $q = \mathcal{O}(e^{-c_0 x})$ *as $x \to +\infty$*
  $\implies u = a_1 + b_1 x + \mathcal{O}(e^{-c_1 x})$ *as $x \to +\infty$ for some $a_1, b_1 \in \mathbb{R}$ and any $c_1 \in (0, c_0)$.*

Proof. The first property can be shown as in [15, Lem. 7.3], while the second as in [3, Lem. 3.2]. $\qquad\square$

An important consequence of Proposition 2.2 and Lemma 2.3 is the next lemma, which may be considered as a sort of *Fredholm alternative* for L.

Lemma 2.4. *Assume that the components of $(H_1, H_2) \in [C(\mathbb{R})]^2$ satisfy*

$$|H_i(x)| \le C e^{-c|x|}, \quad x \in \mathbb{R}, \ i = 1, 2, \tag{2.39}$$

for some constants $c, C > 0$, and one of the orthogonality conditions

$$\int_{-\infty}^{\infty} (V_1' H_1 + V_2' H_2) \, dx = 0 \quad or \quad \int_{-\infty}^{\infty} ((xV_1' + V_1)H_1 + (xV_2' + V_2)H_2) \, dx = 0. \tag{2.40}$$

Then, there exists a solution $(\Phi_1, \Phi_2) \in [C^2(\mathbb{R})]^2$ to (2.16) such that

$$\Phi_1(x) = a_+ + \mathcal{O}(e^{-c'x}), \quad \Phi_2(x) = \mathcal{O}(e^{-c'x}) \qquad as \quad x \to +\infty;$$

$$\Phi_1(x) = \mathcal{O}(e^{c'x}), \qquad \Phi_2(x) = a_- + \mathcal{O}(e^{c'x}) \quad as \quad x \to -\infty. \tag{2.41}$$

for some $a_\pm \in \mathbb{R}$ such that

$$a_+ + a_- = -\frac{1}{2\psi_0} \int_{-\infty}^{\infty} ((xV_1' + V_1)H_1 + (xV_2' + V_2)H_2) \, dx \qquad (2.42)$$

and for any $c' \in (0, c)$.

Proof. Let us begin by assuming that the first orthogonality condition in (2.40) holds. We will construct the desired solution through a limiting process. Motivated by the second assertion of Proposition 2.2, we consider the following sequence of approximate problems:

$$L \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix}, \quad x \in (-n, n); \quad \Phi_i(\pm n) = 0, \ i = 1, 2, \ n \geq 1, \qquad (2.43)$$

where

$$\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} - d_n \begin{pmatrix} V_1' \\ V_2' \end{pmatrix} e^{-c|x|},$$

where

$$d_n = \frac{\int_{-n}^{n} (V_1' H_1 + V_2' H_2) \, dx}{\int_{-n}^{n} [(V_1')^2 + (V_2')^2] \, e^{-c|x|} dx}$$

is chosen so that

$$\int_{-n}^{n} (H_{1,n} V_1' + H_{2,n} V_2') \, dx = 0. \qquad (2.44)$$

We note that (2.39), (2.40) and Lebesgue's dominated convergence theorem yield that

$$d_n \to 0.$$

By the second assertion of Proposition 2.2, if n is sufficiently large, there exists a solution $(\Phi_{1,n}, \Phi_{2,n})$ to (2.43) such that

$$\sum_{i=1}^{2} \left\{ \|\Phi_{i,n}'\|_{L^\infty(-n,n)} + \|\Phi_{i,n}\|_{L^\infty(-n,n)} \right\} \leq C,$$

for some constant $C > 0$ that is independent of n. Hence, thanks again to the standard diagonal-compactness argument, letting $n \to \infty$ in (2.43) (along the appropriate subsequence) yields a bounded solution to (2.16). The asymptotic behaviour (2.41) is a direct consequence of Lemma 2.3. Lastly, relation (2.42) follows at once by testing (2.16) with $(xV_1' + V_1, xV_2' + V_2)$.

The proof in the case of the second orthogonality condition in (2.40) is completely analogous. $\qquad \square$

We can now establish our main result concerning the solvability properties of (2.16).

Proposition 2.3. *Given $(H_1, H_2) \in [C(\mathbb{R})]^2$ satisfying the exponential decay estimate (2.39), there exists a solution $(\Phi_1, \Phi_2) \in [C^2(\mathbb{R})]^2$ to (2.16) such that*

$$\Phi_1(x) = a_+ + bx + \mathcal{O}(e^{-c'x}), \quad \Phi_2(x) = \mathcal{O}(e^{-c'x}) \qquad \qquad as \ x \to +\infty;$$

$$\Phi_1(x) = \mathcal{O}(e^{c'x}), \qquad \qquad \Phi_2(x) = a_- + bx + \mathcal{O}(e^{c'x}) \quad as \ x \to -\infty,$$

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for any $c' \in (0, c)$, where

$$b = -\frac{1}{2\psi_0} \int_{-\infty}^{\infty} (V_1' H_1 + V_2' H_2) \, dx,$$

and a_+, a_- satisfy (2.42).

Proof. The main idea is to search for a solution in the form

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = B \begin{pmatrix} V_1 \\ -V_2 \end{pmatrix} + \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

with $B \in \mathbb{R}$ and $(\Psi_1, \Psi_2) \in [C^2(\mathbb{R})]^2$. The new equation that now needs to be satisfied is

$$L \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + B \begin{pmatrix} 2V_1 V_2^2 \\ -2V_2 V_1^2 \end{pmatrix}.$$

To conclude, we can apply Lemma 2.4 after making the choice

$$B = -\frac{\int_{-\infty}^{\infty} (V_1' H_1 + V_2' H_2) \, dx}{2 \int_{-\infty}^{\infty} (V_1 V_2^2 V_1' - V_2 V_1^2 V_2') \, dx} \overset{(1.10),(1.11)}{=} -\frac{1}{2\psi_0^2} \int_{-\infty}^{\infty} (V_1' H_1 + V_2' H_2) \, dx.$$

$\square$

By applying Proposition 2.3 to the case where the righthand side of (2.16) is the pair (F_1, F_2) (which actually is independent of Λ and satisfies (2.15)), as defined through (2.14), we obtain the existence of a solution $(\hat{\Phi}_1, \hat{\Phi}_2)$ to the system

$$L \begin{pmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.45}$$

such that

$$\hat{\Phi}_1(x) = a_+ + bx + \mathcal{O}(e^{-Dx}), \quad \hat{\Phi}_2(x) = \mathcal{O}(e^{-Dx}) \qquad \text{as } x \to +\infty;$$

$$\hat{\Phi}_1(x) = \mathcal{O}(e^{Dx}), \qquad\qquad \hat{\Phi}_2(x) = a_- + bx + \mathcal{O}(e^{Dx}) \quad \text{as } x \to -\infty, \tag{2.46}$$

for some $a_\pm, b \in \mathbb{R}$ and any $D > 0$ (the expressions for the sum $a_+ + a_-$ and b which are provided by the aforementioned proposition, with (H_1, H_2) in place of (F_1, F_2), will not be needed). In fact, the above relations can be differentiated arbitrary many times.

This allows us to improve our inner approximate solution (2.11):

Definition 1. *We define the inner approximate solution to (1.1) as $(v_{1,in}, v_{2,in})$, with*

$$v_{i,in}(z) = \mu \Lambda^{-\frac{1}{4}} V_i(x) + \mu^{-1} \Lambda^{-\frac{3}{4}} \left[Z_i(x) + \hat{\Phi}_i(x) \right] + B E_i(x), \quad |z| \le (\ln \Lambda) \Lambda^{-\frac{1}{4}}, \ i = 1, 2, \tag{2.47}$$

where x is the stretched variable (2.9), $V_i, Z_i, \hat{\Phi}_i$ are defined through Proposition 1.1, (2.13), (2.45)-(2.46) respectively and

$$(E_1, E_2) = (xV_1' + V_1, xV_2' + V_2) \tag{2.48}$$

is the second element of the kernel of L from (2.8). The constants μ and B will be determined later, subject to the constraints (2.10) and

$$B = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \quad as \ \Lambda \to \infty, \tag{2.49}$$

respectively.

Remark 2.4. *It may appear at first sight that the above inner approximate solution contains two free parameters, μ and B. However, keep in mind that both are present due to the same reason, namely the invariance of (1.10) under scaling. So, essentially there is only one free parameter. An analogous remark applies to the outer approximate solution in (2.4). On the other hand, we stress that, in principle, all the aforementioned parameters should be present when carrying out the formal matched asymptotic analysis.*

2.2.3. *The remainder of the inner approximate solution.* In view of (2.9), (2.10), (2.12), (2.49), and the construction of $(\hat{\Phi}_1, \hat{\Phi}_2)$, we have the validity of the following lemma.

Lemma 2.5. *In equation (1.1), the remainder*

$$R(v_{1,in}, v_{2,in}) = \begin{pmatrix} -v_{1,in}'' + v_{1,in}^3 - v_{1,in} + \Lambda v_{2,in}^2 v_{1,in} \\ -v_{2,in}'' + v_{2,in}^3 - v_{2,in} + \Lambda v_{1,in}^2 v_{2,in} \end{pmatrix}$$

which is left by the solution $(v_{1,in}, v_{2,in})$ of Definition 1 satisfies

$$R(v_{1,in}, v_{2,in}) = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \begin{pmatrix} \Lambda^{\frac{3}{4}} z^3 + 1 \\ e^{-D\Lambda^{\frac{1}{4}} z} \end{pmatrix},$$

for any $D \geq 1$, uniformly on $\left[0, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right]$, as $\Lambda \to \infty$. An analogous estimate holds on $\left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, 0\right]$.

3. Matching the outer and inner approximate solutions

In this section we will 'stitch' together the outer and inner approximate solutions by suitably adjusting the parameters $\xi_1, \xi_2, \tau_1, \tau_2, \mu, B$ in their definitions (recall (2.4) and (2.47)), subject to the constraints (2.5), (2.10) and (2.49). Classical singular perturbation theory dictates that this must be done so that the inner and outer approximations are sufficiently close in the C^1-sense over some intermediate zone satisfying $\Lambda^{-\frac{1}{4}} \ll |z| \ll 1$ (see the section on matched asymptotic expansions in any textbook on the subject or the so called exchange lemmas of the modern geometric singular perturbation theory). This property is of course already satisfied by the first components of the aforementioned approximations in the negative part of the intermediate zone, and the analogous property holds in the positive part. Thus, the task of matching the inner and outer approximate solutions in C^1 over an intermediate zone amounts to satisfying a total of four algebraic equations (one for each of the first two terms of the Taylor expansions of the non-trivial outer approximations). However, in view of Remark 2.4, we essentially have only three free parameters to adjust for this purpose. Fortunately, with some care, this overdetermined issue can be resolved by exploiting the conservation of the hamiltonian of (1.1). Actually, it is more convenient to match them continuously as best as possible at just the two boundary points $\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}$. It turns out that the algebraic system which arises from these considerations, comprising of three equations (one at each boundary point together with one from the explotation of the hamiltonian structure, say at the origin) and containing three unknowns (essentially coming from the translation and scaling invariances of the outer and inner limit problems respectively), is solvable for large Λ thanks to the fact that $\psi_0 \neq 0$. In fact, this type of matching leads

us naturally to building a solution of (1.1)-(1.2) in the same spirit, that is by constructing separately inner and outer genuine solutions which match continuously at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ and share the same hamiltonian constant. In particular, the latter strategy allows us to use directly the last observation in Lemma 2.1 and Proposition 2.1 for this purpose.

