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ON THE GROWTH OF THE ENERGY OF ENTIRE SOLUTIONS TO THE VECTOR ALLEN-CAHN EQUATION

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Abstract. We prove that the energy over balls of entire, nonconstant bounded solutions to the vector Allen-Cahn equation grows faster than $(\ln R)^k R^{n-2}$, for any $k > 0$, as the radius R of the n-dimensional ball tends to infinity. This improves the growth rate of order R^{n-2} if $n \geq 3$ and $\ln R$ if $n = 2$ that follows from the general weak monotonicity formula. Moreover, our estimate may be considered as an approximation to the corresponding rate of order R^{n-1} that is known to hold in the scalar case.

1. **Introduction and statement of the main result.** Consider the semilinear elliptic system

$$\Delta u = W_u(u) \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where the "potential" $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 2$, is sufficiently smooth and *nonnegative*. This system has variational structure, as solutions (in a smooth, bounded domain $\Omega \subset \mathbb{R}^n$) are critical points of the energy

$$E(v; \Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dy$$

(subject to their own boundary conditions), where $|\nabla v|$ is the Euclidean norm of the $m \times n$ matrix ∇v.

In the scalar case, namely $m = 1$, Modica [18] used the so-called P-function technique [23] and intrinsically scalar arguments to show that every entire (i.e. defined in all of $\mathbb{R}^n$), bounded solution to (1) satisfies the pointwise gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \quad \text{in} \quad \mathbb{R}^n, \tag{2}$$

(see also [7] and [12]). Using this, together with Pohozaev type identities, it was subsequently shown in [19] that such solutions satisfy the following *strong monotonicity formula*:

$$\frac{d}{dr}\left(\frac{1}{r^{n-1}} \int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy \right) \geq 0, \quad r > 0, \ x \in \mathbb{R}^n, \tag{3}$$

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where $B(x,r)$ stands for the n-dimensional ball of radius r that is centered at x. In particular, it follows that each entire, bounded and nonconstant solution to the scalar problem satisfies

$$\int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy \geq cr^{n-1}, \quad r>0, \; x \in \mathbb{R}^n, \tag{4}$$

for some constant $c>0$ (depending on x).

In the vector case, that is $m \geq 2$, the analog of the gradient bound (2) does not hold in general, as is indicated by the counterexamples in [11] (see also the comments after (7) herein) and [22] for the cases (7) and (8) respectively. In passing, let us mention that if a solution u satisfied the analog of the gradient bound (2) (see [14] for examples), then it would also satisfy the analog of the strong monotonicity formula (3) (see [1, 25] or Appendix A in [26]). All is not lost, however, as using the fact that every entire solution to (1) satisfies the *weak monotonicity formula*:

$$\frac{d}{dr}\left(\frac{1}{r^{n-2}} \int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy \right) \geq 0, \quad r>0, \; x \in \mathbb{R}^n, \tag{5}$$

(see [1, 5, 9, 21, 25]), and with some more work in the case $n=2$ (see for instance [1, 20]), it follows readily that, given $x \in \mathbb{R}^n$, each nonconstant, entire solution to the system (1) satisfies:

$$\int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy \geq \begin{cases} cr^{n-2} & \text{if } n \geq 3, \\ c\ln r & \text{if } n=2, \end{cases} \tag{6}$$

for all $r>1$ and some constant $c>0$.

Let us mention that the above lower bound is sharp in the case $n=2$. Indeed, for the Ginzburg-Landau system

$$\Delta u = \left(|u|^2-1\right)u, u:\mathbb{R}^n \to \mathbb{R}^m, \text{ (here } W(u)=\frac{\left(1-|u|^2\right)^2}{4} \text{ vanishes on } \mathbb{S}^{m-1}), \tag{7}$$

arising in superconductivity, there are entire bounded solutions, characterized as degree-one vortex solutions, with energy over $B(0,r)$ of order equal to a constant multiple of $\ln r$ as $r \to \infty$, if $n=m=2$ (see [5, 10, 11] and the references therein).

In this note, we will restrict ourselves to the class of potentials that satisfy the following properties: $W \in C^2$ and there exist $N \in \mathbb{N}$ points $a_i \in \mathbb{R}^m$ such that

$$W(a_i)=0, \;\; W>0 \;\text{ in }\; \mathbb{R}^m \setminus \{a_1,\cdots,a_N\} \;\text{ and }\; W_{uu}(a_i)\nu\cdot\nu>0 \;\; \forall\, \nu \in \mathbb{S}^{m-1}, \tag{8}$$

$i=1,\cdots,N$, where $\cdot$ stands for the Euclidean inner product in $\mathbb{R}^m$. For this class of potentials, the system (1) is known as the *vector Allen-Cahn equation* and models multiphase transitions (see [3, 6]).