3.1. Matching $(v_{1,out}, v_{2,out})$ and $(v_{1,in}, v_{2,in})$ continuously at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$.
In view of (1.8), (2.4), (2.5) and the facts that

$$U_1''(0) = 0, \quad U_1'''(0) = -\psi_0, \quad U_1^{(4)}(0) = 0, \tag{3.1}$$

we find that

$$v_{1,out}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \psi_0\left((\ln\Lambda)\Lambda^{-\frac{1}{4}} + \xi_1\right) - \frac{\psi_0}{6}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}} + \xi_1\right)^3$$

$$+ \mathcal{O}\left((\ln\Lambda)^5\Lambda^{-\frac{5}{4}}\right) + \tau_1\psi_0 + \tau_1\mathcal{O}\left((\ln\Lambda)^2\Lambda^{-\frac{1}{2}}\right)$$

as $\Lambda \to \infty$, where the quantities in the Landau symbols are independent of τ_1. In turn, by expanding, we get that

$$v_{1,out}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \psi_0\xi_1 + \psi_0(\ln\Lambda)\Lambda^{-\frac{1}{4}} + \tau_1\psi_0 - \frac{\psi_0}{6}\xi_1^3 - \frac{\psi_0}{2}(\ln\Lambda)\Lambda^{-\frac{1}{4}}\xi_1^2$$

$$- \frac{\psi_0}{6}(\ln\Lambda)^3\Lambda^{-\frac{3}{4}} - \frac{\psi_0}{2}(\ln\Lambda)^2\Lambda^{-\frac{1}{2}}\xi_1$$

$$+ \mathcal{O}\left((\ln\Lambda)^5\Lambda^{-\frac{5}{4}}\right) + \tau_1\mathcal{O}\left((\ln\Lambda)^2\Lambda^{-\frac{1}{2}}\right)$$

as $\Lambda \to \infty$.

On the other side, from (1.13), (2.13), (2.46) and (2.47), we obtain that

$$v_{1,in}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \mu^2\psi_0(\ln\Lambda)\Lambda^{-\frac{1}{4}} + \mu\kappa\Lambda^{-\frac{1}{4}} + B\kappa + 2B\psi_0\mu(\ln\Lambda) + \mu^{-1}\Lambda^{-\frac{3}{4}}a_+$$

$$+ b(\ln\Lambda)\Lambda^{-\frac{3}{4}} - \frac{\psi_0}{6}\mu^2(\ln\Lambda)^3\Lambda^{-\frac{3}{4}} - \frac{1}{2}\kappa\mu(\ln\Lambda)^2\Lambda^{-\frac{3}{4}} + \mathcal{O}(\Lambda^{-\infty})$$

as $\Lambda \to \infty$.

Given B satisfying (2.49), we take

$$\mu = 1, \quad \xi_1 = \psi_0^{-1}\kappa\Lambda^{-\frac{1}{4}} + 2B(\ln\Lambda) + \frac{1}{2}(\ln\Lambda)\Lambda^{-\frac{1}{4}}\xi_1^2, \tag{3.2}$$

which is indeed possible by the implicit function theorem for Λ large (we can even find an explicit formula for ξ_1 by solving the above trinomial). In turn, we choose

$$\tau_1 = \frac{\xi_1^3}{6} + \psi_0^{-1}B\kappa + \psi_0^{-1}a_+\Lambda^{-\frac{3}{4}} + \psi_0^{-1}b(\ln\Lambda)\Lambda^{-\frac{3}{4}}.$$

Then, using that

$$\xi_1 = \psi_0^{-1}\kappa\Lambda^{-\frac{1}{4}} + \mathcal{O}\left((\ln\Lambda)\Lambda^{-\frac{3}{4}}\right) \quad \text{as} \ \Lambda \to \infty,$$

it follows readily that

$$(v_{1,out} - v_{1,in})\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}\left((\ln\Lambda)^5\Lambda^{-\frac{5}{4}}\right) \quad \text{as} \ \Lambda \to \infty. \tag{3.3}$$

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We have

$$(v_{2,out} - v_{2,in})\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = -v_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}\left(\Lambda^{-\infty}\right) \quad \text{as} \ \ \Lambda \to \infty. \qquad (3.4)$$

Analogous considerations apply at $-(\ln \Lambda)\Lambda^{-\frac{1}{4}}$.

3.2. Adjusting the value of the Hamiltonian on the inner approximate solution at $z = 0$.
In this subsection, we will choose B, under the constraint (2.49), so that the value of the Hamiltonian on $(v_{1,in}, v_{2,in})$ at $z = 0$ is equal to the Hamiltonian constant of the expected heteroclinic connection, namely $-\psi_0^2/2$.

Firstly, from (2.9), (2.47) and (2.49), we note that

$$\left[v'_{i,in}(0)\right]^2 = [V'_i(0)]^2 + 2V'_i(0)\left[O(\Lambda^{-\frac{1}{2}}) + 2V'_i(0)B\Lambda^{\frac{1}{4}}\right] + \mathcal{O}(\Lambda^{-1}), \quad i = 1, 2,$$

as $\Lambda \to \infty$, with $O(\Lambda^{-\frac{1}{2}})$ being independent of B. Furthermore, it is clear that

$$v_{i,in}^4(0) = \Lambda^{-1}V_i^4(0) + \mathcal{O}(\Lambda^{-\frac{1}{2}})B^2 + 4\Lambda^{-\frac{3}{4}}V_i^4(0)B + O(\Lambda^{-\frac{3}{2}}) \quad \text{as} \ \Lambda \to \infty,$$

where $O(\Lambda^{-\frac{3}{2}})$ is independent of B, and that

$$\frac{\left(1 - v_{i,in}^2(0)\right)^2}{4} = \frac{1}{4} + O(\Lambda^{-\frac{1}{2}}) + \mathcal{O}(\Lambda^{-\frac{1}{4}})B \overset{(1.8)}{=} \frac{\psi_0^2}{2} + O(\Lambda^{-\frac{1}{2}}) + \mathcal{O}(\Lambda^{-\frac{1}{4}})B$$

as $\Lambda \to \infty$, $i = 1, 2$, where $O(\Lambda^{-\frac{1}{2}})$ is independent of B. Now, using that $V_1(0) = V_2(0)$, $V'_1(0) = -V'_2(0)$ and the hamiltonian identity

$$(V'_1)^2 + (V'_2)^2 - V_1^2V_2^2 = \psi_0^2, \quad x \in \mathbb{R},$$

it follows readily that the sought after equality

$$H(v_{1,in}(0), v_{2,in}(0)) = -\frac{\psi_0^2}{2} \quad \text{(with the obvious notation, keep in mind (1.5), (1.8))}$$

takes the form

$$2\psi_0^2 B\Lambda^{\frac{1}{4}} = O(\Lambda^{-\frac{1}{2}}) + \mathcal{O}(\Lambda^{-\frac{1}{4}})B + \mathcal{O}(\Lambda^{-1}) \quad \text{as} \ \Lambda \to \infty.$$

The above equation clearly has a unique solution

$$B = \mathcal{O}(\Lambda^{-\frac{3}{4}}) \quad \text{as} \ \Lambda \to \infty, \qquad (3.5)$$

as desired (keep in mind that, according to our notation, the term $O(\Lambda^{-\frac{1}{2}})$ above does not contain B).

3.3. A refined inner approximate solution $(w_{1,in}, w_{2,in})$.
For $\tilde{B} \in \mathbb{R}$ to be chosen later, subject to the constraint

$$\tilde{B} = \mathcal{O}(\Lambda^{-1}) \quad \text{as} \ \Lambda \to \infty, \qquad (3.6)$$

we consider the more refined inner approximate solution $(w_{1,in}, w_{2,in})$

$$w_{i,in}(z) = v_{i,in}(z) + \tilde{B}E_i(x), \quad |z| \leq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \ i = 1, 2, \qquad (3.7)$$

where $(v_{1,in}, v_{2,in})$ is defined in Definition 1 and E_i comes from (2.48). Actually, $\tilde{B}$ will turn out to be chosen much smaller than in (3.6).

It is easy to see that the assertion of Lemma 2.5, concerning the remainder of this refined inner solution, continues to hold for $(w_{1,in}, w_{2,in})$.

4. Solution of the inner problem

In this section, we will show that the one-parameter family of refined inner approximate solutions $(w_{1,in}, w_{2,in})$, described in the previous section (parameterized by $\tilde{B}$), can be perturbed smoothly to a one-parameter family of inner genuine solutions to the system (1.1), for large $\Lambda > 0$. Then, we will show that there exists at least one value of $\tilde{B}$, in the range (3.6), for which the corresponding inner genuine solution to (1.1) has a Hamiltonian constant equal to $-\psi_0^2/2$.

4.1. The perturbation argument.

Given $\tilde{B}$ satisfying (3.6), we seek a solution of system (1.1) as

$$(\mathbf{v}_{1,in}, \mathbf{v}_{2,in}) = (w_{1,in}, w_{2,in}) + (\varphi_1, \varphi_2), \quad |z| \leq (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \tag{4.1}$$

with

$$\varphi_i\left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad i = 1, 2.$$

After rearranging terms, we find that (φ_1, φ_2) has to satisfy

$$\begin{cases} \mathcal{L}(\varphi_1, \varphi_2) = -R(w_{1,in}, w_{2,in}) - Q(\varphi_1, \varphi_2) - N(\varphi_1, \varphi_2), \\ \\ \varphi_i\left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad i = 1, 2, \end{cases} \tag{4.2}$$

where

$$\mathcal{L}(\varphi_1, \varphi_2) = \begin{pmatrix} -\varphi_1'' + \Lambda^{\frac{1}{2}}V_2^2(x)\varphi_1 + 2\Lambda^{\frac{1}{2}}V_1(x)V_2(x)\varphi_2 \\ \\ -\varphi_2'' + \Lambda^{\frac{1}{2}}V_1^2(x)\varphi_2 + 2\Lambda^{\frac{1}{2}}V_1(x)V_2(x)\varphi_1 \end{pmatrix},$$

the term $R(w_{1,in}, w_{2,in})$ denotes the remainder which is left by $(w_{1,in}, w_{2,in})$ in (1.1) (analogously to Lemma 2.5),

$$Q(\varphi_1, \varphi_2) =$$
$$\begin{pmatrix} (3w_{1,in}^2 - 1)\varphi_1 + \Lambda\left(w_{2,in}^2 - \Lambda^{-\frac{1}{2}}V_2^2\right)\varphi_1 + 2\Lambda\left(w_{1,in}w_{2,in} - \Lambda^{-\frac{1}{2}}V_1V_2\right)\varphi_2 \\ \\ (3w_{2,in}^2 - 1)\varphi_2 + \Lambda\left(w_{1,in}^2 - \Lambda^{-\frac{1}{2}}V_1^2\right)\varphi_2 + 2\Lambda\left(w_{1,in}w_{2,in} - \Lambda^{-\frac{1}{2}}V_1V_2\right)\varphi_1 \end{pmatrix}$$

and

$$N(\varphi_1, \varphi_2) = \begin{pmatrix} \varphi_1^3 + 3w_{1,in}\varphi_1^2 + \Lambda w_{1,in}\varphi_2^2 + \Lambda\varphi_2^2\varphi_1 + 2\Lambda w_{2,in}\varphi_1\varphi_2 \\ \\ \varphi_2^3 + 3w_{2,in}\varphi_2^2 + \Lambda w_{2,in}\varphi_1^2 + \Lambda\varphi_1^2\varphi_2 + 2\Lambda w_{1,in}\varphi_1\varphi_2 \end{pmatrix}.$$

Concerning the linear operator $\mathcal{L}$, we observe that Proposition 2.2, after a simple re-scaling (recall that $x = \Lambda^{\frac{1}{4}}z$), yields the following.

Corollary 4.1. *Given $\alpha > 0$, there exist $\Lambda_0, C > 0$ such that the boundary value problem*

$$\mathcal{L}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad |z| < (\ln \Lambda)\Lambda^{-\frac{1}{4}}; \quad \varphi_i\left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \ i = 1, 2,$$

where $h_1, h_2 \in C\left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right]$, has a unique solution such that

$$\sum_{i=1}^{2}\left(\Lambda^{-\frac{1}{4}}\|\varphi_i'\|_{L^\infty(I_\Lambda)} + \|\varphi_i\|_{L^\infty(I_\Lambda)}\right) \leq C\Lambda^{-\frac{1}{2}+\alpha}\sum_{i=1}^{2}\|h_i\|_{L^\infty(I_\Lambda)},$$

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where
$$I_\Lambda = \left(-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}} \right),$$
provided that $\Lambda \geq \Lambda_0$.

On the other side, the remainder $R(w_{1,in}, w_{2,in})$ clearly satisfies the thesis of Lemma 2.5. Furthermore, using that
$$|v_{i,in}(z)| \leq C(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \quad |E_i(x)| \leq C(\ln\Lambda),$$
together with the easy to prove estimates
$$\left| v_{i,in}^2(z) - \Lambda^{-\frac{1}{2}}V_i^2(x) \right| + \left| v_{1,in}v_{2,in}(z) - \Lambda^{-\frac{1}{2}}V_1V_2(x) \right| \leq C(\ln\Lambda)^4\Lambda^{-\frac{5}{4}},$$
for $|z| \leq (\ln\Lambda)\Lambda^{-\frac{1}{4}}$, $i = 1, 2$, and (3.6), it follows readily that there exists $C > 0$ such that
$$\sum_{i=1}^{2} \|Q_i(\varphi_1, \varphi_2)\|_{L^\infty(I_\Lambda)} \leq C \sum_{i=1}^{2} \|\varphi_i\|_{L^\infty(I_\Lambda)}, \tag{4.3}$$
for any $\varphi_1, \varphi_2 \in C(\overline{I_\Lambda})$. Moreover, there exists a $C > 0$ such that
$$\|N_i(\varphi_1, \varphi_2)\|_{L^\infty(I_\Lambda)} \leq C \sum_{i=1}^{2} \left\{ \|\varphi_i\|_{L^\infty}^3 + \Lambda^{\frac{3}{4}}(\ln\Lambda)\|\varphi_i\|_{L^\infty}^2 + \Lambda\|\varphi_i\|_{L^\infty}\|\varphi_{i+1}\|_{L^\infty}^2 \right\}, \tag{4.4}$$
$i = 1, 2$, for any $\varphi_1, \varphi_2 \in C(\overline{I_\Lambda})$ (with the obvious notation), and
$$\sum_{i=1}^{2} \|N_i(\varphi_1, \varphi_2) - N_i(\psi_1, \psi_2)\|_{L^\infty(I_\Lambda)} \leq$$
$$C\sum_{i=1}^{2}\left\{ \Lambda\left(\|\varphi_i\|_{L^\infty}^2 + \|\psi_i\|_{L^\infty}^2\right) + \Lambda^{\frac{3}{4}}(\ln\Lambda)\left(\|\varphi_i\|_{L^\infty} + \|\psi_i\|_{L^\infty}\right) \right\} \tag{4.5}$$
$$\times \left(\sum_{i=1}^{2} \|\varphi_i - \psi_i\|_{L^\infty} \right),$$
for any $\varphi_1, \varphi_2, \psi_1, \psi_2 \in C(\overline{I_\Lambda})$.

In view of the above, and paying attention to the dependence on $\tilde{B}$, a standard application of the contraction mapping principle yields the following.

Proposition 4.1. *Given $\alpha \in (0, 1)$, there exists $C > 0$ such that problem (4.2) has a unique solution satisfying*
$$\sum_{i=1}^{2} \left(\Lambda^{-\frac{1}{4}}\|\varphi_i'\|_{L^\infty(I_\Lambda)} + \|\varphi_i\|_{L^\infty(I_\Lambda)} \right) \leq C\Lambda^{-\frac{5}{4}+\alpha},$$
provided that Λ is sufficiently large. Moreover, this solution depends continuously, with respect to the $C^1(\overline{I_\Lambda})$-norm, on $\tilde{B}$ as in (3.6) (for fixed Λ).

We point out that the aforementioned continuous dependence on $\tilde{B}$ can be proven easily as follows. Let $\tilde{B}_n$ satisfy (3.6), for fixed Λ as in the above proposition, and $\tilde{B}_n \to \tilde{B}_\infty$ as $n \to \infty$. We denote by $(\varphi_{1,n}, \varphi_{2,n})$ and $(\varphi_{1,\infty}, \varphi_{2,\infty})$ the solutions of (4.2) corresponding to $\tilde{B}_n$ and $\tilde{B}_\infty$ respectively, as provided by the first part of the above proposition. Then, thanks to Arzela-Ascoli's theorem, passing to a subsequence if necessary, and utilizing the uniqueness assertion of the aforementioned proposition, we find that $\varphi_{i,n} \to \varphi_{i,\infty}$ in $C^1(\overline{I_\Lambda})$ as $n \to \infty$, $i = 1, 2$. Finally, by employing once more the uniqueness property of $(\varphi_{1,\infty}, \varphi_{2,\infty})$, we deduce that the previous convergence holds for the original sequence.

4.1.1. *Some preliminary positivity and monotonicity properties of the inner genuine solution $(\boldsymbol{v}_{1,in}, \boldsymbol{v}_{2,in})$.* It is clear from the construction of the refined inner approximate solution $(w_{1,in}, w_{2,in})$ and Proposition 4.1 that, given any $L > 1$, there exists $c_L > 0$ such that

$$\mathbf{v}_{2,in} \geq c_L \Lambda^{-\frac{1}{4}} \quad \text{and} \quad -\mathbf{v}'_{2,in} \geq c_L \quad \text{on} \quad \left[-(\ln \Lambda)\Lambda^{-\frac{1}{4}}, L\Lambda^{-\frac{1}{4}}\right], \tag{4.6}$$

provided that $\Lambda > 0$ is sufficiently large. On the other side, we observe that $\mathbf{v}_{2,in}$ satisfies a linear equation of the form

$$-v'' + P(z)v = 0 \quad \text{with} \quad P(z) \geq c\Lambda z^2, \quad z \in \left(L\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right), \tag{4.7}$$

with $c > 0$ independent of both Λ, L. Unfortunately, it is not clear to us how to use the maximum principle to deduce the positivity and monotonicity of $\mathbf{v}_{2,in}$ in this remaining interval without too much effort. A possible way would be to show that $\mathbf{v}'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = w'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) < 0$. This last task, however, would require us to keep track of the sharp super-exponential decay of the various functions involved in the construction of $w_{2,in}$. Nevertheless, since

$$\mathbf{v}_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = w_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) \stackrel{(1.13),(2.46)}{=} \Lambda^{-\frac{1}{4}}\mathcal{O}\left(e^{-c(\ln \Lambda)^2}\right) + \Lambda^{-\frac{3}{4}}\mathcal{O}\left(e^{-D(\ln \Lambda)}\right), \tag{4.8}$$

for any $D > 0$, as $\Lambda \to \infty$, we deduce by Proposition 4.1 (used mildly only at $z = \Lambda^{-\frac{1}{4}}$), relation (4.7) and a barrier argument that

$$|\mathbf{v}_{2,in}(z)| \leq C\Lambda^{-\frac{1}{4}}e^{-c\Lambda^{\frac{1}{2}}z^2} + \mathcal{O}(\Lambda^{-\infty}), \quad z \in \left[\Lambda^{-\frac{1}{4}}, (\ln \Lambda)\Lambda^{-\frac{1}{4}}\right]. \tag{4.9}$$

In turn, by (4.7) and a standard interpolation argument, we can easily infer that

$$\mathbf{v}'_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty. \tag{4.10}$$

The above two estimates, and the analogous ones for $\mathbf{v}_{1,in}$, will play a pivotal role in 'extending' $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ to a heteroclinic solution to (1.1)-(1.2), which can then easily be shown to have positive components with the right monotonicity properties.

4.2. Adjusting the Hamiltonian constant of the inner genuine solution.

From the calculations of Subsection 3.2 and Proposition 4.1, it follows that, if α therein is inside $(0, 1/4)$ and $\tilde{B}$ satisfies (3.6), the equation for $\tilde{B}$ such that the Hamiltonian constant of the exact solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ of the inner problem is equal to $-\psi_0^2/2$ has the form

$$2\psi_0^2 \tilde{B}\Lambda^{\frac{1}{4}} + \Lambda^{-1+\alpha}h(\Lambda, \tilde{B}) = 0,$$

where the function h is uniformly bounded in Λ and continuous. Consequently, by the Bolzano-Weistrass theorem, there exists at least one

$$\tilde{B} = \mathcal{O}\left(\Lambda^{-\frac{5}{4}+\alpha}\right) \quad \text{as} \quad \Lambda \to \infty, \tag{4.11}$$

which satisfies the above equation ($\alpha \in (0, 1/4)$ is still as in Proposition 4.1). Clearly, our working assumption (3.6) is satisfied for $\alpha > 0$ sufficiently small.

5. Solution of the outer problem

In this section, we will construct a symmetric solution $(\mathbf{v}_{1,out}, \mathbf{v}_{2,out})$ to system (1.1) outside of the interval I_Λ which, however, agrees on ∂I_Λ with the already constructed solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ of the inner problem (in the C^0 sense) and satisfies the desired asymptotic behaviour in (1.2).