In this case, it was shown recently in [2], by extending the density estimates of [8] to this vector setting, that nonconstant, entire, bounded *minimal* solutions satisfy the analog of (4), that is

$$\liminf_{r\to\infty} \frac{1}{r^{n-1}} \int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy > 0, \quad \forall\, x \in \mathbb{R}^n,$$

(on the other side, under these assumptions, it is easy to show that the quotient inside the above limit is bounded in r). For a simple proof of this result when $n=2$, under weaker assumptions on W, we refer to [24]; in fact, this result says that the first integral, involving the gradient of u, can actually be dropped from the above relation if $n = 2$. More recently, the above lower bound was used in [26] to give a new proof and extensions of the main result of [13] concerning such solutions. In comparison, let us note that the solution mentioned previously in relation to (7) is minimal (see for example the introduction of [10]). For completeness, let us recall, following [2, 13], that an entire solution u of (1) is called minimal if

$$E(u;\Omega) \leq E(u+\varphi;\Omega) \quad \forall\, \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m),$$

for any bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. On the other side, if u is a nonconstant solution which is periodic in each variable, as in [4] for the vector Allen-Cahn equation (the previously mentioned counterexample to (2) of [22] belongs to this class of solutions) or [17] for the Ginzburg-Landau system (7), it is easy to see that

$$\liminf_{r\to\infty} \frac{1}{r^n} \int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy > 0, \quad \forall\, x \in \mathbb{R}^n.$$

Moreover, it was shown recently in [24] that the above relation also holds for nonconstant, entire, radial solutions, as the ones in [16] for the scalar Allen-Cahn equation $\Delta u = u^3 - u$.

It is therefore natural to ask what can be said, besides of the lower bound (6), about the energy growth of arbitrary nonconstant, entire and bounded solutions to the vector Allen-Cahn equation. Our main result is the following improvement of the lower bound (6) for this class of systems.

Theorem 1.1. *Assume that $W \in C^2(\mathbb{R}^m;\mathbb{R})$, $m \geq 2$, satisfies (8). Let $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$, $n \geq 2$, be a nonconstant, entire and bounded solution to the elliptic system (1). Then, for any $x \in \mathbb{R}^n$ and $k > 0$, it holds that*

$$\frac{1}{(\ln r)^k} \frac{1}{r^{n-2}} \int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy \to \infty \quad \text{as} \quad r \to \infty. \tag{9}$$

Our result implies that, in contrast to the Ginzburg-Landau system (7), the lower growth rate in (6) cannot be achieved for any nonconstant, entire and bounded solution of the vector Allen-Cahn equation, for any $n, m \geq 2$. Moreover, it can be considered as an approximation, in some sense, to the corresponding lower bound (4) that holds in the scalar case and in the case of minimal solutions.

Our proof of Theorem 1.1 will be based on the weak monotonicity formula (5) and the following lemma from [21].

Lemma 1.2. *Assume that $W \in C^1(\mathbb{R}^m;\mathbb{R})$, $m \geq 2$, is nonnegative and that $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$ is a nonconstant, entire solution to (1). Then, for every $x \in \mathbb{R}^n$ and any positive numbers $R_0 < R_1$, there exists $r(x) \in (R_0, R_1)$ such that*

$$\begin{aligned} & r(x) \int_{\partial B(x,r(x))} \left| \frac{\partial u}{\partial \nu} \right|^2 dS + 2 \int_{B(x,r(x))} W(u) dy \\ & \leq \frac{1}{\ln(R_1/R_0)} \ln \left(\frac{\tilde{E}(u,x,R_1)}{\tilde{E}(u,x,R_0)} \right) \int_{B(x,r(x))} e(u) dy, \end{aligned} \tag{10}$$

where ν stands for the outer unit normal vector to $\partial B(x, r(x))$, $\frac{\partial u}{\partial \nu} = (\nabla u)\nu \in \mathbb{R}^m$,

$$e(u) \equiv \frac{1}{2}|\nabla u|^2 + W(u)$$

denotes the energy density of u*, and*

$$\tilde{E}(u,x,r) \equiv \frac{1}{r^{n-2}} \int_{B(x,r)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dy.$$