5.1. A refined outer approximate solution $(w_{1,out}, w_{2,out})$.

We first consider a refinement of the outer approximate solution $(v_{1,out}, v_{2,out})$ that was constructed in Subsection 2.1, defined as

$$w_{1,out}(z) = v_{1,out}(z) + \tilde{\tau}_1 U_1'(z + \xi_1), \quad w_{2,out}(z) = \mathbf{v}_{2,in}\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right)\zeta(z),$$

for $z \geq (\ln \Lambda)\Lambda^{-\frac{1}{4}}$; where

$$\tilde{\tau}_1 = -\frac{(v_{1,out} - \mathbf{v}_{1,in})\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right)}{U'\left((\ln \Lambda)\Lambda^{-\frac{1}{4}} + \xi_1\right)} \overset{(2.1),(3.3),\text{Prop}.4.1,(4.11)}{=} \mathcal{O}\left((\ln \Lambda)\Lambda^{-\frac{5}{4}+\alpha}\right),$$

as $\Lambda \to \infty$ ($\alpha \in (0, \frac{1}{4})$ still as in Proposition 4.1), and $\zeta \in C_0^\infty(\mathbb{R})$ is a fixed cutoff function which is equal to one on $[-1, 1]$ (in this regard, keep in mind (3.4), (4.10)). Analogously we define $(w_{1,out}, w_{2,out})$ for $z \leq -(\ln \Lambda)\Lambda^{-\frac{1}{4}}$.

By construction, we have

$$w_{i,out}\left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathbf{v}_{i,in}\left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right), \quad i = 1, 2,$$

and that the asymptotic behaviour (1.2) is still satisfied.

5.2. The remainder of the refined outer approximate solution.

By recalling the proof of Lemma 2.2 and relations (3.4), (4.10), we find that the assertion of the aforementioned lemma continues to hold for $(w_{1,out}, w_{2,out})$; except that in the equations which were satisfied exactly now there is a remainder left, but whose absolute value is bounded by a $\Lambda^{-\infty}$-small number times a fixed, compactly supported function.

5.3. The perturbation argument.

We seek a solution of system (1.1) as

$$(\mathbf{v}_{1,out}, \mathbf{v}_{2,out}) = (w_{1,out}, w_{2,out}) + (\varphi_1, \varphi_2), \quad |z| \geq (\ln \Lambda)\Lambda^{-\frac{1}{4}},$$

$$\varphi_i\left(\pm(\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad \lim_{z \to \pm\infty} \varphi_i(z) = 0, \quad i = 1, 2. \tag{5.1}$$

Proceeding as in Subsection 4.1, we find that now the corresponding linear operator is

$$(\mathcal{L} + Q)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3w_{1,out}^2 - 1)\varphi_1 + \Lambda w_{2,out}^2 \varphi_1 + 2\Lambda w_{1,out} w_{2,out} \varphi_2 \\ -\varphi_2'' + (3w_{2,out}^2 - 1)\varphi_2 + \Lambda w_{1,out}^2 \varphi_2 + 2\Lambda w_{1,out} w_{2,out} \varphi_1 \end{pmatrix}.$$

The invertibility properties of the above operator that we will need are contained in the following proposition.

Proposition 5.1. *Given $m > 1$, there exist $\Lambda_2, C > 0$ such that the boundary value problem*

$$(\mathcal{L} + Q)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \varphi_i\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad \lim_{z \to +\infty} \varphi_i(z) = 0, \quad i = 1, 2, \qquad (5.2)$$

where $f_1, f_2 \in C\left[(\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)$ decay exponentially fast to zero, has a unique solution such that

$$\|\varphi_1'\|_{L^\infty(J_\Lambda)} + \|\varphi_1\|_{L^\infty(J_\Lambda)} \leq C\|f_1\|_{L^\infty(J_\Lambda)} + \Lambda^{-m}\|f_2\|_{L^\infty(J_\Lambda)},$$

and

$$(\ln \Lambda)^{-1}\Lambda^{-\frac{1}{4}}\|\varphi_2'\|_{L^\infty(J_\Lambda)} + \|\varphi_2\|_{L^\infty(J_\Lambda)} \leq \Lambda^{-m}\|f_1\|_{L^\infty(J_\Lambda)} + C(\ln \Lambda)^{-2}\Lambda^{-\frac{1}{2}}\|f_2\|_{L^\infty(J_\Lambda)},$$

where

$$J_\Lambda = \left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right),$$

provided that $\Lambda \geq \Lambda_2$. An analogous estimate holds also in the negative outer region.

Proof. As in Proposition 2.2, it is enough to establish the validity of the asserted a-priori estimates. We note also that the continuous spectrum of $\mathcal{L} + Q$, when defined naturally in $L^2(J_\Lambda) \times L^2(J_\Lambda)$, coincides with the interval $[\Lambda^2 - 1, \infty)$, see [4] or [23], which does not include zero by (1.3).

We will first show that there exist constants $\Lambda_1, C > 0$ such that the following a-priori estimate holds: If $\phi \in C^2(\bar{J}_\Lambda)$ and $f \in C(\bar{J}_\Lambda)$ satisfy

$$-\phi'' + (3w_{1,out}^2 - 1)\phi + \Lambda w_{2,out}^2 \phi = f, \quad z \in J_\Lambda,$$

$$\phi\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}\right) = 0, \quad \lim_{z \to \infty} \phi(z) = 0,$$

for $\Lambda \geq \Lambda_1$, then

$$\|\phi'\|_{L^\infty(J_\Lambda)} + \|\phi\|_{L^\infty(J_\Lambda)} \leq C\|f\|_{L^\infty(J_\Lambda)}.$$

A preliminary observation is that, since

$$\|w_{2,out}\|_{L^\infty(J_\Lambda)} = \mathcal{O}(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty, \qquad (5.3)$$

it is clearly sufficient to show the a-priori estimate

$$\|\phi\|_{L^\infty(J_\Lambda)} \leq C\|f\|_{L^\infty(J_\Lambda)}.$$

To this end, we will argue by contradiction. So, let us suppose that there exist $\Lambda_n \to \infty$, $\phi_n \in C^2(\overline{J_{\Lambda_n}})$ and $f_n \in C(\overline{J_{\Lambda_n}})$ such that

$$-\phi_n'' + (3w_{1,out}^2 - 1)\phi_n + \Lambda_n w_{2,out}^2 \phi_n = f_n, \quad z \in J_{\Lambda_n},$$

$$\phi_n\left((\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}}\right) = 0, \quad \lim_{z \to \infty} \phi_n(z) = 0,$$

while

$$\|\phi_n\|_{L^\infty(J_{\Lambda_n})} = 1 \quad \text{and} \quad \|f_n\|_{L^\infty(J_{\Lambda_n})} \to 0.$$

Let

$$\tilde{\phi}_n(z) = \phi_n\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}}\right), \quad \tilde{f}_n(z) = f_n\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}}\right), \quad z \geq 0.$$

Then, we have that

$$-\tilde{\phi}_n'' + \left[3w_{1,out}^2\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}}\right) - 1\right]\tilde{\phi}_n + \Lambda_n w_{2,out}^2\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}}\right)\tilde{\phi}_n = \tilde{f}_n,$$

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$z \in [0, \infty)$, $\tilde{\phi}_n(0) = 0$, $\lim_{z \to \infty} \tilde{\phi}_n(z) = 0$, while

$$\|\tilde{\phi}_n\|_{L^\infty(0,\infty)} = 1 \quad \text{and} \quad \|\tilde{f}_n\|_{L^\infty(0,\infty)} \to 0.$$

Keeping in mind (5.3), thanks again to standard elliptic estimates and the usual diagonal argument, passing to a subsequence if necessary, and recalling the construction of $w_{1,out}$, we find that

$$\tilde{\phi}_n \to \tilde{\phi}_\infty \quad \text{in} \quad C^1_{loc}[0, \infty),$$

for some $\tilde{\phi}_\infty$ satisfying

$$-\tilde{\phi}''_\infty + \left(3U_1^2(z) - 1\right)\tilde{\phi}_\infty = 0, \quad z > 0; \quad \tilde{\phi}_\infty(0) = 0 \quad \text{and} \quad \|\tilde{\phi}_\infty\|_{L^\infty(0,\infty)} \le 1.$$

Moreover, it is easy to see that $\tilde{\phi}_\infty$ is nontrivial since

$$\left[3w_{1,out}^2 - 1 + \Lambda_n w_{2,out}^2\right]\left(z + (\ln \Lambda_n)\Lambda_n^{-\frac{1}{4}}\right) \to 2, \quad \text{as } z \to \infty, \text{ uniformly in } n,$$

which implies that the points where $|\tilde{\phi}_n|$ attains its maximum cannot escape at infinity. On the other hand, the first case in Lemma 2.1 implies that $\tilde{\phi}_\infty$ is identically equal to zero which is a contradiction.

Applying the previously proven a-priori estimate to the first equation of (5.2), and recalling (5.3), we obtain that

$$\|\varphi_1'\|_{L^\infty(J_\Lambda)} + \|\varphi_1\|_{L^\infty(J_\Lambda)} \le C\|f_1\|_{L^\infty(J_\Lambda)} + \mathcal{O}(\Lambda^{-\infty})\|\varphi_2\|_{L^\infty(J_\Lambda)}. \tag{5.4}$$

The situation in the second equation is considerably simpler. Indeed, observing that

$$w_{1,out}(z) \ge c(\ln \Lambda)\Lambda^{-\frac{1}{4}}, \quad z \in J_\Lambda, \tag{5.5}$$

it follows easily that

$$(\ln \Lambda)^{-1}\Lambda^{-\frac{1}{4}}\|\varphi_2'\|_{L^\infty(J_\Lambda)} + \|\varphi_2\|_{L^\infty(J_\Lambda)} \le$$
$$\mathcal{O}(\Lambda^{-\infty})\|\varphi_1\|_{L^\infty(J_\Lambda)} + C(\ln \Lambda)^{-2}\Lambda^{-\frac{1}{2}}\|f_2\|_{L^\infty(J_\Lambda)}.$$

The assertion of the proposition, in the case of the positive outer region, now follows directly by combining (5.4) and the above relation. The corresponding estimate in the negative outer region follows completely analogously. $\square$

Armed with the above proposition and the observation made in Subsection 5.2, concerning the remainder left by $(w_{1,out}, w_{2,out})$, we can use the contraction mapping principle to capture the desired (φ_1, φ_2) in (5.1) and arrive at the main result of this section.

Proposition 5.2. *If $\Lambda > 0$ is sufficiently large, system (1.1), for $|z| \ge (\ln \Lambda)\Lambda^{-\frac{1}{4}}$, has a solution $(\mathbf{v}_{1,out}, \mathbf{v}_{2,out})$ of the form (5.1) with*

$$\|\varphi_1'\|_{L^\infty\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)} + \|\varphi_1\|_{L^\infty\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)} \le C(\ln \Lambda)^2\Lambda^{-\frac{3}{2}},$$

$$\|\varphi_2'\|_{L^\infty\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)} + \|\varphi_2\|_{L^\infty\left((\ln \Lambda)\Lambda^{-\frac{1}{4}}, \infty\right)} = \mathcal{O}(\Lambda^{-\infty}),$$

and the analogous estimate is valid for $z \le -(\ln \Lambda)\Lambda^{-\frac{1}{4}}$.