This lemma was proven in [21] (see Proposition 25 therein) for the special case of the Ginzburg-Landau system (7), but the proof carries over verbatim to the general case. It is based on the identity

$$\frac{d}{dr}\left(\tilde{E}(u,x,r)\right) = \frac{1}{r^{n-2}} \int_{\partial B(x,r)} \left|\frac{\partial u}{\partial \nu}\right|^2 dS + \frac{2}{r^{n-1}} \int_{B(x,r)} W(u)dy, \tag{11}$$

satisfied by entire solutions of (1), which implies at once the weak monotonicity formula (5) for nonnegative W and is a direct consequence of Pohozaev type identities for systems (see [27] for a complete presentation of these identities in the case of general W). Note also that, in contrast to [21], we have included the boundary integral in the lefthand side of (10) (we have also kept the factor 2). Roughly, the main idea behind (10) is to write (11) as $\frac{d}{dr}f(r) = \frac{1}{r}g(r)$ (with the obvious choices of f, g) and then to desingularize this ordinary differential equation by means of the classical change of variables $s = \ln r$.

The rest of this article is devoted to the proof of Theorem 1.1.

2. **Proof of the main result.**

Proof of Theorem 1.1. Since the problem is translation invariant, without loss of generality, we will carry out the proof for $x = 0$.

Firstly, we note that, since u is bounded, by standard elliptic regularity theory (see [15]), there exists a constant $C_0 > 0$ such that

$$|u| + |\nabla u| \leq C_0 \quad \text{in} \quad \mathbb{R}^n. \tag{12}$$

We will argue by contradiction. So, let us suppose that the assertion (9) does not hold. Then, there would exist constants $k, C_1 > 1$ and a sequence $R_j \to \infty$ such that

$$\tilde{E}(u,0,R_j) \leq C_1(\ln R_j)^k, \quad j \geq 1. \tag{13}$$

On the other side, thanks to (6), we have that

$$\tilde{E}(u,0,R_j^{\frac{1}{2}}) \geq C_2, \quad j \geq 1, \tag{14}$$

for some constant $C_2 > 0$ (we may assume that $C_2 < C_1$). In passing, we note that our motivation for the power $1/2$ comes mainly from the proof of the η-compactness lemma in [20]. By Lemma 1.2, there exist

$$r_j \in (R_j^{\frac{1}{2}}, R_j) \tag{15}$$

such that

$$
\begin{aligned}
r_j \int_{\partial B(0,r_j)} \left|\frac{\partial u}{\partial \nu}\right|^2 dS + 2\int_{B(0,r_j)} W(u)dy \leq & \frac{2}{\ln R_j} \ln\left(\frac{\tilde{E}(u,0,R_j)}{\tilde{E}(u,0,R_j^{\frac{1}{2}})}\right) \int_{B(0,r_j)} e(u)dy \\
\text{using (13), (14)} \leq & \frac{2}{\ln R_j} \ln\left(\frac{C_1(\ln R_j)^k}{C_2}\right) r_j^{n-2}\tilde{E}(u,0,r_j) \\
\text{using (5), (15)} \leq & \frac{2}{\ln R_j} \ln\left(\frac{C_1(\ln R_j)^k}{C_2}\right) r_j^{n-2}\tilde{E}(u,0,R_j) \\
\text{using (13)} \leq & \frac{2}{\ln R_j} \ln\left(\frac{C_1(\ln R_j)^k}{C_2}\right) C_1 r_j^{n-2}(\ln R_j)^k \\
\text{since } [\ln(\ln R)]\,(\ln R)^{-\frac{1}{2}} \to 0 \text{ as } R\to\infty \leq & C_3(\ln R_j)^{k-\frac{1}{2}} r_j^{n-2} \\
\text{using (15)} \leq & C_4(\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\end{aligned} \tag{16}
$$

for some constants $C_3, C_4 > 0$ and all $j \gg 1$.

What we want to do next is to bound the integral of $|\nabla u|^2$ over $B(0,r_j)$. To this end, we consider an auxiliary mapping $F \in C^\infty(\mathbb{R}^m;\mathbb{R}^m)$ such that

$$
F(v) = v - a_i \quad \text{if} \quad |v-a_i| \leq \delta, \quad i = 1,\cdots,N, \tag{17}
$$

for some small $\delta > 0$ (so that the 4δ-neighborhoods of the a_i's are disjoint). We note that such a function can be easily constructed as follows: First, consider a smooth cutoff function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

$$
\zeta(t) = \begin{cases} 1, & |t| \leq \delta, \\ 0, & |t| \geq 2\delta, \end{cases}
$$

and then let

$$
F(v) = \sum_{i=1}^{N} \zeta\left(|v-a_i|\right)(v-a_i), \quad v \in \mathbb{R}^m.
$$