For future reference, we note that a standard barrier argument yields that

$$|\mathbf{v}_{i,out}(z)| \le \mathcal{O}(\Lambda^{-\infty})e^{-c(\ln \Lambda)^{\frac{1}{2}}\Lambda^{\frac{1}{4}}|z|}, \quad (-1)^i z \ge (\ln \Lambda)\Lambda^{-\frac{1}{4}}, \quad i = 1, 2. \tag{5.6}$$

6. Existence and nondegeneracy of the heteroclinic orbit: proof of Theorems 1.1 and 1.2

In this section, we will prove our main result. So far, we have solved exactly the system (1.1) in the inner zone $\left(-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right)$ by $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$, and in the outer zone $|z| > (\ln\Lambda)\Lambda^{-\frac{1}{4}}$ by its continuous extension $(\mathbf{v}_{1,out}, \mathbf{v}_{2,out})$. Furthermore, the asymptotic behaviour in (1.2) is satisfied. In other words, the continuous and piecewise smooth pair which is defined as

$$\mathbf{v}_{i,ap}(z) = \begin{cases} \mathbf{v}_{i,in}(z), & |z| \le (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \\ \mathbf{v}_{i,out}(z), & |z| \ge (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \end{cases} \quad i = 1, 2, \tag{6.1}$$

is an exact solution to problem (1.1)-(1.2) with the exception of the two points $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$. Loosely speaking, the above pair can be considered as a 'caricature' of the desired solution to (1.1)-(1.2) for large Λ. As we will see next, this vague notion can be made precise.

6.1. A gluing argument: The global approximate solution $(\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap})$.

We recall that we have constructed the solution $(\mathbf{v}_{1,in}, \mathbf{v}_{2,in})$ of the inner problem such that its Hamiltonian constant is equal to $-\psi_0^2/2$, which clearly is that of the aforementioned solutions of the outer problems. The main observation is that this implies that the jumps in the derivatives of $\mathbf{v}_{1,ap}, \mathbf{v}_{2,ap}$ are transcendentally small. Indeed, by the equality of the Hamiltonian constants at $(\ln\Lambda)\Lambda^{-\frac{1}{4}}$, and the fact that $\mathbf{v}_{i,in} = \mathbf{v}_{i,out}$, $i = 1, 2$, at that point, we have that

$$\left[\mathbf{v}'_{1,in}\right]^2 + \left[\mathbf{v}'_{2,in}\right]^2 = \left[\mathbf{v}'_{1,out}\right]^2 + \left[\mathbf{v}'_{2,out}\right]^2 \quad \text{at} \quad (\ln\Lambda)\Lambda^{-\frac{1}{4}}.$$

In turn, the corresponding estimates to (4.8), (4.10) for $\mathbf{v}_{1,in}$, and Proposition 5.2, yield that

$$\left[\mathbf{v}'_{1,in}\right]^2 - \left[\mathbf{v}'_{1,out}\right]^2 = \mathcal{O}\left(\Lambda^{-\infty}\right) \quad \text{at} \quad (\ln\Lambda)\Lambda^{-\frac{1}{4}}, \quad \text{as} \quad \Lambda \to \infty.$$

Consequently, since $\mathbf{v}'_{1,in}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) \ge c$ and $\mathbf{v}'_{1,out}\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) \ge c$ (keep in mind Propositions 4.1 and 5.2), we infer that

$$\left(\mathbf{v}'_{1,in} - \mathbf{v}'_{1,out}\right)\left((\ln\Lambda)\Lambda^{-\frac{1}{4}}\right) = \mathcal{O}\left(\Lambda^{-\infty}\right), \quad \text{as} \quad \Lambda \to \infty,$$

as desired. Naturally, the proof of the analogous property for the second components at $-(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ is completely analogous.

We are now ready to define our global C^1-smooth approximate solution to problem (1.1)-(1.2) as

$$\mathbf{w}_{i,ap}(z) = \mathbf{v}_{i,ap}(z) + (s_i)_- e^{-\Lambda^{\frac{1}{4}}\left|z + (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right|} + (s_i)_+ e^{-\Lambda^{\frac{1}{4}}\left|z - (\ln\Lambda)\Lambda^{-\frac{1}{4}}\right|}, \tag{6.2}$$

where the numbers

$$(s_i)_\pm = \mathcal{O}\left(\Lambda^{-\infty}\right) \quad \text{as} \quad \Lambda \to \infty, \quad i = 1, 2, \tag{6.3}$$

are chosen so that $\mathbf{w}_{i,ap}$, $i = 1, 2$, are C^1 at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$. This approximate solution leaves a remainder in (1.1) which is uniformly of order $\mathcal{O}\left(\Lambda^{-\infty}\right)$ (keep in mind, however, that it may have finite jump discontinuities at the two gluing points), while the asymptotic behaviour (1.2) as $z \to \pm\infty$ is fulfilled exactly.

6.2. Perturbing the global approximate solution to a genuine one. Even though $(\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap})$ is an extremely good approximate solution, perturbing it to a genuine one by some type of local inversion argument is a subtle task. Indeed, the associated linearized operator

$$\mathcal{M}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1'' + (3\mathbf{w}_{1,ap}^2 - 1)\varphi_1 + \Lambda \mathbf{w}_{2,ap}^2 \varphi_1 + 2\Lambda \mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\varphi_2 \\ -\varphi_2'' + (3\mathbf{w}_{2,ap}^2 - 1)\varphi_2 + \Lambda \mathbf{w}_{1,ap}^2 \varphi_2 + 2\Lambda \mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\varphi_1 \end{pmatrix} \tag{6.4}$$

is nearly non-invertible because $(\mathbf{w}_{1,ap}', \mathbf{w}_{2,ap}')$ is extremely close to being in the kernel. Nevertheless, we will overcome this difficulty by adapting to our setting a well known *variational Lyapunov-Schmidt method* (see [16] and the references therein).

Naturally, we seek a solution of (1.1)-(1.2) in the form

$$(v_1, v_2) = (\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap}) + (\varphi_1, \varphi_2), \tag{6.5}$$

with fluctuations satisfying the orthogonality condition

$$\int_{-\infty}^{\infty} \left(\mathbf{w}_{1,ap}'\varphi_1 + \mathbf{w}_{2,ap}'\varphi_2 \right) dz = 0. \tag{6.6}$$

The following proposition, concerning the so-called *linear projected problem*, makes it legitimate to apply the aforementioned Lyapunov-Schmidt method.

Proposition 6.1. *Given $\beta > 0$, there exist constants $\Lambda_3, C > 0$ such that if $\Lambda \geq \Lambda_3$ and $(h_1, h_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with $\|(h_1, h_2)\|_* < \infty$, the problem*

$$\mathcal{M}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + c_\Lambda \begin{pmatrix} \mathbf{w}_{1,ap}' \\ \mathbf{w}_{2,ap}' \end{pmatrix}, \tag{6.7}$$

where the linear operator $\mathcal{M}$ is as in (6.4), has a unique solution $(\phi_1, \phi_2) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $c_\Lambda \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \left(\mathbf{w}_{1,ap}'\phi_1 + \mathbf{w}_{2,ap}'\phi_2 \right) dz = 0 \tag{6.8}$$

and

$$\sum_{i=1}^{2} \|\phi_i\|_{L^\infty(\mathbb{R})} \leq C\Lambda^{\frac{\beta}{2}} \|(h_1, h_2)\|_*, \tag{6.9}$$

where $\|\cdot\|_*$ stands for the weighted norm

$$\|(h_1, h_2)\|_* = \sum_{i=1}^{2} \left\| \left(\Lambda^{\frac{1+\beta}{2}} |z|^{2+2\beta} + 1 \right) h_i \right\|_{L^\infty(\mathbb{R})}.$$

Proof. The proof will be divided into three steps.

Step 1. We will first establish the validity of the stronger a-priori estimate

$$\sum_{i=1}^{2} \|\phi_i\|_{L^\infty(\mathbb{R})} \leq C\Lambda^{-\frac{1}{2}} \|(h_1, h_2)\|_*, \tag{6.10}$$

when the constant c_Λ in (6.7) is equal to zero. To this end, as usual, we will argue by contradiction. So, as in Proposition 2.2, let us suppose that there are $\Lambda_n \to \infty$, $(\phi_{1,n}, \phi_{2,n}) \in$

$H^2(\mathbb{R}) \times H^2(\mathbb{R})$, $(h_{1,n}, h_{2,n}) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying

$$
\begin{cases}
-\phi_{1,n}'' + (3\mathbf{w}_{1,ap}^2 - 1)\phi_{1,n} + \Lambda_n \mathbf{w}_{2,ap}^2 \phi_{1,n} + 2\Lambda_n \mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\phi_{2,n} = h_{1,n}, \\
\\
-\phi_{2,n}'' + (3\mathbf{w}_{2,ap}^2 - 1)\phi_{2,n} + \Lambda_n \mathbf{w}_{1,ap}^2 \phi_{2,n} + 2\Lambda_n \mathbf{w}_{1,ap}\mathbf{w}_{2,ap}\phi_{1,n} = h_{2,n},
\end{cases}
\quad z \in \mathbb{R}, \quad (6.11)
$$

and

$$
\int_{-\infty}^{\infty} \left(\mathbf{w}_{1,ap}' \phi_{1,n} + \mathbf{w}_{2,ap}' \phi_{2,n} \right) dz = 0, \tag{6.12}
$$

while

$$
\|\phi_{1,n}\|_{L^\infty(\mathbb{R})} = 1, \quad \|\phi_{2,n}\|_{L^\infty(\mathbb{R})} \le 1, \quad \Lambda_n^{-\frac{1}{2}} \sum_{i=1}^{2} \| \left(\Lambda_n^{\frac{1+\beta}{2}} |z|^{2+2\beta} + 1 \right) h_{i,n}\|_{L^\infty(\mathbb{R})} \to 0. \tag{6.13}
$$

We will first show that

$$
|\phi_{i,n}(z)| \le 2 \frac{\|\phi_{i,n}\|_{L^\infty(\mathbb{R})}}{1 + \Lambda_n^{\frac{1+\beta}{2}} |z|^{2+2\beta}}, \quad (-1)^i z \ge \Lambda_n^{-\frac{1}{4}}, \quad i = 1, 2. \tag{6.14}
$$

To this end, the main observation is that, owing to the first equation in (6.11), $\phi_{1,n}$ satisfies a linear inhomogeneous equation of the form

$$
-\phi_1'' + p(z)\phi_1 = f(z), \quad z \le 0, \tag{6.15}
$$

with

$$
p(z) \ge c\Lambda_n^{\frac{1}{2}} \quad \text{and} \quad |f(z)| \le C \frac{\Lambda_n^{\frac{1}{2}}}{1 + \Lambda_n^{\frac{1+\beta}{2}} |z|^{2+2\beta}}, \quad z \le 0, \tag{6.16}
$$

(as usual, the generic constants $c, C > 0$ do not depend on n). The above relation follows readily from the already established properties of $\mathbf{w}_{1,ap}, \mathbf{w}_{2,ap}$ and (6.13). Let us just point out that special attention should be paid in showing that

$$
|\mathbf{w}_{1,ap}\mathbf{w}_{2,ap}| \le C\Lambda_n^{-\frac{1}{2}} e^{c\Lambda_n^{\frac{1}{4}} z}, \quad z \le -\Lambda_n^{-\frac{1}{4}},
$$

which essentially follows by combining the estimate

$$
c\Lambda^{-\frac{1}{4}} \le \mathbf{w}_{2,ap}(z) \le C(\Lambda^{-\frac{1}{4}} + |z|), \quad z \le 0,
$$

with the analog of (4.9) for $\mathbf{v}_{1,in}$, and (5.6). Analogous considerations apply to $\phi_{2,n}$. The desired estimate (6.14) follows from (6.15)-(6.16), the corresponding relations for $z \ge 0$, and a barrier argument (see also [22, pg. 435]).