For future reference, let us note at this point that, by virtue of (8), the Taylor expansion of W, W_u near each a_i (this is the only place where we need that $W \in C^2$), and (12), there exists a constant $C_5 > 0$ such that

$$
|F(u)\cdot W_u(u)| \leq C_5 W(u), \quad x \in \mathbb{R}^n. \tag{18}
$$

Taking the inner product of (1) with $F(u)$ and integrating by parts the resulting identity over $B(0,r_j)$ yields that

$$
\begin{aligned}
\int_{B(0,r_j)} \operatorname{tr}\left[(\nabla u)\left(F_u(u)\nabla u\right)^\top\right] dy = & \int_{\partial B(0,r_j)} \frac{\partial u}{\partial \nu}\cdot F(u)dS - \int_{B(0,r_j)} F(u)\cdot W_u(u)dy \\
\text{using (12), (18)} \leq & C_6 r_j^{\frac{n-1}{2}} \left(\int_{\partial B(0,r_j)} \left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)^{\frac{1}{2}} \\
& + C_5 \int_{B(0,r_j)} W(u)dy \\
\text{using (16) and that } W \geq 0 \leq & C_6 r_j^{\frac{n-1}{2}} C_4^{\frac{1}{2}} (\ln r_j)^{\frac{k}{2}-\frac{1}{4}} r_j^{\frac{n-3}{2}} + C_5 \frac{C_4}{2}(\ln r_j)^{k-\frac{1}{2}} r_j^{n-2} \\
\leq & C_7 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\end{aligned} \tag{19}
$$

for some constants $C_6, C_7 > 0$ and all $j \gg 1$.

In order to proceed, we consider the "bad" sets

$$\mathcal{A}_j = \{x \in B(0, r_j) \ : \ |u(x) - a_i| > \delta, \quad i = 1, \cdots, N\},$$

wherein we know nothing about the integrand in the lefthand side of (19). It follows from (8) (without using the nondegeneracy assumption at a_i), (12) and the bound for the integral of $W(u)$ from (16) that the n-dimensional Lebesgue measure of $\mathcal{A}_j$ satisfies

$$\mathcal{H}^n(\mathcal{A}_j) \leq C_8 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},$$

for some constant $C_8 > 0$ and all $j \gg 1$. Therefore, making repeated use of (12), recalling (17) and that $|A|^2 = \mathrm{tr}[AA^\top]$ for any matrix $A \in \mathbb{R}^{m\times n}$, we obtain that

$$\begin{aligned}
&\int_{B(0,r_j)} \mathrm{tr}\left[(\nabla u)\left(F_u(u)\nabla u\right)^\top\right] dy \\
&= \int_{B(0,r_j)\setminus\mathcal{A}_j} \mathrm{tr}\left[(\nabla u)\left(F_u(u)\nabla u\right)^\top\right] dy + \int_{\mathcal{A}_j} \mathrm{tr}\left[(\nabla u)\left(F_u(u)\nabla u\right)^\top\right] dy \\
&\geq \int_{B(0,r_j)\setminus\mathcal{A}_j} |\nabla u|^2 dy - C_9 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2} \\
&= \int_{B(0,r_j)} |\nabla u|^2 dy - \int_{\mathcal{A}_j} |\nabla u|^2 dy - C_9 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2} \\
&\geq \int_{B(0,r_j)} |\nabla u|^2 dy - C_{10} (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\end{aligned}$$

for some constants $C_9,\ C_{10} > 0$ and all $j \gg 1$. Hence, it follows from (19) that

$$\int_{B(0,r_j)} |\nabla u|^2 dy \leq C_{11} (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},$$

for some constant $C_{11} > 0$ and all $j \gg 1$.

By combining the above relation with the bound for the integral of $W(u)$ from (16), we arrive at

$$\tilde{E}(u, 0, r_j) \leq C_{12} (\ln r_j)^{k-\frac{1}{2}},$$

for some constant $C_{12} > 0$ and all $j \gg 1$. We have thus reduced the exponent in (13) by $1/2$ (for a new sequence $r_j \to \infty$). Iterating this scheme a finite number of times, we infer that

$$\tilde{E}(u, 0, s_j) \to 0 \quad \text{as} \quad j \to \infty,$$

for some sequence $s_j \to \infty$.

On the other hand, the weak monotonicity formula (5) (recall also (11)) implies that, given $r > 0$, we have

$$\frac{1}{r^{n-2}} \int_{B(0,r)} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} dy \leq \tilde{E}(u, 0, s_j)$$

for sufficiently large j. Hence, letting $j \to \infty$ in the above relation and keeping in mind (8), we conclude that $u \equiv a_i$, for some $i \in \{1, \cdots, N\}$, contradicting the assumption that u is nonconstant. Therefore the theorem is proved. □

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