With this preliminary step, we can now return to showing that a contradiction occurs. Using (6.11), (6.13), (6.14), together with standard elliptic estimates and the familiar diagonal argument, passing to a subsequence if necessary, we find that

$$
\phi_{i,n} \to \phi_{i,\infty} \quad \text{in} \quad C_{loc}^1\left(\mathbb{R} \setminus \{0\}\right), \tag{6.17}
$$

where

$$
\phi_{i,\infty}(z) = 0 \quad \text{for} \quad (-1)^i z > 0, \tag{6.18}
$$

while

$$
-\phi_{i,\infty}'' + \left(3U_i^2(z) - 1\right)\phi_{i,\infty} = 0, \quad (-1)^i z < 0, \quad i = 1, 2.
$$

Since $\phi_{i,\infty}$ is bounded for $z \ne 0$, by Lemma 2.1 we must have that

$$
\phi_{i,\infty}(z) = a_i U_i'(z), \quad (-1)^i z < 0, \quad i = 1, 2, \tag{6.19}
$$

for some $a_1, a_2 \in \mathbb{R}$ (the above a_1, a_2 are not related to those in Proposition 2.2). Moreover, passing to the limit in the orthogonality relation (6.8), with the help of Lebesgue's dominated convergence theorem, we obtain that

$$a_1 \int_0^\infty (U_1')^2 dz + a_2 \int_{-\infty}^0 (U_2')^2 dz = 0. \tag{6.20}$$

Next, motivated by the proof of Proposition 2.2, we wish to show that

$$|\phi_{i,n} - a_i U_i'| \leq \frac{C}{1 + \Lambda_n^{\frac{\beta}{2}} |z|^{2\beta}} + o(1), \quad i = 1, 2, \tag{6.21}$$

uniformly for $0 \leq (-1)^{i+1} z \leq 1$, as $n \to \infty$. We will show this only for $i = 1$, as the case $i = 2$ can be handled similarly. To this end, dropping for the moment some of the subscripts n to relax the notation, we let

$$\psi_1 = \phi_1 - a_1 U_1' \to 0 \quad \text{in} \quad C^1_{loc}(0, \infty), \tag{6.22}$$

and observe that

$$-\psi_1'' + (3\mathbf{w}_{1,ap}^2 - 1)\psi_1 + \Lambda_n \mathbf{w}_{2,ap}^2 \psi_1 = -2\Lambda_n \mathbf{w}_{1,ap} \mathbf{w}_{2,ap} \phi_2 + h_1 - a_1 \Lambda_n \mathbf{w}_{2,ap}^2 U_1'$$

$$-3a_1(\mathbf{w}_{1,ap}^2 - U_1^2)U_1',$$

for $z \geq 0$. It is convenient to write

$$\psi_1(z) = \tilde{\psi}_1(z) + \hat{\psi}_1(z) \quad \text{for} \quad z \in [0, 1], \tag{6.23}$$

where $\tilde{\psi}_1$ is the unique solution of the following boundary value problem:

$$\begin{cases} \tilde{\psi}_1'' = (3\mathbf{w}_{1,ap}^2 - 1)\psi_1 + 3a_1(\mathbf{w}_{1,ap}^2 - U_1^2)U_1', \quad z \in (0, 1), \\ \\ \tilde{\psi}_1(0) = \tilde{\psi}_1(1) = 0. \end{cases}$$

Concerning $\tilde{\psi}_1$, we observe that standard elliptic estimates imply that

$$\|\tilde{\psi}_1\|_{H^2(0,1)} \leq C\|(3\mathbf{w}_{1,ap}^2 - 1)\psi_1 + 3a_1(\mathbf{w}_{1,ap}^2 - U_1^2)U_1'\|_{L^2(0,1)} \to 0,$$

where the last limit holds by virtue of (6.22), the construction of $\mathbf{w}_{1,ap}$ and Lebesgue's dominated convergence theorem. Therefore,

$$\|\tilde{\psi}_1\|_{C^1(0,1)} \to 0. \tag{6.24}$$

Thus, in view of (6.22) and (6.24), we get that the function $\hat{\psi}_1$ in the decomposition (6.23) satisfies

$$\hat{\psi}_1 \to 0 \quad \text{in} \quad C^1_{loc}(0, 1].$$

Then, since $\hat{\psi}_1$ satisfies

$$\hat{\psi}_1'' = \Lambda_n \mathbf{w}_{2,ap}^2 \psi_1 + 2\Lambda_n \mathbf{w}_{1,ap} \mathbf{w}_{2,ap} \phi_2 - h_1 + a_1 \Lambda_n \mathbf{w}_{2,ap}^2 U_1', \quad z \in (0, 1),$$

similarly as in Proposition 2.2 (also keep in mind the derivation of (6.16)), we obtain that

$$|\hat{\psi}_1(z)| \leq \frac{C}{1 + \Lambda_n^{\frac{\beta}{2}} |z|^{2\beta}} + o(1), \quad \text{uniformly on} \quad [0, 1], \quad \text{as} \quad n \to \infty.$$

The desired estimate (6.21) now follows immediately from (6.22), (6.23), (6.24) and the above relation.

On the other side, similarly to the proof of Proposition 2.2, passing to a further subsequence if needed, we get that

$$\phi_i(\Lambda_n^{-\frac{1}{4}}x) \to \mathsf{b}V_i'(x) \quad \text{in} \quad C_{loc}^1(\mathbb{R}), \quad i = 1, 2, \tag{6.25}$$

for some $\mathsf{b} \in \mathbb{R}$.

By the same arguments that led to (2.36)-(2.37), we find from (6.21) and (6.25) that

$$a_1 = \mathsf{b}, \quad a_2 = \mathsf{b}.$$

Hence, in view of (6.20), we get that

$$a_1 = a_2 = \mathsf{b} = 0.$$

Then, by combining (6.14), (6.17), (6.18), (6.19), (6.21) and (6.25), we obtain that

$$\|\phi_{i,n}\|_{L^\infty(\mathbb{R})} \to 0, \quad i = 1, 2,$$

(note also that, as in Proposition 5.1, the points where $|\phi_{i,n}|$ achieves its maximum cannot escape at infinity). The above relation contradicts the first relation in (6.13), and completes the proof of Step 1.

Step 2. We will show that the a-priori estimate (6.10) holds for the full problem (6.7)-(6.8). Testing (6.7) by $(\mathbf{w}_{1,ap}', \mathbf{w}_{2,ap}')$ gives that

$$|c_\Lambda| \leq C\|(h_1, h_2)\|_* + C\left|\left\langle \mathcal{M}\left(\begin{array}{c}\phi_1 \\ \phi_2\end{array}\right), \left(\begin{array}{c}\mathbf{w}_{1,ap}' \\ \mathbf{w}_{2,ap}'\end{array}\right)\right\rangle_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}\right|, \tag{6.26}$$

where we also used that

$$|\mathbf{w}_{i,ap}'| \leq Ce^{-c|z|}, \quad z \in \mathbb{R}, \quad i = 1, 2. \tag{6.27}$$

Unfortunately, since $\mathbf{w}_{1,ap}', \mathbf{w}_{2,ap}'$ are merely in $H^1(\mathbb{R})$, we cannot use directly the self-adjointness of $\mathcal{M}$ in the last term of the above relation to exploit that

$$\left\|\mathcal{M}\left(\begin{array}{c}\mathbf{w}_{1,ap}' \\ \mathbf{w}_{2,ap}'\end{array}\right)\right\|_{L^1(\mathbb{R}\setminus\{\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\})\times L^1(\mathbb{R}\setminus\{\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\})} = \mathcal{O}(\Lambda^{-\infty}).$$

Nevertheless, from (6.2)-(6.3) and the fact that

$$\mathbf{v}_{i,ap}''\left(\left[\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]^-\right) = \mathbf{v}_{i,ap}''\left(\left[\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]^+\right)$$

(keep in mind (6.1)), we find that the jumps of $\mathbf{w}_{i,ap}''$ at $\pm(\ln\Lambda)\Lambda^{-\frac{1}{4}}$ are of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$, $i = 1, 2$. Hence, splitting the integral under consideration into three parts, integrating each one by parts using the self-adjointness of $\mathcal{M}$ and the previous observation to estimate the boundary terms, we reach the bound

$$\left\langle \mathcal{M}\left(\begin{array}{c}\phi_1 \\ \phi_2\end{array}\right), \left(\begin{array}{c}\mathbf{w}_{1,ap}' \\ \mathbf{w}_{2,ap}'\end{array}\right)\right\rangle_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} = \mathcal{O}(\Lambda^{-\infty})\sum_{i=1}^{2}\|\phi_i\|_{L^\infty(\mathbb{R})} \quad \text{as} \quad \Lambda \to \infty.$$

In turn, via (6.26), we obtain that

$$|c_\Lambda| \leq C\|(h_1, h_2)\|_* + \mathcal{O}(\Lambda^{-\infty})\sum_{i=1}^{2}\|\phi_i\|_{L^\infty(\mathbb{R})}.$$

On the other hand, applying the conclusion of Step 1 to (6.7)-(6.8), and keeping in mind (6.27), we get that

$$\sum_{i=1}^{2} \|\phi_i\|_{L^\infty(\mathbb{R})} \le C\Lambda^{-\frac{1}{2}}\|(h_1, h_2)\|_* + C\Lambda^{\frac{\beta}{2}}|c_\Lambda|.$$

The desired a-priori estimate now follows at once by combining the above two relations. We point out that the main reason for using a power weight instead of the more convenient exponential one, as in Proposition 2.2 and Corollary 4.1, was in order to get an efficient estimate in the last term of the above relation.

Step 3. We will establish the existence of a unique solution $(\phi_1, \phi_2) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $c_\Lambda \in \mathbb{R}$ to problem (6.7)-(6.8), given (h_1, h_2) as in the assertion of the proposition. Let $\mathcal{X}$ denote the subspace of $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ which consists of pairs $\Phi = (\phi_1, \phi_2)$ satisfying the orthogonality condition (6.8). The problem (6.7)-(6.8) admits the following weak formulation: find $\Phi \in \mathcal{X}$ such that

$$\langle \mathcal{M}(\Phi), \Psi \rangle_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} = \langle H, \Psi \rangle_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} \quad \forall\ \Psi \in \mathcal{X},$$

(where $H = (h_1, h_2)$). This weak formulation can then be readily put in the operator form

$$\mathbb{M}(\Phi) = \hat{H},$$

where $\mathbb{M} : \mathcal{X} \to \mathcal{X}$ is self-adjoint, and $\hat{H} \in \mathcal{X}$ depends linearly on H. The a-priori estimate of Step 2 implies that, for $\hat{H} = 0$, there is only the trivial solution. Consequently, by the self-adjoint property of $\mathbb{M}$, we infer that the above problem has a solution $\Phi \in \mathcal{X}$ (see also [27, Lem. 4.2]), which is clearly unique. This completes the proof of Step 3 and also of the proposition. $\qquad\square$

Armed with the above proposition, we can apply the contraction mapping theorem in these weighted spaces to show that the *nonlinear projected problem*

$$\begin{cases} -v_1'' + v_1^3 - v_1 + \Lambda v_2^2 v_1 = c_\Lambda \mathbf{w}_{1,ap}', \\ -v_2'' + v_2^3 - v_2 + \Lambda v_1^2 v_2 = c_\Lambda \mathbf{w}_{2,ap}', \end{cases}$$

has a solution (v_1, v_2) and c_Λ such that

$$v_i = \mathbf{w}_{i,ap} + \varphi_i \ \text{ with } \ \varphi_i \in H^2(\mathbb{R}) \ \text{ and } \ \|\varphi_i\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\Lambda^{-\infty}) \ \text{ as } \ \Lambda \to \infty, \ \ i = 1, 2. \quad (6.28)$$

Moreover, the fluctuation (φ_1, φ_2) satisfies the orthogonality condition (6.6), while the constant c_Λ is of order $\mathcal{O}(\Lambda^{-\infty})$ as $\Lambda \to \infty$. Then, elliptic regularity theory imply that the solution is smooth (up to this moment, we know that $v_1', v_2' \in H^2(\mathbb{R})$). To this end, testing the above nonlinear projected problem with (v_1', v_2'), thanks to the gradient structure in the lefthand side, yields that

$$\begin{aligned} 0 &= c_\Lambda \sum_{i=1}^{2} \int_{-\infty}^{\infty} \left[\left(\mathbf{w}_{i,ap}' \right)^2 + \mathbf{w}_{i,ap}' \varphi_i' \right] dz \\ &= c_\Lambda \sum_{i=1}^{2} \int_{-\infty}^{\infty} \left[\left(\mathbf{w}_{i,ap}' \right)^2 - \mathbf{w}_{i,ap}'' \varphi_i \right] dz. \end{aligned}$$

In turn, using the rough estimates

$$\int_{-\infty}^{\infty} \left(\mathbf{w}_{i,ap}' \right)^2 dz \ge c, \quad \left| \mathbf{w}_{i,ap}''(z) \right| \le C\Lambda^{\frac{1}{4}} e^{-c|z|}, \ z \in \mathbb{R}, \ i = 1, 2,$$

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and (6.28), we can conclude that $c_\Lambda = 0$ for Λ sufficiently large, as desired.

6.3. **Proof of Theorem 1.1.**

6.3.1. *The estimates.* The asserted estimates in Theorem 1.1 follow readily by taking into account the construction of the various approximate solutions, the estimates in Propositions 4.1, 5.2, and (6.28) (the latter relation can be differentiated once in the natural way). In particular, for the decay estimates (1.18) and (1.19), keep in mind (4.9) and (5.6) respectively.

6.3.2. *Positivity, monotonicity and decay properties of the heteroclinic orbit.* Armed with the previously proven C^1-uniform estimates for the constructed heteroclinic solution (v_1, v_2), we can complete the qualitative analysis of Subsubsection 4.1.1, and thus the proof of Theorem 1.1.

The main observation is that v_2 satisfies a linear equation of the form

$$-v'' + \tilde{P}(z)v = 0, \quad z \geq L\Lambda^{-\frac{1}{4}},$$

with

$$
\tilde{P}(z) \geq
\begin{cases}
c\Lambda z^2, & z \in (L\Lambda^{-\frac{1}{4}}, \delta), \\[2mm]
c\Lambda, & z \in [\delta, \infty),
\end{cases}
$$

for some fixed small $\delta > 0$ (keep in mind (4.7)). Hence, since $v_2(L\Lambda^{-\frac{1}{4}}) \geq c\Lambda^{-\frac{1}{4}}$, $v_2'(L\Lambda^{-\frac{1}{4}}) \leq -c$ for large Λ (recall (4.6)) and $\lim_{z\to\infty} v_2(z) = 0$, we deduce by the maximum principle that

$$v_2 > 0 \quad \text{and} \quad v_2' < 0 \quad \text{on} \ [L\Lambda^{-\frac{1}{4}}, \infty).$$

In summary, so far we have shown that

$$v_2 > 0 \quad \text{and} \quad v_2' < 0 \quad \text{on} \ \left[-(\ln\Lambda)\Lambda^{-\frac{1}{4}}, \infty\right). \tag{6.29}$$

In fact, by the use of barriers and standard elliptic estimates, it follows readily that

$$v_2(z) - \Lambda^{-\frac{1}{4}}v_2'(z) \leq C\Lambda^{-\frac{1}{4}}e^{-c\Lambda^{\frac{1}{4}}z}, \quad z \geq 0. \tag{6.30}$$

Moreover, the above estimate can be improved for large z: Given any fixed $d > 0$,

$$v_2(z) - \Lambda^{-\frac{1}{2}}v_2'(z) \leq Cv_2(d)e^{-c\Lambda^{\frac{1}{2}}z}, \quad z \geq d.$$

The previously proven C^1-uniform estimates for the convergence of v_2 to U_2 over $\left(-\infty, -(\ln\Lambda)\Lambda^{-\frac{1}{4}}\right]$ guarantee that the same holds on any fixed interval of the form $[-M, \infty)$, provided that Λ is sufficiently large. In particular, $v_2(-M) \to U_2(-M)$ and $v_2'(-M) \to U_2'(-M) < 0$ as $\Lambda \to \infty$. To conclude that v_2 is still decreasing in $(-\infty, -M)$, it is enough to apply the maximum principle to the linear equation that is satisfied by v_2'. Indeed, using the analogous property to (6.29) for v_1, and our previous observations at $-M$, we find that the function $\psi \equiv v_2'$ satisfies

$$-\psi'' + \left(3v_2^2 - 1 + \Lambda v_1^2\right)\psi = -2\Lambda v_2 v_1 v_1' \leq 0, \quad z \leq -M; \quad \psi(-\infty) = 0, \ \psi(-M) < 0,$$

with $3v_2^2 - 1 + \Lambda v_1^2 > 0$ on $(-\infty, -M]$ (having increased the value of M if necessary).

Analogously we argue for showing that

$$v_1' > 0 \quad \text{in} \ \mathbb{R}.$$

Remark 6.1. *A careful inspection of the proofs reveals that the solution provided by Theorem 1.1 depends smoothly on Λ since there is a version of the contraction mapping theorem for operators depending on parameters.*

Remark 6.2. *An effective approach for constructing heteroclinic orbits in singularly perturbed systems of ordinary differential equations is to make use of geometric singular perturbation theory (see [26] and the references therein). In particular, at least heuristically, the blow-up problem (1.10) brings to mind the recent blow-up approach to this theory, used to deal with problems involving loss of normal hyperbolicity (see [32] and the references therein). However, we have not been able to put system (1.1) in the slow-fast form that is required for the aforementioned machinery to apply. In any case, we believe that the approach of the current paper extends in a natural way to deal with analogous problems in the broader context of elliptic systems of partial differential equations.*

6.4. **Nondegeneracy of the heteroclinic: Proof of Theorem 1.2.**

Proof of Theorem 1.2. It has been shown in [4] that the lowest point in the spectrum of $\mathbf{M}$ is 0 which is a simple eigenvalue with (v_1', v_2') as the associated eigenfunction. It was also shown therein that the continuous spectrum of $\mathbf{M}$ coincides with the interval $[\Lambda^2 - 1, \infty)$. So, it is enough to show that the second eigenvalue $\mu > 0$ of $\mathbf{M}$ (should it exist) is bounded away from 0 independently of large Λ. To this end, we will argue by contradiction.

Suppose, to the contrary, that there are $\Lambda_n \to \infty$ such that the second eigenvalue $\mu_n > 0$ of $\mathbf{M}$ exists and satisfies

$$\mu_n \to 0.$$

Hence, there would exist an associated eigenfunction $(\varphi_{1,n}, \varphi_{2,n}) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ such that

$$\begin{cases} -\varphi_{1,n}'' + (3v_1^2 - 1)\varphi_{1,n} + \Lambda_n v_2^2 \varphi_{1,n} + 2\Lambda_n v_1 v_2 \varphi_{2,n} = \mu_n \varphi_{1,n}, \\[2mm] -\varphi_{2,n}'' + (3v_2^2 - 1)\varphi_{2,n} + \Lambda_n v_1^2 \varphi_{2,n} + 2\Lambda_n v_1 v_2 \varphi_{1,n} = \mu_n \varphi_{2,n}, \end{cases} \quad z \in \mathbb{R},$$

$$\|\varphi_{1,n}\|_{L^\infty(\mathbb{R})} + \|\varphi_{2,n}\|_{L^\infty(\mathbb{R})} = 1,$$

and

$$\int_{-\infty}^{\infty} (v_1' \varphi_{1,n} + v_2' \varphi_{2,n}) \, dz = 0.$$

Then, by absorbing $\mu_n \varphi_{i,n}$ in the term $(3v_i^2 - 1)\varphi_{i,n}$, $i = 1, 2$, the proof of Step 1 in Proposition 6.1 goes through to provide a contradiction. $\qquad\square$

7. UNIQUENESS OF SOLUTIONS: PROOF OF THEOREM 1.3

In this section, we will prove Theorem 1.3. The main task will be to establish the uniqueness (modulo translations) of solutions to (1.1)-(1.2), satisfying the natural monotonicity property

$$v_1'(z) > 0, \quad v_2'(z) < 0, \quad z \in \mathbb{R}, \tag{7.1}$$

for any Λ in the range (1.3). In particular, the latter monotonicity property is satisfied by stable solutions with positive components (see [4, Thm. 3.1]), and thus by minimizing ones. To the best of our knowledge, this type of uniqueness was not previously known, even in the case of minimizing solutions (see also [4, Rem. 1.2] and [20, Rem. 4.8]). Once the aforementioned uniqueness property is established, the corresponding assertion of Theorem

1.3, where only one of the inequalities in (7.1) is assumed, will follow immediately thanks to Lemma 7.1 below. We point out that system (1.1) is non cooperative, and that in the case $\Lambda < 1$, uniqueness for a related problem follows from [1].

The main result of this section is the following.

Proposition 7.1. *If $\Lambda > 1$, there exists a unique solution (modulo translations) to (1.1)-(1.2)-(7.1).*

Proof. The proof is based on the key observation that uniqueness holds for $\Lambda = 3$ (see [4] and the references therein) and a continuation argument.

Throughout this proof, we shall assume the 'pinning' condition:
$$v_1(0) = v_2(0). \tag{7.2}$$

Firstly, and for future reference, we note that any solution of (1.1)-(1.2) with $\Lambda > 1$ satisfies
$$v_1^2 + v_2^2 < 1, \quad z \in \mathbb{R}, \tag{7.3}$$
(see [4, Thm. 2.4]).

We next claim that the following localization property holds: Let $\underline{\lambda} > 1$ and $\varepsilon > 0$, then there exists $M > 0$ such that any solution of (1.1)-(1.2)-(7.2) with $\Lambda \geq \underline{\lambda}$ such that
$$v_1'(z) \geq 0, \quad v_2'(z) \leq 0, \quad z \in \mathbb{R}, \tag{7.4}$$
satisfies
$$1 - v_1(z) + v_2(z) < \varepsilon \quad \text{for} \quad z \geq M, \tag{7.5}$$
and the analogous relation for $z \leq -M$. Indeed, in view of the conservation of the Hamiltonian, it is enough to verify that, given $\epsilon > 0$, there exists $L > 0$ so that any such solution satisfies
$$v_1'(z_0) - v_2'(z_0) < \epsilon \quad \text{for some} \quad z_0 \in [0, L].$$
If not, for any $L > 0$, there would exist at least one such solution satisfying
$$v_1'(z) - v_2'(z) \geq \epsilon \quad \text{for } z \in [0, L],$$
i.e.,
$$v_1(L) - v_2(L) \geq \epsilon L,$$
which is clearly not possible for large L by virtue of (7.3) and proves the claim.

In turn, similarly to [10, Thm. 2.8], for any $1 < \underline{\lambda} < \bar{\lambda}$, there exist constants $c, C > 0$ such that any solution of (1.1)-(1.2)-(7.2)-(7.4) with $\Lambda \in [\underline{\lambda}, \bar{\lambda}]$ satisfies
$$\sum_{i=1}^{2} \{|v_i''| + |v_i'| + |v_i - 2 + i|\} \leq Ce^{-cz}, \quad z \geq M, \tag{7.6}$$
and the analogous estimate for $z \leq -M$.

The previous observations have the following interesting implication: Let $(v_{1,n}, v_{2,n})$ be a sequence of solutions of (1.1)-(1.2)-(7.1)-(7.2) with $\Lambda = \Lambda_n$, such that
$$v_{i,n} - v_{i,\infty} \to 0 \text{ in } H^2(\mathbb{R}), \quad i = 1, 2, \text{ and } \Lambda_n \to \Lambda_\infty \in (1, \infty).$$
Then, the limit $(v_{1,\infty}, v_{2,\infty})$ satisfies (1.1)-(1.2)-(7.1)-(7.2) with $\Lambda = \Lambda_\infty$. Indeed, since $C^1(\mathbb{R})$ is continuously imbedded into $H^2(\mathbb{R})$, without loss of generality, it is enough to exclude the scenario where
$$v_{1,\infty}'(z_*) = 0 \quad \text{for some } z_* \in \mathbb{R}. \tag{7.7}$$

To this end, we note that $\varphi \equiv v'_{1,\infty} \geq 0$ satisfies
$$-\varphi'' + P(z)\varphi = -2\Lambda_\infty v_{1,\infty} v_{2,\infty} v'_{2,\infty} \geq 0, \quad z \in \mathbb{R},$$
for some smooth function P. Thus, the above scenario (7.7) cannot happen, as it would violate a version of Hopf's boundary point lemma (see for example [31, Thm. 2.8.4]).

It follows from [4, Thm. 3.1] that the linearized operator of (1.1) about a solution of (1.1)-(1.2)-(7.1)-(7.2), that is $\mathbf{M}$ in (1.22) with $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as its domain, has a one-dimensional kernel spanned by (v'_1, v'_2). We also note that this linear operator is self-adjoint in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and its continuous spectrum is contained in $[\Lambda^2 - 1, \infty)$ (see again [4]). Therefore, by the variational Lyapunov-Schmidt procedure of Proposition 6.1 (in a regular perturbation setting) or a dynamical systems approach (see [33]), and the observation in the previous paragraph, we deduce the following: Each solution $(v_{1,\Lambda_0}, v_{2,\Lambda_0})$ of (1.1)-(1.2)-(7.1)-(7.2), for some $\Lambda_0 > 1$, is contained in a locally unique and smooth for $|\Lambda - \Lambda_0|$ sufficiently small (with respect to variations from $(v_{1,\Lambda_0}, v_{2,\Lambda_0})$ in the $H^2(\mathbb{R}) \times H^2(\mathbb{R})$-norm) branch of solutions of (1.1)-(1.2)-(7.1)-(7.2). In fact, if the aforementioned local uniqueness property failed, there would be such a solution with the associated linearized operator having a nontrivial element (Z_1, Z_2) in its kernel such that $Z_1(0) = Z_2(0)$, which is impossible by the opposite sign of v'_1 and v'_2. We observe next that, thanks to (7.3), (7.5) and (7.6), any solution to (1.1)-(1.2)-(7.1)-(7.2) with $\Lambda \in [\underline{\lambda}, \bar{\lambda}]$ satisfies
$$\|v_1 - v_{1,\Lambda_0}\|_{H^2(\mathbb{R})} + \|v_2 - v_{2,\Lambda_0}\|_{H^2(\mathbb{R})} \leq C,$$
where C depends only on $\underline{\lambda}, \bar{\lambda} > 1$. Therefore, the aforementioned solution branch of (1.1)-(1.2)-(7.1)-(7.2) can be extended smoothly and uniquely for all $\Lambda > 1$.

As was mentioned in the beginning of the proof, it has been observed that for $\Lambda = 3$ there exists a unique solution of (1.1)-(1.2)-(7.1)-(7.2); in fact, this solution can be found explicitly. Indeed, letting $u \equiv v_1 + v_2$ yields that
$$u'' + u - u^3 = 0, \quad z \in \mathbb{R}; \quad u \to 1, \quad z \to \pm\infty,$$
that is $u \equiv 1$ and the aforementioned uniqueness follows at once. Hence, in the case where there was non-uniqueness of solutions to (1.1)-(1.2)-(7.1)-(7.2) for some $\Lambda > 1$, we would have two of the previously described solution branches meeting at some $\Lambda_* > 1$. However, this is not possible from the local uniqueness of the solution branches. $\qquad\square$

Remark 7.1. *We note that, starting the above continuation argument from $\Lambda = 3$, yields a non-variational proof of existence of the heteroclinic solution.*

Concerning the monotonicity condition (7.1), we have the following interesting property which is motivated from [17], where the PDE version of system (1.10) was considered and the concept of *half-monotone* solutions was introduced.

Lemma 7.1. *Assume that (v_1, v_2) is a solution to (1.1)-(1.2) with positive components. Then, $v'_1 > 0$ implies that $v'_2 < 0$, and vice versa.*

Proof. Let us assume that
$$v'_1 > 0, \tag{7.8}$$
(the other case can be treated completely analogously). We note that, in light of (7.3), there exist sequences $\{z_n^\pm\}$ with $z_n^\pm \to \pm\infty$ such that
$$v'_2(z_n^\pm) < 0 \quad \text{for} \quad n \gg 1. \tag{7.9}$$

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Then, in analogy to [17], we let
$$\sigma = \frac{v_2'}{v_2}.$$
A simple computation, using (1.1) and (7.8), gives that
$$(v_2^2 \sigma')' > 2 v_2^4 \sigma, \quad z \in \mathbb{R}.$$
Hence, in view of (7.9), we deduce by the maximum principle that $\sigma < 0$ on $[z_n^-, z_n^+]$ for $n \gg 1$, and the lemma follows. $\qquad \square$

8. Asymptotic behaviour of the energy: Proof of Corollary 1.1

Proof of Corollary 1.1. In view of (1.15), Theorem 1.3 and the discussion in the beginning of Section 7, we infer that the solution in Theorem 1.1 is the only minimizer (modulo translations) with positive components to the problem (1.23). (Actually, a simple reflection argument, using $(|v_1|, |v_2|)$ as a competitor in the energy, yields that minimizers necessarily have positive components). Thus, in order to verify the assertion of Corollary 1.1, it is enough to estimate the energy of the aforementioned solution. For this purpose, a very helpful observation is that, thanks to the conservation of the hamiltonian, we have
$$E_\Lambda(v_1, v_2) = \int_{\mathbb{R}} \left[(v_1')^2 + (v_2')^2 \right] dz.$$
Firstly, making use of (1.20), we find that
$$\int_{(\ln \Lambda) \Lambda^{-\frac{1}{4}}}^{\infty} (v_1')^2 dz = \int_{(\ln \Lambda) \Lambda^{-\frac{1}{4}} + \psi_0^{-1} \kappa \Lambda^{-\frac{1}{4}}}^{\infty} \left[U_1'(s) \right]^2 ds + \mathcal{O}\left((\ln \Lambda) \Lambda^{-\frac{3}{4}} \right)$$
as $\Lambda \to \infty$. In turn, exploiting the fact that $U_1'(s) = \psi_0 + \mathcal{O}(s^2)$ for $0 \le s \le 1$, we can write
$$\int_{(\ln \Lambda) \Lambda^{-\frac{1}{4}}}^{\infty} (v_1')^2 dz = \int_0^{\infty} \left[U_1'(s) \right]^2 ds - \psi_0^2 (\ln \Lambda) \Lambda^{-\frac{1}{4}} - \psi_0 \kappa \Lambda^{-\frac{1}{4}} + \mathcal{O}\left((\ln \Lambda)^3 \Lambda^{-\frac{3}{4}} \right)$$
as $\Lambda \to \infty$. On the other side, we obtain from (1.21) and (1.19) respectively that
$$\int_{-(\ln \Lambda) \Lambda^{-\frac{1}{4}}}^{(\ln \Lambda) \Lambda^{-\frac{1}{4}}} (v_1')^2 dz = \Lambda^{-\frac{1}{4}} \int_{-(\ln \Lambda)}^{(\ln \Lambda)} (V_1')^2 dx + \mathcal{O}\left((\ln \Lambda)^3 \Lambda^{-\frac{3}{4}} \right)$$
and
$$\int_{-\infty}^{-(\ln \Lambda) \Lambda^{-\frac{1}{4}}} (v_1')^2 dz = \mathcal{O}\left(\Lambda^{-\infty} \right) \quad \text{as} \quad \Lambda \to \infty.$$
By adding the above three relations, we arrive at
$$
\begin{aligned}
\int_{-\infty}^{\infty} (v_1')^2 dz &= \int_0^{\infty} \left[U_1'(s) \right]^2 ds + \Lambda^{-\frac{1}{4}} \left\{ \int_{-(\ln \Lambda)}^{0} (V_1')^2 dx + \int_0^{(\ln \Lambda)} \left[(V_1')^2 - \psi_0^2 \right] dx - \psi_0 \kappa \right\} \\
&\quad + \mathcal{O}\left((\ln \Lambda)^3 \Lambda^{-\frac{3}{4}} \right).
\end{aligned}
$$
Obviously, the righthand side increases negligibly if we replace the ends of integration $\pm \ln \Lambda$ with $\pm \infty$ respectively (keep in mind that V_1 is convex and that relation (1.13) can be differentiated). The completely analogous relation holds for the second component. Therefore,

observing that

$$\int_{-\infty}^0 (V_1')^2 dx + \int_0^\infty \left[(V_1')^2 - \psi_0^2 \right] dx = \int_{-\infty}^\infty V_1' \left(V_1' - \psi_0 \right) dx,$$

it remains to verify that

$$\int_{-\infty}^0 \left[U_2'(s) \right]^2 ds + \int_0^\infty \left[U_1'(s) \right]^2 ds = \frac{2\sqrt{2}}{3}, \tag{8.1}$$

(recall also the symmetry property (1.12)). This is indeed the case, as one can determine explicitly the value of each one of the above integrals (in fact, they are equal by symmetry), thanks to the conservation of the hamiltonian of problems (1.6), (1.7) (see for example [20, Lem. 4.1]); we leave the details to the reader. □

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